Riddling

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1 Rational points on a circle (Calculus)

Could there be a circle on the plane where only one point has rational co-ordinates? A point $p = (x, y)$ has rational co-ordinates iff $x, y \in \mathbb{Q}$. What about a sphere? Are there circles with more than one rational point? Are there circles with infinite rational points? Can you find a circle with exactly n rational points on it for each $n \in \mathbb{N}$?

1.1 Solution

Yes, there could. Consider the circle: $(x - r)^2 + y^2 = r^2$ for some irrational r. $(0, 0)$ is a point on this circle and is rational. But from the equation it follows that: $x^2 - 2xr + r^2 + y^2 = r^2$ or $x^2 + y^2 = 2xr$ or $r = (x^2 + y^2)/2x$. If there was a rational solution to this equation where $x \neq 0$ then it would follow that r is rational since it is the result of multiplication, division, addition and subtration of rational numbers and this is a contradiction. This means this equation can only have rational solutions for $x = 0$. If $x = 0$ then $r^2 + y^2 = r^2$ and so $y = 0$ and so $(0, 0)$ is the only such solution.

Same solution applies to a sphere: $(x - r)^2 + y^2 + z^2 = r^2$ for some irrational r. In this case: $r = (x^2 + y^2 + z^2)/2x$. Which means that for $x \neq 0$ any rational soultion would imply a rational r. $(0, 0, 0)$ is therefor the only rational solution.

Yes, there are circles with more than one rational point. Take $x^2 + y^2 = 2$ which is the circle whose center is at $(0, 0)$ and whose radius is $\sqrt{2}$. The points $(1, 1), (-1, 1), (1, -1), (-1, -1)$ are four rational points which are on this circle.

Yes, there are circles with infinite rational points (TODO).

What about for each $n \in \mathbb{N}$? (TODO).

2 Always lead in election (Combinatorics)

An election was voted a perfect tie (even number of electors) and was decided by way of putting all the votes into a hat, mixing them uniformly and counting

them one by one. What is the chance that one of the candidates was leading during the entire counting process?

2.1 Solution

The question could be rephrased as how many graphs that go either one step up or one step down (lets call those binary graphs) do not go down below zero out of the set of all graphs that go from zero to zero. At first lets note that in the following solution n is even and so there is no problem with $n/2$. A first observation is that the set of all graphs going from 0 to 0 in n steps is $\binom{n}{n/2}$.

Let's try to count the graphs that tip below 0. Every graph that tips below 0 will reach -1 at one or more points. Lets find the first place where the graph reaches -1. From that point on the graph climbs one more than it descends since it finally reaches 0. Lets flip it from that point on - meaning switch every climb with descent and every descent with a climb. Now the graph descends one more than it climbs and since it starts from that point at height -1 then it now reaches a height -2 at n instead of the original 0.

The main proposition is that *every* graph which travels from 0 to -2 correspons with a *1-1 correspondance* to graphs that travel from 0 to 0 and reach -1 at some point (convince yourself of this).

If this is so then the number of graphs that go from 0 to -2 is $\binom{n}{n/2+1}$ since we need to choose points at which the graph will go down and we have $n/2+1$ descends. If so then the number of graphs that do not reach -1 is:

$\binom{n}{n/2} - \binom{n}{n/2+1} = \frac{(n)!}{(n/2+1)!(n/2)!} = \frac{1}{n/2+1}\binom{n}{n/2}$.

And now back to the original question "What is the chance that one of the candidates was leading during the entire counting process?". Lets divide the latter result with the former and get:

$\frac{1}{n/2+1}\binom{n}{n/2}/\binom{n}{n/2} = \frac{1}{n/2+1}$

The result is all about http://en.wikipedia.org/wiki/Catalan_number and http://en.wikipedia.org/wiki/Bertrand%27s_ballot_theorem.

One of the interesting results of this question is that $\binom{n}{n/2}$ is divisible by $n/2+1$. One reason is the argument stated above but a more direct number theory based argument may be found. A more general result relating to $\binom{n}{k}$ for any k may also be obtained (TODO).

3 Four Coke bottles on a table (Geometry)

Arrange four Coke bottles on a table so that the distances between each pair of caps will be the same. Assume that you can make a Coke bottle stand on it's head (this is a pretty strong hint!, maybe I should remove it?!?). The length or height of each bottle is h.

3.1 Solution

There are two solutions:

The first solution is to put the bottles in a square where the side of the square is the length of the bottle. Bottles next to each other will stand *in reverse* (if the first is on it's head the the second will stand straight).

The shape created by the bottle caps is, ofcourse, a perfect triangular pyramid with a side of length $d = \sqrt{2h^2}$.

The second solution is to put three bottles on the vertices of an equilateral triangle standing one way. The fourth will be placed in the middle of the triangle and will be standing reversed to the others. The problem here is to calculate the side of the triangle. Lets use the following diagram:

What we are trying to find is d as a function of h. Since H, $d/2$ and d create a right angeled triangle, it follows thae $H^2 + (d/2)^2 = d^2$ and so $H = \sqrt{3}/2d$. Since the size of the inner triangle could be computed two ways then $hH = dH'$ and since $H' = \sqrt{H^2 - (d/2)^2}$ it follows after some math that $d = \sqrt{3/2}h$ which makes d about $1.224h$.

The solution built on this analysis will look like this:

As is evident the key to solving this question is the perfect triangular pyramid as both solutions are based on it and it is the only way to put 4 elements in space being spaced equally apart. In theory you could place the bottles at any height and so infinite solutions to this riddle could be produced by making the side of the perfect pyramid as small or as large as you want. But we are constrained by the table and so the heights at which the caps are will be either 0 or h. The two solutions above seem to be primary examples under this constraint. Minor variations on the above two solutions could be produced by making the bottles whose caps are at table level lay instead of stand on their heads. The first solution is more attractive than the second because it could be performed in practice without relying on measurement tools in order to place the bottles at the calculated distance from each other as per the second solution.

4 Strange Trip (Geometry)

You travel 100 miles north, 100 miles east, and then 100 miles south. You are at the same point that you started from. Describe all the places on earth this could be, if any.

4.1 Solution

At the south pole. Or, 100 miles south of any point on the circle around the north pole that is 100 miles in circumference. Or, 100 miles south of any such circle whose circumference is an integral fraction of 100 miles.

5 Three Cards (Probability)

There are 3 cards: one is all red, one is all blue, and the third is blue on one side and red on the other. The cards are shuffled. You pick one at random and it is blue on the side facing you. What are the chances that it is also blue on the other side?

5.1 Solution

2/3. 2 of the 3 blue sides have blue on the other side. You might think the odds are 50/50 because you could have one of two cards, but you actually get a side at random, not a card, so you are more likely to have the blue/blue card than the red/blue card.

Here is another way to think of it: When you pick a card at random, the chances of it being the same on both sides (either all red or all blue) are 2 out of 3, and this doesn't change just because you see one of the sides. If they did, they'd change no matter which color you saw, and it wouldn't even matter if you looked. So when you do see the blue side, the red/red card is eliminated, but the blue/blue card still has a 2/3 probability.

6 Monty Hall Problem (Probability)

Behind 1 of 3 closed doors is a prize. You pick one of the doors. Monty opens one of the other doors and it is empty. You are given the choice of sticking with your choice or switching to the other unopened door. Should you switch? What are your chances of winning if you do?

6.1 Solution

You should switch. Your chances are 2/3 if you do. The initial chances of your first choice were 1/3, and opening another door without the prize doesn't actually change that, so the remaining door now has a 2/3 chance because the chances of all possibilities must sum to 1.

Many people think the chances are 50/50 because there are two doors left, but that is not correct. If you're not convinced 2/3 is correct, consider a 100 door version: you pick 1 door and Monty opens 98 other doors avoiding the one with the prize. Now you have a 99% chance if you switch.

7 4 Door Monty Hall Problem (Probability)

Behind 1 of 4 closed doors is a prize. You pick door 1. Monty opens the 4th door and it is empty. You switch your choice to door 2. Now monty opens door 3 and it is also empty. Given the choice, should you switch back to door 1, and what are your chances of winning if you do?

7.1 Solution

Yes, your chances are 5/8 if you switch. The initial chances of your first choice were 1/4, and opening another door without the prize doesn't change that, so the remaining 2 doors now have a 3/8 chance because the chances of all possibilities must sum to 1. But then when Monty opens door 3, using the same reasoning, door 2 stays at 3/8 chance and so the remaining door changes to 5/8.

8 Two Envelopes, Also known as the exchange paradox (Probability)

There are two envelopes, one contains twice as much money as the other. You pick one at random and find x dollars inside. What is the "expected value" of the amount in the other envelope? Given the option, should you switch? Does it matter how much you found in that first envelope? Does it matter if you open the envelope?

8.1 Solution

This is a strange paradox because it appears that the expected value is the average of $2x$ and $1/2x$ which are equally probable. The result is therefore $1.25x$ so you should switch. However, this would be true for any value of x you find, so it wouldn't even matter if you open the envelope or not. This same logic suggests that you should also immediately switch back to the original envelope again, which doesn't make sense at all.

The fault is in the wrong working out of the expetency. The x in each option is not the same x since the sums $1/2x$, x and $2x$ do not occur simultaneously. Lets do the calculation more carefully. We will denote the envelopes with the sums $1/2x$ and x. With probability of $1/2$ you got the envelope with $1/2x$ which means you $1/2x$ to gain if you switch. With the same probability of $1/2$ you got the envelope with x in which case you have $1/2x$ to lose if you switch. This means that the expected value of switching is 0 as it should be. No paradox at all. If we had denoted the two envlopes x and $2x$ we would have got the same result: with probability $1/2$ we got the envelope with x in it and can gain x if we switch. With the same probability we got the envelope with $2x$ and stand to lose x if we switch. The overall expected value is again 0 as it should be.

9 100 urns (Mathematics)

There's a room with 100 urns, 2 people standing outside, while they decide on what to do 100 notes with the numbers $1, 2, \ldots, 100$ are randomly placed inside the urns. One of the people from outside will enter the room and will examine all the urns. That person will be allowed to switch at most one pair of notes. That person is then whisked away to a vacation in Mali. The other one will then enter, be given a random number and will be asked to find the note with that number on it. He needs to find it while checking as few urns as possible. Design a strategy for the pair that will allow the second person to look into less than 50 urns to find any number required and in general look at the least number of urns in order to find the note with the required number on it.

9.1   Solution

Lets give a solution which is not good, analyze it and then improve it. Here is the bad solution. Whenever the second person comes in he is given the task of finding the note marked x. He will proceed to look into urn x. If x is there he is done. If not, then some other note y is there. He will process to do the same for y.

Analysis: Can the second person get stuck in an infinite loop? No. There are only 100 urns so in the worst case scenario he will check all 100 urns and find the note x in the last one. Now lets take this further. Lets define an equivalence relation on the urns: two urns x and y are equivalent if there exists a route from x to y. This is indeed an equivalence relation since there is always a route from x to x. If there is a route from x to y then there is also a route from y to x since in our case there is only one route to each urn (only one urn carries the note with it's number). Since starting a route from y must eventually lead back to y (routes do not just end), it must pass through x. All this means is that our 100 urns are divided into cycles of equivalence.

Now for the first players job. When the first player comes along he will either see all cycles with $size <= 50$ in which case he need not do anything (unless he wants to break up the largest cycle to reduce the expected value of urns checked by the second player) or will be confronted with utmost a single cycle with $size > 50$ which means he needs to break just this one cycle.

How does one go about "breaking" a cycle? A cycle is a list of urns $x_1, x_2, \ldots, x_{n-1}, x_n$ that make up a cycle, which means their content is $x_2, x_3, \ldots, x_n, x_1$. In order to break the cycle you need to find the center of the cycle. If the length of the cycle is odd then the center is the middle element if the lengh is even pick any of the central two elements. In our case the longest cycle cannot be of length greater than 100 which means that breaking it up in the center will break it up to pieces less or equal to 50. Lets say the center is in urn k. Switch the content of urns k and n. This will create two cycles: $x_1, x_2, \ldots, x_k$ with content $x_2, x_3, \ldots, x_1$ and another cycle $x_{k+1}, x_{k+2}, \ldots, x_n$ with content $x_{k+2}, x_{k+3}, \ldots, x_{k+1}$.

9.2   Follow up

A similar riddle. In this one you do not need to change the content of any urn. Imagine that 100 prisoners numbered zero to a hundred need to find their own number in the urns above and follow the same algorithm as in the solution. What are the chances that they will all succeed? Phrased a different way we ask what are the chances that given that the notes in the urns were randomly distributed what are the chances that there are no cycles greater than 50?

10   Blue and red martians (Distributed algorithms)

On mars live two types of aliens: ones with blue dots on their head and ones with blue dots. No martian knows what color dot he/she has on their head and

they do not have verbal skills. One day they get told that a new virus is out there that only kills the blue kind and therefor all the blue martians must leave the planet to space where the virus cannot survive. Red martians, however, must not leave the planet since they cannot survive in outer space. Each day a ship, large enough to hold all the martians, will leave at noon carrying martians to outer space. Can just the blue martians leave? if so how? P.S. there is at least one martian with a blue dot on his head and everyone knows so.

10.1   Solution

Since the martians are very smart, if there are N martians then all of them will leave on day N. How is that? The proof is by induction. Imagine that there is only one martian with a blue dot. He sees everyone with red dots on their head and since he knows that there is at least one martian with a blue dot on his/her head he deduces that it is he. He therefor leaves in the first possible occasion, meaning the first day. Now imagine that the proposition is true for N. Now imagine that there are $N+1$ martians. Each of them sees only N blue dots in the crowd. The deduce that there are either N martians with blue dots (excluding themselves) or $N + 1$ blue dotted martians (including themselves). They don't know which. So they wait for day N. If there were indeed N martians all of them would have left on day N. They do not leave. From this each one of the $N + 1$ deduces that the other N martians see N blue dots in the crowd which means that they themselves are blue dotted which means that there are $N + 1$ blue dotted martians which include themselves. This prompts them to leave on the first possible occasion, which is day $N + 1$.

11   The Wardens challenge (Distributed algorithms)

A warden summons all of his prisoners to a talk. "From this day on you will not see each other. I will select, by random and with possible repetitions, one of you each day and will place him or her in a special holding chamber. The holding chamber has a light bulb with a switch. When the day is done I will lead the prisoner back to his or her cell. At any time, any one of who understands that all of you have been in the special holding cell may callout and stop the experiment. If that person is right the you will all go free and if he is wrong you will stay here for life". The prisoners then have time to confer and plan a strategy before they are taken to their cells and are prevented from speaking to one another for the remainder of the experiment. Plan the prisoners strategy.

11.1   Solution

A first solution would be to use a special designated person to act as counter. The counter never flips the light switch on. Every time the counter is called into the holding cell and sees the light he/she adds 1 to a sum he/she is storing in memory as to the number of people that have been in the holding cell and

flips the light off. Every other prisoner will follow the following rules: If you have never flipped the switch on and the switch is off when you enter the room then switch it on. Otherwise do nothing. The counter is the only person to proclaim that everyone has been to the holding cell and obviously has to count himself/herself as well.

The problem with this solution is that it's efficiency is quite low. Time is also wasted when a prisoner that has never been to the cell enters it and sees the light on which prevents him from turning the light on himself and makes him wait for the next time he/she is called to the holding cell.

12 Jumping frogs (Geometric)

Four frogs are sitting on the vertices of a square the side of which is of length 1. In turns, a frog may lep over any other frog jumping twice the distance between them. Could the frogs, after any number of steps, form a square of a size different than 1? Another version would be: can you prove that if, after any number of jumps, the frogs form a square then the size of the square will be 1?

12.1 Solution

If we put the frogs on an infinite grid of size 1 then after each jump all frogs must be located on that grid. So if the frogs can create a square of a size bigger than 1 then that size must be a whole number which is bigger than 1. Now lets note that if the frogs can create a square bigger than 1 then, starting from that square, then can reverse their steps and create a square which is of size 1. This means that the frogs can make the square they form bigger or smaller by some factor at any time. The problem with this is that this is also true for the original sized 1 square: the frogs can make it smaller by that same factor but that is a contradiction to the fact that the frogs always remain on the grid.

13 Liers and Truth tellers (Logic)

In a cicle there be n people, some tall and some short. Tall people always tell the truth while short people always lie. All of them tell you that the two people on their side are of different heights. How many liers and truth tellers are there?

13.1 Solution

Lets see what happens on the circle itself. Pick any spot and look at the two people from that spot onwards. There could be only 4 possibilities: $(t,t)(l,t)(t,l)(l,l)$. If possibility one occurs then the next person must be a lier for the truth teller in the middle to be right, creating (t,t,l). If possibility two occurs then the next person must be a truth teller for the same reaons and we get (l,t,t). If the third option is the one then the next person must be a truth teller so that

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the lier reports, wrongly, that two people of different height are near him. This will give us (t, l, t). Consider these three options, they all combine to form an endless chain of $(t, t, l, t, t, l, t, t, l, \ldots)$ which means that a third of people are liers. Option 4, (l, l) means that the next person is also a lier and so on forever which gives us all liers. To sum up: either a third of people are liers or all of them.

TODO: is there a way to solve this using a formula? I feel there should be . . .

14 $2^{2n} - 1$ (Number theory)

Prove that for each $n \in \mathbb{N} > 0$, $2^{2n} - 1$ is divisible by 3

14.1 Solution

$2^{2n} - 1 = (2^n)^2 - 1^2 = (2^n - 1)(2^n + 1)$. 2^n is not divisible by three (because it's primary decomposition is made up of only 2's. . .). For each three consequitive numbers one must be divisible by 3 and this applies to the following three consequitive numbers: $2^n - 1, 2^n, 2^n + 1$. Since 2^n is not divisible by 3 then either $2^n - 1$ or $2^n + 1$ must be divisible by three and hence, using the equation above, $2^{2n} - 1$ is divisible by 3.

15 25 horses, 3 best (Sorting)

You need to find the 3 fastest horses out of 25. You can only race 5 at a time and you don't have a timer. What's the minimum number of races you need and who will you race?

15.1 Solution

Lets mark all horses as H_1 - H_{25}. First race 5 races H_1-H_5, H_6-H_{10}, H_{11}-H_{15}, H_{16}-H_{20}, H_{21}-H_{25}. Denote the winners as $W_{1,5}$, $W_{6,10}$, $W_{11,15}$, $W_{16,20}$, $W_{21,25}$. Second runners up will be $SRU_{1,5}$, $SRU_{6,10}$, $SRU_{11,15}$, $SRU_{16,20}$, $SRU_{21,25}$. Third runners up will be $TRU_{1,5}$, $TRU_{6,10}$, $TRU_{11,15}$, $TRU_{16,20}$, $TRU_{21,25}$. We will race all the winners and get WW_1 - WW_5.

Now lets use some logic.

WW_1 is the fastest around so we don't need to race him again and he will be the top in the top 3.

Discard anyone in the group that got won by WW_4 and WW_5 since they got beat by at least three horses (WW_1, WW_2, WW_3) so they, and by inference anyone they beat, cannot be in the top 3.

Discard anyone in the group that got won by WW_3 but not WW_3 himself. This is because there are at least two horses better than WW_3 so anyone that WW_3 beat cannot be in the top 3 but WW_3 maybe the third in the entire group. So WW_3 joins the final race.

In the group of horses won by WW_2 we can discard everyone but WW_2 and the second runner up. Third runner up and down have at least 3 horses better than them (SRU_{WW_2}, WW_2 and WW_1). So WW_2 and SRU_{WW_2} joins the race.

In the group of WW_1 we can discard everyone but the second and third runners up. So SRU_{WW_1} and TRU_{WW_1} join the race.

So now we will race WW_3, WW_2, SRU_{WW_2}, SRU_{WW_1}, TRU_{WW_1}. The first two will join WW_1 in being the fastest 3.

Number of races: 7

TODO: how do I prove this is the minimum?

16 Find number of bits in large data set (Computing efficiency)

You get an array of 10000 32 bit integers and need to find out how many bits are turned on in the entire array. You are allowed to do precomputation and prepare any data structure of any size you see fit. What will you do? What will you do in practice?

16.1 Solution

The idea is to prepare a 2^{32} array of integers where for each place n the array contains the number of bits which are on for the number n. Then the computation is just a matter of iteration and summing up.

In practice this is a bad idea since the 2^{32} array is too large and could cause cache misses. Better is to hold an array of size 2^{16} or maybe even 2^8 so as to make a cache hit all the time for L1 cache. The size of the array should depend on the size of cache you want to keep hitting and the tradeoff between cache hit and doing more inside the loop should be determined by tests in practice.

17 Two bits out of three using one (Information theory)

You need to setup a communication protocol to transport 3 bits at a time. Whenever you are given three bits of information you may only pass one bit over the medium to represent that information. The receiver should be able to produce the 3 bits with a maximum of 1 bit of mistake (anywhere).

17.1 Solution

If we put all 8 possible 3 bit combinations in a graph where two combinations are connected via an edge iff there is just 1 bit of information difference between them we see that 000 and 111 are at the edges of the graph and are 1 bit close to $001, 010, 100$ and $011, 101110$ respectively. This means that the entire group of 8 possibilities could be divided into 2 groups of 4 thus: $000, 001, 010, 100$ and

$111, 011, 101, 110$. So when we get 3 bits of information we will send 0 for any member of the first group and 1 for any member of the second. The receiver will guess 000 for 0 and 111 for 1. This means that the receiver will be wrong by at most 1 bit. The chance that the receiver will get the 3rd bit right is 1 in 4 if the input is evenly distributed. We can also reverse the meaning of 0 and 1 and have the decoder guess 111 for 0 and 000 for 1. Actually we are sending 0 when most bits are 0 and 1 when most bits are 1 and since most out of 3 is at least 2 we always get 2 bits right. Note that we can't choose other representatives for the groups, and not even representatives which are reverse of each other (like 001 and 110) since these are not distanced by at most 1 bit from the rest of the group that they are in (for instance 001 has distance 2 from 100).

If we look at the graph then we see that any bi-partition of the graph would work as well. For instance we could divide the graph to $001, 000, 101, 011$ and $110, 100, 010, 111$. The first group the representative will be 001 and the second group representative will be 110. Actually the graph is so symmetric that any two negative bitwise strings would do to partition it so there are 4 possible protocols (8 if reverse counts too).

18   Pigsty for six pigs (Geometry)

A farmer has obtained 13 panels in order to create 6 identical pigsties. He was planning to arrange them like this:

Unfortunately, one of his panels got stolen and he now only has 12 panels. Can you design 6 identical pigsties with 12 panels?

18.1 Solution

19 Three buckets with false info (Boolean logic)

You have three buckets in front of you. One has only black balls in it. One has only white balls in it. The last one has an even mix of black and white. You don't know which is which. Each bucket carries a note. The notes say "black only", "white only" and "mixed". It is known that all three notes are wrong. You need to design an experiment in which you will extract balls from the buckets of your choosing in order to find out which bucket has which content. Your aim is to minimze the number of ball extractions to attain your goal. How many balls will you need to extract in order to ascertain which content is in which bucket?
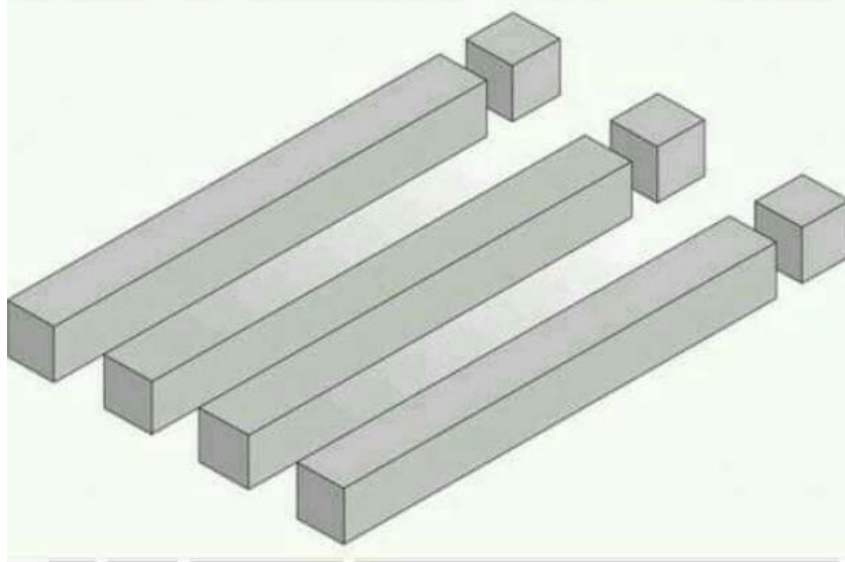
19.1 Solution

Only one ball needs to be extracted from the bucket which is labeled $Mixed$. Since the bucket is marked as $Mixed$ and that is known to be false then the bucket must be either all black or all white. So the single ball we extract tells us which of the two it is. Lets call this color N and it's opposite M. Now we know what this bucket is of color N. One of the two other buckets claims to be of color M. Since it's label is false it must either be N or the true $Mixed$ bucket. We already know which bucket is N and it's not that one. That means that the bucket labeled as M is actually the true $Mixed$ bucket. The last bucket is the bucket of true M.

20 Three or four boxes (Visual illusion)

Are there three or four boxes in the following drawing? How do you explain the illusion?

20.1 Solution

The heart of the illusion is the gradual transformation of floors on the right hand side of the drawing into sides of boxes in the left hand side of the drawing. That is also the reason for the gradual transformation of color of the floors.