CS 401: Homework 4

Due on March 25, 2024 at 11:59pm

Professor Sidiropoulos 9:30am

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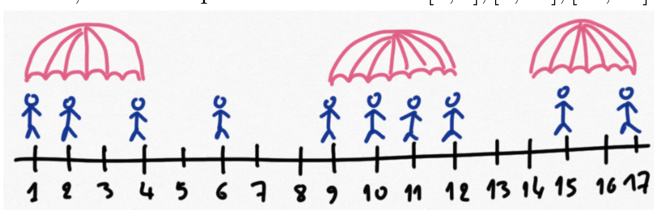
Problem 1: A day at a crowded beach.

You are back at the beach, but now it is too crowded. You have only k umbrellas. You run your optimal greedy algorithm from the previous homework, and you realize that the minimum number of umbrellas needed to cover everyone is larger than k. You decide to try to cover as many people as possible with the k umbrellas you have available.

Formal description: You are given $x_1, \ldots, x_n \in \mathbb{Z}$, and some $k, L \in \mathbb{N}$. You want to find a collection of intervals $I_1, \ldots, I_k \subset \mathbb{R}$, each of length L, such that the number of points in $x_1, \ldots, x_n$ that fall inside $I_1 \cup \cdots \cup I_k$ is maximized. Describe a polynomial-time algorithm for this problem. That is, the running time of your algorithm should be at most polynomial in n. Prove that your algorithm is correct and that it runs in polynomial time.

For example, if the input is $x_1 = 1,\ x_2 = 3,\ x_3 = 4,\ x_4 = 6,\ x_5 = 9, x_6 = 10,\ x_7 = 11,\ x_8 = 12,\ x_9 = 15,\ x_{10} = 17,\ k = 3$, and $L = 3$, then an optimum solution is $[1, 4], [9, 12], [14, 17]$.

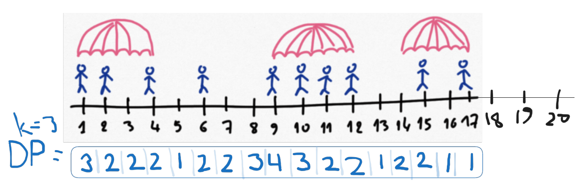


Solution
High Level Algorithm Explanation

I will employ an array DP representing the potential amount of people that will be covered if you were to place an umbrella starting there, that is DP is a table representing the beach and is of size x_n. For example the given scenario would make a DP table of size 17. My algorithm is as follows:

1. Create the DP table of size x_n.
2. Do the following until we go past x_n's index value, start $i = 1$:
   (a) For the current ith beach element, check the interval $[i, i + L]$, count the number of people that can be covered if an umbrella was placed in that interval, insert count into DP$[i]$. The beach should be of size $x_n + L$ to avoid index runtime errors.

   DP should look like this, (using the given scenario):



3. Do the following until $k = 0$:
   (a) Find the current max count in DP, get the index m of where that is in DP.
   (b) Include the interval $[m, m + L]$ in the optimum solution set.
   (c) Do the following for each person convered in the range $[m,\ m + L]$:
      i. Get the current person's index p that is covered by this maximal umbrella.
      ii. Subtract 1 from DP$[p]$ to DP$[p - L]$. (Make sure to not go out of bounds)
      For example 1 person would subtract 1 from $L + 1$ count elements in DP.
   (d) Subtract 1 from k.

Time Complexity Justification

This algorithm has a couple steps, many involving k and L, which change depending on what the user inputs. Let's analyze this algorithm.

Since the beach isn't given to us, we must place each x_i person onto some beach object. This takes $O(n)$ time, where n is the value x_n.

We then create the DP table values by first iterating through the entire beach. Then for each beach i we check $L+1$ positions, and set our DP[i] array value. This takes $O(nL+1) = O(nL)$ time.

After constructing our DP table, we then find the k optimum intervals using a while loop. Inside the while loop we find the max element's index m which will take n time to do. Finally we then check the maximal umbrella's interval $[m, m+L]$, this takes $L+1$ time. For each beach element in that interval we do an inner loop to decrement the previous L DP elements, including the current one, this takes $L+1$ time. All together this portion of the algorithm is $O\left((L+1)^2k\right) = O\left(L^2k\right)$ time.

Combining every together we are left with the following:

$$T(n) = O\left(L^2k + nL + n\right)$$
$$T(n) = O\left(L^2k + nL\right)$$

Assuming L and k must be $\leq n$ (the beach size), this makes our time complexity the following:

$$T(n) = O\left(n^3 + n^2\right)$$
$$T(n) = \boxed{O\left(n^3\right)}$$

Correctness Justification

The problems is asking us to find the intervals that are most optimal for placing an umbrella. My algorithm creates a DP table representing the covered people count if an umbrella was placed from that ith position up to $i+L$. We then select the highest DP index that has the most people covered. Afterwards we update the DP table by essentially making it like the people we just covered don't exist in the DP table, that is the other counts around will change, and this will allow us to pick the next best interval based off the DP table.

Because k, L, and the input people indices must be finite, I can safely assume my algorithm will terminate. The Algorithm selects the best element it can see and does this k times, which aftwards it will terminate. The algorithm checks iterates at most n, k, and L times in combination with the time complexity I listed above. Due to the finite nature of these values, my algorithm will terminate and send the correct intervals as stated for the reasons in my first paragraph.

Problem 1 continued on next page...

Real Code Implementation

```cpp
// g++ -std=c++23 -O2 -Wall P1.cpp -o P1.exe
#include <iostream>
#include <cstdint>
#include <vector>

struct BeachPoint { bool covered = false, person = false; };
std::vector<std::pair<int32_t, int32_t>> ProblemOne(
    const std::vector<int32_t>& xIndices,
    const int32_t L,
    const int32_t n,
    int32_t k)
{
    std::vector<std::pair<int32_t, int32_t>> intervals;
    std::vector<BeachPoint> beach(n + L);  // Ensure enough space.
    std::vector<int32_t> DP(n);
    for (const int32_t& i : xIndices) {
        beach[i].person = true;
    }
    for (int32_t i = 1; i < n; ++i)
    {
        int32_t ithCount = 0;
        for (int32_t j = i; j < i+L+1; ++j) {
            if (beach[j].person) { ithCount++; }
        }
        DP[i] = ithCount;
    }
    while (k--)
    {   // Find max element's index:
        int32_t maxCount = INT32_MIN, m = -1;
        for (int32_t i = 1; i < n; ++i) {
            if (DP[i] > maxCount) {
                maxCount = DP[i];
                m = i;
            }
        }
        intervals.push_back(std::make_pair(m, m+L));
        for (int32_t i = m; i <= m+L; ++i) {
            // Don't decrement [i, i-L] if empty:
            if (!beach[i].person || beach[i].covered) { continue; }
            beach[i].covered = true;
            for (int32_t j = i; j && j >= i-L; --j) {
                DP[j]--;
            }
        }
    }
    return intervals;
}
```

```cpp
int main()
{
    const std::vector<int32_t> inputIndices = {1, 2, 4, 6, 9, 10, 11, 12, 15, 17};
    if (inputIndices.empty()) {
        std::__throw_invalid_argument("There must be at least one person.");
    }
    const int32_t L = 3, k = 3, xn = inputIndices[inputIndices.size() - 1];
    const auto outputIntervals = ProblemOne(inputIndices, L, xn+1, k); // +1: 0 index.
    for (const auto& interval : outputIntervals) {
        std::cout << '[' << interval.first << ", " << interval.second << "]\n";
    }
    return 0;
}
```

Problem 2: Independent Set.

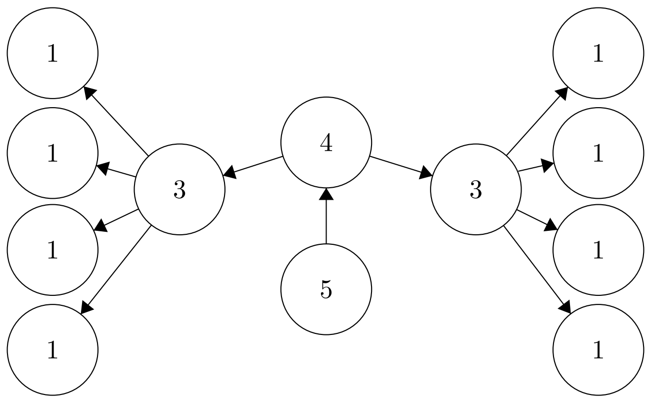
Let $G = (V, E)$ be an undirected graph with n nodes. Suppose that each node $v \in V$ has some integer weight $w(v) \geq 0$. A subset of the nodes is called an independent set if no two of them are joined by an edge.

The goal of this problem is to find an independent set whose total weight is as large as possible.

Part A

Show that the following "heaviest first" algorithm does not always find the maximum weight independent set: While there are nodes in G, add the heaviest remaining node to the independent set and delete it and its neighbors from G.

Solution



A counter example would be as follows, given the graph G in the image I provided, the algorithm would pick 5 first, then remove 4 and itself from G. The algorithm would then pick one of the 3 weight nodes, add it to the total, then remove the 1s connected to it. It would then do the same for the other 3 and 1s. In total the algorithm would return 11 with the set $\{5, 3, 3\} = 11$. But a better total would be 12 with the set $\{4, 1, 1, 1, 1, 1, 1, 1, 1\} = 12$

Part B

Recall that a graph $G = (V, E)$ is a path if its nodes can be written as $v_1, v_2, \ldots, v_n$, with an edge between v_i and v_j if and only if $|i - j| = 1$. Design a dynamic programming algorithm that takes an n-node path G with weights and returns an independent set of maximum weight. Show that your algorithm is correct and that it runs in polynomial time.

Problem 2 continued on next page. . .

Solution

High Level Algorithm Explanation

1. Create an include and exclude array, where each array's ith element shows the potential maximum weight if we include or exclude the ith node from the maximal independent set.
2. Set the 1st element of include to the weight of the 1st node in G, set the 1st element of exclude to 0.
3. For each node i from the 2nd node to the last node in G:
   (a) include[i] is calculated by adding G[i] to the max of the last 2 excluded node weights, this will give us the weight if we include the node at index i in G, checking the excluded previous adjacent node weights will give us the included ith weight.
   (b) exclude[i] is calculated by adding the max of the last include/exclude weights. This will place the max weight in the exclude array, which will be used in the future exclude/include calculations, it represents the max weight if we exclude the current node.
4. From $i =$ the number of nodes in G to the first node (reverse through include / exclude):
   (a) If the include[i] value is $\geq$ to exclude[i], this indicates that the current node yields a higher total weight rather than excluding it, add to the maximal independent set, then skip the next node to retain the independent set property.

Time Complexity Justification

Initialize the two include and exclude arrays, this takes $O(n)$ time, where n is the number of nodes in G.

The algorithm iterates over each node in G to calculate the maximal weights from the include and exclude arrays, this will take $O(n)$ time.

The construction of the returned set requires us to iterate n times to check each ith element of include and exclude, this will take $O(n)$ time.

Putting everything together we get the following:

$$T(n) = O(n + n + n)$$
$$T(n) = \boxed{O(n)}$$

This is polynomial to the size of the input graph G.

Correctness Justification

This algorithm utilizes dynamic programming to compute two arrays: include and exclude. The include array represents the max weight if we include or exclude the current node's weight. At each node i, the algorithm will consider two choices, including or excluding node i from the returned set.

Each ith weight in include is calculated by considering the weight of node i from G plus the maximum of the previous two nodes that exclude their adjacent node weights.

Each ith weight in exclude is calculated by considering the maximum weight independent set of the previous node, whether it includes or excludes the previous node. All together this will give running weight sum if we include or exclude each node in the returned set of nodes.

The algorithm then iterates from the back of the include and exclude arrays. The algorithm adds a node to the returned set if include[i] yields a higher maximal weight if we include it rather than exclude it. It then makes sure to skip the next node in this reverse iteration, keeping our independent node property.

Since the algorithm iterates over the nodes in reverse order, it ensures that adjacent nodes are not included simultaneously in the maximum weight independent set, satisfying the independence property. By selecting nodes greedily based on the include and exclude values, the algorithm ensures that it maximizes the total weight of the independent set.

Problem 2 continued on next page. . .

Real Code Implementation

```cpp
// g++ -std=c++23 -O2 -Wall P2B.cpp -o P2B.exe
#include <algorithm>
#include <iostream>
#include <cstdint>
#include <vector>

std::vector<int32_t> ProblemTwoB(const std::vector<int32_t>& G)
{
    int32_t n = G.size();
    std::vector<int32_t> include(n, 0), exclude(n, 0);
    std::vector<int32_t> independentSet;

    // Base case:
    include[0] = G[0];
    exclude[0] = 0;

    // Preprocess index 1:
    if (n > 1) {
        include[1] = G[1];
        exclude[1] = std::max(G[0], 0);
    }

    // Calculate total weights if we include/exclude each node:
    for (int32_t i = 2; i < n; ++i) {
        include[i] = G[i] + std::max(exclude[i-1], exclude[i-2]);
        exclude[i] = std::max(include[i-1], exclude[i-1]);
    }

    int32_t i = n - 1;
    while (i >= 0)
    {
        if (include[i] >= exclude[i]) {
            independentSet.push_back(i);
            i -= 2;
        } else { i--; }
    }
    return independentSet;
}
```

```cpp
int main()
{
    std::vector<int32_t> G = {
        1, 3, 5, 20, 7, 190, 2000, 1800, -100, -2036, 1
    };
    if (G.empty()) { return -1; }
    std::vector<int32_t> maxIndependentSet = ProblemTwoB(G);

    // Print the index in G and the corresponding weight:
    std::cout << "Maximal independent set: ";
    int32_t count = 0;
    for (int32_t i = maxIndependentSet.size() - 1; i >= 0; --i)
    {
        std::cout << '\n' << maxIndependentSet[i] << ": " << G[maxIndependentSet[i]];
        count += G[maxIndependentSet[i]];
    }
    std::cout << "\nTotal weight: " << count << std::endl;
    return 0;
}
```

Part C

Generalize your algorithm from part (b) to the case where G is an arbitrary tree (that is, G is not necessarily a path). Show that your algorithm is correct and that it runs in polynomial time.

Solution

Tree Generalization

We should keep our include and exclude arrays (as ⟨TreeNode*, List⟩ maps), each with n elements, where n is the number of nodes in tree G.
Set the initial values of include to the root node's value, exclude to 0.

1. Traverse from the root node in a depth-first manner (To map values first).
2. At each node z compute the include and exclude values based on the child nodes of z.
   (a) For the include map value of z, we must consider including z by recursively adding each child's exclude value to the value of z.
   (b) For the exclude map value we will max the include/exclude value of each child to z. Make sure to use a hashmap to retain the values.
   (c) Return up the call chain.
3. Now we must construct our independent set, starting from the root node if the include of the current node i is $\geq$ to exclude[i], then add the node to the returned set, if include was $<$ than exclude, then don't add z to the independent set.
4. Recursively check each child of z

Time Complexity Justification

My algorithm needs to traverse once to create the include and exclude maps, this will take $O(n)$ time, where n is the number of nodes in tree G.

Constructing the maximal independent set requires us to check the include and exclude values of every node, this will take $O(n)$ time.

All together our algorithm's time complexity is as follows:

$$T(n) = O(n + n)$$
$$T(n) = \boxed{O(n)}$$

This is polynomial to the size of the input tree G.

Correctness Justification

The algorithm follows a similar approach as in the path case but applies it to an arbitrary tree structure.

We consider the options for including or excluding each node z in the tree and selecting the optimal combination using the same logic and containers as in the path case. At each layer we must check whether or not that node will yield us a higher maximal weight over excluding it from the returned set.

Real Code Implementation

```cpp
// g++ -std=c++23 -O2 -Wall P2C.cpp -o P2C.exe
#include <unordered_map>
#include <algorithm>
#include <iostream>
#include <cstdint>
#include <vector>

struct TreeNode
{
    TreeNode(int32_t w) : weight(w) {}
    std::vector<TreeNode*> children;
    int32_t weight;
};

void setIncludeExclude(
    std::unordered_map<TreeNode*, int32_t>& include,
    std::unordered_map<TreeNode*, int32_t>& exclude,
    TreeNode* node)
{   // Base cases:
    include[node] = node->weight;
    exclude[node] = 0;
    // Dynamically create node's include/exclude values:
    for (TreeNode* child : node->children)
    {   // Get children values first:
        setIncludeExclude(include, exclude, child);

        // include = exclude layer below it
        include[node] += exclude[child];
        // exclude = max(include[c], exclude[c]) (either or)
        exclude[node] += std::max(include[child], exclude[child]);
    }
}

void construct(
    std::unordered_map<TreeNode*, int32_t>& include,
    std::unordered_map<TreeNode*, int32_t>& exclude,
    std::vector<TreeNode*>& independentSet,
    TreeNode* node)
{
    if (include[node] >= exclude[node]) {
        independentSet.push_back(node);
    }
    for (TreeNode* child : node->children) {
        construct(include, exclude, independentSet, child);
    }
}
```

```
std::vector<TreeNode*> ProblemTwoC(TreeNode* root)
{
    std::unordered_map<TreeNode*, int32_t> include, exclude;

    // Dynamically get the include and exclude values:
    setIncludeExclude(include, exclude, root);

    // Construct maximum weight independent set:
    std::vector<TreeNode*> independentSet;
    construct(include, exclude, independentSet, root);

    return independentSet;
}

int main()
{
    TreeNode* root  = new TreeNode(1111);
    TreeNode* node1 = new TreeNode(-10);
    TreeNode* node2 = new TreeNode(1535);
    TreeNode* node3 = new TreeNode(69);
    TreeNode* node4 = new TreeNode(420);

    root->children  = {node1, node2};
    node1->children = {node3, node4};

    std::vector<TreeNode*> result = ProblemTwoC(root);
    int32_t count = 0;
    std::cout << "Nodes in the maximum weight independent set:" << std::endl;
    for (TreeNode* node : result) {
        std::cout << node->weight << " ";
        count += node->weight;
    }
    std::cout << "\nMax weight: " << count << '\n';
    return 0;
}
```

Problem 3: Counting paths.

Let $G = (V, E)$ be a directed graph with no self-loops. Let $s, t \in V$ be distinct vertices.

Part A

Design an algorithm that decides whether there exist infinitely many paths from s to t in G. Note that a path is allowed to visit the same vertex multiple times. Show that your algorithm is correct and that it runs in polynomial time.

Solution

High Level Algorithm Explanation

1. I will create two sets that contain the visited vertices, and the current vertices on the current path.
2. My algorithm will employ a depth-first search traversal of G starting from vertex s.
3. If a cycle occurs, set a cycle flag to true indicating that a cycle has occured, continue delving deeper, make sure to visit every node at least once.
4. If there are no more nodes to check, check if the cycle flag is set to true then return true if and only if t is present in the visited set, else return false.

Time Complexity Justification

In the worst case, if G is a directed acyclic graph, every vertex and edge must be visted at least once.
This results in a time complexity of:

$$T(V, E) = \boxed{O(V + E)}$$

Given that V is the number of vertices and E is the number of edges in G.
The overall time complexity is polynomial in the size of the input graph G.

Correctness Justification

My algorithm makes use of a well known vertex/edge traversal technique called depth-first search, starting from vertex s. My algorithm maintains a set of visted vertices and the nodes on the current path, this is done to detect cycles. During the recursion if there is a vertex present in the current path, this indicates a cycle has occured, which will set the cycle flag to true. The problem wants us to check if there are infinitely many paths from s to t, this means a cycle between s and t or a cycle before s or a cycle after t. In all cases, because my algorithm makes use of a DFS approach, it will visit every s connected node in the directed graph G.

The algorithm at each recursive step makes sure to backtrack when exploring paths, which helps ensure that each vertex is visted at least once during the DFS traversal. My algorithm handles self loops by checking if that node has been visted before via the path set. My algorithm handles cycles likewise by checking if that node is present in the path set, indicating that our path looped back into itself, indicating a cycle, which indicates infinite paths, if our visited set ever contains vertex t.

Just because there is a cycle in the graph doesn't mean there is infinite paths from s to t, there needs to be at least one valid path from s to t. My algorithm makes use of a cycle flag while traversing and delves into new nodes, this is what allows my algorithm to visit every node, helping us find a single valid path to t. If the cycle flag was set to true, and t is in the visited set, this means we have infinite paths. My algorithm will terminate due to the visited and path set checking if we've seen the vertex and if it's on the current path respectively.

Real Code Implementation

```cpp
// g++ -std=c++23 -O2 -Wall P3A.cpp -o P3A.exe
#include <iostream>
#include <unordered_map>
#include <unordered_set>
#include <vector>

bool ProblemThreeA(
    std::unordered_map<char, std::vector<char>>& G,
    std::unordered_set<char>& visited,
    std::unordered_set<char>& path,
    const char s,
    const char t)
{   // Insert new vertex into containers:
    visited.insert(s);
    path.insert(s);
    bool cycleFound = false;

    // Check if vertex s is in G:
    if (G.find(s) != G.end())
    {   // Retrieve vertex s's neighbors:
        for (const char& neighbor : G[s])
        {
            if (path.find(neighbor) != path.end()    ||    // Cycle detected.
            (visited.find(neighbor) == visited.end() &&    // Not visted, so...
            ProblemThreeA(G, visited, path, neighbor, t))) // Delve deeper.
            {
                cycleFound = true;
            }
        }
    }
    // Backtrack current path and check if there was a cycle AND t has been found:
    path.erase(s);
    return (cycleFound && visited.find(t) != visited.end());
}

int main()
{
    std::unordered_map<char, std::vector<char>> G = {
        {'s', {'a'}},
        {'a', {'y', 'b'}},
        {'b', {'x', 's'}},
        {'t', {'z'}}  // Cycle, but no path to t: False.
    };
    std::unordered_set<char> visited, path;
    const char s = 's', t = 't';
    std::cout << ProblemThreeA(G, visited, path, s, t) << '\n';
    return 0;
}
```

Problem 3 continued on next page. . .

Part B

Design an algorithm that computes the number of paths from s to t in G, assuming there are only finitely many such paths. Note that your algorithm does not have to output the paths. Show that your algorithm is correct and that it runs in polynomial time.

Solution

High Level Algorithm Explanation

1. Create a hashtable called DP, it will have each vertice's number of paths from there to t.
2. Set DP[t] to 1, the value to be returned to each node connected to t.
3. Iterate over every vertex v in G.
   (a) If $v \neq t$, set DP[v]= 0, initialization for v in DP.
   (b) Iterate over each neighboring vertex w from v:
      i. Update the current DP value for v by adding DP[w]'s value to DP[v].
4. Return DP[s].

Time Complexity Justification

Because we are iterating over ever vertex and its edges, the time complexity is as follows:

$$T(V, E) = \boxed{O(V + E)}$$

Given that V is the number of vertices and E is the number of edges in G.
The overall time complexity is polynomial in the size of the input graph G.

Correctness Justification

The algorithm correctly computes the number of paths from s to t in the graph by iteratively updating the DP values. By setting t to 1 initally, the nodes neighboring t will gain 1, then its neighbors will gain the value it has, multiple paths will have intersecting DP values to add, causing the DP value to go up by one for each intersecting neighbor. Returning the value of DP[s] at the end will return the number of paths, that had intersecting nodes on its path towards t. Therefore, the algorithm produces the correct result and runs efficiently, providing an effective solution to the problem. The algorithm will also terminate because the input graph G must be finite.

Real Code Implementation

```cpp
// g++ -std=c++23 -O2 -Wall P3B.cpp -o P3B.exe
#include <unordered_map>
#include <unordered_set>
#include <iostream>
#include <cstdint>
#include <vector>

uint32_t ProblemThreeB(
    std::unordered_map<char, std::vector<char>>& G,
    const char s,
    const char t)
{
    std::unordered_map<char, uint32_t> DP;
    // Base case - t, increment scale factor:
    DP[t] = 1;

    // Iterate over vertices in topological order:
    for (const auto& [v, neighbor] : G)
    {   // Initialization:
        if (v != t) { DP[v] = 0; }
        for (const char& w : neighbor)
        {   // Update dp[v] by adding paths from w to t
            DP[v] += DP[w];
        }
    }
    // Return the number of paths from s to t
    return DP[s];
}

int main()
{
    std::unordered_map<char, std::vector<char>> G = {
        {'s', {'a', 'b'}},
        {'a', {'t'}},
        {'b', {'t'}},
        {'t', {}}
    };
    std::cout << "Number of paths from s to t: " << ProblemThreeB(G, 's', 't') << '\n';
    return 0;
}
```
