# Columbia MA Math Camp

## Set Theory

Vinayak Iyer [a]

July 22, 2020

[a] Material adapted from notes by David Thompson and Xingye Wu

# Motivation of Study Set Theory

- Set theory is one of the fundamental building blocks of mathematics
- Many important concepts later such as relations, functions and sequences are defined using the language of sets
- You will encounter a lot of this during your micro course (and math methods of course)

# Table of Contents

# Small Digression on Proofs

- We went through some basics of logic where we talked about implications like $p \to q$.
- In set theory, you will frequently encounter questions which give you some information and then ask you to show that $p \to q$. → To show.
- The way to show it is to assume that $p$ is true and then use the information to show that $q$ must be true → $p \to q$ is True

(1) Assume $p$ is True
(2) Show $q$ is True

- Sometimes it is easier to prove the contrapositive i.e. try to prove that $\neg q \to \neg p$.
    - Assume that $\neg q$ is true and then proceed to show that $\neg p$ must be true.

(3) Conclude that $p \to q$ is True

# Table of Contents

# Common Notation

$\in$ : belongs to

$\forall$ : for all

$\exists$ : there exists

$\forall \to$ for all.

- $\in$: **"in"**; e.g. $x \in \mathbb{N}$ means $x$ is a **natural number**.
- $\forall$: **"for all"**; e.g. $\forall x \in \mathbb{N}$ means for all natural numbers $x$.
- $\exists$: **"there exists"**; e.g. $\forall x \in \mathbb{N}, \exists y \in \mathbb{Z}$ such that $x + y = 0$
- $!$: **"unique"**; typically used in conjunction with $\exists$
- $\Rightarrow$: **"implies"**; e.g. $A \Rightarrow B$ means $A$ implies $B$.
  - We have been using $\to$. We will switch to $\implies$ from now as this is more commonly used.

$\to$ $\boxed{\Rightarrow}$

## Basic Concepts

- A set is a collection of objects and each individual object is called an **element**
- Lowercase letters are used for elements and Capital letters for sets.
- The notation $x \in X$ means that the object $x$ **is an element** of the set $X$.
- A set is typically written in curly brackets $\{1, 2, 3\}$
  - The order of the elements listed does not matter
- For more complicated sets we use **"set-builder"** notation, e.g.

$$\{x \in \mathbb{N} \mid x^2 < 100\} \equiv \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}.$$

- The item before the vertical line defines the domain of our search.
- In the example above, we are searching for natural numbers which satisfy the requirement to the right of the vertical line

## Common Sets

$\mathbb{R}_{++}$ $(0, \infty)$ $\mathbb{Z}_{++}$ 2 $\frac{4}{2}$ $\frac{8}{4}$

$\mathbb{R}_{+}$ $[0, \infty)$ $\mathbb{R}$ $(-\infty, \infty)$

- **Common sets:**
  - $\mathbb{N}$: natural numbers $\{0, 1, 2, ...\}$
  - $\mathbb{Z}$: integers $\{..., -2, -1, 0, 1, 2, ...\}$
  - $\mathbb{Q}$: rational numbers; all numbers of form $\frac{p}{q}$ with $p, q \in \mathbb{Z}, q \neq 0$
  - $\mathbb{R}$: real numbers; most of econ happens here

$\mathbb{Z}_+$ refers to positive integers. $\{0, 1, 2, 3, 4 \ldots \}$

$\sqrt{2}$ rational number?

- We do allow a set to contain **no element** at all, and we call it the **empty set** denoted by $\emptyset$
  - The empty set $\emptyset$ is a subset of every set. (Why?)

$\pi$, $e$, $\sqrt{2}$ are not rational numbers.

$\sqrt{2} \in \mathbb{Q}$ ?

$\sqrt{2}$ cannot be expressed in the form $\frac{p}{q}$ where $p, q \in \mathbb{Z}$

# Comparing sets - Inclusion

- $A$ is a **subset** of $B$ if every element of $A$ is an element of $B$; write $A \subseteq B$ or $\boxed{B \supseteq A}$
  - In other words, $\boxed{x \in A \implies x \in B\ \forall x}$ $[A \subseteq B]$.

$\underbrace{A = \{1, 2\}}_{}$ $A \subseteq B$ .
$B = \{1, 2, 3\}$

- Two sets are **equal** if they contain exactly the same elements. $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$. then $A = B$
- A set $A$ is a **proper subset** of $B$ if $A \subseteq B$ and $A \neq B$. This is sometimes written $A \subset B$ or $A \subsetneq B$

(1) $\forall x \in A$, $x \in B$ $A \subseteq B$
(2) $\exists x \in B$ that $\notin A$ $B \not\subseteq A$

$A \subseteq B$ ?

  - Not all sets are comparable. Give me an example?

$A = \{2, 3\}$

If p $\rightarrow$ q

- Set inclusion is **transitive**: $A \subseteq B$ and $B \subseteq C$ implies $A \subseteq C$.

$B = \{1, 2\}$

- For finite sets, the **cardinality** of a set $|A|$ is the number of elements of $A$

$|A| = 2$ $A = \{1, 2\}$ $B = \{1, 2, 3\}$

Is $B \subseteq A$ ? $\boxed{\text{No because } \exists x \in B \text{ that } \notin A}$

# How to prove Set Inclusion is Transitive?

$p \to q$

## Lemma 1.1

If $A \subseteq B$ & $B \subseteq C$ then $A \subseteq C$.

*Set inclusion is transitive i.e.* $\boxed{A \subseteq B \text{ and } B \subseteq C}$ *implies* $A \subseteq C$

## Proof.

x was completely arbitrary

We want to show $A \subseteq C$. By definition, this means that we need to show for any $x \in A$ it must be that $x \in C$. Take any $x \in A$. By definition of $A \subseteq B$ and because $x \in A$, we have that $x \in B$. Again by the definition of $B \subseteq C$, we have $x \in C$. $\Rightarrow A \subseteq C$. □

(1) Assume that $\boxed{A \subseteq B \text{ \& } B \subseteq C}$. We need to show $A \subseteq C$.

(2) To show $A \subseteq C$, we need to show $\forall x \in A$, then $x \in C$

(3) So take any $x \in A$. This implies $x \in B$ as $A \subseteq B$.

(4) We know $B \subseteq C$. $\Rightarrow x \in B \Rightarrow x \in C$. Hence $\forall x \in A \Rightarrow x \in C$.

# Constructing new sets

- The **union** of $A$ and $B$, $A \cup B$ is the collection of elements in $A$ or $B$ (or both)

$$\boxed{A \cup B := \{x \mid x \in A \text{ or } x \in B\}}$$

$A = \{1,2,3\}$

$B = \{3,4\}$

$A \cup B = \{1,2,3,4\}$

- The **intersection** of $A$ and $B$, $A \cap B$ is the collection of elements that belong to both $A$ and $B$

$$A \cap B := \{x : x \in A \text{ and } x \in B\}$$

$A \cap B = \{3\}$

If $A \cap B = \emptyset$, then we say that $A$ and $B$ are disjoint.

- The **difference** of $A$ and $B$, $A \setminus B$ or $A - B$, is the collection of elements in $A$ and not in $B$.

$$A - B := \{x : x \in A \,\&\, x \notin B\}$$

$A - B = \{1,2\}$

Can take unions and intersections of big collections of sets, typically indexed by an **index set**. Most common index set is $\mathbb{N}$, e.g.

$$\bigcup_{i \in \{0,1,2,\ldots\}} A_i$$

$$\boxed{A_1 \cup A_2 \cup A_3 \cup A_4 \ldots}$$

$$\bigcup_{i \in \{1,2,3\ldots\}} A_i$$

# Some properties - I

$p \leftrightarrow q$ Need to show $\underline{p \rightarrow q}$ & $\underline{q \rightarrow p}$

## Lemma 1.2

$A \cup B = B$ *iff* $A \subseteq B$ (*Note that this is a equivalence because of the iff*)

$\leftrightarrow$ (1) iff (2) if & only if (3)

**Proof.**
We show subset containment both ways i.e. both the $\implies$ and the $\impliedby$.
Let's first prove the $\boxed{\impliedby \text{ side.}}$

$\Leftarrow$ $\rightarrow$ $A \subseteq B$ $\Leftarrow$ $\implies$ $A \cup B = B$

- Suppose $A \subseteq B$ . WTS that $A \cup B = B$. This means that we need to show both $A \cup B \subseteq B$ and $B \subseteq A \cup B$. Let us first show $A \cup B \subseteq B$.
- Take any $x \in A \cup B$. WTS $x \in B$. Because $x \in A \cup B$, then, by definition $\cup$, either $x \in A$ or $x \in B$
- If $x \in A$, then by definition of $x \subseteq B$ we have $x \in B$. So either case, we have $x \in B$. Thus $A \cup B \subseteq B$ is proved. Proving $B \subseteq A \cup B$ is left as an exercise.

Now let us prove the other direction i.e. $\boxed{\implies}$

$A \cup B = B$ then $A \subseteq B$.

$\forall x \in A \implies x \in B$

- Given $A \cup B = B$, WTS $A \subseteq B$.
- Take any $x \in A$, WTS that $x \in B$. By definition of $A \cup B$ and $x \in A$, we have $x \in A \cup B$. Because $A \cup B = B$, we have $x \in B$. Thus $\implies$ is proved.

$p \longrightarrow q$

$\boxed{\text{To Prove}}$: $A \subseteq B \implies A \cup B = B$.

(1) Suppose $A \subseteq B$ is true.

(2) We want to show $\boxed{A \cup B = B}$

$A \cup B \subseteq B$ $\qquad$ $\boxed{B \subseteq A \cup B}$

$\boxed{x \in A \cup B \text{ then } x \in B}$

Take any $x \in A \cup B$

$\Rightarrow$ $x \in A$ or $x \in B$

But we know $A \subseteq B$

For $x \in A \Rightarrow x \in B$

If $x \in B$, then we are done because we wanted to show $x \in B$.

But if $x \in A$, we know $x \in B$ because $A \subseteq B$.

$\Rightarrow$ $x \in A \cup B \Rightarrow x \in B$

$x \in B$ then $\boxed{x \in A \cup B}$

$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

$x \in B \Rightarrow x \in A \cup B$

(by definition of $A \cup B$)

### Lemma 1.3

*Intersection is distributive with respect to the union:*

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Want to prove :- $p \to q$

(1) Assume $p$ is true

| $p$ | $q$ | $p \to q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

### Proof.

We show subset containment both ways.

$\subseteq$

- Let $x \in A \cap (B \cup C)$.
- By definition, $x \in A$ and $x \in B \cup C$, so $x \in B$ or $x \in C$.
- Thus $x \in A \cap B$ or $A \cap C$, so $\boxed{x \in (A \cap B) \cup (A \cap C)}$.

(1) $x \in A$

(2) $x \in B$ OR $x \in C$.

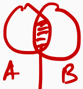
$A \subseteq B$

$\boxed{\forall x \in A \Rightarrow x \in B}$

Now the other direction

$\supseteq$

- Let $x \in (A \cap B) \cup (A \cap C)$.
- By definition, $x \in A \cap B$ or $x \in A \cap C$, so $x \in A$ and $x \in B$ or $x \in C$.
- Thus $\boxed{x \in A \cap (B \cup C)}$




$x \in A \cap B$

$\Rightarrow x \in A \;\&\; x \in B$



$\square$

# Proof of General Statement of Distributive Property

Note that we don't need to limit the proof before to finite unions. We can prove it for an infinite unions.

**Lemma 1.4**
*Prove that the intersection is distributive wrt to the union i.e.*

$$\boxed{B \cap \left( \bigcup_{i \in \mathbb{N}} A_i \right) = \bigcup_{i \in \mathbb{N}} (B \cap A_i)}$$

$\bigcup_{i \in \mathbb{N}} A_i = A_1 \cup A_2 \cup A_3 \cup \ldots$

**Proof :** To do this, we need to show that both the $\subseteq$ and $\supseteq$ to show that both sets are equal.

$\boxed{\subseteq}$: Take any $x \in B \cap \left( \bigcup_{i \in \mathbb{N}} A_i \right)$. By definition of $\cap$, this implies that $x \in B$ and $x \in \bigcup_{i \in \mathbb{N}} A_i$. By definition of $\bigcup_{i \in \mathbb{N}} A_i$, $\exists i \in \mathbb{N}$, such that $x \in A_i$. So we have for that $i$, $x \in B \cap A_i$. Therefore $x \in \bigcup_{i \in \mathbb{N}} (B \cap A_i)$

$$B \cap \left( \bigcup_{i \in \mathbb{N}} A_i \right) \subseteq \bigcup_{i \in \mathbb{N}} (B \cap A_i)$$

(1) Take any $x \in B \cap \left( \bigcup_{i \in \mathbb{N}} A_i \right)$ $\qquad A_1, A_2, \ldots$

$\Rightarrow x \in B$ AND $x \in \bigcup_{\boxed{i \in \mathbb{N}}} A_i$

$\boxed{j \in \mathbb{N}.}$

$\Rightarrow \exists\, j \in \mathbb{N}$, s.t. $x \in A_j$

$\Rightarrow \underline{x \in B \cap} A_j \qquad i = j$

$$\Rightarrow \boxed{x \in \bigcup_{i \in \mathbb{N}} (B \cap A_i)}$$

# Proof Continued...

$\supseteq$ : Take any $x \in \bigcup_{i \in \mathbb{N}} (B \cap A_i)$. By definition, $\exists i \in \mathbb{N}$ such that $x \in B \cap A_i$.

Therefore we have that $x \in B$ and $x \in A_i$. We can conclude then that $x \in \bigcup A_i$ and therefore $x \in B \cap \left( \bigcup_{i \in \mathbb{N}} A_i \right)$

# Complements and DeMorgan's Laws

We normally think of sets living in some larger space $\Omega$. The **complement** of a set $A$, $A^c$, is the collection of elements not in $A$.

$$\boxed{A^c := \Omega \setminus A}$$

$\Omega = \{\text{oranges, apples, bananas}\}$

$A = \{\text{oranges, apples}\}$

Further, the complement of the complement of a set is the set itself.

$$\boxed{(A^c)^c = A}$$

$A^c = \{\text{bananas}\}$

$$\boxed{(A^c)^c \subseteq A \;\&\; A \subseteq (A^c)^c}$$

Complements play nicely with unions and intersections. The following are DeMorgan's Laws for 2 sets $A$ and $B$.

2 sets

$$
\begin{aligned}
(1)\quad (A \cap B)^c &= A^c \cup B^c \\
(2)\quad (A \cup B)^c &= A^c \cap B^c
\end{aligned}
$$

# General statement of DeMorgan's Laws


$x \notin A$ & $x \notin B$

**Lemma 1.5**
*Let $A_i$ be a collection of sets. We have that :*

$\subseteq$ $=$

(a) $$\left(\bigcup_{i\in\mathbb{N}} A_i\right)^c = \bigcap_{i\in\mathbb{N}} \boxed{A_i^c}$$

Prove.

LHS $\subseteq$ RHS

RHS $\subseteq$ LHS

(b) $$\left(\bigcap_{i\in\mathbb{N}} A_i\right)^c = \bigcup_{i\in\mathbb{N}} A_i^c$$

**Proof :** We'll prove only the first statement. As usual we need to prove both $\subseteq$ and $\supseteq$.

$\subseteq$ : Take any $x \in \left(\bigcup_{i\in\mathbb{N}} A_i\right)^c = \Omega \setminus \left(\bigcup_{i\in\mathbb{N}} A_i\right)$. This implies $x \in \Omega$ and $\boxed{x \notin \left(\bigcup_{i\in\mathbb{N}} A_i\right)}$.

Therefore $\boxed{x \notin A_i}$ $\forall i \in \mathbb{N}$ which implies that $\boxed{x \in A_i^c \ \forall i}$. Thus, $\bigcap_{i\in\mathbb{N}} A_i^c$.

Defn of $\cap$

$x \in A \cap B \iff x \in A$ & $x \in B$

$x \in A_1^c$ & $x \in A_2^c$ & $x \in A_3^c$ ...

$\implies x \in \bigcap_{i\in\mathbb{N}} A_i^c$


$\Omega$.

$x \in A^c$

$x \in B^c$

$\implies x \in A^c \cap B^c$

$$\left(\bigcap_{i\in\mathbb{N}} A_i\right)^c = \bigcup_{i\in\mathbb{N}} A_i^c$$

$\subseteq$ :

$$x \in \left(\bigcap_{i\in\mathbb{N}} A_i\right)^c$$

$$\Rightarrow x \notin \bigcap A_i$$

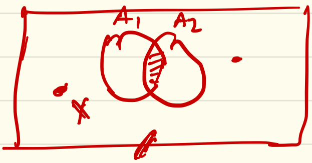
$$\Rightarrow \exists \text{ some } i \text{ s.t. } x \notin A_i$$

$x \notin \bigcap_{i\in\mathbb{N}} A_i \Rightarrow \underbrace{x \in A_i \ \forall i}$. Not true

$\Rightarrow$ There must be some $A_j$ s.t. $x \notin A_j$

$\Rightarrow x \in A_j^c$

$\Rightarrow x \in \bigcup_{i\in\mathbb{N}} A_i^c$




$x \in (A_1 \cap A_2)^c$

$\Rightarrow x \notin A_1$

$\Rightarrow x \in A_1^c$

$\Rightarrow x \in \boxed{A_1^c \cup A_2^c}$

# Proof Continued...

$$\bigcap_{i \in \mathbb{N}} A_i^c \subseteq \left( \bigcup_{i \in \mathbb{N}} A_i \right)^c$$

WTS: All elements in the left side set belongs to the right hand set

$\supseteq$ : Take any $x \in \bigcap_{i \in \mathbb{N}} A_i^c$.

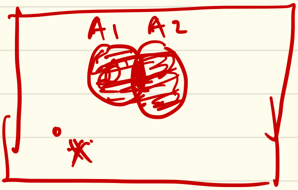
By definition, we have that $x \in A_i^c \ \forall i \in \mathbb{N}$. This implies $x \in \Omega$ and $x \notin A_i$ for any $i \in \mathbb{N}$. So we have $x \in \Omega$ and $x \notin \bigcup_{i \in \mathbb{N}} A_i$ which implies that $x \in \left( \bigcup_{i \in \mathbb{N}} A_i \right)^c$

$\boxed{x \in \bigcap_{i \in \mathbb{N}} A_i^c} \Rightarrow x \in A_i^c \ \forall i \in \mathbb{N}$

$\Rightarrow x \notin A_i \ \forall i \in \mathbb{N}$

$\Rightarrow x \notin \bigcup A_i \Rightarrow x \in \left( \bigcup_{i \in \mathbb{N}} A_i \right)^c$

$x \notin A_1$ AND $x \notin A_2$




$x \notin A_1$ & $x \notin A_2$

$x \notin A_1 \cup A_2$ $\qquad$ $x \notin A_1 \cap A_2$

# Cartesian Products

The **Cartesian product** is our last common method for generating new sets: take all pairs from $A$ and $B$:

$$\boxed{A \times B} \equiv \{(a, b) | a \in A, b \in B\}$$

$A \times B$

Typically work in $\mathbb{R}^n$:

$\mathbb{R} \times \mathbb{R}$

$$\mathbb{R}^n \equiv \{(a_1, ..., a_n) | a_i \in \mathbb{R}\ \forall i = 1, ..., n\}$$

$\mathbb{R}^2 = \{ (x,y) |$

Note order matters: $\boxed{(2, 1) \neq (1, 2)}$

$\mathbb{R} \times \mathbb{R}$ $\quad x \in \mathbb{R}$ & $y \in \mathbb{R} \}$

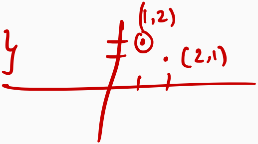
$(1,2) \in \mathbb{R}^2$

$(2,1)$

$(a, b)$ is known as an ordered pair where the element $a \in A$ and $b \in B$

So in other words $A \times B \neq B \times A$. So the order in which you take the Cartesian product matters (unlike in union or intersection!)

$n = 2$ $\quad \mathbb{R}^2 = \{ (a_1, a_2) | a_1 \in \mathbb{R}, a_2 \in \mathbb{R} \}$

$A = \{ \text{Apples}, \text{Oranges}, \text{Bananas} \}$

$B = \{ 1, 2 \}$

$A \times B = \{ (\text{Apples}, 1), (\text{Apples}, 2), (\text{Oranges}, 1), (\text{Oranges}, 2), (\text{Bananas}, 1), (\text{Bananas}, 2) \}$

$\{ (\text{Bananas}, 2), (\text{Apples}, 2) \}$

$(a, b)$

$a \rightarrow \{ \text{Apples}, \text{Oranges}, \text{Bananas} \}$

$b \rightarrow \{1, 2\}$

$A \times B \neq B \times A$

---

$A = \{1, 2, 3\}$

$A = \{1, 1\}$

$B = \{1, 2\}$

$A \times B = \{ (1,1), (1,2), (2,1), (2,2), (3,1), (3,2) \}$

$\parallel$

$B \times A = \{ (1,1), (1,2), \boxed{(1,3)}, (2,1), (2,2), (2,3) \}$

$(1,3) \in B \times A$ but $\notin A \times B$

# Table of Contents

# Functions

A **function** $f : \boxed{X} \to \boxed{Y}$ from a set $X$ to a set $Y$ is a rule that assigns **exactly one** element $y = f(x)$ in $Y$ to each $\boxed{x \text{ in } X}$.

$y = f(x)$ f: Rule. $X \to Y$.

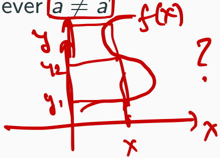

**Terminology:**

- The **domain** of $f$ is $X$ the **codomain** of $f$ is $Y$
- The **image** or **range** of $f$ is the collection of values $f$ takes :

$$f(X) \equiv \{f(x)|x \in X\} \subseteq Y$$

- A function $f$ is **one-to-one**, or **injective**, if $f(a) \neq f(a')$ whenever $\boxed{a \neq a'}$
- A function $f$ is **onto**, or **surjective**, if the range of $f$ is $Y$.
- A function $f$ is **bijective** if it is one-to-one and onto

injective & surjective.




NOT A VALID FUNCTION

$$f : X \to Y$$

**Example:-**

$\Rightarrow$ $X = \{1, 2, 3\}$ — X is the domain

$Y = \{4, 5, 6, \boxed{7}\}$ — Y is the codomain

$f(1) = 4$, $f(2) = 5$, $f(3) = 6$

Is this a valid function?

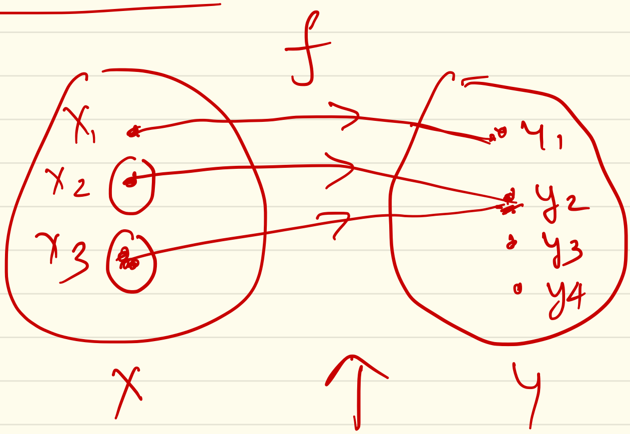
**Image of f:** $f(X) = \{f(x) \mid x \in X\}$

$= \{4, 5, 6\} \subseteq Y$

f is injective!! But not surjective!

---

**Injective Function:-**




$x_2 \neq x_3$

BUT

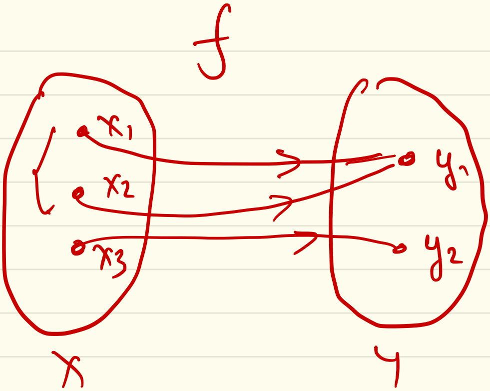
$f(x_2) = f(x_3)$

Is this an injective function?

So NOT INJECTIVE!

AND ~~so~~ not surjective!

Example:




Is this surjective?

Yes! Because every element in Y has some $x \in X$ which maps / is assigned to it!

$$f(X) = Y$$

---

$X = \{1, 2\}$ $\quad$ $Y = \{3, 4\}$

$f(1) = 3$ $\quad$ $f(2) = 4$

This is both injective AND surjective

$\Rightarrow$ Bijective.

# Functions (cont.)

$x^2 - 4 = 0$

$f(x) = 0$

We're frequently interested in inverting equations like $f(x) = 0$.

$x: f(x) = 0$

- If a function is injective, it admits an **inverse function** $f^{-1} : \text{range}(f) \to X$ defined by

  $y$ s.t. $f(x) = y$.

  $$\boxed{f^{-1}(y) = x} \Leftrightarrow f(x) = y$$

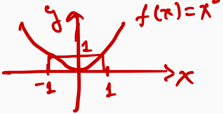
  $f: X \to Y$

  $f^{-1}: \text{Range}(f) \to X$

- For any function, we can define the **inverse image** of a set $S \subseteq Y$:

  $$\boxed{f^{-1}(S) = \{x \in X | f(x) \in S\}}$$

- Example: $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2$.
  - Not injective because $f(1) = f(-1)$; inverse function does not exist
  - But, inverse image is defined: $f^{-1}(\{1\}) = \{-1, 1\}$




$f(1) = f(-1) = 1$

Example:

$$X = \{1, 2\} \qquad Y = \{3, 4\}$$

$$f(1) = 3 \qquad f(2) = 3$$

Is $f^{-1}$ a valid function?

$f^{-1}: \text{Range}(f) \to X$ $\qquad$ $\text{Range}(f) = \{3\}$

$$f^{-1}(3) = \{1, 2\}$$

It's NOT A VALID FUNCTION!

Because $f$ is not an injective function!

Instead $f(2) = 4$

Then $f^{-1}$ is valid.

$$f^{-1}(3) = 1 \quad \& \quad f^{-1}(4) = 2$$

---

Example:- $\qquad X = \{1, 2, 3\} \qquad Y = \{3, 4, 5, 6\}$

$$f(1) = 3 \qquad f(2) = 5 \qquad f(3) = 5$$

What is the inverse image of $S = \{5\}$.

$f^{-1}(\{5\}) = \{2, 3\}$ $\qquad$ $x \in X$ s.t $f(x) \in S$.

$\{5\}$

Inverse Image of $S = \{3, 5\}$

$$f^{-1}(\{3,5\}) = \{1, 2, 3\}$$

$$f^{-1}(\{4,6\}) = \{\phi\} \,/\, \phi$$

There are no elements $x \in X$ s.t

$f(x) \in S$

# Relations

A **relation** $R$ from a set $A$ to a set $B$ is a subset of $A \times B$. If $(a, b) \in R$, we say $a$ and $b$ are **in relation**, sometimes written $aRb$.

$1R3$ & $2R3$

- A relation describes pairs of elements that are (surprise, surprise) related in some fashion
- To give a sense of how general this is, note every function is a relation: $aRb \Leftrightarrow f(a) = b$. (Formal definition of functions in terms of relations?)
- What is the difference between functions and relations? 2 things...

$X = \{1,2\}$
$Y = \{3,4\}$
$f(1) = 3 \quad f(2) = 3$

Here are a few examples. We'll take $\boxed{A = B = \mathbb{R}.}$

- The "=" relation: In this case, $R = \{(x, x), x \in \mathbb{R}\}$.
- The "<" relation: here, $R = \{(x, y) | x < y\}$.
- The "≤" relation: here, $R = \{(x, y) | x \leq y\}$.

$R \subseteq X \times Y$

$R = \{(1,3), (2,3)\}$

$X \times Y = \{(1,3)(2,3),..\}$

Relations could also be really abstract. Take $P$ to be the set of professors and $S$ to be the set of students. We can define the *advising* relation $R$ if student $s$ is advised by professor $p$ and denote it by $\boxed{pRs}$

Every Function is a Relation
but NOT VICE VERSA!

# Relations (cont.)

subset of $A \times A$

$R_1 = \{(A,B), (B,D), (D,A)\}$

NOT TRANSITIVE!!

A relation from $A$ to itself is called a **binary relation**. Binary relations can satisfy a number of properties, such as:

if $(x,y) \in R$ & $(y,z) \in R$ then $(x,z)$ must $\in R$

- A relation is **transitive** if $xRy$ and $yRz$ implies $xRz$ for all $x, y, z \in A$
- A relation is **symmetric** if $xRy$ implies $yRx$ for all $x, y \in A$
- A relation is **complete** if $xRy$ or $yRx$ for all $x, y \in A$

Relevance for econ? Preferences in consumer theory

$X = \{$Apples, Oranges, Bananas$\}$.

$X \times X$.

(1) Apples over Bananas

(2) Bananas " Oranges

(3) Apples over Oranges.

$R = \{$(Apples, Bananas), (Bananas, Oranges), (Apples, Oranges)$\}$.

$R$ is Transitive.

$X = \{A, B, O\}$.

$$R_2 = \boxed{\{(A,O),\ (A,B)\}}$$

(1) Is this preference relation complete?
(2) " " " " transitive?

(1) No! Because no relation between B & O

$R_2$ is not a complete binary relation!!

(2) Antecedent in the definition of transitivity is NOT SATISFIED $\Rightarrow$ it is vacuously / trivially transitive.

$(B, A)$

BRA & ARO then we should have had BRO

$X = \{A, B, O\}$

$R_4 = \{(A,B),\ (B,A)\}$ is symmetric!

# Table of Contents

# Motivation

- Econ isn't math, but it doesn't hurt to be formal when we can
- You'll see a lot of proofs in coursework and papers - even if you're not writing proofs day-to-day, important to know how to read and evaluate them

# Proof Methods

Most proofs we will encounter are conditional statements of the form "if $A$ then $B$". Notation/terminology:

- $A$ implies $B$
- $A \Rightarrow B$

Three general approaches to proving such statements

- **Direct proof:** Assume $A$, show that $B$ holds
- **Proof by contrapositive:** Assume $\neg B$, show $\neg A$
- **Proof by contradiction:** Assume $A$ and $\neg B$ and derive a logical contradiction

# Direct Proofs

**Example:** if $n$ and $m$ are even integers, $n + m$ is even

- Since $n$ and $m$ are even, $n = 2k$ and $m = 2j$ for some integers $j, k$
- Therefore $n + m = 2k + 2j = 2(k + j)$ is even

# Proofs by contraposition

An **implication** is equivalent to its **contrapositive**: $A \Rightarrow B$ if and only if $\neg B \Rightarrow \neg A$

- How would you disprove the statement $A \Rightarrow B$? What about $\neg B \Rightarrow \neg A$?
- In set-theoretic language, $A \subseteq B \Leftrightarrow B^c \subseteq A^c$

**Example:** if $n^2$ is even, then $n$ is even

- Assume $n$ is odd
- Then $n = 2k + 1$ for some integer $k$
- Then $n^2 = 4k^2 + 4k + 1$ is odd

# Proof by contradiction

**Proof by contradictions** are powerful but can be weird

- Like with contraposition, we assume $\neg B$. We also assume $A$, so we have more "ammo" than we do with direct/contrapositive proofs
- The end of the proof is less clear - we need to show that $A$ and $\neg B$ produce a logical inconsistency

**Example:** If $x$ is rational and $y$ is irrational, then $x + y$ is irrational

- Assume $x + y$ is rational. Then $x + y = \frac{p}{q}$ for some integers $p, q$.
- Since $x$ is rational, $x = \frac{a}{b}$ for some integers $a, b$.
- Therefore $y = \frac{p}{q} - \frac{a}{b} = \frac{pb-aq}{qb}$ is rational, a contradiction

Generally best to use direct or contrapositive proofs (many "contradiction" proofs are contrapositives in disguise)

# Common types of proofs

**"If and only if"**: $A \Leftrightarrow B$

- Approach: Two separate proofs! Show $A \Rightarrow B$ and $B \Rightarrow A$

**"For all"** : (e.g. "for all $x \in X$, property $P$ holds")

- Approach: take an arbitrary element of $X$ and show that $P$ holds

**"There exists"**: (e.g. "there exists an $x \in X$ such that property $P$ holds")

- Approach: constructive or nonconstructive. Example : Intermediate Value Theorem.

**Uniqueness proofs** (e.g. "there exists a unique $x$ satisfying $P$")

- Typically take two elements $x_1$ and $x_2$ satisfying $P$ and show they must be the same

# General guidance

- Always have a roadmap in mind. Be clear about what you need to prove, and how you're planning to do it
- Understand your actors: what objects are you working with, and what do we know about them (either from assumptions or previous results)
    - Be comfortable with definitions
- If you're stuck, try a different approach (e.g. contradiction)
- The conditions of a theorem are clues - if you haven't used one yet you're likely missing something
- Proofs are rarely presented the way they're derived (this is true of almost all endeavors, academic or otherwise)