CS301 :: Midterm 2

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80/80 Score

November 2, 2023

Problem 1. Multiple Choice

Solution ::

a) Which of the following means that a grammar is ambiguous?
There is a string with multiple parse trees in the grammar.

b) Which of the following is not a member of the 7-tuple for a TM?
$F =$ The set of final states.

c) T/F. Deterministic and Non-deterministic Pushdown Automata decided the same same set of languages. False.

d) Which is a valid rule in a right-linear single-step grammar with $a \in \Sigma$ and $A, S \in V$?
$S \to aA$.

e) T/F. All finite languages are context-free.
True.

f) Which of the following languages is not context-free?
$\{0^a 1^b 2^c \mid a \leq b, b \leq c\}$

g) Context-Free languages are not closed which of the following operations?
Intersection ($\cap$).

h) Which of the following rules is not context-free?
$bSb \to bab$.

i) Which is not a possible outcome of computation in a Turing Machine?
Contradiction.

j) T/F. All Turing-recognizable languages are Turing-decidable.
False.

Problem 2. Short Answer

a) What are the elements of the 6-tuple for a PDA?
Solution ::

$$\begin{aligned}
Q &= \text{Set of states.}\\
\Sigma &= \text{Alphabet set of input characters.}\\
\Gamma &= \text{Alphabet set of stack characters.}\\
\delta &= \text{Transition function:}\\
&\quad Q \times (\Sigma \cup \{\epsilon\}) \times (\Gamma \cup \{\epsilon\}) \longrightarrow P(Q \times (\Gamma \cup \{\epsilon\}))\\
q_0 &= \text{Starting state.}\\
&\quad q_0 \in Q\\
F &= \text{Set of final states}\\
&\quad F \subseteq Q
\end{aligned}$$

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b) Given CFGs $G_1 = (V_1, \Sigma, R_1, S_1)$ and $G_2 = (V_2, \Sigma, R_2, S_2)$ that decide languages L_1 and L_2 respectively, describe how to construct a CFG which accepts the concatenation of L_1 and L_2. You may either describe how to construct the CFG that accepts their concatenation, or give the 4-tuple for the CFG that accepts their concatenation.
Solution ::

$$\begin{aligned}
S_0 &= \text{New starting state.}\\
V &= V_1 \cup V_2 \cup \{S_0\}\\
\Sigma &= \Sigma_1 = \Sigma_2\\
R &= R_1 \cup R_2 \cup \{S_0 \longrightarrow S_1 S_2\}\\
S &= S_0
\end{aligned}$$

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c) Give an example of a language that is context-free but not regular.
Solution ::

$$L = \{0^n 1^n \mid n \geq 0\}$$

Problem 3. Context-Free Grammars
Give a Context-free grammar for the following language L.
$\Sigma = \{a, b, c, d\}$

$$L = \{a^i b^j c^k d^k \mid 0 < i < j, k \geq 1\}$$

Solution ::

$$S \longrightarrow aXbcYd \tag{1}$$
$$Y \longrightarrow \epsilon \mid cYd \tag{2}$$
$$X \longrightarrow aXb \mid B \tag{3}$$
$$B \longrightarrow bB \mid b \tag{4}$$

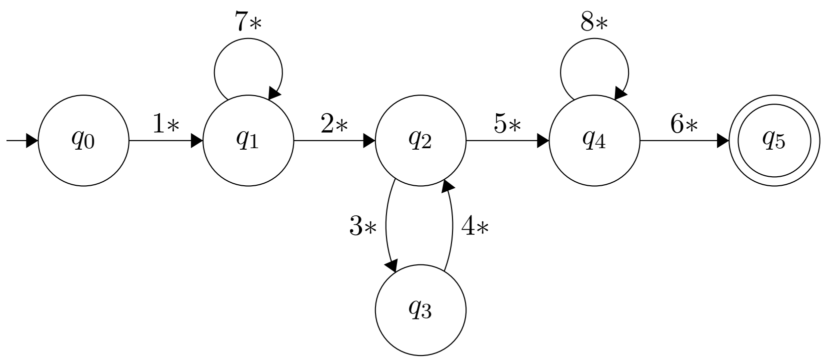
Problem 4. PDA Construction

Create the PDA which decided the following language L.

$\Sigma = \{a, b, c\}$

$$L = \{a^n b^{2n} c^m \mid n \geq 0, m \geq 1\}$$

Solution ::



$$\Gamma = \{\$, A\}$$
$$1* = \epsilon,\ \epsilon \longrightarrow \$$$
$$2* = \epsilon,\ \epsilon \longrightarrow \epsilon$$
$$3* = b,\ A \longrightarrow \epsilon$$
$$4* = b,\ \epsilon \longrightarrow \epsilon$$
$$5* = c,\ \epsilon \longrightarrow \epsilon$$
$$6* = \epsilon,\ \$ \longrightarrow \epsilon$$
$$7* = a,\ \epsilon \longrightarrow A$$
$$8* = c,\ \epsilon \longrightarrow \epsilon$$

Problem 5. Chomsky Normal Form
Convert the following grammar to CNF.

$$\begin{aligned} S &\longrightarrow xAy \mid AB \\ A &\longrightarrow AB \mid xA \\ B &\longrightarrow yB \mid \epsilon \end{aligned}$$

Solution ::

START ::

$$S_0 \longrightarrow S \tag{5}$$
$$S \longrightarrow xAy \mid AB \tag{6}$$
$$A \longrightarrow AB \mid xA \tag{7}$$
$$B \longrightarrow yB \mid \epsilon \tag{8}$$

BIN ::

$$S_0 \longrightarrow S \tag{9}$$
$$S \longrightarrow xS_1 \mid AB \tag{10}$$
$$S_1 \longrightarrow Ay \tag{11}$$
$$A \longrightarrow AB \mid xA \tag{12}$$
$$B \longrightarrow yB \mid \epsilon \tag{13}$$

DEL ::

$$S_0 \longrightarrow S \tag{14}$$
$$S \longrightarrow xS_1 \mid AB \mid A \tag{15}$$
$$S_1 \longrightarrow Ay \tag{16}$$
$$A \longrightarrow AB \mid A \mid xA \tag{17}$$
$$B \longrightarrow yB \mid y \tag{18}$$

UNIT ::

$$S_0 \longrightarrow XS_1 \mid AB \mid xA \tag{19}$$
$$S \longrightarrow XS_1 \mid AB \mid xA \tag{20}$$
$$S_1 \longrightarrow Ay \tag{21}$$
$$A \longrightarrow AB \mid xA \tag{22}$$
$$B \longrightarrow yB \mid y \tag{23}$$

TERM ::

$$S_0 \longrightarrow U_0S_1 \mid AB \mid U_0A \tag{24}$$
$$S \longrightarrow U_0S_1 \mid AB \mid U_0A \tag{25}$$
$$S_1 \longrightarrow AU_1 \tag{26}$$
$$A \longrightarrow AB \mid U_0A \tag{27}$$
$$B \longrightarrow U_1B \mid Y \tag{28}$$
$$U_0 \longrightarrow x \tag{29}$$
$$U_1 \longrightarrow y \tag{30}$$

Problem 6. Non-Context-Free Proof
Prove that the following language L is not context-free.

$$L = \{0^i 1^j 2^i \,|\, j < i\}$$

Solution ::

Suppose for the sake of contradiction, that L is context-free. Then, by definition, there must be a CFG G with pumping length p that generates it.
Let $s = 0^{p+1}1^p2^{p+1}$
By the pumping length, s can be partitioned into u, v, x, y, z such that $|vxy| \leq p$ and $|vy| \geq 1$. We proceed on cases for v and y:

- v/y are contained within the 1s. If we were to set vxy to k 1s for some $k \geq 1$ such that $|vxy| \leq p$. By the pumping lemma if we pump up let's say $i = 10$, we would generate ≥ 10 1s which makes our total 1s atleast $p + 10$ which makes our inequality $j < i$ false. This new string is $\notin L$, a contradiction.

- v/y are contained within only the 0s or 2s. If we were to set vxy to k 0s (or 2s), for some $k \geq 1$ such that $|vxy| \leq p$. By the pumping lemma if we pump up or down i, i.e. $i \neq 1$, $i \geq 0$, the # of 0s would not match the $p + 1$ 2s or vice versa. Similarily this occurs when vxy is contained within 0s and 1s or 1s and 2s, a contradiction.

All cases result in a contradiction of the pumping lemma.
$\therefore L$ is not context-free.