MATH 210 :: Homework 15

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November 27, 2022

Problem 1.
$C = [0, 1] \times [0, 1]$, Square.

$$\oint \left(3^{\cos(x)} + 6y^2\right) dx + \left(\sin(5^y) + (6x^3)\right) dy$$

Solution ::

$$\begin{aligned}
\oint_{\partial D} Pdx + Qdy &= \iint_D (Q_x - P_y)\, dA && (1)\\
P &= 3^{\cos(x)} + 6y^2 && (2)\\
P_y &= 12y && (3)\\
Q &= \sin(5^y) + 16x^3 && (4)\\
Q_x &= 48x^2 && (5)\\
\int_0^1 \int_0^1 48x^2 - 12y\, dxdy &= 16x^3 - 12yx \Big|_0^{x=1} && (6)\\
&= 16 - 12y && (7)\\
\int_0^1 16 - 12y\, dy &= 16y - 6y^2 \Big|_0^{y=1} && (8)\\
&= 16 - 6 && (9)\\
&= 6 && (10)
\end{aligned}$$

Problem 2.
Centered at $(4, 0)$, $0 < r < 3$.

$$\oint_C (x-4)^3 dy - y^3 dx$$

Solution ::

$$\oint_{C'} x^3 dy - y^3 dx \tag{11}$$

$$\oint_{\partial D'} P dx' + Q dy' = \iint_{D'} (Q_x - P_y)\, dA' \tag{12}$$

$$P = -y^3 \tag{13}$$

$$P_y = -3y^2 \tag{14}$$

$$Q = x^3 \tag{15}$$

$$Q_x = 3x^2 \tag{16}$$

$$\iint_{D'} 3x^2 - (-3y^2) dA' = \iint_{D'} 3x^2 + 3y^2 dA' \tag{17}$$

$$\tag{18}$$

Convert to polar coordinates:

$$\int_0^{2\pi} \int_0^3 3r^2 r\, dr d\theta = \frac{3r^4}{4}\bigg|_0^{r=3} = \frac{243}{4} \tag{19}$$

$$\int_0^{2\pi} \frac{243}{4} d\theta = \frac{243\theta}{4}\bigg|_0^{\theta=2\pi} \tag{20}$$

$$= \frac{243(2\pi)}{4} \tag{21}$$

$$= \frac{243\pi}{2} \tag{22}$$

Problem 3.
Ellipse:

$$\frac{x^2}{1} + \frac{y^2}{16} = 1$$

Solution ::

Use:

$$\text{area}(D) = \frac{1}{2} \oint_C x\,dy - y\,dx$$

Simply:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Let:

$$x = a\cos(\theta) \tag{23}$$

$$dx = -a\sin(\theta)\,d\theta \tag{24}$$

$$y = b\sin(\theta) \tag{25}$$

$$dy = b\cos(\theta)\,d\theta \tag{26}$$

So:

$$\text{area}(D) = \frac{1}{2} \oint_C (a\cos(\theta))(b\cos(\theta)\,d\theta) - (b\sin(\theta))(-a\sin(\theta)\,d\theta) \tag{27}$$

$$= \frac{1}{2}ab \cdot \int_0^{2\pi} (\cos^2(\theta) + \sin^2(\theta))\,d\theta \tag{28}$$

$$= \frac{1}{2}ab \cdot \int_0^{2\pi} 1\,d\theta = \frac{1}{2}ab \cdot \theta \Big|_0^{2\pi} \tag{29}$$

$$= \frac{1}{2}ab(2\pi) \tag{30}$$

$$= \pi ab \tag{31}$$

So:

$$a = \sqrt{1} = 1 \tag{32}$$

$$b = \sqrt{16} = 4 \tag{33}$$

$$1 \cdot 4 \cdot \pi = 4\pi \tag{34}$$