Supplementary Material: Explanation of angle of linear polarization estimator error

Joaquin Rodriguez, Lew-Fock-Chong Lew-Yan-Voon, Renato Martins, and Olivier Morel
ImViA, Université de Bourgogne, Le Creusot, France

This is supplementary material of our RAL/IROS 2022 submission entitled **"A practical calibration method for RGB micro-grid polarimetric cameras"**. We demonstrate why the error of the AoLP estimator has a sine-like shape. This demonstration is auxiliary to the submitted paper and the explanation was not included in the main paper due to space restrictions. The shape shown in Fig. 1 is a plot of the circular average of the AoLP calculated from the intensity measurements of a 50×50 region around the center of the sensor of a real camera with a 16mm lens. For each AoLP, ten images of a light are captured and averaged. The reason is to reduce the influence of the noise in the estimations.

I. Preliminaries

The polarization state of a light source is defined by its 4D Stokes vector $\mathbf{S}$. If the polarization analyzer has only linear filters, then only the first three components of this vector can be measured. This is the case of our camera, thus, S_3 will not be considered. In other words, we are interested in only the linear Stokes vector:

$$\mathbf{S} = \begin{bmatrix} S_0 \\ S_1 \\ S_2 \end{bmatrix} \tag{1}$$

where S_0 is the total intensity of the incoming light, S_1 is the amount of this light that is polarized linearly, in the horizontal and vertical directions, and S_2 is the polarized light in the $\pm 45°$ directions. From this vector, the angle of linear polarization α and the degree of linear polarization ρ can be computed as:

$$\alpha = \frac{1}{2} atan\left(\frac{S_2}{S_1}\right) \quad \rho = \frac{\sqrt{S_1^2 + S_2^2}}{S_0} \tag{2}$$

In order to measure the Stokes vector, and therefore, its linear components, at least three measurements with a linear polarizer at three different orientations are required. In the case of a RGB DoFP sensor, a set of 2×2 pixels with the same color filter can be used for this purpose. Each pixel of this set has a linear polarization filter oriented at 0°, 45°, 90°, and 135°. If these filters are perfectly placed at these orientations, and the pixel qualities are ideal, then the Stokes vector can be computed from these measurements as:

$$\begin{bmatrix} S_0 \\ S_1 \\ S_2 \end{bmatrix} = \begin{bmatrix} \dfrac{I_0 + I_{45} + I_{90} + I_{135}}{2} \\ I_0 - I_{90} \\ I_{45} - I_{135} \end{bmatrix}, \tag{3}$$

where I_0, I_{45}, I_{90}, and I_{135} are respectively, the intensity measurements of the pixels whose filter has an orientation of 0°, 45°, 90°, and 135°. This equation can be demonstrated by using Eq. (10), included in the main paper, when the camera is ideal ($T_i = 0.5$, $P_i = 1.0$, and $\theta_i = i$, for $i = \{0°, 45°, 90°, 135°\}$). As a consequence, the identity shown in Eq. (3) allows to measure the AoLP α directly from the super-pixel intensities.

Nevertheless, in the general case, the pixels are not ideal, and the filter orientations are not perfect. Therefore, using Eqs. (2) and (3) to estimate α brings an error that must be quantified. In order to demonstrate this error, we need to remember two important set of properties:

- Sine and cosine properties:

$$\begin{aligned} sin\left(\tfrac{\pi}{2} + \theta\right) &= cos\left(\theta\right) & sin\left(\tfrac{3\pi}{2} + \theta\right) &= -cos\left(\theta\right) \\ cos\left(\tfrac{\pi}{2} + \theta\right) &= -sin\left(\theta\right) & cos\left(\tfrac{3\pi}{2} + \theta\right) &= sin\left(\theta\right) \\ sin\left(\pi + \theta\right) &= -sin\left(\theta\right) & cos\left(\pi + \theta\right) &= -cos\left(\theta\right) \end{aligned} \tag{4}$$

- Taylor expansion of sine and cosine functions up to order 2, around $\theta = 0$:

$$\begin{aligned} sin\left(\theta\right) &\simeq \theta \\ cos\left(\theta\right) &\simeq 1 - \frac{\theta^2}{2} \end{aligned} \tag{5}$$

Additionally, when a pixel is not ideal, the relationship between the Stokes vector and its measured intensity is given

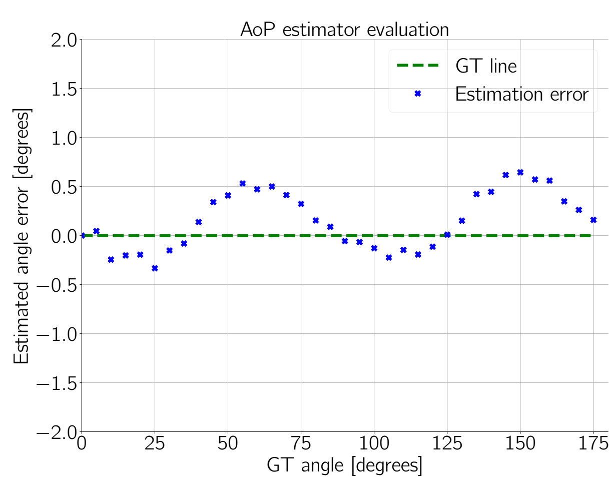

Fig. 1. AoLP estimator error plot.

by Eq. (6) in the main paper. This equation has been copied here for convenience:

$$I_i = T_i \begin{bmatrix} \frac{1}{P_i} & cos\left(2\theta_i\right) & sin\left(2\theta_i\right) \end{bmatrix} \begin{bmatrix} S_0 \\ S_1 \\ S_2 \end{bmatrix}, \quad (6)$$

where I_i is the measured pixel intensity, T_i is the pixel gain, P_i is a factor that models the non-ideality of the pixel micro-filter, and θ_i is the micro-filter orientation, for $i = \{0°, 45°, 90°, 135°\}$.

II. Demonstration

To demonstrate the error equation in the estimated AoLP, we compute the error in the measured Stokes components when the pixels are considered ideal.

$$\begin{aligned} \hat{S_1} &= I_0 - I_{90} \\ \hat{S_2} &= I_{45} - I_{135} \end{aligned} \quad (7)$$

Considering the pixel model of Eq. (6), and assuming that the filter orientation has an error $\Delta\theta_i$ we obtain:

$$\hat{S_1} = \Big[T_0 \begin{bmatrix} \frac{1}{P_0} & cos\left(2\Delta\theta_0\right) & sin\left(2\Delta\theta_0\right) \end{bmatrix} - T_{90} \begin{bmatrix} \frac{1}{P_{90}} & cos\left(\pi + 2\Delta\theta_{90}\right) & sin\left(\pi + 2\Delta\theta_{90}\right) \end{bmatrix} \Big] \begin{bmatrix} S_0 \\ S_1 \\ S_2 \end{bmatrix} \quad (8)$$

Using the sine and cosine properties, and grouping terms gives:

$$\hat{S_1} = \begin{bmatrix} A & B & C \end{bmatrix} \begin{bmatrix} S_0 \\ S_1 \\ S_2 \end{bmatrix}, \quad (9)$$

with

$$A = \frac{T_0}{P_0} - \frac{T_{90}}{P_{90}}$$

$$B = T_0 cos\left(2\Delta\theta_0\right) + T_{90} cos\left(2\Delta\theta_{90}\right)$$

$$C = T_0 sin\left(2\Delta\theta_0\right) + T_{90} sin\left(2\Delta\theta_{90}\right)$$

Since the angle error can be considered close to zero, then the corresponding Taylor expansions in Eq. (5) can be used to replace the sine and cosine functions. Moreover, by doing the matricial multiplication we obtain:

$$\hat{S_1} = AS_0 + \left[T_0 + T_{90} - 2T_0\Delta\theta_0^2 - 2T_{90}\Delta\theta_{90}^2\right] S_1 + \left[2T_0\Delta\theta_0 + 2T_{90}\Delta\theta_{90}\right] S_2. \quad (10)$$

If the angle errors are between $[-10°, 10°]$, the corresponding range in radians is $[-0.1745, 0.1745]$. Thus, if we square this range, we obtain a range $[0, 0.03]$. The orientation errors due to manufacturing problems have values less than the given example, therefore, the second order variables can be ignored.

$$\hat{S_1} = AS_0 + G^{'} S_1 + K_1 S_2. \quad (11)$$

with:

$$G^{'} = T_0 + T_{90}$$

$$K_1 = 2T_0\Delta\theta_0 + 2T_{90}\Delta\theta_{90}.$$

Similarly, $\hat{S_2}$ can be obtained as a function of the Stokes components.

$$\hat{S_2} = DS_0 - K_2 S_1 + G^{''} S_2. \quad (12)$$

where:

$$D = \frac{T_{45}}{P_{45}} - \frac{T_{135}}{P_{135}}$$

$$G^{''} = T_{45} + T_{135}$$

$$K_2 = 2T_{45}\Delta\theta_{45} + 2T_{135}\Delta\theta_{135}.$$

It follows that the estimated AoLP $\hat{\alpha}$ is equal to:

$$\hat{\alpha} = \frac{1}{2} atan\left(\frac{\hat{S_2}}{\hat{S_1}}\right) = \frac{1}{2} atan\left(\frac{DS_0 - K_2 S_1 + G^{''} S_2}{AS_0 + G^{'} S_1 + K_1 S_2}\right) \quad (13)$$

Remembering that $S_1 = S_0 \rho cos\left(2\alpha\right)$, and $S_2 = S_0 \rho sin\left(2\alpha\right)$, where ρ is the degree of linear polarization, and α is the angle of linear polarization of the incoming light, Eq. (13) becomes Eq. (14).

$$\hat{\alpha} = \frac{1}{2} atan\left(\frac{D - K_2 \rho cos\left(2\alpha\right) + G^{''} \rho sin\left(2\alpha\right)}{A + G^{'} \rho cos\left(2\alpha\right) + K_1 \rho sin\left(2\alpha\right)}\right) \quad (14)$$

This equation converges to the true AoLP α if the pixels and the filters are ideal, i.e., $P_i = 1$, $T_i = 0.5$, and $\Delta\theta_i = 0$, for $i = \{0, 45, 90, 135\}$. In a real case, slight deviations from these values will appear. The sources of these deviations are the manufacturing process of the sensor, and the lens added to the camera. As mentioned in the main paper, considering a small region around the center of the sensor reduces the deviations caused by the lens.

Analyzing this equation, it is possible to conclude that:

- The deviations in the pixel parameters will make the other polarization parameters to influence the AoLP measurement.
- The deviations in the orientations of the micro-polarizers, denoted by $\Delta\theta_i$, for $i = \{0, 45, 90, 135\}$ will introduce an error based on the value of the complementary Stokes parameter (for the measurement of S_1, a deviation in the orientation of the micro-polarizers will introduce an error based on the value of S_2, and an error based on S_1 in the measurement of S_2).
- For measurements of the same light at different AoLP, the deviations in the non-ideality factor P_i and the gain T_i will produce a constant shift in both, numerator and denominator, of Eq. (14).

- The values of K_1 and K_2 should be close to zero, and this can happen in two situations: either the pixel parameters are almost ideal and therefore the orientation errors are almost zero, or the pixels orientation error are almost the same, but in opposite directions. The second case can be understood by looking at the definitions of these variables. For instance, $K_1 = 2T_0 \Delta\theta_0 + 2T_{90}\Delta\theta_{90}$, where T_0 and T_{90} are positive numbers, and in general, they are close to 0.5. Therefore, if $\Delta\theta_0 \simeq -\Delta\theta_{90}$, then K_1 will have a very tiny value. Similarly, K_2 will have a tiny value when $\Delta\theta_{45} \simeq -\Delta\theta_{135}$. Finally, it can be seen that the errors in the orientations can be compensated if they are in opposite directions.

In all the cases, the errors will produce sine-like functions, since they will change the ratio of the sine to the cosine functions. Nevertheless, the effect of each parameter to the final shape of the error is different. The error in the orientations can change only the minimum and maximum values in the estimation error function, and the factors T_i and P_i can change the position of these extreme values.

III. Experiments

In this section, the error function has been computed for several set of samples. The samples to which the functions are fitted have been captured using the RGB polarization camera with the following lenses:

- Lens 1: Fuji-film HF8XA-5M - F1.6/8mm
- Lens 2: Fuji-film HF12XA-5M - F1.6/12mm
- Lens 3: Fuji-film HF16XA-5M - F1.6/16mm
- Lens 4: Fuji-film HF25XA-5M - F1.6/25mm

Additionally, all the lenses have been correctly focused on the light source used, and their F-number have been set to 3, which is higher than 2.8. This configuration have been done to comply with the recommendations given by [1].

To run this experiment, the AoLP estimator as described in the main paper has been implemented. Then, with a uniform unpolarized light source and a rotative linear polarization filter, a linearly polarized light is obtained. The position of the filter is changed in the range $[0°, 180°]$, with a step of $5°$. The reference angle of linear polarization of each sample have been measured from the rotative mount of the linear filter. Additionally, the AoLP is estimated with the implemented algorithm for each of these samples. Finally, the error between the reference value and the estimation is computed and plotted in Figs. 2 to 5. To avoid a constant shift in the measurements due to misalignment, since the first reference angle is zero, the first estimated angle have been subtracted from all the estimations.

By using a least-square optimizer, the pixel parameters have been found for each set of samples taken with the lens. These parameters are shown in Tab. I. For creating this data, the degree of linear polarization was supposed to be $\rho = 0.97$.

As shown in Figs. 2 to 5, taking the measurements of the AoLP from the center pixels and doing their circular average, produces an estimation of the true AoLP with a maximum

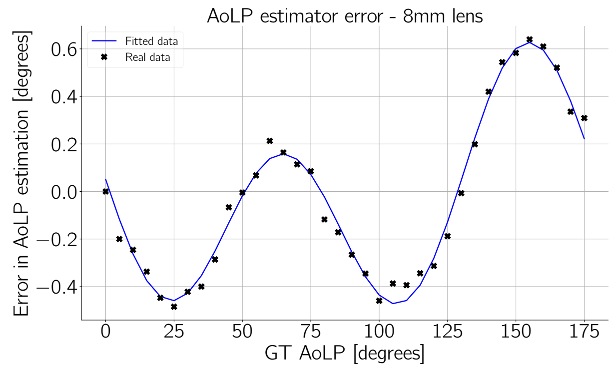

Fig. 2. Error in the AoLP and curve fitted of Eq. (14) for the lens 1

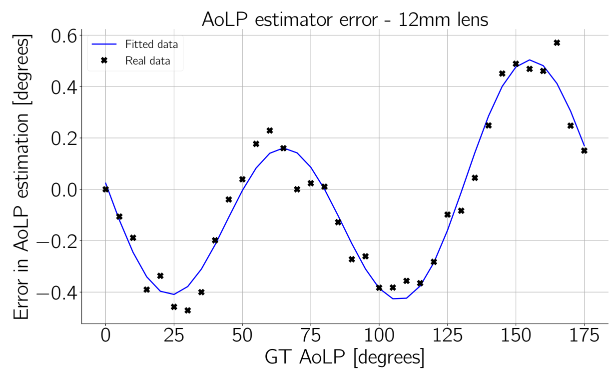

Fig. 3. Error in the AoLP and curve fitted of Eq. (14) for the lens 2

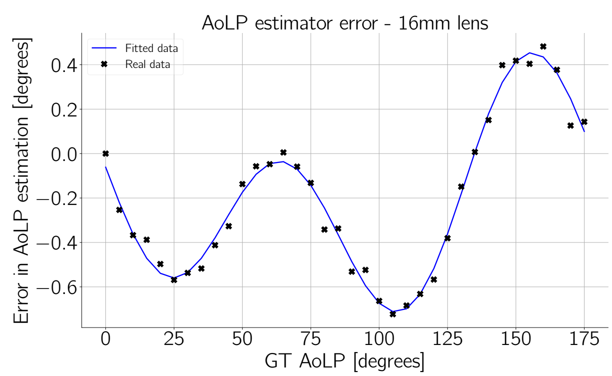

Fig. 4. Error in the AoLP and curve fitted of Eq. (14) for the lens 3

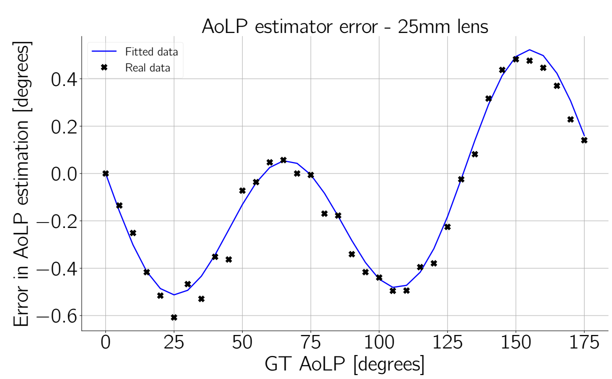

Fig. 5. Error in the AoLP and curve fitted of Eq. (14) for the lens 4

Lens model	Parameter	$i = \{0, 45, 90, 135\}$
T_i	Lens 1	$[0.525, 0.542, 0.548, 0.499]$
	Lens 2	$[0.53, 0.58, 0.57, 0.49]$
	Lens 3	$[0.53, 0.48, 0.49, 0.51]$
	Lens 4	$[0.52, 0.49, 0.44, 0.45]$
P_i	Lens 1	$[1.009, 0.992, 1.067, 0.922]$
	Lens 2	$[0.96, 1.06, 1.03, 0.91]$
	Lens 3	$[1.04, 0.88, 0.97, 0.95]$
	Lens 4	$[1.09, 1.05, 0.95, 0.98]$
$\Delta\theta_i$	Lens 1	$[0.109, 0.114, -0.192, 0.092]$
	Lens 2	$[-0.42, -0.94, 0.37, 1.31]$
	Lens 3	$[1.47, -1.07, -1.32, 1.54]$
	Lens 4	$[0.07, 0.17, 0.07, 0.16]$

TABLE I

PARAMETERS OBTAINED BY NON-LINEAR OPTIMIZATION FOR EQ. (14).

error of $0.65°$. This upper limit is valid for all the tested lenses.

Tab. I shows all the pixel parameters obtained by least-squared optimization of Eq. (14) with the real data. From this table it is possible to confirm the effective pixel values are not far away from the ideal ones. Particularly, the maximum orientation error is $\Delta\theta_0 = 1.47°$. Nonetheless, as explained in the previous section, this error is compensated by the complementary pixel orientation which is $\Delta\theta_{90} = -1.32°$. Additionally, the values exposed in this table show that the different lenses influence the pixel parameters. Indeed, the figures have similar shapes, but the corresponding maximum values are not the same, and they are located at different positions. This is because the corresponding pixel parameters have changed for each case.

As a conclusion, it is possible to confirm that using the ideal values of the pixels parameters introduces an error in the estimation of the AoLP, and this error depends on the measured angle, and the actual pixel parameters. Even though the pixels can be ideal, placing a lens between the light to measure and the sensor will introduce an error in the measured Stokes vector that is reflected as a change in the effective pixel parameter values.

To minimize these deviations, a better estimation of the Stokes vector must be used. This better estimation can be obtained if the deviation between the used pixel parameters and their true value is reduced. These values can be found by doing calibration. From the comparison done in the main paper, in which the super-pixel calibration is compared with our method, it follows that our method is the best option since a similar accuracy is obtained with a simpler experimental set-up.

REFERENCES

[1] Connor Lane, David Rode, and Thomas Roesgen. Calibration of a polarization image sensor andinvestigation of influencing factors. Applied Optics, 61, 10 2021.