

Complex hyperbolic 2×2 rotations

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Let $V \in \mathbb{C}^{2 \times 2}$ be J -unitary and let $A \in \mathbb{C}^{2 \times 2}$ be a Hermitian positive semidefinite matrix, where

$$A = \begin{bmatrix} a_{11} & \overline{a_{21}} \\ a_{21} & a_{22} \end{bmatrix}, \quad J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad V = \begin{bmatrix} \cosh \phi & e^{-i\beta} \sinh \phi \\ e^{i\beta} \sinh \phi & \cosh \phi \end{bmatrix}. \quad (1)$$

Then, $V^* = V$ and $V^*JV = VJV^* = J$. Find β and ϕ such that $V^*AV = \Xi$,

$$\Xi = \begin{bmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{bmatrix} = \begin{bmatrix} \cosh \phi & e^{-i\beta} \sinh \phi \\ e^{i\beta} \sinh \phi & \cosh \phi \end{bmatrix} \begin{bmatrix} a_{11} & \overline{a_{21}} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \cosh \phi & e^{-i\beta} \sinh \phi \\ e^{i\beta} \sinh \phi & \cosh \phi \end{bmatrix}. \quad (2)$$

Dividing (2) by $\cosh^2 \phi > 0$ gives, with $\xi'_i = \xi_i / \cosh^2 \phi$,

$$\begin{bmatrix} 1 & e^{-i\beta} \tanh \phi \\ e^{i\beta} \tanh \phi & 1 \end{bmatrix} \begin{bmatrix} a_{11} & \overline{a_{21}} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 & e^{-i\beta} \tanh \phi \\ e^{i\beta} \tanh \phi & 1 \end{bmatrix} = \begin{bmatrix} \xi'_1 & 0 \\ 0 & \xi'_2 \end{bmatrix}. \quad (3)$$

Multiplying the first two matrices in (3) gives

$$\Xi' = \begin{bmatrix} a_{11} + a_{21}e^{-i\beta} \tanh \phi & \overline{a_{21}} + a_{22}e^{-i\beta} \tanh \phi \\ a_{11}e^{i\beta} \tanh \phi + a_{21} & \overline{a_{21}}e^{i\beta} \tanh \phi + a_{22} \end{bmatrix} \begin{bmatrix} 1 & e^{-i\beta} \tanh \phi \\ e^{i\beta} \tanh \phi & 1 \end{bmatrix},$$

with the final multiplication producing, elementwise,

$$\xi'_1 = a_{11} + (2\Re(a_{21}e^{-i\beta}) + a_{22} \tanh \phi) \tanh \phi, \quad (4)$$

$$0 = \overline{a_{21}} + (a_{11} + a_{22} + a_{21}e^{-i\beta} \tanh \phi)e^{-i\beta} \tanh \phi, \quad (5)$$

$$0 = a_{21} + (a_{11} + a_{22} + \overline{a_{21}}e^{i\beta} \tanh \phi)e^{i\beta} \tanh \phi, \quad (6)$$

$$\xi'_2 = a_{22} + (2\Re(a_{21}e^{-i\beta}) + a_{11} \tanh \phi) \tanh \phi, \quad (7)$$

since $z + \bar{z} = 2\Re z$. Evidently, $\xi_i \in \mathbb{R}$. If $a_{21} = 0$ then $\tanh \phi = 0$, and vice versa, if $\tanh \phi = 0$, then from (5) (or (6), which is the complex conjugate of (5)) it follows $a_{21} = 0$. Therefore, assume in the following that $\tanh \phi \neq 0$.

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Multiplying (5) by $e^{i\beta}$ and (6) by $e^{-i\beta}$, it follows

$$0 = \overline{a_{21}}e^{i\beta} + (a_{11} + a_{22}) \tanh \phi + a_{21}e^{-i\beta} \tanh^2 \phi, \quad (8)$$

$$0 = a_{21}e^{-i\beta} + (a_{11} + a_{22}) \tanh \phi + \overline{a_{21}}e^{i\beta} \tanh^2 \phi. \quad (9)$$

By extracting the common middle term on the right hand sides of (8) and (9),

$$-(a_{11} + a_{22}) \tanh \phi = \overline{a_{21}}e^{i\beta} + a_{21}e^{-i\beta} \tanh^2 \phi, \quad (10)$$

$$-(a_{11} + a_{22}) \tanh \phi = a_{21}e^{-i\beta} + \overline{a_{21}}e^{i\beta} \tanh^2 \phi, \quad (11)$$

it follows that the right hand sides of (10) and (11) have to be equal,

$$\overline{a_{21}}e^{i\beta} + a_{21}e^{-i\beta} \tanh^2 \phi = a_{21}e^{-i\beta} + \overline{a_{21}}e^{i\beta} \tanh^2 \phi, \quad (12)$$

while at the same time being the complex conjugates of one another. Therefore, both sides in (12) are real, what (10) and (11) also suggest. With

$$\beta = \arg(a_{21}), \quad \text{i.e.,} \quad e^{i\beta} = \frac{a_{21}}{|a_{21}|}, \quad (a_{21} = 0 \implies \beta = 0), \quad (13)$$

this condition is always fulfilled. If a_{21} is real in (1), then $e^{i\beta} = \text{sign } a_{21}$.

Now, (8) and (9) become

$$|a_{21}|(1 + \tanh^2 \phi) = -(a_{11} + a_{22}) \tanh \phi, \quad (14)$$

or, after rearranging (14),

$$\frac{-|a_{21}|}{a_{11} + a_{22}} = \frac{\tanh \phi}{1 + \tanh^2 \phi} = \frac{1}{2} \tanh(2\phi). \quad (15)$$

Note that $a_{11} \geq 0$ and $a_{22} \geq 0$ by definition, so $a_{11} + a_{22} = 0$ if and only if $a_{11} = 0$ and $a_{22} = 0$, in which case $a_{21} = 0$, and thus (15) does not even have to be evaluated in this degenerate case to compute $\tanh \phi = 0$.

The proper sequence of checks on the inputs a_{11} , a_{22} , and a_{21} is thus

1. if $a_{11} < 0$ then ERROR; else
2. if $a_{22} < 0$ then ERROR; else
3. if $a'_{21} = 0$ then $\tanh \phi = 0$,

where a'_{21} is a_{21} , scaled as in (20). Else, proceed to compute $\tanh(2\phi)$. If it is detected that the computed $\tanh(2\phi) \leq -1$, then ERROR, since the input does not define a positive semidefinite matrix.

Alternatively, the hyperbolic cotangent of 2ϕ might be computed as:

$$\coth(2\phi) = \frac{a_{11} + a_{22}}{-2|a_{21}|}, \quad (16)$$

but this can easily cause overflow for small $|a_{21}|$. On the other hand, underflow of $\tanh(2\phi)$, as long as it is not exactly zero, will cause no trouble in the further computation, and might preserve at least some information, possibly resulting in a non-zero $\tanh \phi$ as well.

There are two solutions for the quadratic equation for $\tanh \phi$ from (15), only one of which obeys $|\tanh \phi| < 1$,

$$\tanh \phi = \frac{1 - \sqrt{1 - \tanh^2(2\phi)}}{\tanh(2\phi)}. \quad (17)$$

However, if the numerator and the denominator in (17) are multiplied by $1 + \sqrt{1 - \tanh^2(2\phi)}$, a more stable form emerges,

$$\begin{aligned} \tanh \phi &= \frac{1 - \sqrt{1 - \tanh^2(2\phi)}}{\tanh(2\phi)} \cdot \frac{1 + \sqrt{1 - \tanh^2(2\phi)}}{1 + \sqrt{1 - \tanh^2(2\phi)}} \\ &= \frac{1 - (1 - \tanh^2(2\phi))}{\tanh(2\phi) \left(1 + \sqrt{1 - \tanh^2(2\phi)}\right)} \\ &= \frac{\tanh(2\phi)}{1 + \sqrt{1 - \tanh^2(2\phi)}} \approx \frac{\tanh(2\phi)}{1 + \sqrt{\text{fma}(-\tanh(2\phi), \tanh(2\phi), 1)}}. \end{aligned} \quad (18)$$

Since $|\tanh(2\phi)| < 1$, from (18) it follows that $|\tanh \phi| < 1$ as well. Now,

$$\begin{aligned} \cosh \phi &= \frac{1}{\sqrt{1 - \tanh^2 \phi}} \approx \text{rsqrt}(\text{fma}(-\tanh \phi, \tanh \phi, 1)), \\ \sinh \phi &= \tanh \phi \cdot \cosh \phi. \end{aligned} \quad (19)$$

If $\tanh \phi$ is regarded as a computed floating-point value, then $|\tanh \phi| \leq 1^-$, where 1^- is the first floating-point predecessor of unity. Otherwise, the computed $|\tanh(2\phi)|$ would have to be unity, what has already been ruled out. Thus,

$$\tanh^2 \phi < |\tanh \phi| \leq 1^- \implies 1 - \tanh^2 \phi \geq 1 - 1^-.$$

Since $1 - 1^-$ is a floating-point value in the normal range, its square root is as well, so $\cosh \phi$ cannot overflow in (19), and neither can $\sinh \phi$.

The input data has to be prescaled by the highest possible power of two, 2^s , to get $a'_{11} = 2^s a_{11}$, $a'_{22} = 2^s a_{22}$, and $a'_{21} = 2^s a_{21}$, such that

$$\max\{a'_{11}, a'_{22}\} \leq \nu/2, \quad \max\{|\Re a'_{21}|, |\Im a'_{21}|\} \leq \nu/2, \quad (20)$$

where ν is the largest finite floating-point value. This way no quantity in (15) will overflow¹, and the chances of dealing with subnormal values will be minimized. See the scaling of A and the computation of $e^{i\beta}$ in:

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¹not even $2|a'_{21}|$, since A is positive semidefinite, so $2|a'_{21}| \leq 2\sqrt{a'_{11}a'_{22}} \leq \nu$, but the upper bound of $\nu/4$ instead of $\nu/2$ in (20) might be safer