Increasing Convergence of Three-Channel Scattering Amplitudes

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Introduction

Quantum chromodynamics (QCD) is the theory of the strong nuclear force, which is essential to understanding nuclear physics, but it is difficult to predict interactions at low energies. Here, we use Lattice Quantum Chromodynamics as a brute force computational technique, with a finite and discrete spacetime. This is a powerful model to solve three-body scattering problems.

We are using:

- Infinite-Volume (IV) Formalism: an idealized framework where the system spans infinitely in all spatial directions, which allows us to ignore boundary effects and simplify calculations
- Scattering theory: scattering amplitudes can help us predict the behavior of non-colliding particles, and allows for the derivation of perturbative and non-perturbative relationships between observables

Below is the integral equation we solved:

$$d_S^{(u,u)}(p,k;\epsilon,N) = -G_S(p,k;\epsilon) - ig^2\, G_S(p,q;\epsilon)\,\rho_{\varphi b}(E)\, d_S^{(u,u)}(q,k;\epsilon,N)$$
$$-\sum_{n=0}^{N-1} \frac{\Delta k'\, k_{n'}^2}{(2\pi)^2\, \omega_{k_{n'}}} G_S(p,k'_n;\epsilon)\Delta\mathcal{M}_2(k'_n;\epsilon)\, d_S^{(u,u)}(k'_n,k;\epsilon,N)\,.$$

This has been published with a straight contour and an epsilon value proportional to some value k_max, below the 3 body threshold where a 2 body bound state can be created. This poster details some additional studies at a fixed scattering length (a = 2). We 1) solve integral equations and compute scattering amplitudes above the threshold, 2) examine its convergence with various epsilon, and 3) study a relevant cross section for the two body bound state.

Solving Integral Equations

Computing these scattering amplitudes involves solving complex integral equations. We do this by discretizing our integral equation, which yields a linear equation that can be implemented and solved in Python. In IV formalism, deviation from unitarity is quantified by the following:

$$\Delta\rho_{\varphi b}(E;N) \equiv \left| \frac{\mathrm{Im}\left[\mathcal{M}_{\varphi b}^{-1}(E;N)\right] + \rho_{\varphi b}(E)}{\rho_{\varphi b}(E)} \right| \times 100$$

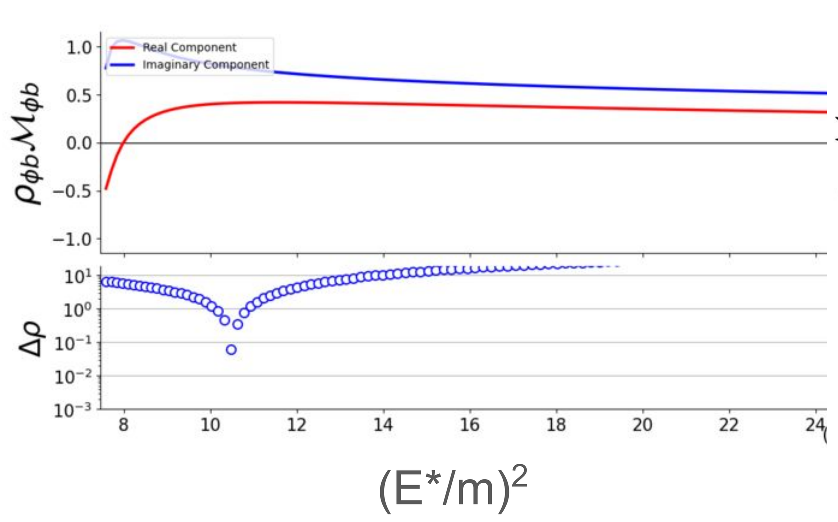
We attempt to drive this term closer and closer to zero as we solve the integral equation. This comes as a percent error from the scattering matrix unitarity condition.

We increase the convergence of previous three-channel scattering amplitudes in the presence of a two-body bound state by:

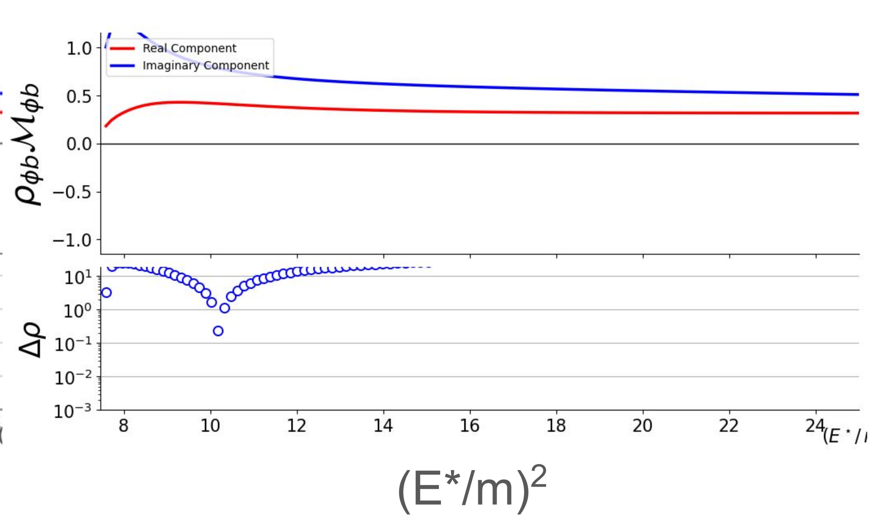
1. Different epsilon values for our iε prescription
2. Implementing various straight contours to avoid poles
3. Comparing the epsilon values for various samplings of the contour to check for convergence

3-Body Scattering Amplitudes above Threshold

Scattering Amplitudes for ϵ = η (kmax proportionality)



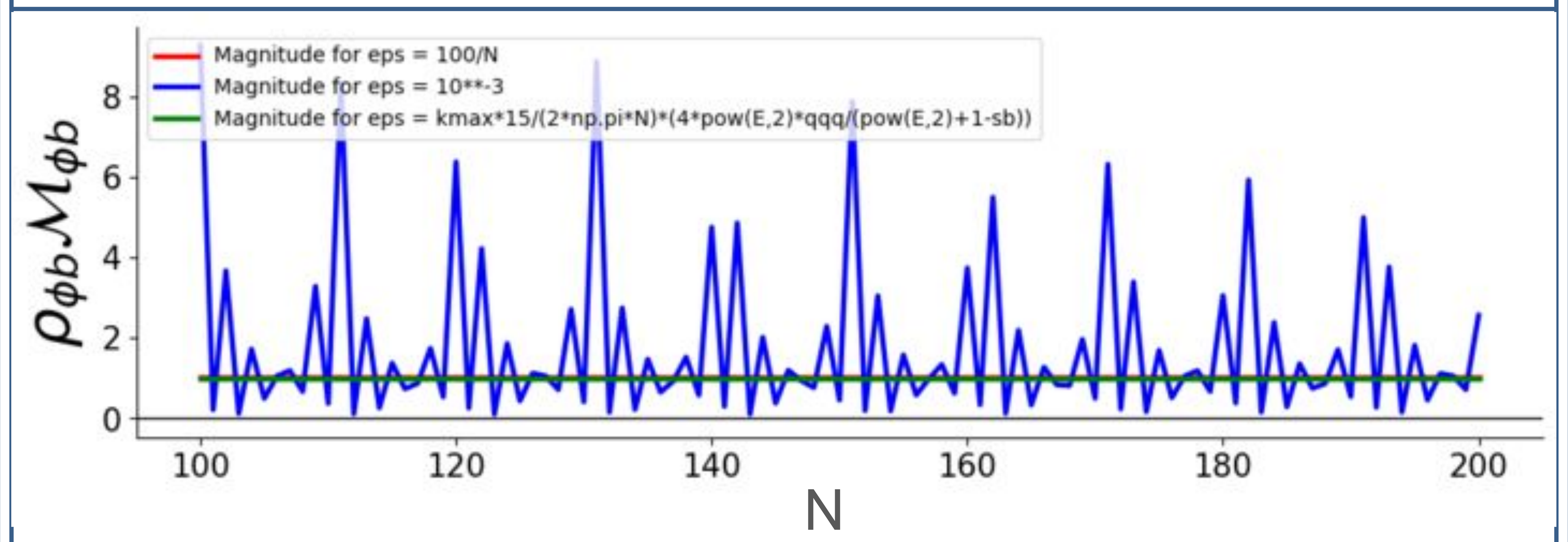
Scattering Amplitudes for ϵ = 100/N



Convergence of Scattering Amplitudes

From the scattering amplitude plots, we can see that the convergence for the eta and 100/N epsilon values are quite similar which is to be expected for well converging epsilons. To directly compare the epsilons, we created a convergence plot below. Here we plotted the scattering amplitude magnitude on the y-axis against the number of contour points sampled (N), for a fixed E = 3m value.

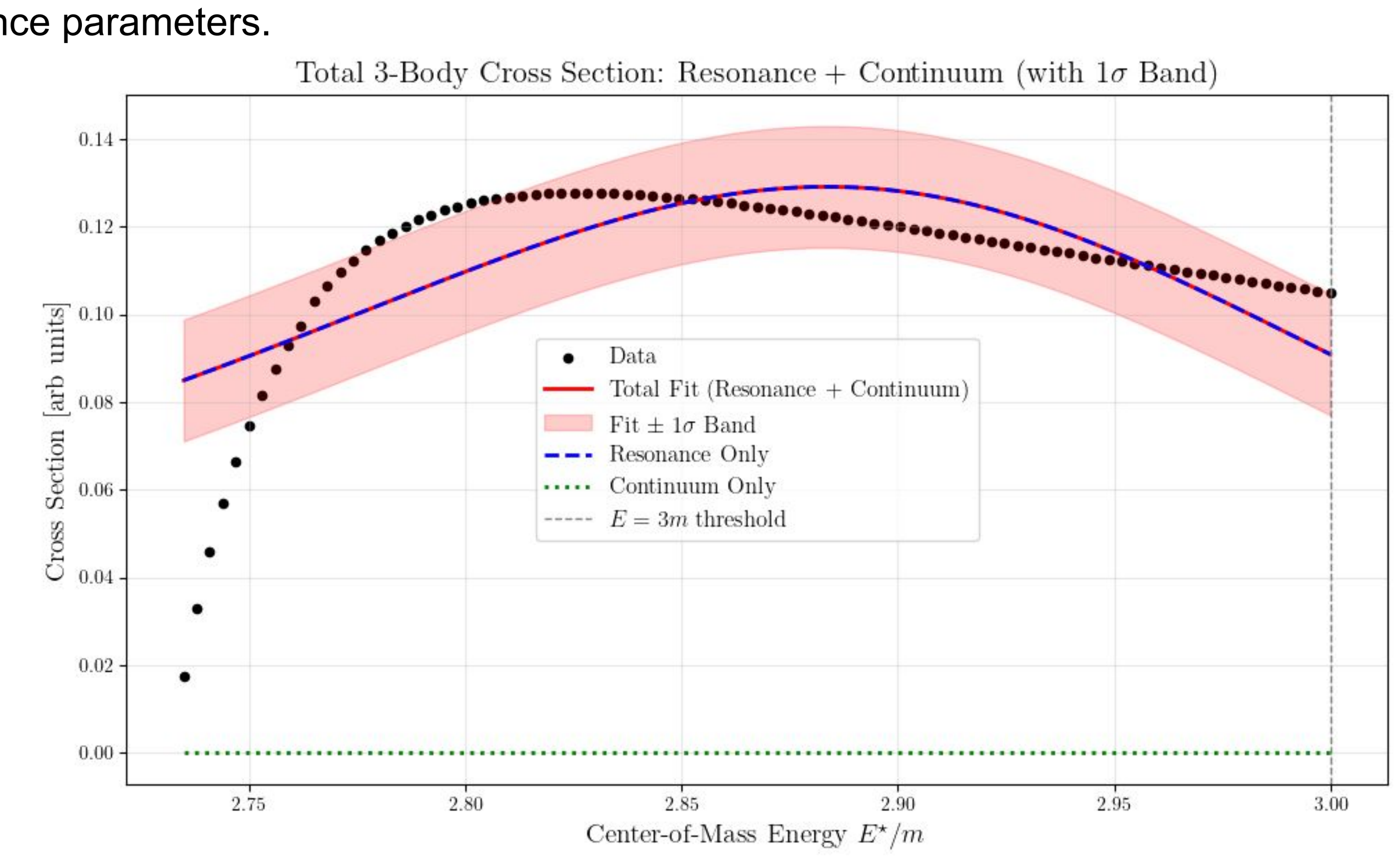
Convergence Plot with 3 different epsilons at E = 3m



For comparison against the 100/N and eta (term proportional to kmax) epsilon values, we also plotted a poorly converging epsilon (10^{-3}). This plot highlights the stable convergence of the 100/N and eta terms, while showing that an arbitrary epsilon that is too small to avoid the pole will not yield good convergence. It is evident with the increase of N that with enough sampling of the contour, even that epsilon may converge with the other two plots. Future studies can focus on verifying this behavior numerically.

Comparative Study

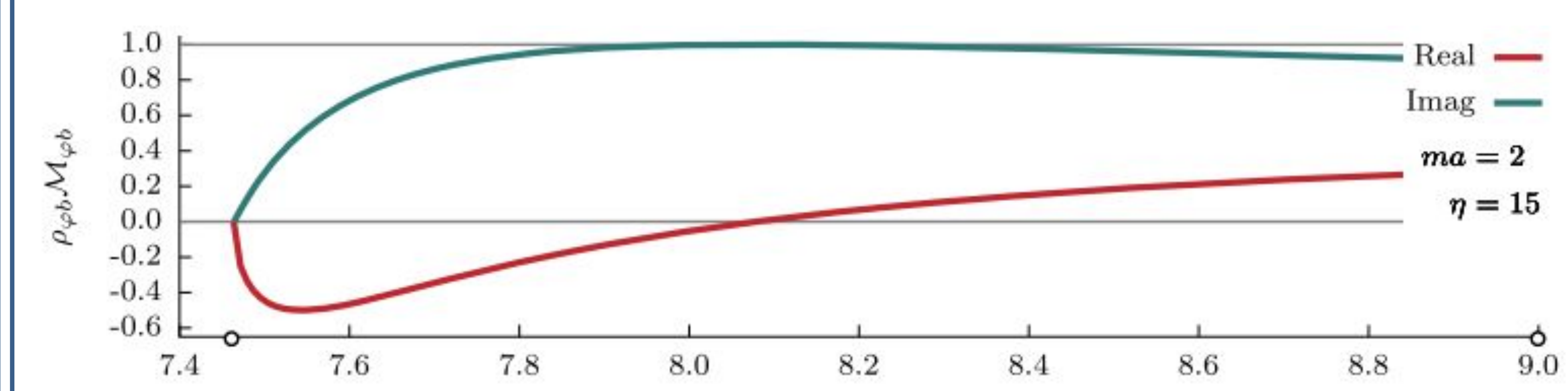
To characterize scattering processes in few-body systems, we extract the three-body cross section as a function of the total center-of-mass energy. Cross sections provide direct access to the strength and structure of interactions, capturing both resonant contributions from intermediate bound states and nonresonant continuum scattering. We model the total cross section as a relativistic Breit-Wigner resonance, including a phase-space suppression factor, combined with a continuum background active above the threshold. Fitting simulated data below the three-body breakup threshold (E*<3m), we isolate the dominant resonance contribution and determine the resonance parameters.



The best-fit peak location is E0≈2.896 with a width Γ≈0.375 m The fit quality is excellent, with χ2/d.o.f≈0, and the extracted model accurately reproduces the observed energy dependence. This analysis demonstrates the extraction of scattering information from real-time amplitudes and motivates future extension to systems with larger continuum effects.

Discussion

In Jackura et al., the integral equation was solved for energies below threshold: that is, E*<3m.



As shown earlier, we computed the amplitudes above this threshold. And studied the 2 body bound state (below E*=3m) in the form of a cross section. These studies are particularly relevant to the study of 3 body amplitudes as a whole: a prime example we give is the study by Briceño et al., that utilizes these infinite volume amplitudes as a baseline for evaluating a quantum computing method for extracting finite-volume amplitudes in a Minkowski signature.

Conclusion

Using Lattice QCD methods, we aimed to increase the convergence of three-channel scattering amplitudes in the presence of a two-body bound state. Through testing with various epsilon values, implementing integration contours, and using discrete approximation methods, we were able to show that the error in our calculations can be driven to near zero.

In our work, we reduced the error by increasing the convergence behavior of three-channel scattering amplitudes. This was achieved by optimizing the epsilon values chosen and by implementing more stable numerical methods. Our analysis demonstrated that both the eta and 100/N epsilon approaches lead to consistent convergence patterns. Then, by examining scattering amplitudes above the E* = 3m threshold, we validated our methods against known results. These improvements provide a foundation for more accurate extractions of scattering information, and enable further studies with Lattice methods. Future research may explore further optimization of contour sampling strategies and broader generalizations to complex three-body systems.

Acknowledgements

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References

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[2] Briceño, Raúl A., et al. "Role of boundary conditions in quantum computations of scattering observables." Physical Review D 103.1 (2021): 014506.