Complex hyperbolic 2×2 rotations

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Let V be J-unitary, $H = GJG^*$ (Hermitian), and $A = G^*G$, where

$$A = \begin{bmatrix} a_{11} & \overline{a_{21}} \\ a_{21} & a_{22} \end{bmatrix}, \quad J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad V = \begin{bmatrix} \cosh\phi & \mathrm{e}^{-\mathrm{i}\beta}\sinh\phi \\ \mathrm{e}^{\mathrm{i}\beta}\sinh\phi & \cosh\phi \end{bmatrix}. \tag{1}$$

Then, $V^* = V$ and $V^*JV = VJV^* = J$. Find β and ϕ such that $V^*AV = \Xi$,

$$\Xi = \begin{bmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{bmatrix} = \begin{bmatrix} \cosh\phi & \mathrm{e}^{-\mathrm{i}\beta}\sinh\phi \\ \mathrm{e}^{\mathrm{i}\beta}\sinh\phi & \cosh\phi \end{bmatrix} \begin{bmatrix} a_{11} & \overline{a_{21}} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \cosh\phi & \mathrm{e}^{-\mathrm{i}\beta}\sinh\phi \\ \mathrm{e}^{\mathrm{i}\beta}\sinh\phi & \cosh\phi \end{bmatrix}. \tag{2}$$

Dividing (2) by $\cosh^2\phi > 0$ gives, with $\xi_i' = \xi_i/\cosh^2\phi$,

$$\begin{bmatrix} 1 & \mathrm{e}^{-\mathrm{i}\beta}\tanh\phi \\ \mathrm{e}^{\mathrm{i}\beta}\tanh\phi & 1 \end{bmatrix} \begin{bmatrix} a_{11} & \overline{a_{21}} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 & \mathrm{e}^{-\mathrm{i}\beta}\tanh\phi \\ \mathrm{e}^{\mathrm{i}\beta}\tanh\phi & 1 \end{bmatrix} = \begin{bmatrix} \xi_1' & 0 \\ 0 & \xi_2' \end{bmatrix}. \tag{3}$$

Multiplying the first two matrices in (3) gives

$$\Xi' = \begin{bmatrix} a_{11} + a_{21}\mathrm{e}^{-\mathrm{i}\beta}\tanh\phi & \overline{a_{21}} + a_{22}\mathrm{e}^{-\mathrm{i}\beta}\tanh\phi \\ a_{11}\mathrm{e}^{\mathrm{i}\beta}\tanh\phi + a_{21} & \overline{a_{21}}\mathrm{e}^{\mathrm{i}\beta}\tanh\phi + a_{22} \end{bmatrix} \begin{bmatrix} 1 & \mathrm{e}^{-\mathrm{i}\beta}\tanh\phi \\ \mathrm{e}^{\mathrm{i}\beta}\tanh\phi & 1 \end{bmatrix},$$

with the final multiplication producing, elementwise,

$$\begin{aligned} \xi_1' &= a_{11} + (2\Re(a_{21}\mathrm{e}^{-\mathrm{i}\beta}) + a_{22}\tanh\phi)\tanh\phi, && (4) \\ 0 &= \overline{a_{21}} + (a_{11} + a_{22} + a_{21}\mathrm{e}^{-\mathrm{i}\beta}\tanh\phi)\mathrm{e}^{-\mathrm{i}\beta}\tanh\phi, && (5) \\ 0 &= a_{21} + (a_{11} + a_{22} + \overline{a_{21}}\mathrm{e}^{\mathrm{i}\beta}\tanh\phi)\mathrm{e}^{\mathrm{i}\beta}\tanh\phi, && (6) \\ \xi_2' &= a_{22} + (2\Re(a_{21}\mathrm{e}^{-\mathrm{i}\beta}) + a_{11}\tanh\phi)\tanh\phi, && (7) \end{aligned}$$

since $z + \bar{z} = 2\Re z$. Evidently, $\xi_i \in \mathbb{R}$. If $a_{21} = 0$ then $\tanh\phi = 0$, and vice versa, if $\tanh\phi = 0$, then from (5) (or (6), which is the complex conjugate of (5)) it follows $a_{21} = 0$. Therefore, assume in the following that $\tanh\phi \neq 0$.

Multiplying (5) by $\mathrm{e}^{\mathrm{i}\beta}$ and (6) by $\mathrm{e}^{-\mathrm{i}\beta}$, it follows

$$\begin{aligned} 0 &= \overline{a_{21}}\mathrm{e}^{\mathrm{i}\beta} + (a_{11} + a_{22})\tanh\phi + a_{21}\mathrm{e}^{-\mathrm{i}\beta}\tanh^2\phi, && (8) \\ 0 &= a_{21}\mathrm{e}^{-\mathrm{i}\beta} + (a_{11} + a_{22})\tanh\phi + \overline{a_{21}}\mathrm{e}^{\mathrm{i}\beta}\tanh^2\phi. && (9) \end{aligned}$$

By extracting the common middle term on the right hand sides of (8) and (9),

$$-(a_{11}+a_{22})\tanh\phi = \overline{a_{21}}\mathrm{e}^{\mathrm{i}\beta} + a_{21}\mathrm{e}^{-\mathrm{i}\beta}\tanh^2\phi, \tag{10}$$

$$-(a_{11}+a_{22})\tanh\phi = a_{21}\mathrm{e}^{-\mathrm{i}\beta} + \overline{a_{21}}\mathrm{e}^{\mathrm{i}\beta}\tanh^2\phi, \tag{11}$$

it follows that the right hand sides of (10) and (11) have to be equal,

$$\overline{a_{21}}\mathrm{e}^{\mathrm{i}\beta} + a_{21}\mathrm{e}^{-\mathrm{i}\beta}\tanh^2\phi = a_{21}\mathrm{e}^{-\mathrm{i}\beta} + \overline{a_{21}}\mathrm{e}^{\mathrm{i}\beta}\tanh^2\phi, \tag{12}$$

while at the same time being the complex conjugates of one another. Therefore, both sides in (12) are real, what (10) and (11) also suggest. With

$$\beta = \arg(a_{21}), \quad \text{i.e.,} \quad \mathrm{e}^{\mathrm{i}\beta} = \frac{a_{21}}{|a_{21}|}, \quad (a_{21} = 0 \implies \beta = 0), \tag{13}$$

this condition is always fulfilled. Specifically, if a_{21} is real, then $\mathrm{e}^{\mathrm{i}\beta} = \operatorname{sign} a_{21}$.

Now, (8) and (9) become

$$|a_{21}|(1+\tanh^2\phi) = -(a_{11}+a_{22})\tanh\phi, \tag{14}$$

or, after rearranging (14),

$$\frac{-|a_{21}|}{a_{11}+a_{22}} = \frac{\tanh\phi}{1+\tanh^2\phi} = \frac{1}{2}\tanh(2\phi). \tag{15}$$

Note that $a_{11} \ge 0$ and $a_{22} \ge 0$ by definition, so $a_{11}+a_{22} = 0$ if and only if $a_{11} = 0$ and $a_{22} = 0$, in which case $a_{21} = 0$ (since A is a Grammian), and thus (15) does not even have to be evaluated in this degenerate case to compute $\tanh\phi = 0$. Another issue occurs if somehow $2|a_{21}| \ge (a_{11}+a_{22})$, i.e., if the computed $\tanh(2\phi) \le -1$.

Alternatively, the hyperbolic cotangent of twice the angle might be computed as:

$$\coth(2\phi) = \frac{a_{11}+a_{22}}{-2|a_{21}|}, \tag{16}$$

but this can easily cause overflow for small $|a_{21}|$. On the other hand, underflow of $\tanh(2\phi)$, as long as it is not exactly zero, will cause no trouble in the further computation, and might preserve at least some information, possibly resulting in a non-zero $\tanh\phi$ as well.

There are two solutions for the quadratic equation for $\tanh\phi$ from (15), only one of which obeys $|\tanh\phi| < 1$,

$$\tanh\phi = \frac{1-\sqrt{1-\tanh^2(2\phi)}}{\tanh(2\phi)}. \tag{17}$$

However, if the numerator and the denominator in (17) are multiplied by $1+\sqrt{1-\tanh^2(2\phi)}$, a more stable form emerges

$$\begin{aligned}
\tanh\phi &= \frac{1-\sqrt{1-\tanh^2(2\phi)}}{\tanh(2\phi)}\cdot\frac{1+\sqrt{1-\tanh^2(2\phi)}}{1+\sqrt{1-\tanh^2(2\phi)}}\\
&= \frac{1-(1-\tanh^2(2\phi))}{\tanh(2\phi)\left(1+\sqrt{1-\tanh^2(2\phi)}\right)}\\
&= \frac{\tanh(2\phi)}{1+\sqrt{1-\tanh^2(2\phi)}}\approx\frac{\tanh(2\phi)}{1+\sqrt{\mathrm{fma}(-\tanh(2\phi),\tanh(2\phi),1)}}.
\end{aligned}\tag{18}$$

Having computed $\tanh\phi$, it follows that

$$\begin{aligned}
\cosh\phi &= \frac{1}{\sqrt{1-\tanh^2\phi}}\\
&\approx \mathrm{rsqrt}(\mathrm{fma}(-\tanh\phi,\tanh\phi,1)),\\
\sinh\phi &= \tanh\phi\cdot\cosh\phi.
\end{aligned}\tag{19}$$

TODO: Scaling of A and the accurate computation of $\mathrm{e}^{\mathrm{i}\beta}$ similarly to `https://doi.org/10.1016/j.cam.2024.116003`