Helium User Manual

Harvard-MIT Math Tournament

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November 3, 2016

Helium is a grading system for math contests. It was developed for HMMT in October of 2016 following inspiration from the Stanford Math Tournament's new Atomic Grader.

Helium is built using Django to be compatible with HMMT's then-existing structure. It features scan-based grading, a rapid-fire score input system, as well as support for direct score entry. Conflict resolution is handled automatically, as well as generating of results according to the HMMT scoring algorithm. The Django admin interface is built in.

This documentation is split into several parts. Chapters are dedicated to graders (day-of volunteers), officers (trusted staff members with extra responsibilities), system administrators (who need to set up the system before the tournament), and finally other developers (who wish to play with code), with each section more technical and detailed than the last.

The appendix consists of a list of terminology which will be used throughout the manual.

Contents

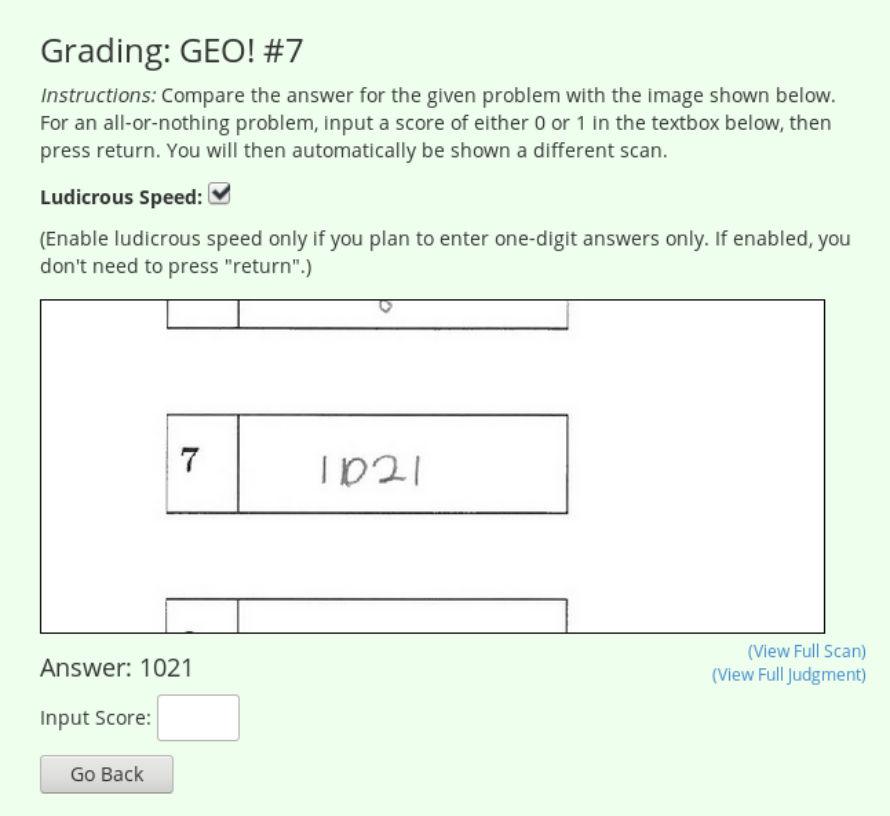
1 Instructions for Graders

This details the tasks which should be performed by day-of graders. Most graders will only need to read this chapter of the manual, and maybe the terminology section (Appendix A).

The two functions that you care about are "Grade Scans" and "Classical Grader"

§1.1 Scan grading

This is pretty self-explanatory.



- Try to select one of the 10 problems from an exam at random, so that we don't end up with everyone problem 1 or something.
- For all-or-nothing problems, type 0 for an incorrect answer and 1 for a correct answer.
- Press "go back" if you made a mistake and want to fix it.

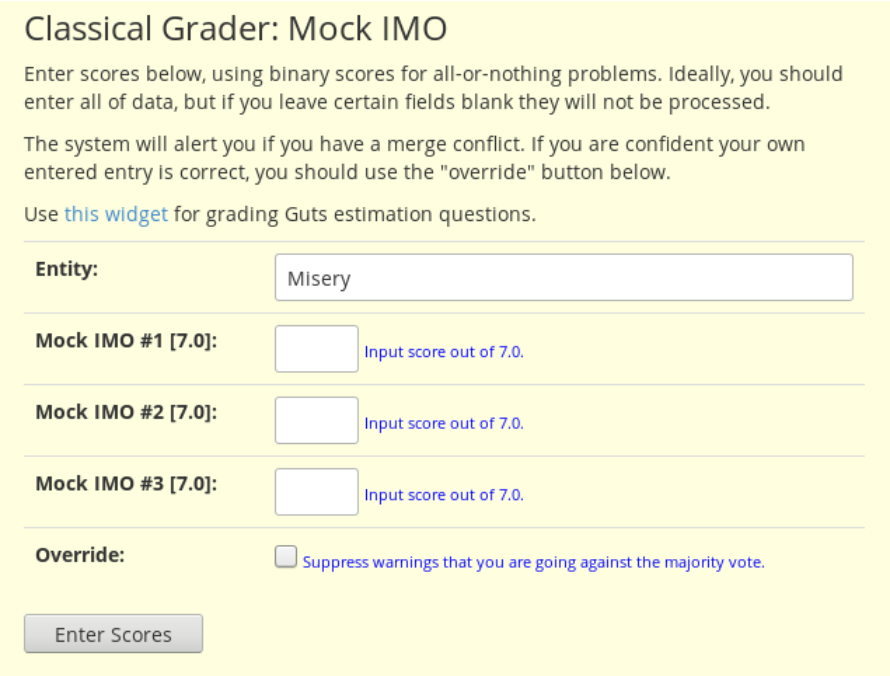
It's possible that you might have some problems where scores entered are not just 0 or 1. That's fine too.

The option "ludicrous speed" is enabled by default (checkbox above), as long as the problem you are grading is not all-or-nothing). This means that you don't have to press enter. If you prefer to take things more slowly, or for some reason the problem you are grading accepts scores other than 0 or 1, you should disable this option.

If configured, the background of the page may be tinted with the color of the test, so that you don't accidentally grade the wrong test.

§1.2 Classical grader

Most importantly: *do not use the classical grader on problems graded by scan*! (The interface disables this by default.)



Use the classical grader for tests which are not graded by scan (for example, the Guts round). There are two ways you can do this:

- Grade by exam, meaning you are grading many problems on that exam.
- Grade by problem, meaning that you are only grading a particular problem (for example a February proof-based team round problem).

Afterwards, this is self-explanatory; select the entity you want, enter their scores, and submit!

This time, Helium will warn you if you are submitting a score that goes against a consensus for that problem (for example if Alice enters 1 and then Bob tries to enter 0 later, Helium will warn Bob). If you are confident you are correct, you can use the override option. However ideally when this happens you should find the person who submitted the wrong answer before and check with them first.

Whenever you select an entity, the scores you previously entered for that entity will be automatically displayed.

If configured, the background of the page may be tinted with the color of the test, so that you don't accidentally grade the wrong test.

§1.3 Viewing Full Scans

When doing scan grading, you can press "view full page" in order to see the entire scanned page instead of just the cut-out. Just click the "(View Full Scan)" button near the corner of the image.

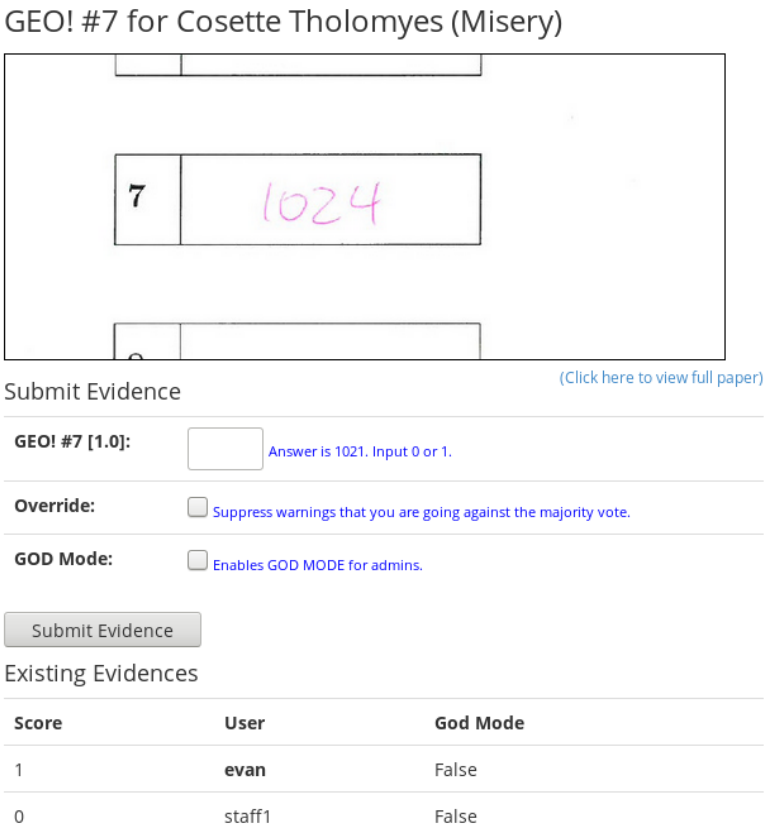
On this page there is also a switch where you can flag the page for administrative attention, if there is something wrong with it (for example, it is blank, or rotated, or illegible, etc.). This will make the scan appear in a queue for administrators and will prevent the scan from being given to regular users.

§1.4 Guts round

- Guts round scoring is done using the classical grader.
- For the "estimation problems" you will have to compute the score to assign to the problems at the end. So a "Guts estimation calculator", self-explanatory, has been provided.
- The old Babbage Guts scoreboard should still be working (it now pulls data from Helium).

§1.5 Viewing conflicts

You can see any evidences you submitted which conflicted with other graders by pressing "View Your Grading Conflicts". This allows you to also open any particular verdict, and change your decision on it.



Own Grading Conflicts

Entity	Problem	Score	Num Grades	View
Erik Fantasia (Opera)	GEO! #1	?	2	Open
Cosette Tholomyes (Misery)	GEO! #7	?	2	Open
Erik Fantasia (Opera)	GEO! #9	?	2	Open
Christine Daae (Opera)	GEO! #10	?	2	Open

This shouldn't really be necessary for scan grading, because verdicts are settled with a sufficiently large majority vote anyways. That is, if you mis-grade a scan, it will eventually sort itself out.

§1.6 Progress grading

There are some progress reports on how far grading is going for each problem. You can use this if you're not sure which problems needs more help. (Officers may also use this so they can panic as they see things aren't getting done, or something like that.)

Grading Progress

Problem	Done	Pending	Unread	Conflict	Missing
GEO! #1	15	0	0	1	0
GEO! #2	16	0	0	0	0
GEO! #3	16	0	0	0	0
GEO! #4	16	0	0	0	0
GEO! #5	16	0	0	0	0
GEO! #6	16	0	0	0	0
GEO! #7	15	0	0	1	0
GEO! #8	16	0	0	0	0
GEO! #9	15	0	0	1	0
GEO! #10	15	0	0	1	0

§1.7 Sneak peek at results

Some of you may be curious at how teams or students are doing. While I can't endorse this, the interface lets you do so: you can poke around the results page, or look up student papers, etc. If you do look through this, note that:

- Nothing you see is final, and
- All results are confidential.

§1.8 Do not use "Match Scans" or "Upload Scans"

Unless instructed to do so. Instructions for this are provided in the next chapter, because ideally they should mostly be done by superusers.

2 Instructions for officers

This assumes you have super-user privileges. So it contains some higher-level administration tasks.

§2.1 Uploading scans

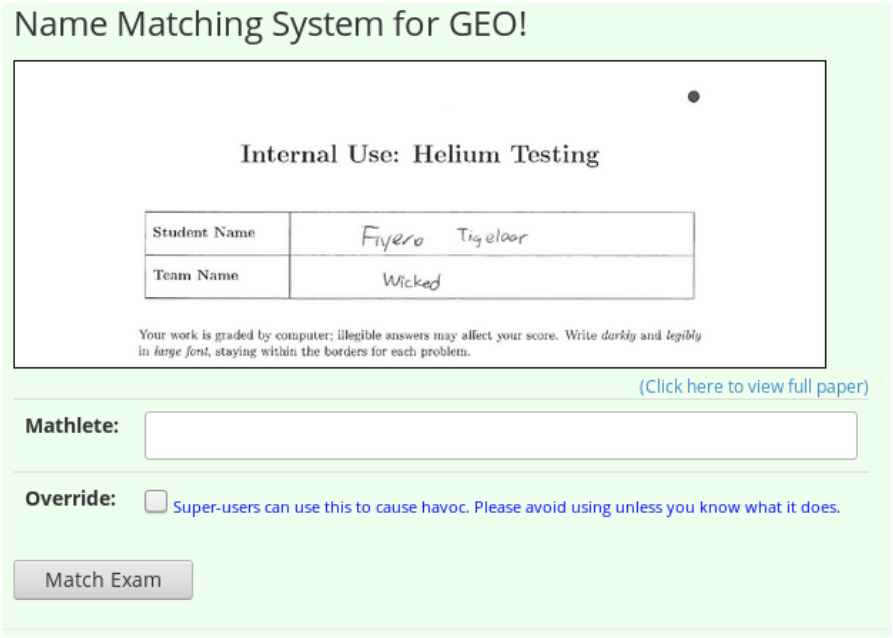
Self-explanatory, but some gotchas:

- For safety, uploaded filenames *must* be distinct. (This is to prevent someone from accidentally submitting the form twice, or submitting the same PDF twice.) If your scanner has silly filenames, fix those. Also, filenames should end with either "pdf" or "PDF".
- At the moment we don't have image processing, so please make sure all scans are correctly rotated.
- *Please*, make sure you are uploading the scans to the correct contest! There are some ways to deal with such mistakes but better if they can be avoided in the first place.

When you upload a scan, several problem scribbles, exam scribbles, and verdicts are immediately created corresponding to the items in the PDF file. However since the system does not yet know who is who, the verdicts and problem scribbles will not have an entity attached to them. The "match scans" functionality will then pair them up.

§2.2 Matching scans

Matching scans is more error-prone and not double-checked. People who should do this should be ones who are a little familiar with all the teams attending and so on (e.g. TD's and registration director at HMMT).



Interface usage: a name will be shown.

- Find the corresponding entity, and submit.
- If all goes well, next entity will be shown.

Easy. When a match of a paper is successful, all the problem scribbles, exam scribbles, and verdicts attached to the paper are immediately updated to reflect the new name.

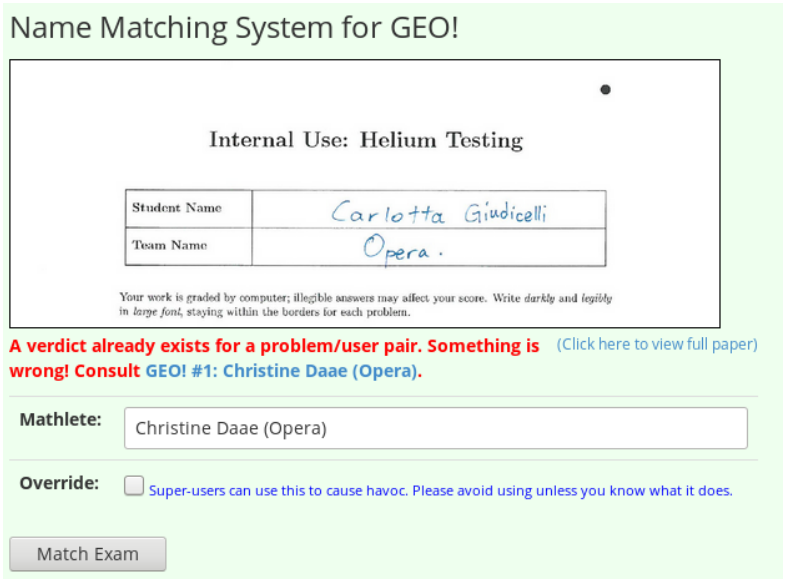
The comments in Section 1.3 apply equally well here.

§2.2.1 Error handling

This is where it gets tricky. Consider the following situation:

> You match a geometry exam to a student "Christine Daae". Later on, you try to match another geometry exam to the same student "Christine Daae".

This is obviously not okay, and the system will complain. An error will appear linking to the first *verdict* it finds for a geometry problem taken by Christine Daae.



At this point you need to fix the error. There are a couple things you could do.

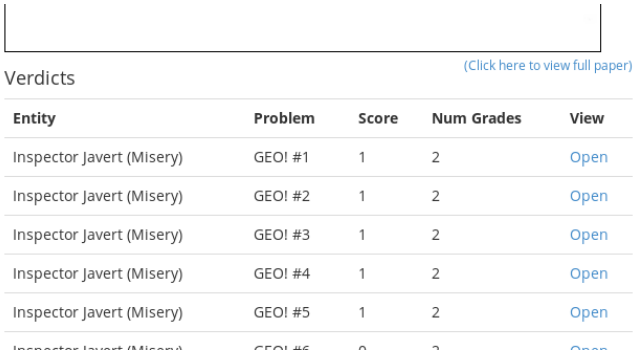
- If you made the mistake this time, then just re-submit the form with the correct name. Easy.
- If you made the mistake the *first* time around, then you need to go back and fix it.

  You can do this by clicking on the verdict provided by the system, and then opening the exam scribble for that verdict. Alternatively, the "Find Papers" interface will let you find the previous paper as well. This will bring you to a match form where you can change the name to the correct one.
- As a last resort, the "Override" switch (visible only to super-users) will forcibly assign the paper, and retroactively delete all verdicts and problems scribbles to the contrary. This is a Bad Idea™ and you should only do it if you really understand Helium well and have a compelling reason to do so.

§2.3 Find papers

You can locate specific papers by selecting "Find Papers" from the index page. Some ways to do so:

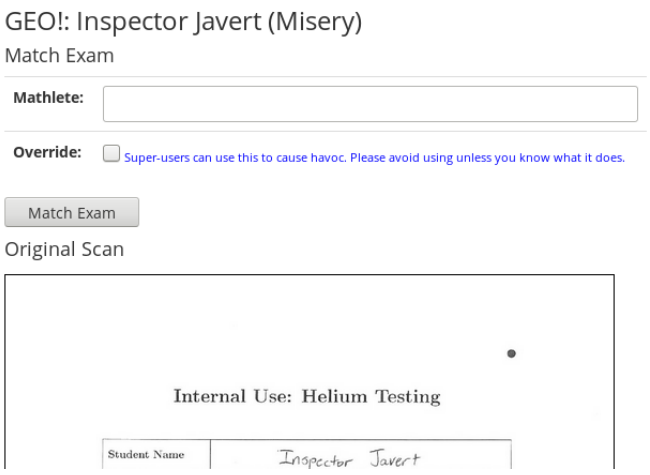
- You can look up by exam and entity.
- You can scroll through the pages of the PDF files.
- A list of papers which need moderator attention will appear here (see Section 1.3).


(Click here to view full paper)

Verdicts

Entity	Problem	Score	Num Grades	View
Inspector Javert (Misery)	GEO! #1	1	2	Open
Inspector Javert (Misery)	GEO! #2	1	2	Open
Inspector Javert (Misery)	GEO! #3	1	2	Open
Inspector Javert (Misery)	GEO! #4	1	2	Open
Inspector Javert (Misery)	GEO! #5	1	2	Open

If the paper has an exam scribble associated, that will also be displayed, along with a form to change the name associated to it (in case of error).

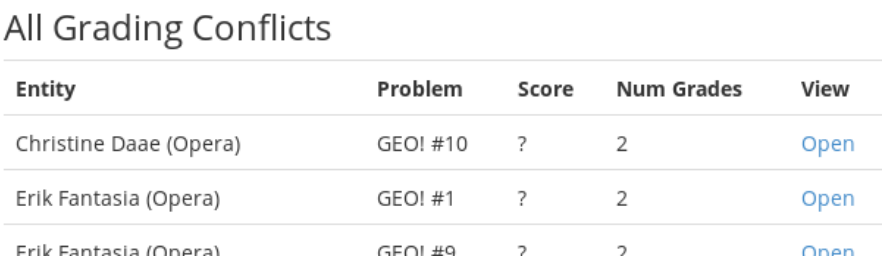


§2.4 Conflict resolution

First, I should describe the logic behind verdicts. For each exam there are two parameters:

- A **minimum majority** for the verdict to be valid, defined as a constant n such that an n-to-1 majority suffices.
- If valid, a **minimum number of graders** needed before the verdict considers itself done. (So setting this number to 2 is double-grading, 3 is triple-grading, etc. and the default is 3.)

Officers get an extra panel called "View All Conflicts". This lets you see the grading conflicts for *everyone*.


All Grading Conflicts

Entity	Problem	Score	Num Grades	View
Christine Daae (Opera)	GEO! #10	?	2	Open
Erik Fantasia (Opera)	GEO! #1	?	2	Open
Erik Fantasia (Opera)	GEO! #9	?	2	Open

Given a conflict, you can open the verdict page corresponding to it, at which point you can submit your own evidence. If you like being final authority, there should also be an option to submit this evidence in "God mode" which will settle the verdict once and for all.

§2.5 Finishing results

Once grading is all said and done, you can download the results at the bottom of the page. Administrators can also download full results and a full score spreadsheet.

Note that algorithmic scoring is not done automatically, since it is a long and intensive operation. To run it, you should use the management command "Run Algorithmic Scoring", after all verdicts are settled. Alternatively (and maybe better), you can use `python manage.py algscore` from SSH.

§2.6 Other Administration

The Django admin interface can be used if there is some operation you need to do not supported by the existing views.

§2.6.1 Management

In addition, to operations that might be useful but in theory should not be necessary:

- "Compute Verdicts" will ask every verdict (and there will be many!) to re-compute its judgment. Normally, verdicts only re-compute their judgment when they get new evidence. This is needed if e.g. you change the thresholds for grading.
- "Update scribbles" is similar, it updates all problem scribbles and verdicts with matched names. This should basically never be necessary unless you done goof.

Both of these commands actually just run `manage.py` commands, so they can also be done from SSH.

§2.6.2 Someone uploaded PDF to wrong exam!

Well, that's a bummer. You should

- Delete the relevant PDF file from the Django admin interface. This will kill of the exam and problem scribbles too.
- Delete all the verdicts that no have neither an entity or problem scribble.

That should mostly save things. No priomises though.

3 Instructions for Helium administrator

Here are some tasks you need to do to set up Helium to work before the tournament.

§3.1 Create the exams

You need to produce the exams. Remember to input colors corresponding to how the answer sheets will be tinted.

Right now the only way to do this is in the Django admin by hand, but eventually we will support importing these from HMMT Problems database. Sorry, this is a pain, I know.

§3.2 Import entities from registration

In Babbage a command called "import entities from reg" will do this for you. It is careful, and will never import the same entity twice.

§3.3 Printing and checking scans

You should make sure that a large number of answer sheets are printed, and in different colors between tests to avoid errors.

Unfortunately, right now the scan regions (i.e. the cut-outs) are hard-coded into `scanimage.py`. So before the tournament you should check the scanners being used and verify that the cut-out regions are compatible with the scanner.

The file `scanimage.py` has a facility to make this easier. If you run

```
python /path/to/scanimage.py FILENAME.pdf
```

then it will generate (in the current directory) the images corresponding to the cut-outs. Thus you can test this locally and make adjustments to the regions if needed.

§3.4 Test data

§3.4.1 Location

Some test data has been provided for you. You can look for it in the following places:

- `:/helium/static/batchX-samplescans.pdf`
- `:/helium/static/batchY-samplescans.pdf`
- `:/helium/fixtures/*`

The easiest thing to do is to load the two fixtures in the "setup" directory, which will populate the exams and entities tables. The sample scans correspond exactly to the entities in these fixtures.

You can also load fixtures in the `:/helium/fixtures/scenarios` directly which are snapshots of the entire database at various points in time in the test run (after matching

exams, after grading, after algorithmic scoring, etc.). But note that the `/media/` folder is NOT included.

Warning: the scenarios folder contains a superuser with name and password "evan" as well as a staff user with name and password "staff1". Consequently, be careful to NOT run these on production, as we will then have some very insecure user accounts!

§3.4.2 Synopsis of test data

Synopsis of test data:

- There is a single scan-based exam, called "GEO!", and there is a single non-scanned exam, called "Mock IMO". The former is 10 problems scored algorithmically. The latter is 3 problems worth 7 points each.
- There are 16 mathletes split into three teams of five, corresponding to characters from *Lés Miserables*, *Wicked*, and the *Phantom of the Opera*, plus one unaffiliated individual named "Evan Chen".
- The two sample batches contain all their "GEO!" exams. The answers to that exam corresponding exactly to the real answers to the HMMT February 2016 Geometry test.

4 Notes for software team

This contains low-level technical notes on some of the non-obvious moving parts behind the whole system. This is intended to help you extend Helium in some way.

Most of the code I tried to comment or `help_text` well. But if anything is still confusing, that's probably my fault. Feel free to contact me at `chen.evan6@gmail.com`.

§4.1 Personal request

I have a personal request before you start editing, which is that my name remains intact in `LICENSE.txt`, this documentation, and other places (for example file headers). You are free of course to add your own name should you make substantial contributions.

The reason I ask this is because this system represents well over 100 volunteered hours of my free time pooled into over 300 commits. I built the entire system from scratch with no assistance from anyone else. Since I didn't collect any payment for this work, the least I ask is that I am recognized for it.

Thanks!

§4.2 Moving parts

When possible I tried to keep everything in Helium self-contained, not relying on even files outside the directory.

§4.2.1 Helium dependencies

Here is a partial list of things externally that Helium relies on; it may be incomplete. All this is at time of writing; other people move things around all the time so God knows if we will even have `core/settings.py` in a few years.

- `requirements.txt` in the root directory has some Helium specific additions. In particular `wand` which does image processing in Python.
- Helium needs to host its media on an external S3 server, because scan images need to be distributed. So currently I have an S3 bucket called `heliummedia`. The connection is in `core/settings.py`, The S3 bucket itself lives inside AWS along with the entire HMMT application.
- Image installation is done through the EB extension `01yum`. This installs `imagemagick`, `freetype`, `ghostscript`, which is used for the image processing.
- `core/settings.py` should also have some basic applications installed, such as
  - 'django.contrib.admin',
  - 'django.contrib.auth',
  - 'django.contrib.sessions',
  - 'django.contrib.messages',
  - 'django.contrib.staticfiles',
  - 'storages'.

§4.2.2 Things depending on Helium

- The Guts scoreboard reads its data from Helium. So if you make changes to Helium, check there too.
- Registration import into Helium, as described before.

§4.2.3 Other nice things to know

Some not terribly Helium-specific things, but you should still know:

- Logging configuration (not specific to Helium, but nice!) is done by the EB extension `00logging` and configured in `core/settings.py`.
- Not Helium-specific, but the EB extension `05managepy` is responsible for running `collectstatic` and `migrate`.

§4.3 A wishlist

Some possible to-do things to do:

- Image processing: auto align scan regions and cut-outs.
- Auto import exams from HMMT Problems Database.
- Proof grading (long shot).

A Terminology

The grading system is itself known by the name **Helium**. Each event will be referred to as a **contest**.

§A.1 Participants

In each contest there are several participants. The students are referred to as **mathletes**. These students are often grouped into **teams**. A mathlete may have no team; in this case we say they are an **unaffiliated individual**, or sometimes just "individual" for short although this is confusing.

An **entity**[1] refers to someone, either a mathlete or a team, who takes an exam.

A **user** refers to a staff member of the contest (not a student or coach). A **superuser** refers to a higher-level administrator of the contest; for example a head grader, or problem czar, or tournament director, or software director. To be more exact this is the set of users with administrative privileges on the Django website.

§A.2 Exams and problems

A contest consists of several **exams**[2], each of which has several **problems**, which are numbered 1, 2, ..., n in each exam. These exams are **taken** by entities, who must attempt to solve the problems on the exam.

A problem has the following properties:

- The problem has a **weight**, which is the number of points a contestant earns for solving it.
- The problem may allow **partial** marks, or it may be an "all-or-nothing" problem.

Exams may be "individual" or "team" exams, depending on which types of entities take them.

§A.3 Scans

A **problem scribble** refers to the scan of a contestant's answer to a problem. An **exam scribble** refers to the scan of a contestant's entire answer sheet for an exam. You won't see these two terms much in the user interface, but they are used internally in the source code just about everywhere.

A **paper** refers to the set of answers for a contestant to any given exam, regardless of whether the paper was fed into a scanner or not.

§A.4 Grading

Suppose an entity E attempts a problem P, Helium then needs to give a **verdict** to the pair (E, P), which consists of a score assigned to the pair (E, P). This represents the judgment handed down by Helium to E for this problem P.

[1] This abstraction is useful because an exam doesn't care so much who takes it.

[2] Helium deliberately avoids the word "test", since this may be confused with "testing" and so on

To generate this verdict, one or more users will input a score for the pair (E, P). Each such score by a grader is termed an **evidence** contributing towards the verdict, and we say the user **submits** this evidence. The verdict is **valid** if there is a clear consensus what the final score should be (by default this is at least a 3-to-1 majority, but this may be adjusted per exam). The verdict is **done** once it is valid and there is enough evidence.

There is one possible exception: super-users may submit a evidence in **God mode**. When this occurs, it overrides any and all evidence submitted by regular users.

§A.5 Algorithmic scoring

At HMMT a complex algorithm is used for assigning the weights to the individual tests. Details of this algorithm are available at

`https://www.hmmt.co/static/scoring-algorithm.pdf`

A copy of this algorithm is also included in an appendix at the end of this document.

Let $\mathcal{A}$ be the set of exams scored using this algorithm. Each problem on an exam in $\mathcal{A}$ obtains a weight which is called a **beta value**, denoted $3 \le \beta \le 8$. Moreover, each entity taking one or more exams in $\mathcal{A}$ is then assigned an **alpha value**, which represents their strength, and is denoted $\alpha \in [0, \infty)$. These α and β values are stored in database, since they are quite hard to compute.

At HMMT the alpha values are used to aggregate individual scores across all tests, and thus used to rank individuals.

B HMMT Algorithmic Scoring Details

§B.1 Synopsis

§B.1.1 Input

Each exam contains at least one **problem**. The exams are taken by several **contestants**.

This algorithm takes several **observations**: i.e. it takes as inputs pairs (c, p) of contestants and problems, and for each such pair is told whether or not c **solved** p. During the actual tournament, every problem is attempted by every contestant; however, the algorithm will not use this assumption.

§B.1.2 Output

Based on this, the algorithm outputs:

- An **strength** $\alpha_c \in [0, \infty)$ for each competitor c, and
- A **weight** $\beta_p \in [3, 8]$ for each problem p, which represents its difficulty.

Note that the strength is *not* the same as the score of a contestant. In particular, as described below, each contestant is given a single strength value across all exams.

§B.1.3 Usage in contest

The calculation at the actual HMMT is done in one pass. For example, at HMMT February 2016, there were three exams with 10 problems each and 700 contestants. Thus there were $700 \cdot 10 \cdot 3 = 21000$ observations as input. Note that the algorithm is applied only once! For example the observations on the Algebra test affect the weights of the Geometry test.

For each particular 10-problem **exam**, the **score** of a contestant on that exam is the sum of the difficulties β_p for each problem that they solve. The score of a contestant is used to determine their rankings within each individual test. Scores across different exams are not comparable.

For individual sweepstakes, the strength α_c is used instead (in order to ensure that no particular exam dominates in determining the aggregate individual score). Contestants are ranked based on the value of α_c assigned to them. When determining team sweepstakes, the individual contribution (out of 800) is proportional to the sum of α_c across the team members c, scaled so that the strongest teams earns the maximal 800 points.

§B.2 Design

§B.2.1 Advantages

The scoring algorithm is designed to meet the following criterion:

- The system provides a careful way to **compare scores across tests**, leading to less noise in the aggregate individual and team rankings.
- The system **factors all possible inputs**, rather than for example merely the top 10 contestants or otherwise, as occurred in previous HMMT tournaments.

- The system is **resistant to preconceived difficulty**. Originally, problem czars were forced to estimate the difficulty of every problem by assigning it a weight; this leads to possible noise in the results. The algorithmic system determines the difficulty based on the actual performance.
- The system is **hard to exploit**, in part because it takes inputs from all contestants, and in part because it is so complicated that a contestant is probably better off thinking about the exam problems.

§B.2.2 Qualitative criteria

The following additional criteria are satisfied:

- The strength of each contestant is nonnegative,
- The strength of a contestant is unbounded, but not infinite even if they solve every problem.
- The strength of a contestant is zero if they solve no problems.
- Problem weights lie in the interval $(3, 8)$.
- The distribution functions are smooth.

§B.2.3 Choice of priors

In light of this, we select the following parameters, which describe how the α_c and β_p should be interpreted.

- The strength $\alpha = \alpha_c$ of each contestant c is *a priori* distributed in $[0, \infty)$ according to
$$\mathbf{P}(\alpha) = \exp(-\alpha). \tag{B.1}$$
- The weight (difficulty) $\beta = \beta_p$ of each problem p is *a priori* distributed in $[3, 8]$ according to
$$\mathbf{P}(\beta) = \exp\left(\frac{5}{(\beta-3)(8-\beta)}\right). \tag{B.2}$$

Now we relate the strength $\alpha = \alpha_c$ of contestant c to the difficulty $\beta = \beta_p$ of problem p by postulating that the probability that c solves p is

$$\mathbf{P}\left(c \text{ solves } p \mid \alpha_c = \alpha \text{ and } \beta_p = \beta\right) = \frac{\exp(-\beta/\alpha)}{1+\exp(-\beta/\alpha)}. \tag{B.3}$$

Assuming (B.1), (B.2), (B.3) and holding fixed the set of observations of the tournament:

> *The algorithm outputs the unique set of α_c and β_p for which the outcome of the tournament was most likely.*

The rest of the document describes how to actually compute the maximum.

§B.3 Algorithm Description

The algorithm is a slightly modified version of the Rasch model.

§B.3.1 Pseudocode

Let us for brevity denote the function in (B.3) by

$$w(\alpha, \beta) \stackrel{\text{def}}{=} \frac{\exp(-\beta/\alpha)}{1+\exp(-\beta/\alpha)}.$$

Note that $w(\alpha, \beta) \in (0, 1)$,

Now consider the following systems of equations, whose origin we explain below. First, for every $\alpha = \alpha_c$ we have the equation

$$0 = -1 + \sum_{p \text{ solved by } c} \beta_p \alpha^{-2} - \sum_{p \text{ took by } c} \beta_p \alpha^{-2} w(\alpha, \beta_p). \tag{B.4}$$

Moreover, for every $\beta = \beta_p$ we have the equation

$$0 = \frac{1}{(\beta-3)^2} - \frac{1}{(8-\beta)^2} + \sum_{c \text{ taken by } p} \alpha_c^{-1} w(\alpha_c, \beta) - \sum_{c \text{ solved by } p} \alpha_c^{-1}. \tag{B.5}$$

The algorithm binary searches for a set of $\vec{\alpha}$ and $\vec{\beta}$ which simultaneously satisfy (B.4) and (B.5). It is proved below that this set of parameters is unique; this is the output of the algorithm.

§B.3.2 Derivation

We show the derivation of equations (B.4) and (B.5).

Fix a set of observations for the tournament. Now consider a choice of $\vec{\alpha}$ and $\vec{\beta}$ is known. Then the probability of the observations is distributed according to

$$F(\vec{\alpha}, \vec{\beta}) = \prod_c \mathbf{P}(\alpha_c) \prod_p \mathbf{P}(\beta_p) \prod_{c \text{ solved } p} w(\alpha_c, \beta_p) \prod_{c \text{ missed } p} (1 - w(\alpha_c, \beta_p)) \tag{B.6}$$

assuming the events are independent.

We now claim that:

Proposition B.3.1

If a choice of $\vec{\alpha}$ and $\vec{\beta}$ optimizes F, then it is a solution to (B.4) and (B.5). In other words, the outputted $\vec{\alpha}$ and $\vec{\beta}$ are those for which the observed outcome was most probable.

Proof. Note that

$$\prod_{c \text{ solved } p} w(\alpha_c, \beta_p) \prod_{c \text{ missed } p} (1 - w(\alpha_c, \beta_p)) = \prod_{c \text{ solved } p} e^{-\beta_p/\alpha_c} \prod_{c \text{ took } p} \frac{1}{1+e^{-\beta_p/\alpha_c}}.$$

From this, and plugging in (B.1) and (B.2), we get

$$F(\vec{\alpha}, \vec{\beta}) = \prod_c e^{-\alpha_c} \prod_p e^{-\frac{5}{(8-\beta_p)(\beta_p-3)}} \prod_{c \text{ solved } p} e^{-\beta_p/\alpha_c} \prod_{c \text{ took } p} \frac{1}{1+e^{-\beta_p/\alpha_c}}.$$

It is equivalent to maximize $\log F$, which is

$$\log F = -\sum_c \alpha_c - \sum_p \frac{5}{(8-\beta_p)(\beta_p-3)} - \sum_{c \text{ solved } p} \frac{\beta_p}{\alpha_c} + \sum_{c \text{ took } p} \log \frac{1}{1+e^{-\beta_p/\alpha_c}}.$$

Then right-hand sides of (B.4) and (B.5) are the *partial derivatives* of $\log F$ with respect to α_c and β_p for each c and p. The partial derivatives equal zero at any local maximum of $\log F$, as desired. □

In fact, one can actually check that the right-hand sides of (B.4) and (B.5) are monotonic in α and β respectively. Consequently, F is convex. This implies:

Proposition B.3.2

The function $\log F$ has a unique point at which all partial derivatives vanish, and that point is a maximum for F.

This also implies that one can numerically solve the equations by the following binary search procedure. We let $\vec{\alpha}_0$ be an arbitrary point. Then we let $\vec{\beta}_n$ be the solution to (B.5) given $\vec{\alpha}_{n-1}$. Notice that one can actually solve for each component β_p using binary search separately, as each equation of the form (B.5) involves only a single β_p. Similarly, we can let $\vec{\alpha}_n$ be the solution to (B.4) given $\vec{\beta}_{n-1}$ in the same fashion. The pairs $(\vec{\alpha}_n, \vec{\beta}_n)$ will converge to the maximum since we are looking at partial derivatives of a concave function, so we now have an iterative procedure for computing the output.