MATH 210: Homework 12

Due on November 2, 2024 at 8:00am

Professor Smith 1:00pm

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Problem 1

$0 \leq \rho \leq R$, $0 \leq \theta \leq 2\pi$, $0 \leq \phi \leq \pi$

$$\int_0^{\pi}\int_0^{2\pi}\int_0^{R} \rho^2 \sin(\phi)\ \mathrm{d}\rho\mathrm{d}\theta\mathrm{d}\phi$$

Solution

$$\int_0^{R} \rho^2\ \mathrm{d}\rho = \frac{\rho^3}{3}\bigg|_0^R = \int_0^{2\pi} \frac{R^3}{3}\sin(\phi)\ \mathrm{d}\theta = \frac{2\pi R^3\sin(\phi)}{3}$$

$$\begin{aligned}\frac{2\pi R^3}{3}\cdot\int_0^{\pi}\sin(\phi)\ \mathrm{d}\phi &= -\frac{2\pi R^2\cos(\phi)}{3}\bigg|_0^{\phi=\pi} \\ &= -\frac{2\pi R^3(-1)}{3} + \frac{2\pi R^3(1)}{3} \\ &= \boxed{\frac{4\pi R^3}{3}}\end{aligned}$$

Problem 2

$0 \leq \rho \leq 2$, $0 \leq \theta \leq 2\pi$, $0 \leq \phi \leq \pi/6$

$$\int_0^{\frac{\pi}{6}}\int_0^{2\pi}\int_0^{2} \rho^2 \sin(\phi)\ \mathrm{d}\rho\mathrm{d}\theta\mathrm{d}\phi$$

Solution

$$\int_0^{2} \rho^2\ \mathrm{d}\rho = \frac{\rho^3}{3}\bigg|_0^2 = \frac{8}{3}$$

$$\int_0^{2\pi} 1\ \mathrm{d}\theta = 2\pi$$

$$\int_0^{\frac{\pi}{6}} \frac{16\pi\sin(\phi)}{3}\ \mathrm{d}\phi = \frac{16\pi}{3}\cdot\int\sin(\phi)\ \mathrm{d}\phi = \cos(\pi/6) = \frac{\sqrt{3}}{2}$$

$$\begin{aligned}\frac{16\pi\cos(\phi)}{3}\bigg|_0^{\phi=\pi/6} &= -\frac{16\pi\frac{\sqrt{3}}{2} + 16\pi}{3} \\ &= \boxed{-\frac{8\sqrt{3}\pi + 16\pi}{3}}\end{aligned}$$

Problem 3

$x = \frac{u}{v+5}$, $y = \frac{uv}{v+5}$

$$\iint_D 5x + y \text{ d}A$$

Solution

$5x + y = 3$, $5x + y = 6$, $y = x$, $y = 2x$

$$(5x + y = 3) = \left(5\left(\frac{u}{v+5}\right) + \frac{uv}{v+5} = 3\right)$$
$$\frac{u(5+v)}{v+5} = 3 \text{ or } u = 3$$
$$5x + y = 6,\ u = 6$$
$$y = x : \frac{uv}{v+5} = \frac{u}{v+5},\ uv = u,\ v = 1$$
$$y = 2x : \left(\frac{uv}{v+5} = 2\left(\frac{u}{v+5}\right)\right) = \frac{uv}{u} = \frac{2u}{u},\ v = 2$$
$$u = 3,\ u = 6,\ v = 1,\ v = 2,\ [3,6] \times [1,2]$$
$$\frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \cdot \frac{\partial y}{\partial u}$$
$$\begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} = \begin{bmatrix} \frac{1}{v+5} & \frac{-u}{(v+5)^2} \\ \frac{v}{v+5} & \frac{5u}{(v+5)^2} \end{bmatrix}$$

$$= \left(\frac{1}{v+5} \cdot \frac{5u}{(v+5)^2}\right) - \left(\frac{-u}{(v+5)^2} \cdot \frac{v}{v+5}\right)$$
$$= \frac{5u}{(v+5)^3} - \frac{-uv}{(v+5)^3}$$
$$= \frac{5u + uv}{(v+5)^3}$$
$$\text{Jacobian} = \frac{u}{(v+5)^2}$$
$$5x + y = 3 \text{ or } u$$
$$\int_1^2 \int_3^6 \frac{u^2}{(v+5)^2} \text{ d}u\text{d}v = \int_3^6 u^2 \text{ d}u \cdot \int_1^2 \frac{1}{(v+5)^2} \text{ d}v$$
$$\int_3^6 u^2 \text{ d}u = \frac{u^3}{3}\bigg|_3^6 = 72 - 9 = 63$$
$$\int_1^2 \frac{1}{(v+5)^2} \text{ d}v = \frac{-1}{v+5}\bigg|_1^2 = \frac{1}{42}$$
$$63 \cdot \frac{1}{42} = \frac{63}{42} = \boxed{\frac{3}{2}}$$