In the name of God

School of Electrical and Computer Engineering

Signals and Systems

Computer exercise 2

Delivery deadline: 23 May Professor: Dr. Rabiei

Fourier analysis in one dimension

Table of Content

FFT/DTFT

Intro

In this section, we introduce you to Fourier analysis in MATLAB and how to obtain Fourier coefficients. First, we will introduce you to implementing Fourier transform from scratch, then with the ready function FFT in MATLAB.

To get the Fourier coefficients of an actual signal, we want to introduce you to the two concepts of complex exponential signal and inner product. A complex exponential signal is defined as follows:

$$\text{Complex Exponential Waves} \rightarrow \quad Ae^{i2\pi ft}$$

In the following, we will examine the reason for choosing the mixed exponential signal.

In general, the inner product of two vectors of equal length define as below:

$$\mathrm{dot}(v_1, v_2) \triangleq v_1.v_2 = \sum_{i=1}^{l} v_1[i]\, v_2[i]$$

Which has a direct relationship with the similarity of two vectors. The basic problem with the current inner product is that the larger the elements of the two vectors, the greater the inner product value and is not limited to a specific interval. To have a criterion for comparing different inner products, we need to normalize two vectors:

$$v_{1_{\text{scaled}}} = \frac{v_1}{{}^{1}|v_1|} = \overline{v_1}\ , v_{2_{\text{scaled}}} = \frac{v_2}{|v_2|} = \overline{v_2}$$

Where the size of $\overline{v_1}$ and $\overline{v_2}$equals to 1.

And the inner scaled product will be as follows:

$$\mathrm{dot}(v_1, v_2)_{\text{scaled}} = \frac{v_1.v_2}{|v_1||v_2|}$$

Now to examine the similarity of the two vectors v_2 and $v_1$with v_3,will have :

$$A = \frac{\mathrm{dot}(v_1, v_2)_{\text{scaled}}}{\mathrm{dot}(v_1, v_3)_{\text{scaled}}} = \frac{v_1.v_2}{v_1.v_3}\frac{|v_3|}{|v_2|} = \frac{v_1.\overline{v_2}}{v_1.\overline{v_3}}$$

[1] Norm

Therefore, to check the similarity of vectors w_J , $J = 1,2, \dots , n$,with reference vectors such as v_1, it Is enough to calculate $v_1 . \overline{w_J}$ and compare their values.

Now the above concepts can be used for the .similarity of two signalsSupposeyou have a discrete signal v_1 and you want to compare that to discrete signals w_J , $J = 1,2, \dots , n$. In the following, we will examine an example :

Suppose a reference signal with a sampling rate of 1000 (The number of samples per second of the signal is 1000) and is defined in the time interval [-1,1] with the following equation:

$$x(t) = \sin(10\,\pi\, t + \theta)\, e^{-t^2}$$

In the second step, we provide two desired signals for the inner product in the above signal:

1. Simple sine signal : $\sin(2\pi f t)$
2. complex exponential signal : $e^{i2\pi f t}$

A) Draw a figure, as shown in Figure 1, that the reference signal is located in the first column, and the values obtained from the inner product of the above reference signal and the two desired signals for the frequencies listed in the interval [2: 0.5: 10] are located in the second column. (*Note that the output of the inner product generally has a real and imaginary part, and you should utilize the amplitude of the work)

B) What information do the diagram in the second row and the second column of Figure 1 give about the desired signal?

C) According to the results obtained in Figure 1, and the points made above, argue why it is problematic to choose a sine instead of a complex exponential signal

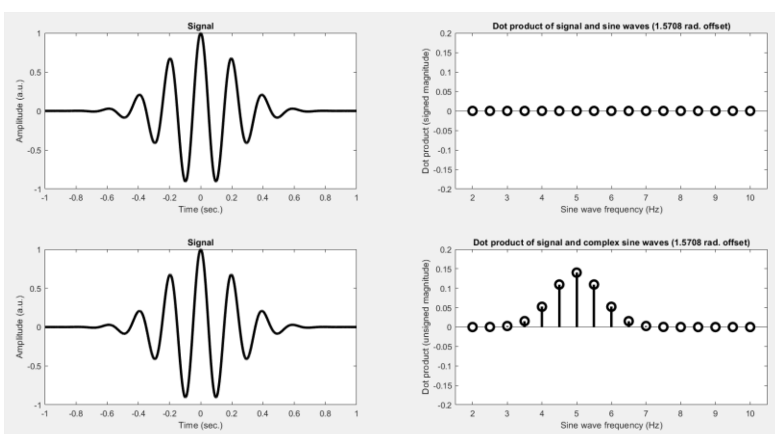

Figure1

Simulation

Guideline

Now that you are familiar with the basic concepts of complex exponential signal and inner product, it is time to calculate the Fourier coefficients of an arbitrary signal and plot it in the frequency domain. Consider the signal $x(t)$:

$$x(t) = 2.5\sin(8\,\pi\,t) + 1.5\sin(13\,\pi\,t)$$

The time in the interval [0 2] and the sampling rateequal 1000.

Recall the Fourier transform relation as a function $x(t)$:

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-i2\pi ft}\,dt$$

In the discrete field and by sampling the signal $x(t)$, the above relationship is as follows:

$$X[f]_{{}^{2}i} = {}^{3}\mathrm{dot}(\,x\left[{}^{4}t\right],\, e^{-{}^{5}i\;2\;\pi\;{}^{6}f_i\;{}^{7}t_f}\,)$$

According to the initial assumption in this section, the x (t) signal was sampled at a rate of 1000, so the maximum value of the frequency can be equal to 1000, but the main question is whether all the frequencies in the range of [0 1000], are the valid frequencies?

To answer this question, we must first acquaint you with the Nyquist-Shannon Theorem. Suppose you have an alternating continuous signal. To work and analyze all signals, we need to turn them into a discrete form. The exact concept of discretization in signal and system means sampling. You need to save a limited number of signal points. Since a signal periodicity contains all of our signal information, we focus on sampling a given signal periodicity. If you were asked how many points at equal intervals in a signal cycle should be selected to preserve the general signal information, what would you say?

You may first ask what is meant by the general information of a signal? In response, it should be said that the most important features in a signal cycle are the maximum and minimum signal points and how the signal changes continuously between these two points, etc.

First of all, you may answer that the more points you select in a signal period, the more information will be stored, so perhaps the right question is what is the minimum number of points?

According to Mr. Shannon, at least 2 points are a relatively good answer, but there is no information given about the type of these two points. For example, the best choice for two points

[2] The ith element of vector X[f]
[3] Dot Product
[4] Discretized time
[5] Imaginary Number
[6] The ith frequency of Complex Exponential wave : $f_i \in [1\!:\!npts]$
[7] Fourier Time : $t_f = \frac{[0:npts-1]}{npts}$ (*Normalized Time Vector)

in most signals is the minimum and maximum points, but in general, this number is a theoretical value that can be generalized to all signals regardless of whether the value is not exactly correct.

Now if the average distance for 2 points in a periodicity is equal to half the periodicity, by increasing the number of points, this distance will be reduced, which will result in the following equation:

$$^{8}\mathrm{SR} \geq 2\ ^{9}(\mathrm{Signal\ Maximum\ Frequency})$$

$$\mathrm{Equivalently} \rightarrow \ \ \mathrm{Signal\ Maximum\ Frequency} \leq \frac{\mathrm{SR}}{2}$$

The highest or fastest frequency in a signal is called the Nyquist frequency.

Now back to the first question, since the signal x (t) was sampled at 1000, the maximum allowable frequency would be half of 1000 or 500 Hz:

$$\mathrm{f} = \mathrm{linspace}\left(0, \frac{\mathrm{SR}}{2}, \left\lfloor \frac{\mathrm{npts}}{2} \right\rfloor + 1\right)$$

In contrast to the positive frequency, there is a negative frequency. Frequencies in the range $(\frac{\mathrm{SR}}{2}, \mathrm{SR}]$ are called Negative frequencies. In the Fourier transform of a real signal, there is a negative frequency corresponding to each positive frequency. Why?

As you know, the real signal given can be written as a group of sine or cosine terms. Now consider another form of Euler relationship:

$$\cos(2\,\pi\,k\,t) = \frac{e^{i\,2\,\pi\,k\,t} + e^{-\,i\,2\,\pi\,k\,t}}{2}$$

The phrase $\frac{e^{-\,i\,2\,\pi\,k\,t}}{2}$ contains positive frequencies and the phrase $\frac{e^{i\,2\,\pi\,k\,t}}{2}$ contains the negative frequencies of your true signal, and each contains half the amplitude of your true signal, so if you just want to eliminate the negative frequencies and show all frequencies up to the Nyquist frequency, you need to Multiply the obtained Fourier coefficients by 2. (First Fourier transform correction factor)

Also, the values corresponding to the negative frequencies mirror the values corresponding to the positive frequency.

The following is a diagram of the Fourier transform of an unknown signal with a sampling rate of 1000 in Figure 2:

[8] Sampling Rate
[9] Nyquist Frequency

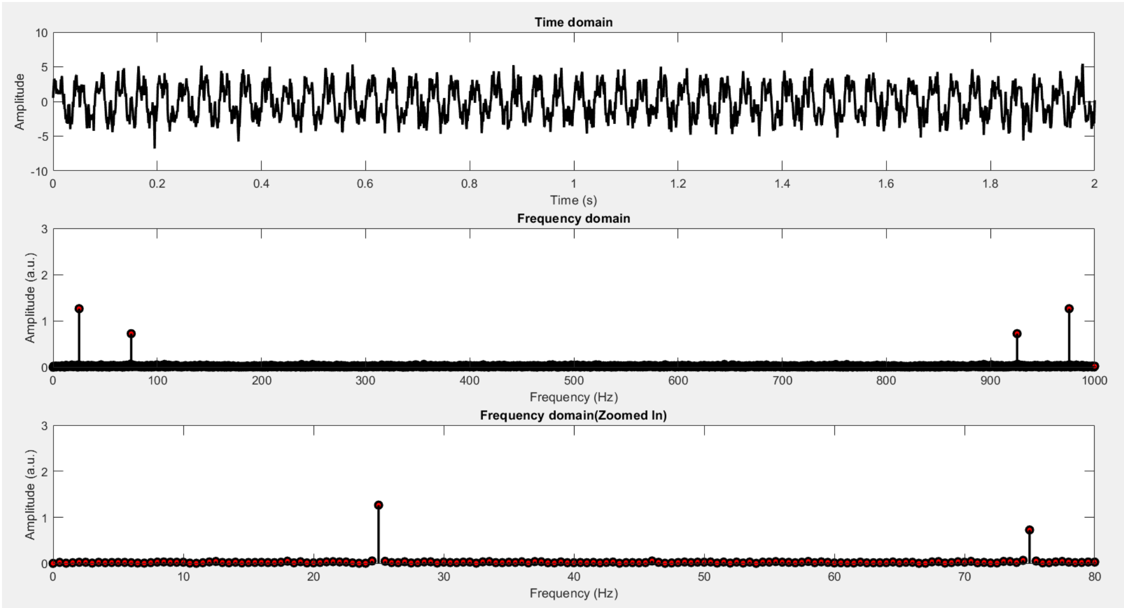

Figure 2

As can be seen in the frequency domian, the value of the Nyquist frequency is 500 and the positive frequencies are in the range [0 500] and the negative frequencies are in the range [500 1000] and the values of these two are mirror relative to the the Nyquist frequency. The third diagram shows an enlarged view of the Fourier transform of the signal, which shows the main frequencies of the signal at 25 and 75 Hz.

As mentioned earlier, frequencies larger than Nyquist are not valid, and as shown above, there are two frequencies with values greater than 900 that are definitely not our signal frequencies but are created symmetrically.

In some applications, the values assigned to positive and negative frequencies should be numerically symmetric, so we need to shift the Nyquist frequency to zero:

$$\mathrm{f_{shifted}} = \mathrm{linspace}\left(-\frac{\mathrm{SR}}{2}, \frac{\mathrm{SR}}{2}, \mathrm{npts}\right)$$

Also, shift the positive frequency as the value of the Nyquist frequency to the right and the negative frequency as the value of the Nyquist frequency to the left, and actually replace the positive and negative frequencies. The fftshift command in MATLAB can do this, create an array of numbers from 1 to 10 and then check the output of the fftshift on the array.

$$\mathrm{X_{new}}[\mathrm{f_{shifted}}]_{\mathrm{i}} = \mathrm{fftshift}(\mathrm{X_{old}}[\mathrm{f}]_{\mathrm{i}})$$

Figure 3 shows the new Fourier transform diagrams of the previously unknown signal:

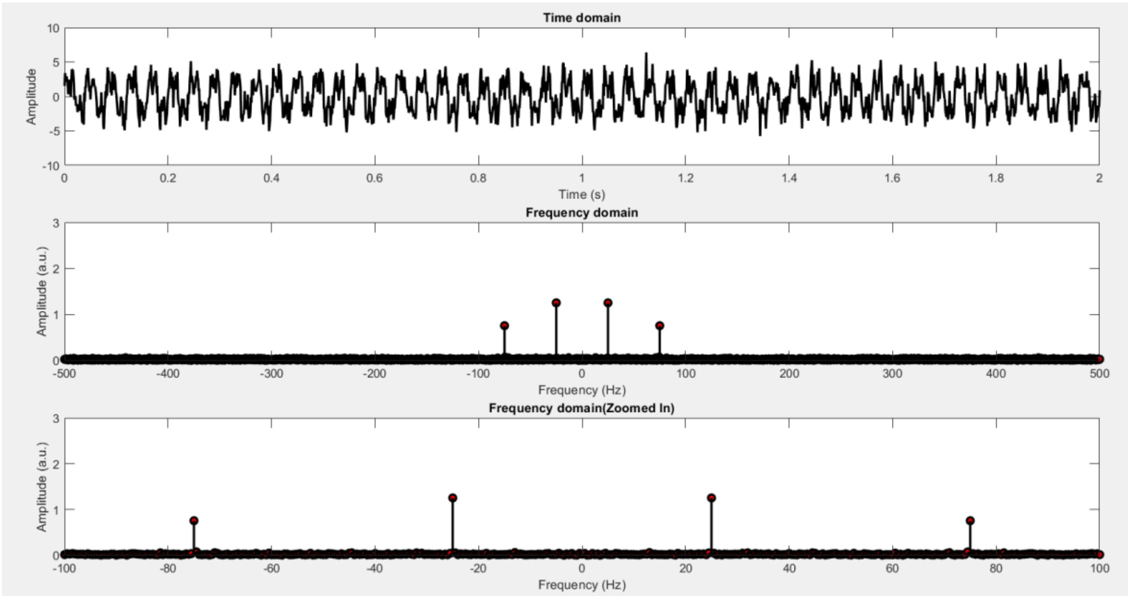

Figure 3(first view)

As shown in the figure above, the signal's new frequencies are ± 25 , ± 75.

In many applications, the negative frequency is eliminated and only the Fourier coefficients are plotted for the positive frequencies up to the allowable frequency (NQ) (see the scale of the Fourier coefficients in Figure 4):

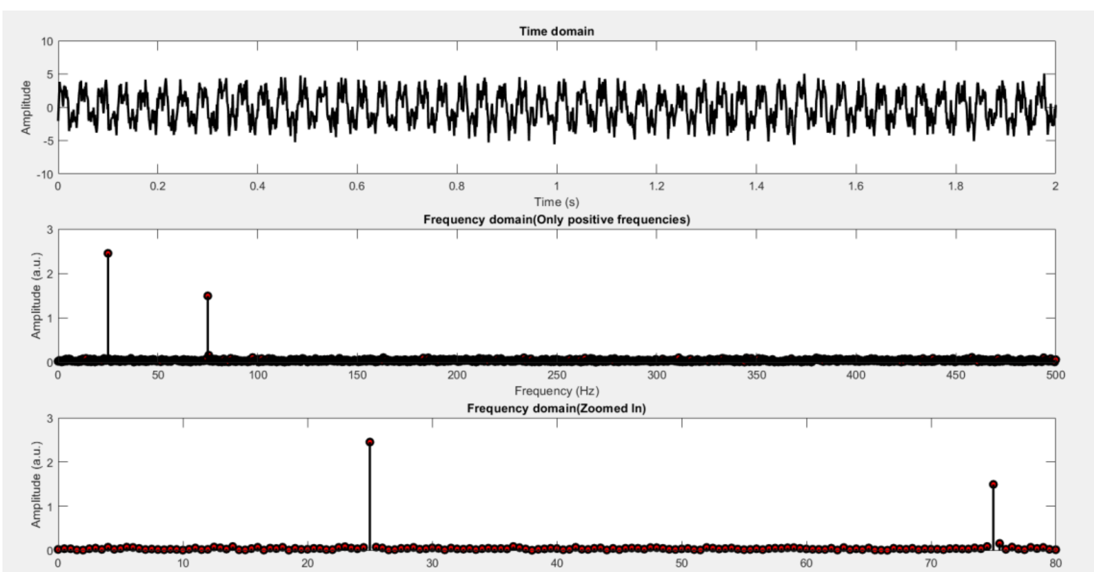

Figure 4(Second view)

The last point is obtained in the scale of Fourier coefficients. In explaining the positive and negative frequencies, the first correction factor was mentioned. The second correction factor refers to a point about the inner product. As explained, the array containing the Fourier coefficients of the signal is obtained by the inner product between two discrete signals. The problem with the inner product of two vectors is that the larger the length of the two vectors, the larger the dot product of the two vectors, but our goal is to obtain the Fourier coefficients of the signal regardless of the number of sampled points. So we divide the obtained Fourier coefficients by the number of sampled points (second correction factor).

MATLAB Implementation

A)

As explained, the correct choice of sampling rate is very important because otherwise the signal loses its properties in discretization and is no longer detectable. Suppose we first have a continuous sine signal with a frequency of 10 Hz in the interval [0 1]. Since it is not possible to generate a continuous signal in MATLAB, assume a sampling rate of as large as 1000. Now for 4 values of sampling rate 15,20,50,100, Discretize your continuous signal (assuming the first sampling point is at t = 0), and plot the continuous signal and the sampled signal as shown in the subplot in Figure 5:

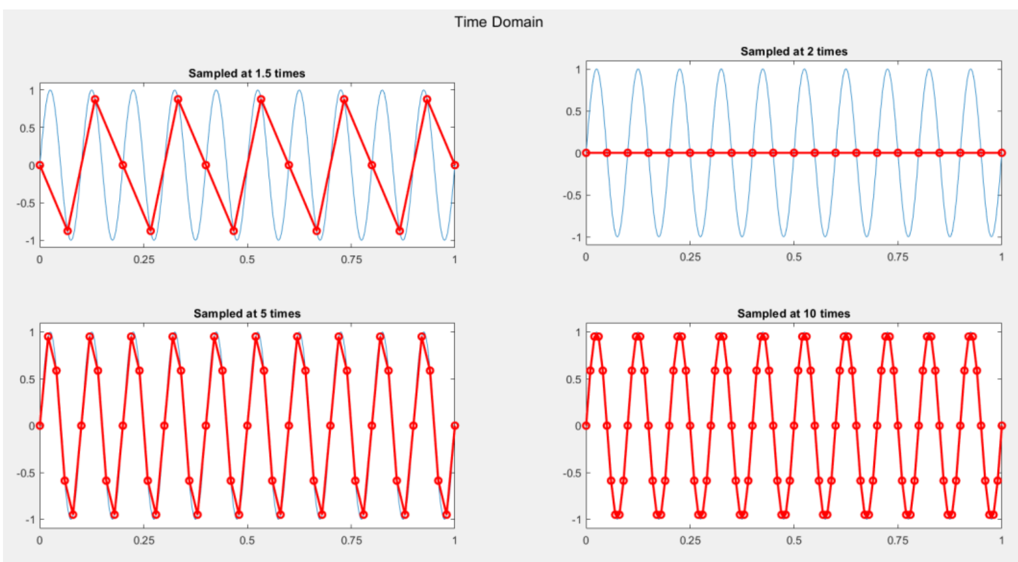

Figure5

In another figure, plot the Fourier transform of all 4 types of discrete signals to the Nyquist frequency as shown in Figure 6. For all 4 graphs, set the xlim values to 50 and the ylim values to 1.5.

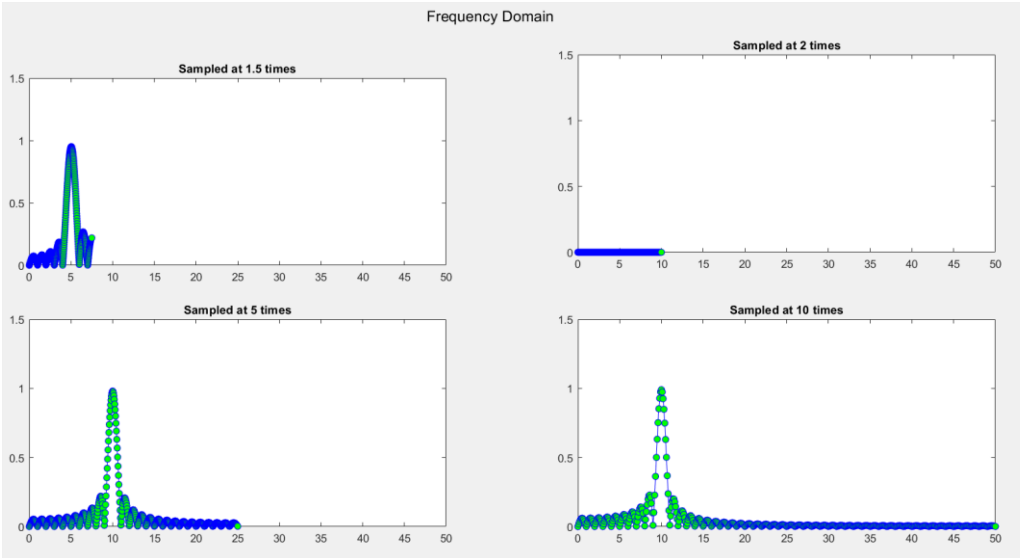

Figure 6

* Note: In this question, you are allowed to use the ready-made FFT function.

B)

Consider the signal x(t), the one defined in the guideline section. Using the described equations, implement the Fourier transform of the given signal and plot the obtained Fourier coefficients in the frequency domain. (* You are not allowed to use the FFT function here.)

C)

Consider the sine signal $x(t) = 1.5 + 2.5 * \sin(8\,\pi\, t)$ this time. As in part B, plot the obtained Fourier coefficients of the signal. Are the coefficients obtained correctly? where is the problem from? Modify the coefficients by applying a new condition.

Frequency resolution, Zero Padding (time domain)

In the previous section, you were introduced to a concept called the Sampling Interval, in which the time interval between two consecutive points was sampled. In contrast, there is a concept that the distance between two consecutive points is obtained in the frequency domain[10] Which is related to two values of sampling rate and the number of sampled points.

A)

Consider the signal $x(t) = sin^2_{(0.5\pi t)}$ in the time period [-2 2] with a sampling rate of 10. First, create a subplot with 2 rows and 2 columns as shown in Figure 7, and in the first column, plot the discrete signal and the obtained Fourier coefficients. Next, add an array of length 40 with zero values to the discrete signal. Then plot the new signal and its Fourier coefficients in the second

[10] Frequency Resolution

column. (Plot the obtained Fourier coefficients to the Nyquist frequency and set xlim for these two graphs in the range [0 10].)

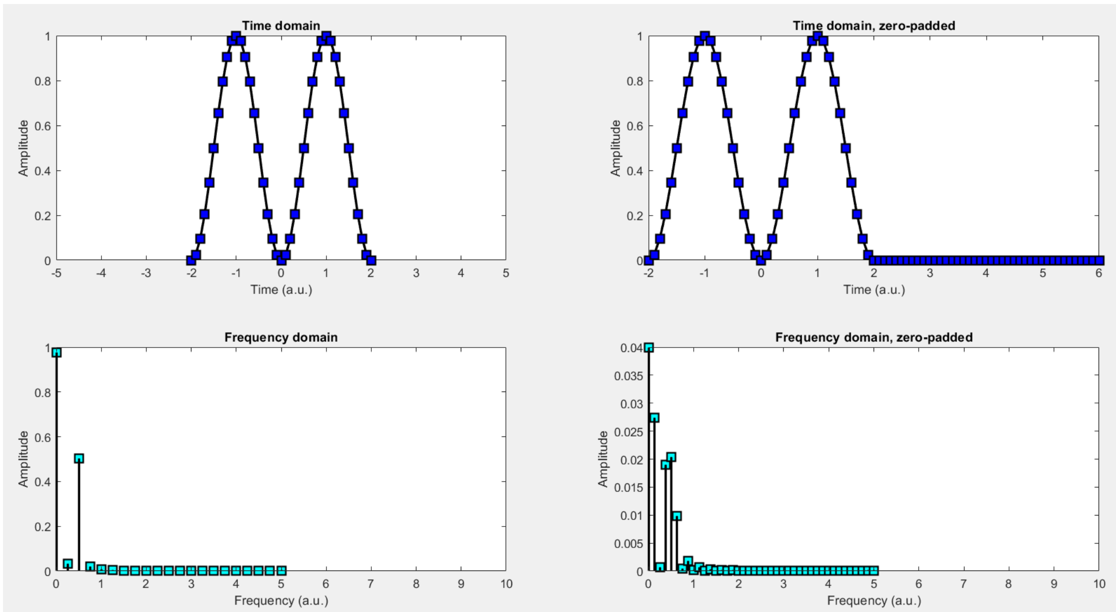

Figure7

B)

According to the obtained figure, with the increase in the number of sampling points, what change was made in the signal frequency domain?

By examining the output obtained, provide a formula for Frequency Resolution that relates to the number of sampling points and the sampling rate.

C)

Consider the following popular signal[11]:

We intend to create three different types of sampling at different time intervals from the above signal.

1. The first signal is in the range [0 1] and with a sampling rate of 100.

2. The second signal is in the range [0 10] and with a sampling rate of 100.

3. The third signal is in the range [0 1] and with a sampling rate of 50.

[11] Morlet Wavelet

Next, create a subplot with two rows as shown in Figure 8. In the first row, plot three discrete signals in a graph, and in the second row, plot the Fourier coefficients of three signals in a graph.

Compare the Frequency Resolution of these signals from figure 8 and sort them from small to large.

Verify the results obtained through the relation obtained from Section B.

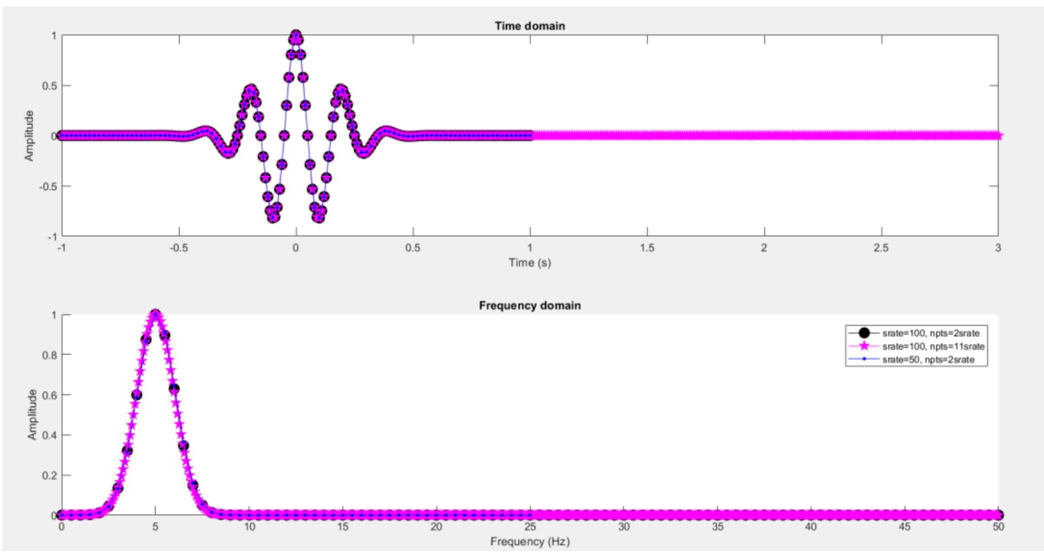

Figure 8

IDTFT/IFFT

In this section, we are going to deal with the Inverse Fourier transform. Recall the Inverse Fourier relation:

$$X(t) = \int_{-\infty}^{\infty} X(f)\, e^{i2\pi ft} df$$

A)

According to the discrete relation of Fourier Transform in the previous section and the above formula, offer a discrete relation to obtain x(t) from X(f).

B)

Load the signal inside a. mat file in the project folder in the MATLAB environment. Then first take the Fourier Transform from the obtained signal and then get the Inverse Fourier transform of the obtained coefficients. Finally, plot the two original and retrieved signals in a figure like a Figure 9. (In this section you are not allowed to use the fft and ifft functions and you must implement each separately).

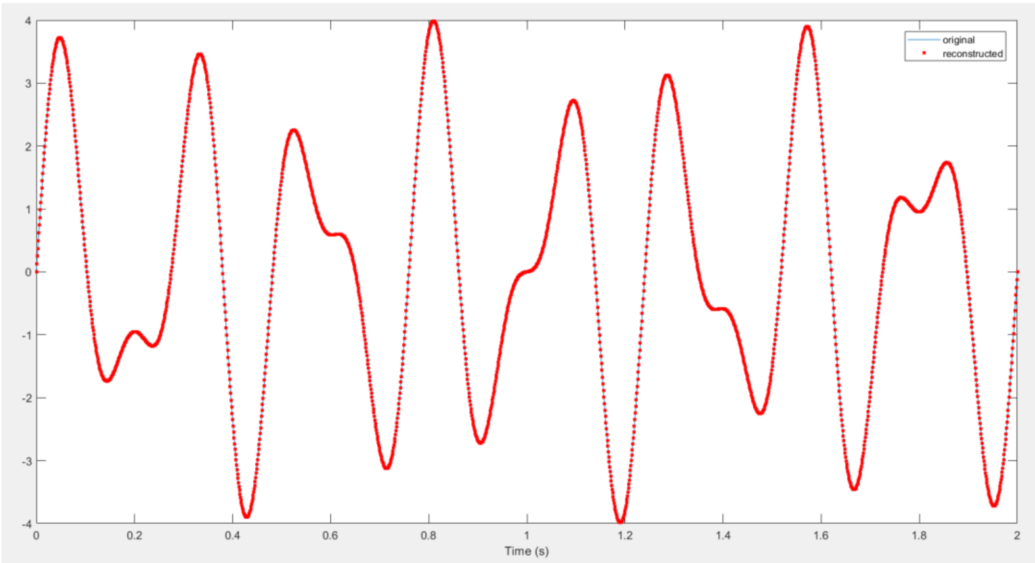

Figure 9

C)

In this section, we examine the effect of t_f in $e^{i\,2\,\pi\,f_i\,t_f}$ on retrieving the signal given in section B. In the introduction section, it was mentioned that t_f should be normalized and includes the range [0 1], but we did not mention a reason for that. For the three values in the range $\alpha = [0\ 35\ 50]$, shift the value of t_f as follows:

$$t_{f_{new}} = \frac{(a : npts + \alpha - 1)}{npts}$$

Create a subplot for each of the values above, as in Figure 10, with the original and retrieved signal in the first row. In the second line, draw the magnitude of the Fourier coefficients and in the third row, draw the phase of the Fourier Transform of the signal:

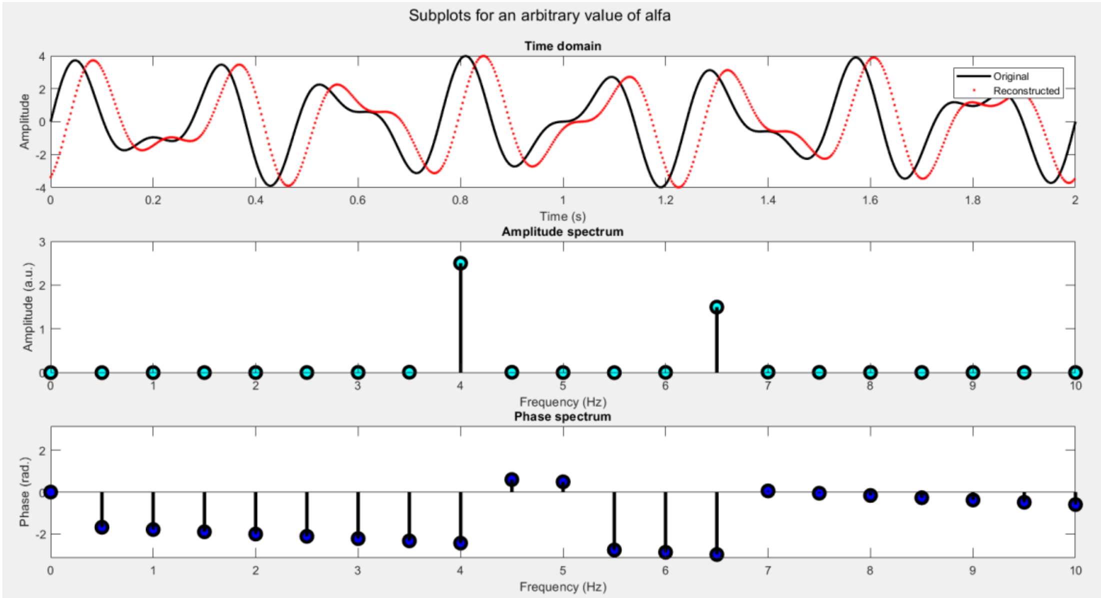

Figure 10

What is the effect of the parameter on the retrieval, Fourier coefficients, and the phase of the signal? What do you conclude?

Voice Analysis / Spectrogram

The project file contains 10 audio files in .wav format which are the mobile phone's numbers from 0 to 9. First, read each of the given audio files using the audioread command in MATLAB:

```
[u,Fs]=audioread('key2.wav');
```

Where u is the audio signal data and Fs is the sampling frequency. Save the Fourier coefficients for each number as you learned in the previous sections.

Next, we want to get acquainted with the spectrogram diagram and analyze it. In the following, the spectrogram of an unknown data with its Fourier transform is shown in Figure 11:

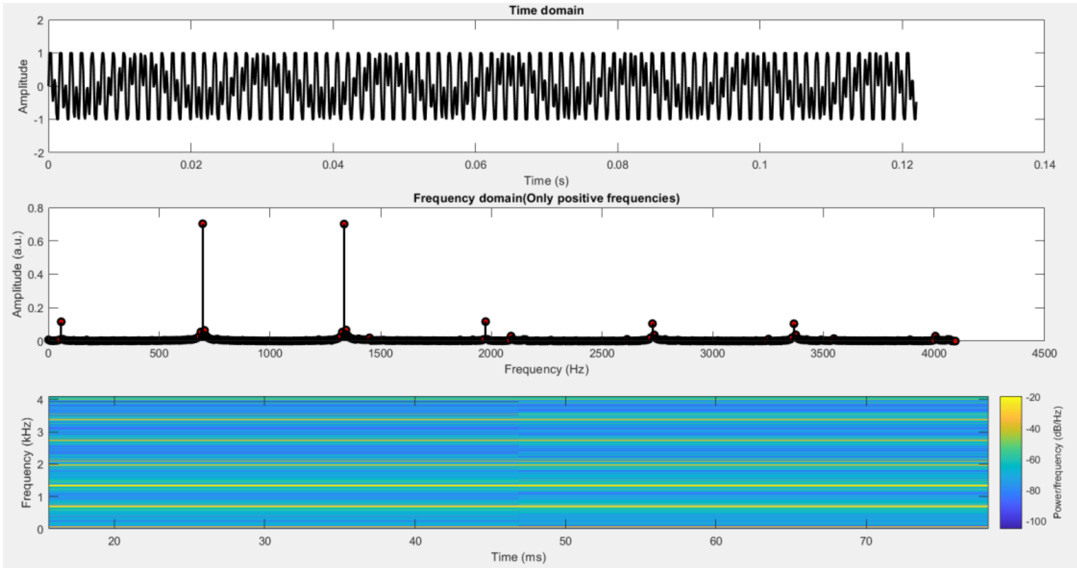

Figure 11

In the spectrogram diagram, there is three important information about the signal which are displayed in two dimensions. As you can see in the third row of the figure 11, in the first and second dimensions, there are discrete values of time and signal frequency, and the information in the third dimension is shown as the corresponding energy at a specific time and frequency in the diagram. The energy values at each point of the spectrogram are shown as a color spectrum next to the graph. Because these values are normalized, their values in decibels are negative. The larger the Fourier coefficients corresponding to a frequency in the signal, the higher the energy contained in that frequency, the more positive it's energy in decibel scale, and the closer it will be to yellow in the frequency spectrum.

Use the following command to draw a spectrogram of a signal in MATLAB:

$$\mathrm{Spectrogram}(\mathrm{signal}, 256, [\ ], \mathrm{npts}, \mathrm{SR}, '\mathrm{yaxis}')$$

A)

For the three arbitrary numbers, plot the three main signal diagrams, the Fourier transform, and the spectrogram, as shown in Figure 11.

B)

Find the two dominant frequencies for each audio file from the spectrogram diagram and report it in the form of a table, then estimate a signal for each audio file by it's dominant frequencies and store the obtained results.

C)

In the next step, we want to utilize the estimated signals for each audio file to generate a custom phone number.

(For example, create (09121151869), Listen to the generated sound and insert the output audio in the project delivery file.)

Use the MATLAB sound function to listen to the output signal (* Note to place the sampling frequency of the audio files as the second argument).

```
sound(u_estiamte,Fs)
```

To create a series of cell phone numbers, it is necessary to have some space between the two digits so that we can hear the sound of each digit clearly when hearing the phone number. To do this, add a zero matrix of a specified length (greater than the length of the data matrix) between the data of both digits. for example:

$$\text{Phone Number} = [u1_{\text{estimate}}; \text{Zeros}(n,1)\ ;\ u2_{\text{estimate}}\ ; \text{Zeros}(n,1), \ldots]$$

Final Points

1. Finally, place the delivery code for each question in a .m file. Also, in each .m file, separate the different parts of each question by cells.
2. The evaluation of your work is determined by having a proper report, so be sure to include the output of the graphs and the answers to the different parts in your report. Exercises without a report will not be awarded a grade.
3. The Computer exercise aims to help you learn new concepts, so you will get a low score if there is an unjustifiable similarity in your reports or codes.
4. If you have any questions or concerns about the exercise, you can contact me via email (sh.vassef@ut.ac.ir) or in the telegram group.

Good luck.