Complex hyperbolic 2×2 rotations

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March 24, 2025

1 Formulas

Let $V \in \mathbb{C}^{2\times 2}$ be J-unitary and let $A \in \mathbb{C}^{2\times 2}$ be a Hermitian positive semidefinite matrix such that $2|a_{21}| < a_{11} + a_{22}$, where

$$A = \begin{bmatrix} a_{11} & \overline{a_{21}} \\ a_{21} & a_{22} \end{bmatrix}, \quad J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad V = \begin{bmatrix} \cosh\phi & \mathrm{e}^{-\mathrm{i}\beta}\sinh\phi \\ \mathrm{e}^{\mathrm{i}\beta}\sinh\phi & \cosh\phi \end{bmatrix}. \tag{1}$$

Then, $V^* = V$ and $V^*JV = VJV^* = J$. Find β and ϕ such that $V^*AV = \Xi$,

$$\Xi = \begin{bmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{bmatrix} = \begin{bmatrix} \cosh\phi & \mathrm{e}^{-\mathrm{i}\beta}\sinh\phi \\ \mathrm{e}^{\mathrm{i}\beta}\sinh\phi & \cosh\phi \end{bmatrix} \begin{bmatrix} a_{11} & \overline{a_{21}} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \cosh\phi & \mathrm{e}^{-\mathrm{i}\beta}\sinh\phi \\ \mathrm{e}^{\mathrm{i}\beta}\sinh\phi & \cosh\phi \end{bmatrix}. \tag{2}$$

Dividing (2) by $\cosh^2\phi > 0$ gives, with $\xi_i' = \xi_i / \cosh^2\phi$,

$$\begin{bmatrix} 1 & \mathrm{e}^{-\mathrm{i}\beta}\tanh\phi \\ \mathrm{e}^{\mathrm{i}\beta}\tanh\phi & 1 \end{bmatrix} \begin{bmatrix} a_{11} & \overline{a_{21}} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 & \mathrm{e}^{-\mathrm{i}\beta}\tanh\phi \\ \mathrm{e}^{\mathrm{i}\beta}\tanh\phi & 1 \end{bmatrix} = \begin{bmatrix} \xi_1' & 0 \\ 0 & \xi_2' \end{bmatrix}. \tag{3}$$

Multiplying the first two matrices in (3) gives

$$\Xi' = \begin{bmatrix} a_{11} + a_{21}\mathrm{e}^{-\mathrm{i}\beta}\tanh\phi & \overline{a_{21}} + a_{22}\mathrm{e}^{-\mathrm{i}\beta}\tanh\phi \\ a_{11}\mathrm{e}^{\mathrm{i}\beta}\tanh\phi + a_{21} & \overline{a_{21}}\mathrm{e}^{\mathrm{i}\beta}\tanh\phi + a_{22} \end{bmatrix} \begin{bmatrix} 1 & \mathrm{e}^{-\mathrm{i}\beta}\tanh\phi \\ \mathrm{e}^{\mathrm{i}\beta}\tanh\phi & 1 \end{bmatrix},$$

with the final multiplication producing, elementwise,

$$\begin{aligned} \xi_1' &= a_{11} + (2\Re(a_{21}\mathrm{e}^{-\mathrm{i}\beta}) + a_{22}\tanh\phi)\tanh\phi, && (4) \\ 0 &= \overline{a_{21}} + (a_{11} + a_{22} + a_{21}\mathrm{e}^{-\mathrm{i}\beta}\tanh\phi)\mathrm{e}^{-\mathrm{i}\beta}\tanh\phi, && (5) \\ 0 &= a_{21} + (a_{11} + a_{22} + \overline{a_{21}}\mathrm{e}^{\mathrm{i}\beta}\tanh\phi)\mathrm{e}^{\mathrm{i}\beta}\tanh\phi, && (6) \\ \xi_2' &= a_{22} + (2\Re(a_{21}\mathrm{e}^{-\mathrm{i}\beta}) + a_{11}\tanh\phi)\tanh\phi, && (7) \end{aligned}$$

*with suggestions from Vjeran Hari

since $z + \bar{z} = 2\Re z$. Evidently, $\xi_i \in \mathbb{R}$. If $a_{21} = 0$ then $\tanh\phi = 0$, and vice versa, if $\tanh\phi = 0$, then from (5) (or (6), which is the complex conjugate of (5)) it follows $a_{21} = 0$. Therefore, assume in the following that $\tanh\phi \neq 0$.

Multiplying (5) by $\mathrm{e}^{\mathrm{i}\beta}$ and (6) by $\mathrm{e}^{-\mathrm{i}\beta}$, it follows

$$\begin{aligned} 0 &= \overline{a_{21}}\mathrm{e}^{\mathrm{i}\beta} + (a_{11} + a_{22})\tanh\phi + a_{21}\mathrm{e}^{-\mathrm{i}\beta}\tanh^2\phi, \qquad (8)\\ 0 &= a_{21}\mathrm{e}^{-\mathrm{i}\beta} + (a_{11} + a_{22})\tanh\phi + \overline{a_{21}}\mathrm{e}^{\mathrm{i}\beta}\tanh^2\phi. \qquad (9)\end{aligned}$$

By extracting the common middle term on the right hand sides of (8) and (9),

$$\begin{aligned} -(a_{11} + a_{22})\tanh\phi &= \overline{a_{21}}\mathrm{e}^{\mathrm{i}\beta} + a_{21}\mathrm{e}^{-\mathrm{i}\beta}\tanh^2\phi, \qquad (10)\\ -(a_{11} + a_{22})\tanh\phi &= a_{21}\mathrm{e}^{-\mathrm{i}\beta} + \overline{a_{21}}\mathrm{e}^{\mathrm{i}\beta}\tanh^2\phi, \qquad (11)\end{aligned}$$

it follows that the right hand sides of (10) and (11) have to be equal,

$$\overline{a_{21}}\mathrm{e}^{\mathrm{i}\beta} + a_{21}\mathrm{e}^{-\mathrm{i}\beta}\tanh^2\phi = a_{21}\mathrm{e}^{-\mathrm{i}\beta} + \overline{a_{21}}\mathrm{e}^{\mathrm{i}\beta}\tanh^2\phi, \qquad (12)$$

while at the same time being the complex conjugates of one another. Therefore, both sides in (12) are real, what (10) and (11) also suggest. With

$$\beta = \arg(a_{21}), \quad \text{i.e.,} \quad \mathrm{e}^{\mathrm{i}\beta} = \frac{a_{21}}{|a_{21}|}, \quad (a_{21} = 0 \implies \beta = 0), \qquad (13)$$

this condition is always fulfilled. If a_{21} is real in (1), then $\mathrm{e}^{\mathrm{i}\beta} = \operatorname{sign} a_{21}$.

Now, (8) and (9) become

$$|a_{21}|(1 + \tanh^2\phi) = -(a_{11} + a_{22})\tanh\phi, \qquad (14)$$

or, after rearranging (14),

$$\frac{-|a_{21}|}{a_{11} + a_{22}} = \frac{\tanh\phi}{1 + \tanh^2\phi} = \frac{1}{2}\tanh(2\phi). \qquad (15)$$

Note that $a_{11} \geq 0$ and $a_{22} \geq 0$ by definition, so $a_{11} + a_{22} = 0$ if and only if $a_{11} = 0$ and $a_{22} = 0$, in which case $a_{21} = 0$, and thus (15) does not even have to be evaluated in this degenerate case to compute $\tanh\phi = 0$.

The proper sequence of checks on the inputs a_{11}, a_{22}, and a_{21} is thus

1. if $a_{11} < 0$ then ERROR; else

2. if $a_{22} < 0$ then ERROR; else

3. if $a'_{21} = 0$ then $\tanh\phi = 0$,

where a'_{21} is a_{21}, scaled as in (20). Else, proceed to compute $\tanh(2\phi)$.

If the computed $\tanh(2\phi) \leq -1$, then setting $\tanh(2\phi) = -1$ allows the computation to proceed, to get $\cosh\phi = \infty$ and $|\sinh\phi| = \infty$. The input in this case is invalid, and then it is up to the caller to provide a remedy.

Alternatively, the hyperbolic cotangent of 2ϕ might be computed as:

$$\coth(2\phi) = \frac{a_{11} + a_{22}}{-2|a_{21}|}, \tag{16}$$

but this can easily cause overflow for small $|a_{21}|$. On the other hand, underflow of $\tanh(2\phi)$, as long as it is not exactly zero, will cause no trouble in the further computation, and might preserve at least some information, possibly resulting in a non-zero $\tanh\phi$ as well.

There are two solutions for the quadratic equation for $\tanh\phi$ from (15), only one of which obeys $|\tanh\phi| < 1$,

$$\tanh\phi = \frac{1 - \sqrt{1 - \tanh^2(2\phi)}}{\tanh(2\phi)}. \tag{17}$$

However, if the numerator and the denominator in (17) are multiplied by $1 + \sqrt{1 - \tanh^2(2\phi)}$, a more stable form emerges,

$$\begin{aligned}\tanh\phi &= \frac{1 - \sqrt{1 - \tanh^2(2\phi)}}{\tanh(2\phi)} \cdot \frac{1 + \sqrt{1 - \tanh^2(2\phi)}}{1 + \sqrt{1 - \tanh^2(2\phi)}} \\ &= \frac{1 - (1 - \tanh^2(2\phi))}{\tanh(2\phi)\left(1 + \sqrt{1 - \tanh^2(2\phi)}\right)} \\ &= \frac{\tanh(2\phi)}{1 + \sqrt{1 - \tanh^2(2\phi)}} \approx \frac{\tanh(2\phi)}{1 + \mathrm{sqrt}(\mathrm{fma}(-\tanh(2\phi), \tanh(2\phi), 1))}.\end{aligned} \tag{18}$$

Since $|\tanh(2\phi)| < 1$, from (18) it follows that $|\tanh\phi| < 1$ as well. Now,

$$\begin{aligned}\cosh\phi &= \frac{1}{\sqrt{1 - \tanh^2\phi}} \approx \mathrm{rsqrt}(\mathrm{fma}(-\tanh\phi, \tanh\phi, 1)), \\ \sinh\phi &= \tanh\phi \cdot \cosh\phi.\end{aligned} \tag{19}$$

If $\tanh\phi$ is regarded as a computed floating-point value, then $|\tanh\phi| \leq 1^-$, where 1^- is the first floating-point predecessor of unity. Otherwise, the computed $|\tanh(2\phi)|$ would have to be unity, what has already been ruled out. Thus,

$$\tanh^2\phi < |\tanh\phi| \leq 1^- \implies 1 - \tanh^2\phi \geq 1 - 1^-.$$

Since $1 - 1^-$ is a floating-point value in the normal range, its square root is as well, so $\cosh\phi$ cannot overflow in (19), and neither can $\sinh\phi$.

The input data has to be prescaled by the highest possible power of two, 2^s, to get $a'_{11} = 2^s a_{11}$, $a'_{22} = 2^s a_{22}$, and $a'_{21} = 2^s a_{21}$, such that

$$\max\{a'_{11}, a'_{22}\} \le \nu/2, \quad \max\{|\Re a'_{21}|, |\Im a'_{21}|\} \le \nu/2, \tag{20}$$

where ν is the largest finite floating-point value. This way no quantity in (15) will overflow, and the chances of dealing with subnormal values will be minimized. See the scaling of A and the computation of $\mathrm{e}^{\mathrm{i}\beta}$ in:
`https://doi.org/10.1016/j.cam.2024.116003`

2 Accuracy

Let the computed values, unlike the exact ones, be underlined in the following, let ε be the machine precision, and assume hypot is correctly rounded.

Also, assume that all inputs have been scaled exactly to a'_{11}, a'_{22}, a'_{21}.

Lemma 1. *If* $|\tanh(2\phi)| \le 40/41$ *then* $|\tanh\phi| \le 4/5$*, and vice versa.*

Proof. From (15),

$$|\tanh(2\phi)| = \frac{2|\tanh\phi|}{1 + \tanh^2\phi} \le \frac{40}{41} \implies 40\tanh^2\phi - 82|\tanh\phi| + 40 \ge 0.$$

This inequality is valid for $|\tanh\phi| \le 4/5$. □

Lemma 2. *Barring inexact underflow,* $\underline{|a'_{21}|} = |a'_{21}|(1+\epsilon_1)|$*,* $|\epsilon_1| \le \varepsilon$*.*

Proof. From the assumption that hypot is correctly rounded. □

Lemma 3. *With* $|\epsilon_2| \le \varepsilon$*,* $|\epsilon_3| \le \varepsilon$*, and* ϵ_d *such that*

$$1 + \epsilon_d = \frac{1+\epsilon_1}{1+\epsilon_2}(1+\epsilon_3), \tag{21}$$

barring inexact underflow it holds

$$\underline{\tanh(2\phi)} = \frac{-2|a'_{21}|(1+\epsilon_1)}{(a'_{11} + a'_{22})(1+\epsilon_2)}(1+\epsilon_3) = \tanh(2\phi)(1+\epsilon_d).$$

Proof. From Lemma 2. □

Note that

$$\frac{(1-\varepsilon)^\gamma}{1+\varepsilon} \le 1 + \epsilon_d \le \frac{(1+\varepsilon)^\gamma}{1-\varepsilon}, \tag{22}$$

where $\gamma = 2$ if a'_{21} is complex, and $\gamma = 1$ if a'_{21} is real (since then $\epsilon_1 = 0$).

Lemma 4. *Assume that* $|\tanh(2\phi)| \le 40/41$. *Then*

$$1 - (\underline{\tanh(2\phi)})^2 = (1 - \tanh^2(2\phi))(1 + \epsilon_4),$$

where

$$|\epsilon_4| \le \frac{1600}{81}|\epsilon'_d|, \quad \epsilon'_d = (2 + \epsilon_d)\epsilon_d.$$

Proof. Let $y = 1 - \tanh^2(2\phi) \ge 81/1681$. Then, from Lemma 3,

$$1 - (\underline{\tanh(2\phi)})^2 = 1 - \tanh^2(2\phi)(1 + \epsilon_d)^2 = y - \tanh^2(2\phi)\epsilon'_d,$$

where $(1 + \epsilon_d)^2 = 1 + \epsilon'_d$, i.e., $\epsilon'_d = (2 + \epsilon_d)\epsilon_d$.

Using the definition of y, find ϵ_4 such that

$$y(1 + \epsilon_4) = y + y\epsilon_4 = y - \tanh^2(2\phi)\epsilon'_d = y + (y - 1)\epsilon'_d.$$

Therefore, by subtracting y from these equalities,

$$y\epsilon_4 = (y - 1)\epsilon'_d \implies \epsilon_4 = \frac{y - 1}{y}\epsilon'_d.$$

By taking the lower bound for y, and thus the upper bound for $|y - 1|/|y|$, it follows that

$$|\epsilon_4| \le \frac{1600}{81}|\epsilon'_d|,$$

i.e., $|\epsilon_4| \lessapprox 19.753|\epsilon'_d|$. □

Lemma 5. *With* $|\epsilon_5| \le \varepsilon$ *and* $|\epsilon_6| \le \varepsilon$,

$$\mathrm{sqrt}(\mathrm{fma}(-\underline{\tanh(2\phi)}, \underline{\tanh(2\phi)}, 1)) = \sqrt{1 - \tanh^2(2\phi)}(1 + \epsilon_7),$$

where

$$1 + \epsilon_7 = \sqrt{(1 + \epsilon_4)(1 + \epsilon_5)}(1 + \epsilon_6).$$

Proof. From Lemma 4 and the definition of fma,

$$\mathrm{fma}(-\underline{\tanh(2\phi)}, \underline{\tanh(2\phi)}, 1) = (1 - (\underline{\tanh(2\phi)})^2)(1 + \epsilon_5).$$

□

Lemma 6. *Let* $x = 1 + \sqrt{1 - \tanh^2(2\phi)}$. *Then,*

$$1 + \mathrm{sqrt}(\mathrm{fma}(-\underline{\tanh(2\phi)}, \underline{\tanh(2\phi)}, 1)) = x(1 + \epsilon_8),$$

where $|\epsilon_8| \le |\epsilon_7|/2$.

Proof. From Lemma 5 it follows

$$1 + \mathrm{sqrt}(\mathrm{fma}(-\underline{\tanh(2\phi)}, \underline{\tanh(2\phi)}, 1)) = 1 + (x-1)(1+\epsilon_7) = x + \epsilon_7(x-1).$$

Using $50/41 \le x \le 2$, since $1 - \tanh^2(2\phi) \ge 81/1681$, find ϵ_8 such that

$$x(1+\epsilon_8) = x + x\epsilon_8 = x + \epsilon_7(x-1).$$

From the last two equalities subtraction of x gives

$$\epsilon_8 = \frac{x-1}{x}\epsilon_7.$$

Therefore,

$$\frac{9}{50}|\epsilon_7| \le |\epsilon_8| \le \frac{1}{2}|\epsilon_7|.$$

□

Lemma 7. *The denominator in* (18) *is computed as*

$$(1 + \sqrt{1 - \tanh^2(2\phi)})(1+\epsilon_{10}),$$

where, with $|\epsilon_9| \le \varepsilon$ *due to the rounding in the final addition,*

$$1 + \epsilon_{10} = (1+\epsilon_8)(1+\epsilon_9).$$

Proof. From Lemma 6. □

Theorem 1. *With* $|\epsilon_{11}| \le \varepsilon$*,*

$$\underline{\tanh\phi} = \tanh\phi(1+\epsilon_t), \quad 1 + \epsilon_t = \frac{1+\epsilon_d}{1+\epsilon_{10}}(1+\epsilon_{11}). \tag{23}$$

Proof. From (18), Lemma (3), and Lemma (7). □

Lemma 8. *With* $\tanh\phi \le 4/5$ *due to Lemma 1,*

$$1 - (\underline{\tanh\phi})^2 = (1 - \tanh^2\phi)(1+\epsilon_{12}),$$

where

$$|\epsilon_{12}| \le \frac{16}{9}|\epsilon_t'|, \quad \epsilon_t' = (2+\epsilon_t)\epsilon_t.$$

Proof. As for Lemma 4, using $z = 1 - \tanh^2\phi \ge 9/25$ instead of y. □

Theorem 2. *With* $|\epsilon_{13}| \le \varepsilon$ *and* $|\epsilon_{14}| \le \varepsilon$*,*

$$\underline{\cosh\phi} = \cosh\phi(1+\epsilon_c),$$

where, since rsqrt *is assumed to be correctly rounded,*

$$1+\epsilon_c = \frac{1+\epsilon_{14}}{\sqrt{(1+\epsilon_{12})(1+\epsilon_{13})}}. \tag{24}$$

Proof. From (19), Lemma 8, and the definition of fma,

$$\mathrm{fma}(-\underline{\tanh\phi}, \underline{\tanh\phi}, 1) = (1-(\underline{\tanh\phi})^2)(1+\epsilon_{13}).$$

□

Theorem 3. *With* $|\epsilon_{15}| \le \varepsilon$*,*

$$\underline{\sinh\phi} = \sinh\phi(1+\epsilon_s), \quad 1+\epsilon_s = (1+\epsilon_t)(1+\epsilon_c)(1+\epsilon_{15}). \tag{25}$$

Proof. From (19), Lemma 8, and Theorem 2. □

This completes the analysis if a'_{21} is real. Otherwise, with $|\epsilon_{16}| \le \varepsilon$ and $|\epsilon_{17}| \le \varepsilon$, and barring inexact underflow,

$$\underline{\Re e^{\mathrm{i}\beta}} = \frac{\Re a'_{21}(1+\epsilon_{16})}{|a'_{21}|(1+\epsilon_1)} = \Re e^{\mathrm{i}\beta}(1+\epsilon'_{\Re}), \quad \underline{\Im e^{\mathrm{i}\beta}} = \frac{\Re a'_{21}(1+\epsilon_{17})}{|a'_{21}|(1+\epsilon_1)} = \Im e^{\mathrm{i}\beta}(1+\epsilon'_{\Im}).$$

Theorem 4. *With* $|\epsilon_{18}| \le \varepsilon$ *and* $|\epsilon_{19}| \le \varepsilon$*,*

$$\underline{\Re e^{\mathrm{i}\beta}\sinh\phi} = \Re e^{\mathrm{i}\beta}\sinh\phi(1+\epsilon_{\Re}), \quad \underline{\Im e^{\mathrm{i}\beta}\sinh\phi} = \Im e^{\mathrm{i}\beta}\sinh\phi(1+\epsilon_{\Im}),$$

where

$$1+\epsilon_{\Re} = (1+\epsilon'_{\Re})(1+\epsilon_s)(1+\epsilon_{18}), \quad 1+\epsilon_{\Im} = (1+\epsilon'_{\Im})(1+\epsilon_s)(1+\epsilon_{19}). \tag{26}$$

Proof. From Theorem 3. □

Upper bounds on $|\epsilon_d|$, $|\epsilon_t|$, $|\epsilon_c|$, $|\epsilon_s|$, $|\epsilon_{\Re}|$, and $|\epsilon_{\Im}|$ can be obtained symbolically in the terms of ε and γ.