# Complex hyperbolic $2 \times 2$ rotations

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Let $V \in \mathbb{C}^{2\times 2}$ be $J$-unitary and let $A \in \mathbb{C}^{2\times 2}$ be a Hermitian positive semidefinite matrix, where

$$A = \begin{bmatrix} a_{11} & \overline{a_{21}} \\ a_{21} & a_{22} \end{bmatrix}, \quad J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad V = \begin{bmatrix} \cosh\phi & \mathrm{e}^{-\mathrm{i}\beta}\sinh\phi \\ \mathrm{e}^{\mathrm{i}\beta}\sinh\phi & \cosh\phi \end{bmatrix}. \tag{1}$$

Then, $V^* = V$ and $V^*JV = VJV^* = J$. Find $\beta$ and $\phi$ such that $V^*AV = \Xi$,

$$\Xi = \begin{bmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{bmatrix} = \begin{bmatrix} \cosh\phi & \mathrm{e}^{-\mathrm{i}\beta}\sinh\phi \\ \mathrm{e}^{\mathrm{i}\beta}\sinh\phi & \cosh\phi \end{bmatrix} \begin{bmatrix} a_{11} & \overline{a_{21}} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \cosh\phi & \mathrm{e}^{-\mathrm{i}\beta}\sinh\phi \\ \mathrm{e}^{\mathrm{i}\beta}\sinh\phi & \cosh\phi \end{bmatrix}. \tag{2}$$

Dividing (2) by $\cosh^2\phi > 0$ gives, with $\xi_i' = \xi_i / \cosh^2\phi$,

$$\begin{bmatrix} 1 & \mathrm{e}^{-\mathrm{i}\beta}\tanh\phi \\ \mathrm{e}^{\mathrm{i}\beta}\tanh\phi & 1 \end{bmatrix} \begin{bmatrix} a_{11} & \overline{a_{21}} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 & \mathrm{e}^{-\mathrm{i}\beta}\tanh\phi \\ \mathrm{e}^{\mathrm{i}\beta}\tanh\phi & 1 \end{bmatrix} = \begin{bmatrix} \xi_1' & 0 \\ 0 & \xi_2' \end{bmatrix}. \tag{3}$$

Multiplying the first two matrices in (3) gives

$$\Xi' = \begin{bmatrix} a_{11} + a_{21}\mathrm{e}^{-\mathrm{i}\beta}\tanh\phi & \overline{a_{21}} + a_{22}\mathrm{e}^{-\mathrm{i}\beta}\tanh\phi \\ a_{11}\mathrm{e}^{\mathrm{i}\beta}\tanh\phi + a_{21} & \overline{a_{21}}\mathrm{e}^{\mathrm{i}\beta}\tanh\phi + a_{22} \end{bmatrix} \begin{bmatrix} 1 & \mathrm{e}^{-\mathrm{i}\beta}\tanh\phi \\ \mathrm{e}^{\mathrm{i}\beta}\tanh\phi & 1 \end{bmatrix},$$

with the final multiplication producing, elementwise,

$$\begin{aligned} \xi_1' &= a_{11} + (2\Re(a_{21}\mathrm{e}^{-\mathrm{i}\beta}) + a_{22}\tanh\phi)\tanh\phi, && (4) \\ 0 &= \overline{a_{21}} + (a_{11} + a_{22} + a_{21}\mathrm{e}^{-\mathrm{i}\beta}\tanh\phi)\mathrm{e}^{-\mathrm{i}\beta}\tanh\phi, && (5) \\ 0 &= a_{21} + (a_{11} + a_{22} + \overline{a_{21}}\mathrm{e}^{\mathrm{i}\beta}\tanh\phi)\mathrm{e}^{\mathrm{i}\beta}\tanh\phi, && (6) \\ \xi_2' &= a_{22} + (2\Re(a_{21}\mathrm{e}^{-\mathrm{i}\beta}) + a_{11}\tanh\phi)\tanh\phi, && (7) \end{aligned}$$

since $z + \bar{z} = 2\Re z$. Evidently, $\xi_i \in \mathbb{R}$. If $a_{21} = 0$ then $\tanh\phi = 0$, and vice versa, if $\tanh\phi = 0$, then from (5) (or (6), which is the complex conjugate of (5)) it follows $a_{21} = 0$. Therefore, assume in the following that $\tanh\phi \neq 0$.

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Multiplying (5) by $\mathrm{e}^{\mathrm{i}\beta}$ and (6) by $\mathrm{e}^{-\mathrm{i}\beta}$, it follows

$$0 = \overline{a_{21}}\mathrm{e}^{\mathrm{i}\beta} + (a_{11} + a_{22})\tanh\phi + a_{21}\mathrm{e}^{-\mathrm{i}\beta}\tanh^2\phi, \tag{8}$$
$$0 = a_{21}\mathrm{e}^{-\mathrm{i}\beta} + (a_{11} + a_{22})\tanh\phi + \overline{a_{21}}\mathrm{e}^{\mathrm{i}\beta}\tanh^2\phi. \tag{9}$$

By extracting the common middle term on the right hand sides of (8) and (9),

$$-(a_{11} + a_{22})\tanh\phi = \overline{a_{21}}\mathrm{e}^{\mathrm{i}\beta} + a_{21}\mathrm{e}^{-\mathrm{i}\beta}\tanh^2\phi, \tag{10}$$
$$-(a_{11} + a_{22})\tanh\phi = a_{21}\mathrm{e}^{-\mathrm{i}\beta} + \overline{a_{21}}\mathrm{e}^{\mathrm{i}\beta}\tanh^2\phi, \tag{11}$$

it follows that the right hand sides of (10) and (11) have to be equal,

$$\overline{a_{21}}\mathrm{e}^{\mathrm{i}\beta} + a_{21}\mathrm{e}^{-\mathrm{i}\beta}\tanh^2\phi = a_{21}\mathrm{e}^{-\mathrm{i}\beta} + \overline{a_{21}}\mathrm{e}^{\mathrm{i}\beta}\tanh^2\phi, \tag{12}$$

while at the same time being the complex conjugates of one another. Therefore, both sides in (12) are real, what (10) and (11) also suggest. With

$$\beta = \arg(a_{21}), \quad \text{i.e.,} \quad \mathrm{e}^{\mathrm{i}\beta} = \frac{a_{21}}{|a_{21}|}, \quad (a_{21} = 0 \implies \beta = 0), \tag{13}$$

this condition is always fulfilled. Specifically, if $a_{21}$ is real, then $\mathrm{e}^{\mathrm{i}\beta} = \operatorname{sign} a_{21}$.

Now, (8) and (9) become

$$|a_{21}|(1 + \tanh^2\phi) = -(a_{11} + a_{22})\tanh\phi, \tag{14}$$

or, after rearranging (14),

$$\frac{-|a_{21}|}{a_{11} + a_{22}} = \frac{\tanh\phi}{1 + \tanh^2\phi} = \frac{1}{2}\tanh(2\phi). \tag{15}$$

Note that $a_{11} \geq 0$ and $a_{22} \geq 0$ by definition, so $a_{11} + a_{22} = 0$ if and only if $a_{11} = 0$ and $a_{22} = 0$, in which case $a_{21} = 0$, and thus (15) does not even have to be evaluated in this degenerate case to compute $\tanh\phi = 0$.

The proper sequence of checks on the inputs $a_{11}$, $a_{22}$, and $a_{21}$ is thus

1. if $a_{11} < 0$ then ERROR; else
2. if $a_{22} < 0$ then ERROR; else
3. if $a'_{21} = 0$ then $\tanh\phi = 0$,

where $a'_{21}$ is $a_{21}$ scaled as described below. Else, proceed to compute $\tanh(2\phi)$. If it is detected that the computed $\tanh(2\phi) \leq -1$, then ERROR, since the input does not define a positive semidefinite matrix.

Alternatively, the hyperbolic cotangent of $2\phi$ might be computed as:

$$\coth(2\phi) = \frac{a_{11} + a_{22}}{-2|a_{21}|}, \tag{16}$$

but this can easily cause overflow for small $|a_{21}|$. On the other hand, underflow of $\tanh(2\phi)$, as long as it is not exactly zero, will cause no trouble in the further computation, and might preserve at least some information, possibly resulting in a non-zero $\tanh\phi$ as well.

There are two solutions for the quadratic equation for $\tanh\phi$ from (15), only one of which obeys $|\tanh\phi| < 1$,

$$\tanh\phi = \frac{1 - \sqrt{1 - \tanh^2(2\phi)}}{\tanh(2\phi)}. \tag{17}$$

However, if the numerator and the denominator in (17) are multiplied by $1 + \sqrt{1 - \tanh^2(2\phi)}$, a more stable form emerges,

$$\begin{aligned} \tanh\phi &= \frac{1 - \sqrt{1 - \tanh^2(2\phi)}}{\tanh(2\phi)} \cdot \frac{1 + \sqrt{1 - \tanh^2(2\phi)}}{1 + \sqrt{1 - \tanh^2(2\phi)}} \\ &= \frac{1 - (1 - \tanh^2(2\phi))}{\tanh(2\phi)\left(1 + \sqrt{1 - \tanh^2(2\phi)}\right)} \\ &= \frac{\tanh(2\phi)}{1 + \sqrt{1 - \tanh^2(2\phi)}} \approx \frac{\tanh(2\phi)}{1 + \sqrt{\mathrm{fma}(-\tanh(2\phi), \tanh(2\phi), 1)}}. \end{aligned} \tag{18}$$

Since $|\tanh(2\phi)| < 1$, from (18) it follows that $|\tanh\phi| < 1$ as well. Now,

$$\begin{aligned} \cosh\phi &= \frac{1}{\sqrt{1 - \tanh^2\phi}} \approx \mathrm{rsqrt}(\mathrm{fma}(-\tanh\phi, \tanh\phi, 1)), \\ \sinh\phi &= \tanh\phi \cdot \cosh\phi. \end{aligned} \tag{19}$$

If $\tanh\phi$ is regarded as a computed, floating-point value, then $|\tanh\phi| \le 1^-$, where $1^-$ is the first floating-point predecessor of unity. Otherwise, the computed $|\tanh(2\phi)|$ would have to be unity, what has already been ruled out. Thus,

$$\tanh^2\phi < |\tanh\phi| \le 1^- \implies 1 - \tanh^2\phi \ge 1 - 1^-.$$

Since $1 - 1^-$ is a floating-point value in the normal range, its square root is as well, so $\cosh\phi$ cannot overflow in (19), and neither can $\sinh\phi$.

The input data has to be prescaled by the highest possible power of two, $2^s$, to get $a'_{11}=2^s a_{11}$, $a'_{22}=2^s a_{22}$, and $a'_{21}=2^s a_{21}$, such that

$$\max\{a'_{11},a'_{22}\}\le\nu/2,\quad \max\{|\Re a'_{21}|,|\Im a'_{21}|\}\le\nu/2, \tag{20}$$

where $\nu$ is the largest finite floating-point value. This way no quantity in (15) will overflow[1], and the chances of dealing with subnormal values will be minimized. See the scaling of $A$ and the computation of $\mathrm{e}^{\mathrm{i}\beta}$ in:
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[1] not even $2|a'_{21}|$, since $A$ is positive semidefinite, so $2|a'_{21}|\le 2\sqrt{a'_{11}a'_{22}}\le\nu$, but the upper bound of $\nu/4$ instead of $\nu/2$ in (20) might be safer