MATH 210: Homework 11

Due on November 1, 2022 at 11:59pm

Professor Smith 1:00pm

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Problem 1

$U = \{0 \leq z \leq 36 - x^2 - y^2\}$, $x = r\cos(\theta)$, $y = r\sin(\theta)$

$$\int_0^{2\pi}\int_0^6\int_0^{36-r^2} r\ \mathrm{d}z\mathrm{d}r\mathrm{d}\theta$$

Solution

$$\int_0^{36-r^2} r\ \mathrm{d}z = rz\Big|_0^{36-r^2} = 36r - r^3$$

$$\int_0^6 36r - r^3\ \mathrm{d}r = 18r^2 - \frac{r^4}{4}\Big|_0^6 = 648 - 324 = 324$$

$$\int_0^{2\pi} 324\ \mathrm{d}\theta = 324\theta\Big|_0^{2\pi} = \boxed{648\pi}$$

Problem 2

$U = \{0 \leq x^2 + y^2 \leq 1, 0 \leq z \leq 5 - x - y\}$

$$\int_0^{2\pi}\int_0^1\int_0^{5-r\cos(\theta)-r\sin(\theta)} r\ \mathrm{d}z\mathrm{d}r\mathrm{d}\theta$$

Solution

$$\int_0^{5-r\cos(\theta)-r\sin(\theta)} r\ \mathrm{d}z = rz\Big|_0^{5-r\cos(\theta)-r\sin(\theta)} = 5r - r^2\cos(\theta) - r^2\sin(\theta)$$

$$\int_0^1 5r - r^2\cos(\theta) - r^2\sin(\theta)\ \mathrm{d}r = \frac{5r^2}{2} - \frac{r^2}{3}\cos(\theta) - \frac{r^2}{3}\sin(\theta)\Big|_0^1$$

$$\int_0^{2\pi} \frac{5}{2} - \frac{\cos(\theta)}{3} - \frac{\sin(\theta)}{3}\ \mathrm{d}\theta = 5\pi - 0 - 0 = \boxed{5\pi}$$

Problem 3

$0 \leq \rho \leq 3$, $0 \leq \theta \leq \frac{\pi}{2}$, $0 \leq \phi \leq \frac{\pi}{2}$, $y = \rho \sin(\theta) \sin(\phi)$, 1st octant.

$$\iiint_V y \ \mathrm{d}V$$

Solution

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^3 \rho \sin(\theta) \sin(\phi) \rho^2 \sin(\phi) \ \mathrm{d}\rho \mathrm{d}\phi \mathrm{d}\theta = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^3 \sin^2(\phi) \sin(\theta) \rho^3 \ \mathrm{d}\rho \mathrm{d}\phi \mathrm{d}\theta$$

$$\int \rho^3 \ \mathrm{d}\rho = \left. \frac{\rho^4}{4} \right|_0^3 = \frac{81}{4}$$

$$\frac{81}{4} \cdot \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{\sin^2(\phi) \sin(\theta)}{4} \ \mathrm{d}\phi \mathrm{d}\theta = \int \sin^2(\phi) \ \mathrm{d}\phi$$

$$\frac{81 \sin(\theta) \phi}{8} - \frac{81 \sin(\theta) \sin(2\phi)}{16} = \left. \frac{81 \sin(\theta) \left(\phi - \frac{\sin(2\phi)}{2} \right)}{8} \right|_{\frac{\pi}{2}}^{0}$$

$$= \int_0^{\frac{\pi}{2}} \frac{81\pi \sin(\theta)}{16} \ \mathrm{d}\theta = -\cos(\theta)$$

$$-\left. \frac{81\pi \cos(\theta)}{16} \right|_0^{\frac{\pi}{2}} = 0 - \left(-\frac{81\pi}{16} \right) = \boxed{\frac{81\pi}{16}}$$