

Complex hyperbolic 2×2 rotations

Vedran Novaković*

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1 Formulas

Let $V \in \mathbb{C}^{2 \times 2}$ be J -unitary and let $A \in \mathbb{C}^{2 \times 2}$ be a Hermitian positive definite matrix, where

$$A = \begin{bmatrix} a_{11} & \overline{a_{21}} \\ a_{21} & a_{22} \end{bmatrix}, \quad J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad V = \begin{bmatrix} \cosh \phi & e^{-i\beta} \sinh \phi \\ e^{i\beta} \sinh \phi & \cosh \phi \end{bmatrix}. \quad (1)$$

Then, $V^* = V$ and $V^* J V = V J V^* = J$. Find β and ϕ such that $V^* A V = \Xi$,

$$\Xi = \begin{bmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{bmatrix} = \begin{bmatrix} \cosh \phi & e^{-i\beta} \sinh \phi \\ e^{i\beta} \sinh \phi & \cosh \phi \end{bmatrix} \begin{bmatrix} a_{11} & \overline{a_{21}} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \cosh \phi & e^{-i\beta} \sinh \phi \\ e^{i\beta} \sinh \phi & \cosh \phi \end{bmatrix}. \quad (2)$$

Dividing (2) by $\cosh^2 \phi > 0$ gives, with $\xi'_i = \xi_i / \cosh^2 \phi$,

$$\begin{bmatrix} 1 & e^{-i\beta} \tanh \phi \\ e^{i\beta} \tanh \phi & 1 \end{bmatrix} \begin{bmatrix} a_{11} & \overline{a_{21}} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 & e^{-i\beta} \tanh \phi \\ e^{i\beta} \tanh \phi & 1 \end{bmatrix} = \begin{bmatrix} \xi'_1 & 0 \\ 0 & \xi'_2 \end{bmatrix}. \quad (3)$$

Multiplying the first two matrices in (3) gives

$$\Xi' = \begin{bmatrix} a_{11} + a_{21} e^{-i\beta} \tanh \phi & \overline{a_{21}} + a_{22} e^{-i\beta} \tanh \phi \\ a_{11} e^{i\beta} \tanh \phi + a_{21} & \overline{a_{21}} e^{i\beta} \tanh \phi + a_{22} \end{bmatrix} \begin{bmatrix} 1 & e^{-i\beta} \tanh \phi \\ e^{i\beta} \tanh \phi & 1 \end{bmatrix},$$

with the final multiplication producing, elementwise,

$$\xi'_1 = a_{11} + (2\Re(a_{21} e^{-i\beta}) + a_{22} \tanh \phi) \tanh \phi, \quad (4)$$

$$0 = \overline{a_{21}} + (a_{11} + a_{22} + a_{21} e^{-i\beta} \tanh \phi) e^{-i\beta} \tanh \phi, \quad (5)$$

$$0 = a_{21} + (a_{11} + a_{22} + \overline{a_{21}} e^{i\beta} \tanh \phi) e^{i\beta} \tanh \phi, \quad (6)$$

$$\xi'_2 = a_{22} + (2\Re(a_{21} e^{-i\beta}) + a_{11} \tanh \phi) \tanh \phi, \quad (7)$$

*with suggestions from Vjeran Hari

since $z + \bar{z} = 2\Re z$. Evidently, $\xi_i \in \mathbb{R}$. If $a_{21} = 0$ then $\tanh \phi = 0$, and vice versa, if $\tanh \phi = 0$, then from (5) (or (6), which is the complex conjugate of (5)) it follows $a_{21} = 0$. Therefore, assume in the following that $\tanh \phi \neq 0$.

Multiplying (5) by $e^{i\beta}$ and (6) by $e^{-i\beta}$, it follows

$$0 = \overline{a_{21}}e^{i\beta} + (a_{11} + a_{22})\tanh \phi + a_{21}e^{-i\beta}\tanh^2 \phi, \quad (8)$$

$$0 = a_{21}e^{-i\beta} + (a_{11} + a_{22})\tanh \phi + \overline{a_{21}}e^{i\beta}\tanh^2 \phi. \quad (9)$$

By extracting the common middle term on the right hand sides of (8) and (9),

$$-(a_{11} + a_{22})\tanh \phi = \overline{a_{21}}e^{i\beta} + a_{21}e^{-i\beta}\tanh^2 \phi, \quad (10)$$

$$-(a_{11} + a_{22})\tanh \phi = a_{21}e^{-i\beta} + \overline{a_{21}}e^{i\beta}\tanh^2 \phi, \quad (11)$$

it follows that the right hand sides of (10) and (11) have to be equal,

$$\overline{a_{21}}e^{i\beta} + a_{21}e^{-i\beta}\tanh^2 \phi = a_{21}e^{-i\beta} + \overline{a_{21}}e^{i\beta}\tanh^2 \phi, \quad (12)$$

while at the same time being the complex conjugates of one another. Therefore, both sides in (12) are real, what (10) and (11) also suggest. With

$$\beta = \arg(a_{21}), \quad \text{i.e.,} \quad e^{i\beta} = \frac{a_{21}}{|a_{21}|}, \quad (a_{21} = 0 \implies \beta = 0), \quad (13)$$

this condition is always fulfilled. If a_{21} is real in (1), then $e^{i\beta} = \text{sign } a_{21}$.

Now, (8) and (9) become

$$|a_{21}|(1 + \tanh^2 \phi) = -(a_{11} + a_{22})\tanh \phi, \quad (14)$$

or, after rearranging (14),

$$\frac{-|a_{21}|}{a_{11} + a_{22}} = \frac{\tanh \phi}{1 + \tanh^2 \phi} = \frac{1}{2}\tanh(2\phi). \quad (15)$$

Note that $a_{11} \geq 0$ and $a_{22} \geq 0$ by definition, so $a_{11} + a_{22} = 0$ if and only if $a_{11} = 0$ and $a_{22} = 0$, in which case $a_{21} = 0$, and thus (15) does not even have to be evaluated in this degenerate case to compute $\tanh \phi = 0$.

The proper sequence of checks on the inputs a_{11} , a_{22} , and a_{21} is thus

1. if $a_{11} < 0$ then ERROR; else
2. if $a_{22} < 0$ then ERROR; else
3. if $a'_{21} = 0$ then $\tanh \phi = 0$,

where a'_{21} is a_{21} , scaled as in (20). Else, proceed to compute $\tanh(2\phi)$.

If the computed $\tanh(2\phi) \leq -1$, then setting $\tanh(2\phi) = -1$ allows the computation to proceed, to get $\cosh \phi = \infty$ and $|\sinh \phi| = \infty$. The input in this case is invalid, and then it is up to the caller to provide a remedy.

Alternatively, the hyperbolic cotangent of 2ϕ might be computed as:

$$\coth(2\phi) = \frac{a_{11} + a_{22}}{-2|a_{21}|}, \quad (16)$$

but this can easily cause overflow for small $|a_{21}|$. On the other hand, underflow of $\tanh(2\phi)$, as long as it is not exactly zero, will cause no trouble in the further computation, and might preserve at least some information, possibly resulting in a non-zero $\tanh \phi$ as well.

There are two solutions for the quadratic equation for $\tanh \phi$ from (15), only one of which obeys $|\tanh \phi| < 1$,

$$\tanh \phi = \frac{1 - \sqrt{1 - \tanh^2(2\phi)}}{\tanh(2\phi)}. \quad (17)$$

However, if the numerator and the denominator in (17) are multiplied by $1 + \sqrt{1 - \tanh^2(2\phi)}$, a more stable form emerges,

$$\begin{aligned} \tanh \phi &= \frac{1 - \sqrt{1 - \tanh^2(2\phi)}}{\tanh(2\phi)} \cdot \frac{1 + \sqrt{1 - \tanh^2(2\phi)}}{1 + \sqrt{1 - \tanh^2(2\phi)}} \\ &= \frac{1 - (1 - \tanh^2(2\phi))}{\tanh(2\phi) \left(1 + \sqrt{1 - \tanh^2(2\phi)}\right)} \\ &= \frac{\tanh(2\phi)}{1 + \sqrt{1 - \tanh^2(2\phi)}} \approx \frac{\tanh(2\phi)}{1 + \text{sqrt}(\text{fma}(-\tanh(2\phi), \tanh(2\phi), 1))}. \end{aligned} \quad (18)$$

Since $|\tanh(2\phi)| < 1$, from (18) it follows that $|\tanh \phi| < 1$ as well. Now,

$$\begin{aligned} \cosh \phi &= \frac{1}{\sqrt{1 - \tanh^2 \phi}} \approx \text{rsqrt}(\text{fma}(-\tanh \phi, \tanh \phi, 1)), \\ \sinh \phi &= \tanh \phi \cdot \cosh \phi. \end{aligned} \quad (19)$$

If $\tanh \phi$ is regarded as a computed floating-point value, then $|\tanh \phi| \leq 1^-$, where 1^- is the first floating-point predecessor of unity. Otherwise, the computed $|\tanh(2\phi)|$ would have to be unity, what has already been ruled out. Thus,

$$\tanh^2 \phi < |\tanh \phi| \leq 1^- \implies 1 - \tanh^2 \phi \geq 1 - 1^-.$$

Since $1 - 1^-$ is a floating-point value in the normal range, its square root is as well, so $\cosh \phi$ cannot overflow in (19), and neither can $\sinh \phi$.

The input data has to be prescaled by the highest possible power of two, 2^s , to get $a'_{11} = 2^s a_{11}$, $a'_{22} = 2^s a_{22}$, and $a'_{21} = 2^s a_{21}$, such that

$$\max\{a'_{11}, a'_{22}\} \leq \nu/2, \quad \max\{|\Re a'_{21}|, |\Im a'_{21}|\} \leq \nu/2, \quad (20)$$

where ν is the largest finite floating-point value. This way no quantity in (15) will overflow¹, and the chances of dealing with subnormal values will be minimized. See the scaling of A and the computation of $e^{i\beta}$ in:

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2 Accuracy

Let the computed values, unlike the exact ones, be underlined in the following, let ε be the machine precision, and assume hypot is correctly rounded.

Also, assume that all inputs have been scaled exactly to a'_{11} , a'_{22} , a'_{21} .

Lemma 1. *If $|\tanh(2\phi)| \leq 40/41$ then $|\tanh \phi| \leq 4/5$, and vice versa.*

Proof. From (15),

$$|\tanh(2\phi)| = \frac{2|\tanh \phi|}{1 + \tanh^2 \phi} \leq \frac{40}{41} \implies 40 \tanh^2 \phi - 82|\tanh \phi| + 40 \geq 0.$$

This inequality is valid for $|\tanh \phi| \leq 4/5$. □

Lemma 2. *Barring inexact underflow, $\underline{|a'_{21}|} = |a'_{21}|(1 + \epsilon_1)$, $|\epsilon_1| \leq \varepsilon$.*

Proof. From the assumption that hypot is correctly rounded. □

Lemma 3. *With $|\epsilon_2| \leq \varepsilon$, $|\epsilon_3| \leq \varepsilon$, and ϵ_d such that*

$$1 + \epsilon_d = \frac{1 + \epsilon_1}{1 + \epsilon_2}(1 + \epsilon_3), \quad (21)$$

barring inexact underflow it holds

$$\underline{\tanh(2\phi)} = \frac{-2|a'_{21}|(1 + \epsilon_1)}{(a'_{11} + a'_{22})(1 + \epsilon_2)}(1 + \epsilon_3) = \tanh(2\phi)(1 + \epsilon_d).$$

Proof. From Lemma 2. □

¹not even $2|a'_{21}|$, since A is positive definite, so $2|a'_{21}| \leq 2\sqrt{a'_{11}a'_{22}} \leq \nu$, but the upper bound of $\nu/4$ instead of $\nu/2$ in (20) might be safer

Note that

$$\frac{(1 - \varepsilon)^\gamma}{1 + \varepsilon} \leq 1 + \epsilon_d \leq \frac{(1 + \varepsilon)^\gamma}{1 - \varepsilon}, \quad (22)$$

where $\gamma = 2$ if a'_{21} is complex, and $\gamma = 1$ if a'_{21} is real (since then $\epsilon_1 = 0$).

Lemma 4. *Assume that $|\tanh(2\phi)| \leq 40/41$. Then*

$$1 - (\tanh(2\phi))^2 = (1 - \tanh^2(2\phi))(1 + \epsilon_4),$$

where

$$|\epsilon_4| \leq \frac{1600}{81}|\epsilon'_d|, \quad \epsilon'_d = (2 + \epsilon_d)\epsilon_d.$$

Proof. Let $y = 1 - \tanh^2(2\phi) \geq 81/1681$. Then, from Lemma 3,

$$1 - (\tanh(2\phi))^2 = 1 - \tanh^2(2\phi)(1 + \epsilon_d)^2 = y - \tanh^2(2\phi)\epsilon'_d,$$

where $(1 + \epsilon_d)^2 = 1 + \epsilon'_d$, i.e., $\epsilon'_d = (2 + \epsilon_d)\epsilon_d$.

Using the definition of y , find ϵ_4 such that

$$y(1 + \epsilon_4) = y + y\epsilon_4 = y - \tanh^2(2\phi)\epsilon'_d = y + (y - 1)\epsilon'_d.$$

Therefore, by subtracting y from these equalities,

$$y\epsilon_4 = (y - 1)\epsilon'_d \implies \epsilon_4 = \frac{y - 1}{y}\epsilon'_d.$$

By taking the lower bound for y , and thus the upper bound for $|y - 1|/|y|$, it follows that

$$|\epsilon_4| \leq \frac{1600}{81}|\epsilon'_d|,$$

i.e., $|\epsilon_4| \lesssim 19.753|\epsilon'_d|$. □

Lemma 5. *With $|\epsilon_5| \leq \varepsilon$ and $|\epsilon_6| \leq \varepsilon$,*

$$\text{sqrt}(\text{fma}(-\tanh(2\phi), \tanh(2\phi), 1)) = \sqrt{1 - \tanh^2(2\phi)(1 + \epsilon_7)},$$

where

$$1 + \epsilon_7 = \sqrt{(1 + \epsilon_4)(1 + \epsilon_5)}(1 + \epsilon_6).$$

Proof. From Lemma 4 and the definition of fma ,

$$\text{fma}(-\tanh(2\phi), \tanh(2\phi), 1) = (1 - (\tanh(2\phi))^2)(1 + \epsilon_5).$$

□

Lemma 6. *Let $x = 1 + \sqrt{1 - \tanh^2(2\phi)}$. Then,*

$$1 + \text{sqrt}(\text{fma}(\underline{-\tanh(2\phi)}, \underline{\tanh(2\phi)}, 1)) = x(1 + \epsilon_8),$$

where $|\epsilon_8| \leq |\epsilon_7|/2$.

Proof. From Lemma 5 it follows

$$1 + \text{sqrt}(\text{fma}(\underline{-\tanh(2\phi)}, \underline{\tanh(2\phi)}, 1)) = 1 + (x - 1)(1 + \epsilon_7) = x + \epsilon_7(x - 1).$$

Using $50/41 \leq x \leq 2$, since $1 - \tanh^2(2\phi) \geq 81/1681$, find ϵ_8 such that

$$x(1 + \epsilon_8) = x + x\epsilon_8 = x + \epsilon_7(x - 1).$$

From the last two equalities subtraction of x gives

$$\epsilon_8 = \frac{x - 1}{x} \epsilon_7.$$

Therefore,

$$\frac{9}{50} |\epsilon_7| \leq |\epsilon_8| \leq \frac{1}{2} |\epsilon_7|.$$

□

Lemma 7. *The denominator in (18) is computed as*

$$(1 + \sqrt{1 - \tanh^2(2\phi)})(1 + \epsilon_{10}),$$

where, with $|\epsilon_9| \leq \varepsilon$ due to the rounding in the final addition,

$$1 + \epsilon_{10} = (1 + \epsilon_8)(1 + \epsilon_9).$$

Proof. From Lemma 6. □

Theorem 1. *With $|\epsilon_{11}| \leq \varepsilon$,*

$$\underline{\tanh \phi} = \tanh \phi(1 + \epsilon_t), \quad 1 + \epsilon_t = \frac{1 + \epsilon_d}{1 + \epsilon_{10}}(1 + \epsilon_{11}). \quad (23)$$

Proof. From (18), Lemma (3), and Lemma (7). □

Lemma 8. *With $\tanh \phi \leq 4/5$ due to Lemma 1,*

$$1 - (\underline{\tanh \phi})^2 = (1 - \tanh^2 \phi)(1 + \epsilon_{12}),$$

where

$$|\epsilon_{12}| \leq \frac{16}{9} |\epsilon'_t|, \quad \epsilon'_t = (2 + \epsilon_t) \epsilon_t.$$

Proof. As for Lemma 4, using $z = 1 - \tanh^2 \phi \geq 9/25$ instead of y . \square

Theorem 2. With $|\epsilon_{13}| \leq \varepsilon$ and $|\epsilon_{14}| \leq \varepsilon$,

$$\underline{\cosh \phi} = \cosh \phi(1 + \epsilon_c),$$

where, since `rsqrt` is assumed to be correctly rounded,

$$1 + \epsilon_c = \frac{1 + \epsilon_{14}}{\sqrt{(1 + \epsilon_{12})(1 + \epsilon_{13})}}. \quad (24)$$

Proof. From (19), Lemma 8, and the definition of `fma`,

$$\text{fma}(-\underline{\tanh \phi}, \underline{\tanh \phi}, 1) = (1 - (\underline{\tanh \phi})^2)(1 + \epsilon_{13}).$$

\square

Theorem 3. With $|\epsilon_{15}| \leq \varepsilon$,

$$\underline{\sinh \phi} = \sinh \phi(1 + \epsilon_s), \quad 1 + \epsilon_s = (1 + \epsilon_t)(1 + \epsilon_c)(1 + \epsilon_{15}). \quad (25)$$

Proof. From (19), Lemma 8, and Theorem 2. \square

This completes the analysis if a'_{21} is real. Otherwise, with $|\epsilon_{16}| \leq \varepsilon$ and $|\epsilon_{17}| \leq \varepsilon$, and barring inexact underflow,

$$\underline{\Re e^{i\beta}} = \frac{\Re a'_{21}(1 + \epsilon_{16})}{|a'_{21}|(1 + \epsilon_1)} = \Re e^{i\beta}(1 + \epsilon'_{\Re}), \quad \underline{\Im e^{i\beta}} = \frac{\Im a'_{21}(1 + \epsilon_{17})}{|a'_{21}|(1 + \epsilon_1)} = \Im e^{i\beta}(1 + \epsilon'_{\Im}).$$

Theorem 4. With $|\epsilon_{18}| \leq \varepsilon$ and $|\epsilon_{19}| \leq \varepsilon$,

$$\underline{\Re e^{i\beta} \sinh \phi} = \Re e^{i\beta} \sinh \phi(1 + \epsilon_{\Re}), \quad \underline{\Im e^{i\beta} \sinh \phi} = \Im e^{i\beta} \sinh \phi(1 + \epsilon_{\Im}),$$

where

$$1 + \epsilon_{\Re} = (1 + \epsilon'_{\Re})(1 + \epsilon_s)(1 + \epsilon_{18}), \quad 1 + \epsilon_{\Im} = (1 + \epsilon'_{\Im})(1 + \epsilon_s)(1 + \epsilon_{19}). \quad (26)$$

Proof. From Theorem 3. \square

Upper bounds on $|\epsilon_d|$, $|\epsilon_t|$, $|\epsilon_c|$, $|\epsilon_s|$, $|\epsilon_{\Re}|$, and $|\epsilon_{\Im}|$ can be obtained symbolically in the terms of ε and γ .