

Columbia MA Math Camp

Introduction to Propositional Logic

Vinayak Iyer

July 22, 2020

Introduction

Outline of Today's Class

- Introduction to Propositional Logic
- Introduction to Set Theory
- Overview of how to write proofs

But first, more important stuff :

- How to approach classes
- Rubinstein Interview 1
- Rubinstein Interview 2

Before we start...

- This is meant to prepare a **basic** foundation of mathematics that **will** help you in your classes
 - Not meant to teach you advanced mathematics
- Your goal should be to understand the concepts well. The more you use this time to really grasp concepts, the easier it will be in your classes
- I **will** try and go slow, but please let me know if I am going too fast.
- Please interrupt me and ask questions! You **will** benefit and so will others!

Why do we need to learn Propositional Logic?

- In economics, we rely on a lot of proofs. You will encounter proofs in most of your classes.
- So it is useful to go through some foundations of logic which will help you in your class

Propositions

Definition

A **proposition** is a statement that can be either **True (T)** or **False (F)**

Examples :

- *The earth is flat* - **False**
- *March has 31 days* - **True**
- *You should maintain a work-life balance* - **Not a proposition**

Notation : Lower case letters are often used to represent propositions.

Example :

- *p* : *"The earth is flat"*
- *q* : *"March has 31 days"*

Logical Operators (Connectives)

Definition

A connective is a symbol that combine propositions. Propositions separated by connectives make a compund proposition.

There are 3 basic connectives :

1. **Conjunction** : “ p and q ” and is denoted by \wedge

• “The Earth is flat and March has 31 days”

2. **Disjunction** : “ p or q ” and denoted by \vee

• “The Earth is flat or March has 31 days”

Note : The meaning of \vee is *inclusive* in the sense that the either one statement **is true** or both can be true

3. **Negation** : “ $\neg p$ ” is the negation of the statement p



Truth Values

- We now must determine the **truth values** of compound statements.
(What were compound statements?)
- Note that the truth or falsity of a compound statement depends on the truth or falsity of the individual statements.
- We do this through a **truth table** which lists all possible combinations of the truth value of a compound statement based on the truth value of the individual components.

Truth Table
for "And" connective

$p \wedge q$		
p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Truth Table
for the "OR" connective

$p \vee q$		
p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

p	$\neg p$
T	F
F	T

Examples of Evaluating Compound Propositions

Example

Suppose p, q, r are 3 statements. Suppose \underline{p} and \underline{r} are true while \underline{q} is false.

Evaluate whether or not the following statements are true.

$$\underline{\underline{p \wedge q}} \Rightarrow .$$

- $\underline{\neg(p \wedge q)}$:- Since p is True and q is False $\Rightarrow p \wedge q$ is False
 $\Rightarrow \underline{\neg(p \wedge q)}$ is True

- $\underline{(\neg p) \wedge (\neg q)}$:- $(\neg p)$ is False and $(\neg q)$ is True $\Rightarrow (\neg p) \wedge (\neg q)$ is False

- $\underline{p \vee ((\neg q) \wedge r)}$:- (?)

$$p \vee (\underline{(\neg q) \wedge r})$$

p is True

$\neg q$ is True
 r is True $\Rightarrow \neg q \wedge r$ is True

$$\Rightarrow p \vee (\neg q \wedge r) \text{ is True}$$

Filling out a Truth Table

In the previous slide I specified the truth values of the individual statements. If I had not specified the truth values, then we would have to fill out the truth table based on all possible combinations of p , q and r . So let's fill it out for $p \vee ((\neg q) \wedge r)$.

p	q	r	$p \vee (\neg q \wedge r)$
T	T	T	
T	T	F	
T	F	T	
T	F	F	
F	T	T	
F	T	F	
F	F	T	
F	F	F	

$$2^3 = 8$$

Conditional Propositions

Definition

The conditional operator ***if*** is used in a compound statement of the form ‘‘*if p then q*’’ and is denoted by $p \rightarrow q$. It is also called the logical implication sometimes.

$$p \rightarrow q$$

- p is called the antecedent or hypothesis
- q is called the consequent or conclusion
- $p \rightarrow q$ is also read as “*p implies q*”

Example :

- p : “*You got an A*”
- q : “*I give you a dollar*”
- $p \rightarrow q$: “*If you get an A, then I will give you a dollar*”

Truth Table for \rightarrow

Like the other logical operators, the **logical implication** (\rightarrow) also has a truth table.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

$q \rightarrow p$

T	T	T
T	F	F
F	T	T
F	F	T

$p \rightarrow q$ is True whenever p is false.

- The last two are by definition. To understand why, let us go back to the previous example.

If p then q

- $p \rightarrow q$:- "If you get an A, then I will give you a dollar"

- Statement is true if I keep my promise and false if I don't.
- If you don't get an A, then irrespective of whether or not I give you a dollar, I haven't broken my promise!

Example

Example

Suppose p , q and r are statements such that \underline{p} , \underline{r} are true and $\underline{\underline{q}}$ is false.

Evaluate the following compound statements.

p is True, q is False, $p \rightarrow q$ is False.

1. $\neg(p \rightarrow q)$:- From the truth table, since p is true and q is false $\Rightarrow p \rightarrow q$ is False and the negation of a False statement is True. Hence

$\boxed{\neg(p \rightarrow q) \text{ is True}}$

False

2. $(p \wedge q) \rightarrow r$:- $(p \wedge q)$ is False. This implies that $\underline{(p \wedge q) \rightarrow r}$ is a True statement.

$\underline{q \text{ is False}}, \text{ that means } \underline{p \wedge q \text{ is False}}$

3. $(p \rightarrow q) \rightarrow r$:- (?)

$p \rightarrow q$ is False $\Rightarrow (p \rightarrow q) \rightarrow r$ is True

Necessary and Sufficient Conditions

- You will come across these terms a LOT in economics!
- Consider the logical statement $p \rightarrow q$
 - p is said to be a **sufficient condition** for q
 - If the statement "*If you get an A, then I will give you a dollar*" is true, then knowing that you got an A is sufficient to know that you also received a dollar.
 - q is said to be a **necessary condition** for p
 - Similarly, for the statement above to be true, then getting a dollar is necessarily true if you get an A.
- Of course p being sufficient for q does not mean it is necessary and vice versa.

More examples of Necessary vs Sufficient Conditions

If P then q $p \rightarrow q$ is True

1. Consider the statement "If it rains, then it is cloudy" Rains Cloudy

- Then knowing that it is raining is sufficient to know that it is cloudy. (But being cloudy is not sufficient for rain!)
- It is necessary for it to be cloudy for rain to occur. In other words, if it is not cloudy, then it cannot rain! (It is not necessary for it to rain if it is cloudy!) q is necessary for p.

2. How does this relate to economics? Most of you must have heard the phrases "**First Order Condition**" and "**Second Order Condition**" right?

- The FOC when maximizing a function $f(x)$ is given by $f'(x) = 0$. For x to be an interior maximum, it is necessary that $f'(x) = 0$ (but not sufficient!) If x is an interior max, then $f'(x) = 0$
- The SOC sufficient condition is generally given by the statement that if x is such that $f'(x) = 0$ and $f''(x) < 0$, then x is a maximizer. But this is not necessary! - Find the maxima of



$$f'(x) = 0$$

$$f(x) = -x^4$$

$$f''(x) = -12x^2$$

$$f''(0) = 0$$

$$f(x) = -x^4$$

$$= -x^4$$

$$f''(x) = 0$$

$$x=0$$

sufficient condition
for x to be a max

Some statements can be both necessary and sufficient

- Consider the conditional statement given by :
 - "A number being 2,4,6 or 8 → it is an even number"
 - As per the discussion previously, knowing a number being 2,4,6 or 8 is **sufficient** to know that it is even. But it is **not necessary** that number be 2,4,6 or 8 for it to be even
 - Now consider the alternate statement :

p is sufficient for q but NOT necessary.

 - "A number being 2,4,6 or 8 → it is a positive even number < 10"
 - Now a number being 2,4,6 or 8 is **both necessary and sufficient** for it to be a positive even number < 10
 - In this case, we can actually write "A number being 2,4,6 or 8
 \leftrightarrow it is an even number < 10"
- p \leftrightarrow q*

Definition

We write $p \leftrightarrow q$ or p if and only if q when p is **both necessary and sufficient** for each other.

What does the truth table look like? When is $p \leftrightarrow q$ true?

Converse and Contrapositive

Definition

For a conditional statement $p \rightarrow q$

It's easier to prove. $\neg q \rightarrow \neg p$

- The **converse** is defined as $q \rightarrow p$
- The **contrapositive** is defined as $(\neg q) \rightarrow (\neg p)$ → **Contrapositive**

Properties :

1. If $p \rightarrow q$ is true, that **does not** mean that $q \rightarrow p$ is true!

1.1 When p is False and q is True, then $p \rightarrow q$ is True but $q \rightarrow p$ is False

2. The contrapositive $(\neg q) \rightarrow (\neg p)$ is **equivalent** to $p \rightarrow q$. (Why?)

Question : Does $p \rightarrow q$ imply $\neg p \rightarrow \neg q$?

(1) If it rains then it is cloudy.

(2) If it does not rain then it is not cloudy.

Example by Rubinstein!

Q6. My paper has just been rejected. What should I do?

I have a lot of experience with the mental state you must be in, so I have three pieces of advice:

- (a) Don't read the referee reports. They are likely to depress you. Even if they are potentially useful, you are not in a state of mind to benefit from them.
- (b) Find comfort in my motto: "A paper that has not been rejected should not be published." But beware of the faulty logic in assuming that "every paper that has been rejected should be published."

Figure 1: Words of Wisdom from Rubinstein

If A paper that has not been rejected then it should not be published

$$P \rightarrow q$$

- What is the faulty logic? $p \rightarrow q$ DOES NOT IMPLY $\neg p \Rightarrow \neg q$
- Contrapositive is that "A paper that has been published must have been rejected"

$$p \rightarrow q \neq \neg p \rightarrow \neg q$$
$$(\neg p \rightarrow \neg q)$$

If a paper has been rejected then it should be published

Examples of Converse and Contrapositive Statements

Consider the statement "*If a paper that has been published, then it must have been rejected*"

- **Converse :** *If a paper has been rejected, then it must be published*
- **Contrapositive :** *If a paper has not been rejected, then it must not be published*

Note : The contrapositive has the same truth value as the original statement.

Logical Equivalence

Definition

Two statements p and q are **logically equivalent** denoted by $p \leftrightarrow q$ when the truth values in **all** rows in the truth table are the same.

For example, show that the statements $(p \rightarrow q)$ and $(\neg p \vee q)$ are logically equivalent

p	q	$p \rightarrow q$	$\neg p \vee q$
T	T	1	1
T	F		
F	T		
F	F		

are logically
equivalent

DeMorgan's Law

Fact

Consider 2 propositions p and q . Then the statements $\neg(p \vee q)$ and $(\neg p \wedge \neg q)$ are logically equivalent i.e.

$$\neg(p \vee q) \leftrightarrow (\neg p \wedge \neg q)$$

$$(A \cup B)^c = A^c \cap B^c$$

Proof.

p	q	$\neg(p \vee q)$	$(\neg p \wedge \neg q)$
T	T	F	F
T	F	F	F
F	T	F	F
F	F	T	T

□

Tautology

Definition

A **tautology** is a formula that is always true i.e. for every combination of truth values of its components, the compound statement is true.

Example

Show that $(p \rightarrow q) \vee (q \rightarrow p)$ is a tautology.

P	Q	$P \rightarrow Q$	$Q \rightarrow P$	$(P \rightarrow Q) \vee (Q \rightarrow P)$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

Final Example

Example

Suppose x is a real number. Consider the statement :

If $x^2 = 4$, then $x = 2$

If $x^2 = 4$, then $x = \pm 2$

Construct the **converse** and **contrapositive** of the statement. Determine the truth or falsity of the original, converse and the contrapositive using your knowledge of algebra. In the converse statement, identify the **necessary and sufficient statements**.

Proof.

The converse of the statement is "If $x = 2$, then $x^2 = 4$ ". The contrapositive is "If $x \neq 2$, then $x^2 \neq 4$ ".

Truth value : The original statement is false since $(-2)^2 = 4$. The contrapositive has the same truth value as the original, hence is also false, while the converse is true i.e. If $x = 2$, then $x^2 = 4$.

$p \rightarrow q$
 p is sufficient for q

Knowing that $x = 2$ is sufficient to know that $x^2 = 4$, while x^2 being equal to 4 is necessary for x being equal to 2.

q is necessary for p