Complex hyperbolic 2×2 rotations

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March 21, 2025

1 Formulas

Let $V \in \mathbb{C}^{2\times 2}$ be J-unitary and let $A \in \mathbb{C}^{2\times 2}$ be a Hermitian positive semidefinite matrix, where

$$A = \begin{bmatrix} a_{11} & \overline{a_{21}} \\ a_{21} & a_{22} \end{bmatrix}, \quad J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad V = \begin{bmatrix} \cosh\phi & \mathrm{e}^{-\mathrm{i}\beta}\sinh\phi \\ \mathrm{e}^{\mathrm{i}\beta}\sinh\phi & \cosh\phi \end{bmatrix}. \tag{1}$$

Then, $V^* = V$ and $V^*JV = VJV^* = J$. Find β and ϕ such that $V^*AV = \Xi$,

$$\Xi = \begin{bmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{bmatrix} = \begin{bmatrix} \cosh\phi & \mathrm{e}^{-\mathrm{i}\beta}\sinh\phi \\ \mathrm{e}^{\mathrm{i}\beta}\sinh\phi & \cosh\phi \end{bmatrix} \begin{bmatrix} a_{11} & \overline{a_{21}} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \cosh\phi & \mathrm{e}^{-\mathrm{i}\beta}\sinh\phi \\ \mathrm{e}^{\mathrm{i}\beta}\sinh\phi & \cosh\phi \end{bmatrix}. \tag{2}$$

Dividing (2) by $\cosh^2\phi > 0$ gives, with $\xi_i' = \xi_i/\cosh^2\phi$,

$$\begin{bmatrix} 1 & \mathrm{e}^{-\mathrm{i}\beta}\tanh\phi \\ \mathrm{e}^{\mathrm{i}\beta}\tanh\phi & 1 \end{bmatrix} \begin{bmatrix} a_{11} & \overline{a_{21}} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 & \mathrm{e}^{-\mathrm{i}\beta}\tanh\phi \\ \mathrm{e}^{\mathrm{i}\beta}\tanh\phi & 1 \end{bmatrix} = \begin{bmatrix} \xi_1' & 0 \\ 0 & \xi_2' \end{bmatrix}. \tag{3}$$

Multiplying the first two matrices in (3) gives

$$\Xi' = \begin{bmatrix} a_{11} + a_{21}\mathrm{e}^{-\mathrm{i}\beta}\tanh\phi & \overline{a_{21}} + a_{22}\mathrm{e}^{-\mathrm{i}\beta}\tanh\phi \\ a_{11}\mathrm{e}^{\mathrm{i}\beta}\tanh\phi + a_{21} & \overline{a_{21}}\mathrm{e}^{\mathrm{i}\beta}\tanh\phi + a_{22} \end{bmatrix} \begin{bmatrix} 1 & \mathrm{e}^{-\mathrm{i}\beta}\tanh\phi \\ \mathrm{e}^{\mathrm{i}\beta}\tanh\phi & 1 \end{bmatrix},$$

with the final multiplication producing, elementwise,

$$\xi_1' = a_{11} + (2\Re(a_{21}\mathrm{e}^{-\mathrm{i}\beta}) + a_{22}\tanh\phi)\tanh\phi, \tag{4}$$
$$0 = \overline{a_{21}} + (a_{11} + a_{22} + a_{21}\mathrm{e}^{-\mathrm{i}\beta}\tanh\phi)\mathrm{e}^{-\mathrm{i}\beta}\tanh\phi, \tag{5}$$
$$0 = a_{21} + (a_{11} + a_{22} + \overline{a_{21}}\mathrm{e}^{\mathrm{i}\beta}\tanh\phi)\mathrm{e}^{\mathrm{i}\beta}\tanh\phi, \tag{6}$$
$$\xi_2' = a_{22} + (2\Re(a_{21}\mathrm{e}^{-\mathrm{i}\beta}) + a_{11}\tanh\phi)\tanh\phi, \tag{7}$$

*with suggestions from Vjeran Hari

since $z + \bar{z} = 2\Re z$. Evidently, $\xi_i \in \mathbb{R}$. If $a_{21} = 0$ then $\tanh\phi = 0$, and vice versa, if $\tanh\phi = 0$, then from (5) (or (6), which is the complex conjugate of (5)) it follows $a_{21} = 0$. Therefore, assume in the following that $\tanh\phi \neq 0$.

Multiplying (5) by $\mathrm{e}^{\mathrm{i}\beta}$ and (6) by $\mathrm{e}^{-\mathrm{i}\beta}$, it follows

$$\begin{aligned} 0 &= \overline{a_{21}}\mathrm{e}^{\mathrm{i}\beta} + (a_{11} + a_{22})\tanh\phi + a_{21}\mathrm{e}^{-\mathrm{i}\beta}\tanh^2\phi, && (8)\\ 0 &= a_{21}\mathrm{e}^{-\mathrm{i}\beta} + (a_{11} + a_{22})\tanh\phi + \overline{a_{21}}\mathrm{e}^{\mathrm{i}\beta}\tanh^2\phi. && (9)\end{aligned}$$

By extracting the common middle term on the right hand sides of (8) and (9),

$$\begin{aligned} -(a_{11} + a_{22})\tanh\phi &= \overline{a_{21}}\mathrm{e}^{\mathrm{i}\beta} + a_{21}\mathrm{e}^{-\mathrm{i}\beta}\tanh^2\phi, && (10)\\ -(a_{11} + a_{22})\tanh\phi &= a_{21}\mathrm{e}^{-\mathrm{i}\beta} + \overline{a_{21}}\mathrm{e}^{\mathrm{i}\beta}\tanh^2\phi, && (11)\end{aligned}$$

it follows that the right hand sides of (10) and (11) have to be equal,

$$\overline{a_{21}}\mathrm{e}^{\mathrm{i}\beta} + a_{21}\mathrm{e}^{-\mathrm{i}\beta}\tanh^2\phi = a_{21}\mathrm{e}^{-\mathrm{i}\beta} + \overline{a_{21}}\mathrm{e}^{\mathrm{i}\beta}\tanh^2\phi, \tag{12}$$

while at the same time being the complex conjugates of one another. Therefore, both sides in (12) are real, what (10) and (11) also suggest. With

$$\beta = \arg(a_{21}), \quad \text{i.e.,} \quad \mathrm{e}^{\mathrm{i}\beta} = \frac{a_{21}}{|a_{21}|}, \quad (a_{21} = 0 \implies \beta = 0), \tag{13}$$

this condition is always fulfilled. If a_{21} is real in (1), then $\mathrm{e}^{\mathrm{i}\beta} = \operatorname{sign} a_{21}$.

Now, (8) and (9) become

$$|a_{21}|(1 + \tanh^2\phi) = -(a_{11} + a_{22})\tanh\phi, \tag{14}$$

or, after rearranging (14),

$$\frac{-|a_{21}|}{a_{11} + a_{22}} = \frac{\tanh\phi}{1 + \tanh^2\phi} = \frac{1}{2}\tanh(2\phi). \tag{15}$$

Note that $a_{11} \geq 0$ and $a_{22} \geq 0$ by definition, so $a_{11} + a_{22} = 0$ if and only if $a_{11} = 0$ and $a_{22} = 0$, in which case $a_{21} = 0$, and thus (15) does not even have to be evaluated in this degenerate case to compute $\tanh\phi = 0$.

The proper sequence of checks on the inputs a_{11}, a_{22}, and a_{21} is thus

1. if $a_{11} < 0$ then ERROR; else
2. if $a_{22} < 0$ then ERROR; else
3. if $a'_{21} = 0$ then $\tanh\phi = 0$,

where a'_{21} is a_{21}, scaled as in (20). Else, proceed to compute $\tanh(2\phi)$. If it is detected that the computed $\tanh(2\phi) \le -1$, then ERROR, since the input does not define a positive semidefinite matrix.

Alternatively, the hyperbolic cotangent of 2ϕ might be computed as:

$$\coth(2\phi) = \frac{a_{11} + a_{22}}{-2|a_{21}|}, \tag{16}$$

but this can easily cause overflow for small $|a_{21}|$. On the other hand, underflow of $\tanh(2\phi)$, as long as it is not exactly zero, will cause no trouble in the further computation, and might preserve at least some information, possibly resulting in a non-zero $\tanh\phi$ as well.

There are two solutions for the quadratic equation for $\tanh\phi$ from (15), only one of which obeys $|\tanh\phi| < 1$,

$$\tanh\phi = \frac{1 - \sqrt{1 - \tanh^2(2\phi)}}{\tanh(2\phi)}. \tag{17}$$

However, if the numerator and the denominator in (17) are multiplied by $1 + \sqrt{1 - \tanh^2(2\phi)}$, a more stable form emerges,

$$\begin{aligned}\tanh\phi &= \frac{1 - \sqrt{1 - \tanh^2(2\phi)}}{\tanh(2\phi)} \cdot \frac{1 + \sqrt{1 - \tanh^2(2\phi)}}{1 + \sqrt{1 - \tanh^2(2\phi)}} \\ &= \frac{1 - (1 - \tanh^2(2\phi))}{\tanh(2\phi)\left(1 + \sqrt{1 - \tanh^2(2\phi)}\right)} \\ &= \frac{\tanh(2\phi)}{1 + \sqrt{1 - \tanh^2(2\phi)}} \approx \frac{\tanh(2\phi)}{1 + \mathrm{sqrt}(\mathrm{fma}(-\tanh(2\phi), \tanh(2\phi), 1))}.\end{aligned} \tag{18}$$

Since $|\tanh(2\phi)| < 1$, from (18) it follows that $|\tanh\phi| < 1$ as well. Now,

$$\begin{aligned}\cosh\phi &= \frac{1}{\sqrt{1 - \tanh^2\phi}} \approx \mathrm{rsqrt}(\mathrm{fma}(-\tanh\phi, \tanh\phi, 1)), \\ \sinh\phi &= \tanh\phi \cdot \cosh\phi.\end{aligned} \tag{19}$$

If $\tanh\phi$ is regarded as a computed floating-point value, then $|\tanh\phi| \le 1^-$, where 1^- is the first floating-point predecessor of unity. Otherwise, the computed $|\tanh(2\phi)|$ would have to be unity, what has already been ruled out. Thus,

$$\tanh^2\phi < |\tanh\phi| \le 1^- \implies 1 - \tanh^2\phi \ge 1 - 1^-.$$

Since $1-1^-$ is a floating-point value in the normal range, its square root is as well, so $\cosh\phi$ cannot overflow in (19), and neither can $\sinh\phi$.

The input data has to be prescaled by the highest possible power of two, 2^s, to get $a'_{11}=2^s a_{11}$, $a'_{22}=2^s a_{22}$, and $a'_{21}=2^s a_{21}$, such that

$$\max\{a'_{11},a'_{22}\}\le\nu/2,\quad \max\{|\Re a'_{21}|,|\Im a'_{21}|\}\le\nu/2, \tag{20}$$

where ν is the largest finite floating-point value. This way no quantity in (15) will overflow[1], and the chances of dealing with subnormal values will be minimized. See the scaling of A and the computation of $\mathrm{e}^{\mathrm{i}\beta}$ in:
`https://doi.org/10.1016/j.cam.2024.116003`

2 Accuracy

Let the computed values, unlike the exact ones, be underlined in the following, let ε be the machine precision, and assume hypot is correctly rounded.

Also, assume that all inputs have been scaled exactly to a'_{11}, a'_{22}, a'_{21}.

Lemma 1. *If $|\tanh(2\phi)|\le 40/41$ then $|\tanh\phi|\le 4/5$, and vice versa.*

Proof. From (15),

$$|\tanh(2\phi)|=\frac{2|\tanh\phi|}{1+\tanh^2\phi}\le\frac{40}{41}\implies 40\tanh^2\phi-82|\tanh\phi|+40\ge 0.$$

This inequality is valid for $|\tanh\phi|\le 4/5$. $\square$

Lemma 2. *Barring inexact underflow, $\underline{|a'_{21}|}=|a'_{21}|(1+\epsilon_1)|$, $|\epsilon_1|\le\varepsilon$.*

Proof. From the assumption that hypot is correctly rounded. $\square$

Lemma 3. *With $|\epsilon_2|\le\varepsilon$, $|\epsilon_3|\le\varepsilon$, and ϵ_4 such that*

$$1+\epsilon_4=\frac{1+\epsilon_1}{1+\epsilon_2}(1+\epsilon_3),$$

barring inexact underflow it holds

$$\underline{\tanh(2\phi)}=\frac{-2|a'_{21}|(1+\epsilon_1)}{(a'_{11}+a'_{22})(1+\epsilon_2)}(1+\epsilon_3)=\tanh(2\phi)(1+\epsilon_4).$$

[1]not even $2|a'_{21}|$, since A is positive semidefinite, so $2|a'_{21}|\le 2\sqrt{a'_{11}a'_{22}}\le\nu$, but the upper bound of $\nu/4$ instead of $\nu/2$ in (20) might be safer

Proof. From Lemma 2. Note that

$$\frac{(1-\varepsilon)^{\gamma}}{1+\varepsilon} \le 1+\epsilon_4 \le \frac{(1+\varepsilon)^{\gamma}}{1-\varepsilon}, \tag{21}$$

where $\gamma = 2$ if a'_{21} is complex, and $\gamma = 1$ if a'_{21} is real (since then $\epsilon_1 = 0$). $\square$

Lemma 4. *Assume that* $|\tanh(2\phi)| \le 40/41$. *Then*

$$1 - (\underline{\tanh(2\phi)})^2 = (1 - \tanh^2(2\phi))(1+\epsilon_5),$$

where

$$|\epsilon_5| \le \frac{1600}{81}|\epsilon'_4|, \quad \epsilon'_4 = (2+\epsilon_4)\epsilon_4. \tag{22}$$

Proof. Let $y = 1 - \tanh^2(2\phi) \ge 81/1681$. Then, from Lemma 3,

$$1 - (\underline{\tanh(2\phi)})^2 = 1 - \tanh^2(2\phi)(1+\epsilon_4)^2 = y - \tanh^2(2\phi)\epsilon'_4,$$

where $(1+\epsilon_4)^2 = 1 + \epsilon'_4$, i.e., $\epsilon'_4 = (2+\epsilon_4)\epsilon_4$.

Using the definition of y, find ϵ_5 such that

$$y(1+\epsilon_5) = y + y\epsilon_5 = y - \tanh^2(2\phi)\epsilon'_4 = y + (y-1)\epsilon'_4.$$

Therefore, by subtracting y from these equalities,

$$y\epsilon_5 = (y-1)\epsilon'_4 \implies \epsilon_5 = \frac{y-1}{y}\epsilon'_4.$$

By taking the lower bound for y, and thus the upper bound for $|y-1|/|y|$, it follows that

$$|\epsilon_5| \le \frac{1600}{81}|\epsilon'_4|,$$

i.e., $|\epsilon_5| \lessapprox 19.753|\epsilon'_4|$. $\square$

Lemma 5. *With* $|\epsilon_6| \le \varepsilon$ *and* $|\epsilon_7| \le \varepsilon$,

$$\mathrm{sqrt}(\mathrm{fma}(-\underline{\tanh(2\phi)}, \underline{\tanh(2\phi)}, 1)) = \sqrt{1-\tanh^2(2\phi)}(1+\epsilon_8),$$

where

$$1+\epsilon_8 = \sqrt{(1+\epsilon_5)(1+\epsilon_6)}(1+\epsilon_7).$$

Proof. From Lemma 4 and the definition of fma,

$$\mathrm{fma}(-\underline{\tanh(2\phi)}, \underline{\tanh(2\phi)}, 1) = (1 - (\underline{\tanh(2\phi)})^2)(1+\epsilon_6).$$

$\square$

Lemma 6. *Let* $x = 1 + \sqrt{1 - \tanh^2(2\phi)}$*. Then,*

$$1 + \mathrm{sqrt}(\mathrm{fma}(-\underline{\tanh(2\phi)}, \underline{\tanh(2\phi)}, 1)) = x(1 + \epsilon_9),$$

where $|\epsilon_9| \le |\epsilon_8|/2$*.*

Proof. From Lemma 5 it follows

$$1 + \mathrm{sqrt}(\mathrm{fma}(-\underline{\tanh(2\phi)}, \underline{\tanh(2\phi)}, 1)) = 1 + (x - 1)(1 + \epsilon_8) = x + \epsilon_8(x - 1).$$

Using $50/41 \le x \le 2$, since $1 - \tanh^2(2\phi) \ge 81/1681$, find ϵ_9 such that

$$x(1 + \epsilon_9) = x + x\epsilon_9 = x + \epsilon_8(x - 1).$$

From the last two equalities subtraction of x gives

$$\epsilon_9 = \frac{x - 1}{x}\epsilon_8.$$

Therefore,

$$\frac{9}{50}|\epsilon_8| \le |\epsilon_9| \le \frac{1}{2}|\epsilon_8|.$$

□

Lemma 7. *The denominator in* (18) *is computed as*

$$(1 + \sqrt{1 - \tanh^2(2\phi)})(1 + \epsilon_{11}),$$

where, with $|\epsilon_{10}| \le \varepsilon$ *due to the rounding in the final addition,*

$$1 + \epsilon_{11} = (1 + \epsilon_9)(1 + \epsilon_{10}).$$

Proof. From Lemma 6. □

Theorem 1. *With* $|\epsilon_{12}| \le \varepsilon$*,*

$$\underline{\tanh\phi} = \tanh\phi(1 + \epsilon_t), \quad 1 + \epsilon_t = \frac{1 + \epsilon_4}{1 + \epsilon_{11}}(1 + \epsilon_{12}). \tag{23}$$

Proof. From (18), Lemma (3), and Lemma (7). □

Lemma 8. *With* $\tanh\phi \le 4/5$ *due to Lemma 1,*

$$1 - (\underline{\tanh\phi})^2 = (1 - \tanh^2\phi)(1 + \epsilon_{13}),$$

where

$$|\epsilon_{13}| \le \frac{16}{9}|\epsilon_t'|, \quad \epsilon_t' = (2 + \epsilon_t)\epsilon_t. \tag{24}$$

Proof. As for Lemma 4, using $z = 1 - \tanh^2\phi \geq 9/25$ instead of y. $\square$

Theorem 2. *With* $|\epsilon_{14}| \leq \varepsilon$ *and* $|\epsilon_{15}| \leq \varepsilon$,

$$\underline{\cosh\phi} = \cosh\phi(1+\epsilon_c),$$

where, since rsqrt *is assumed to be correctly rounded,*

$$1+\epsilon_c = \frac{1+\epsilon_{15}}{\sqrt{(1+\epsilon_{13})(1+\epsilon_{14})}}. \tag{25}$$

Proof. From (19), Lemma 8, and the definition of fma,

$$\mathrm{fma}(-\underline{\tanh\phi}, \underline{\tanh\phi}, 1) = (1-(\underline{\tanh\phi})^2)(1+\epsilon_{14}).$$

$\square$

Theorem 3. *With* $|\epsilon_{16}| \leq \varepsilon$,

$$\underline{\sinh\phi} = \sinh\phi(1+\epsilon_s), \quad 1+\epsilon_s = (1+\epsilon_t)(1+\epsilon_c)(1+\epsilon_{16}). \tag{26}$$

Proof. From (19), Lemma 8, and Theorem 2. $\square$

This completes the analysis if a'_{21} is real. Otherwise, with $|\epsilon_{17}| \leq \varepsilon$ and $|\epsilon_{18}| \leq \varepsilon$, and barring inexact underflow,

$$\underline{\Re e^{\mathrm{i}\beta}} = \frac{\Re a'_{21}(1+\epsilon_{17})}{|a'_{21}|(1+\epsilon_1)} = \Re e^{\mathrm{i}\beta}(1+\epsilon'_{\Re}), \quad \underline{\Im e^{\mathrm{i}\beta}} = \frac{\Re a'_{21}(1+\epsilon_{18})}{|a'_{21}|(1+\epsilon_1)} = \Im e^{\mathrm{i}\beta}(1+\epsilon'_{\Im}).$$

Theorem 4. *With* $|\epsilon_{19}| \leq \varepsilon$ *and* $|\epsilon_{20}| \leq \varepsilon$,

$$\underline{\Re e^{\mathrm{i}\beta}\sinh\phi} = \Re e^{\mathrm{i}\beta}\sinh\phi(1+\epsilon_{\Re}), \quad \underline{\Im e^{\mathrm{i}\beta}\sinh\phi} = \Im e^{\mathrm{i}\beta}\sinh\phi(1+\epsilon_{\Im}),$$

where

$$1+\epsilon_{\Re} = (1+\epsilon'_{\Re})(1+\epsilon_s)(1+\epsilon_{19}), \quad 1+\epsilon_{\Im} = (1+\epsilon'_{\Im})(1+\epsilon_s)(1+\epsilon_{20}). \tag{27}$$

Proof. From Theorem 3. $\square$

TODO: Upper bounds on $|\epsilon_t|$, $|\epsilon_c|$, $|\epsilon_s|$, $|\epsilon_{\Re}|$, and $|\epsilon_{\Im}|$ can be obtained symbolically in the terms of ϵ and γ.