Columbia MA Math Camp

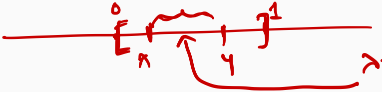
Convexity

Vinayak Iyer

August 2018

Table of Contents

Convexity and Quasiconvexity



$\lambda \in [0,1]$

$\lambda x + (1-\lambda) y$

Definition 1.1
Let $S \subseteq \mathbb{R}^n$. We say S is convex *if for all $x, y \in S$ and $\lambda \in [0,1]$:*

$$\lambda x + (1-\lambda) y \in S$$

Yes! Take any 2 points $x, y \in [0,1]$ & $\lambda \in [0,1] \Rightarrow \lambda x + (1-\lambda) y \in [0,1]$

Is the set $S = [0,1]$ convex? What about $S = [0,1)$? What about $S = [0,1) \cup [2,3]$? $S = \{1, 2, 3, \ldots\}$? → No!

Yes!

Notes :

- In other words, the convex combination of 2 vectors in a set belongs to the same set.
- The intersection of convex sets is convex
- The union of convex sets need not be convex

$\lambda x + (1-\lambda) y \notin [0,1) \cup [2,3]$

Convex Sets (cont..)

For finitely many vectors $x_1, x_2, \ldots, x_n$, a **convex combination** is a vector $\sum_{i=1}^{n} \lambda_i x_i$ for scalars $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}_+$ such that $\sum_{i=1}^{n} \lambda_i = 1$

Proposition 1.1
Suppose $S \subseteq \mathbb{R}^n$. The set S is convex iff any convex combination of $x_1, x_2, \ldots, x_n \in S$ is also in S.

Proof:

($\Longleftarrow$) is trivial based on the definition of convex sets.

($\Longrightarrow$) If $n = 1$, the statement is trivial.

If $n = 2$, the statement is true by the definition of convexity.

Suppose it is true for $n = k$. This implies that for any set of k vectors $x_1, x_2, \ldots, x_k$, $\sum_{i=1}^{k} \lambda_i x_i \in S$ for all $\lambda_i \geq 0$ such that $\sum \lambda_i = 1$.

$\lambda_1 x_1 + \lambda_2 x_2 \ldots + \lambda_k x_k \in S$
where $\sum \lambda_i = 1$, $\lambda_i \geq 0$

Proof by Induction

Show that it is true for $n=1$ & 2.

Assume it is true for $n=k$.

Show that is true for $n=k+1$

Proof continued...

WTS :- $\lambda_1 x_1 + \lambda_2 x_2 + \dots \lambda_{k+1} x_{k+1} \in S$.

Now consider $n = k+1$. We need to show that $\overbrace{\sum_{i=1}^{k+1} \lambda_i x_i \in S}$.   where $\sum_{i=1}^{k+1} \lambda_i = 1$

We can rewrite this as :

Qn: Is $\sum_{i=1}^{k} \lambda_i = 1$? No!

$$\sum_{i=1}^{k+1} \lambda_i x_i = \sum_{i=1}^{k} \lambda_i x_i + \lambda_{k+1} x_{k+1}$$

$$= \left(\sum_{i=1}^{k} \lambda_i\right) \left(\overbrace{\sum_{i=1}^{k} \frac{\lambda_i}{\sum_{i=1}^{k} \lambda_i} x_i}\right) + \lambda_{k+1} x_{k+1}$$

$\bar{x} \in S$ $\bar{x} = \sum^{k} \frac{\lambda_i}{\Sigma \lambda_i} x_i$

$$= \underbrace{\left(\sum_{i=1}^{k} \lambda_i\right)}_{1-\lambda_{k+1}} \bar{x} + \lambda_{k+1} x_{k+1} \quad \text{(since it is true for } n = k \text{ i.e. } \sum_{i=1}^{k} \frac{\lambda_i}{\sum_{i=1}^{k} \lambda_i} x_i \in S)$$

$\in S$ (Since it is true for $n = 2$)

λ_1	λ_2	λ_3
0.3	0.3	0.4

$\Sigma \lambda_i = 1$

$\lambda_1 + \lambda_2 = .6$

Divide both sides by .6

$\Rightarrow \frac{\lambda_1}{\lambda_1+\lambda_2} + \frac{\lambda_2}{\lambda_1+\lambda_2} = 1$

Definition 1.2
A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex *if for any $x_1, x_2 \in \mathbb{R}^n$ and any $\lambda \in (0,1)$:*

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$$

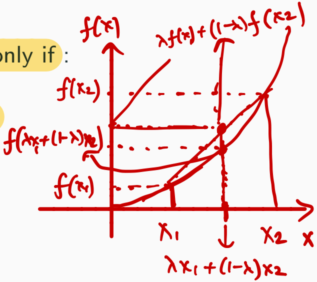
function evaluated at convex combⁿ of x_1 & x_2 $\leq$ convex comb of the function.

- If the inequality is strict, f is **strictly convex**
- If the inequality is reversed, f is **concave**

Another characterization: A function f is **convex** if and only if:

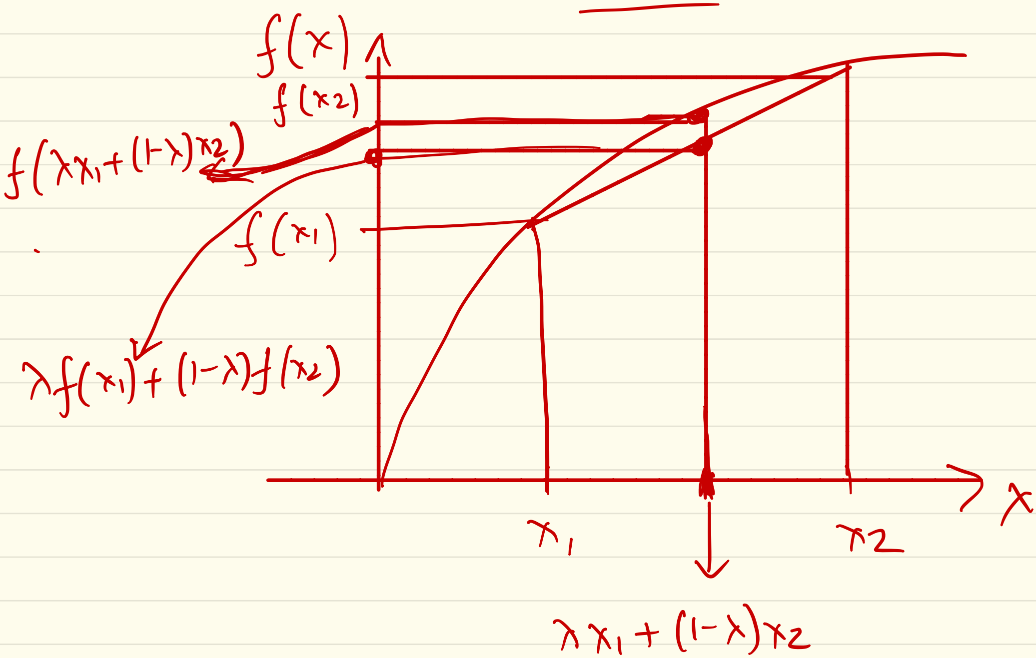
$$\{(x,y) \in \mathbb{R}^n \times \mathbb{R} | y \geq f(x)\}$$

is **convex**

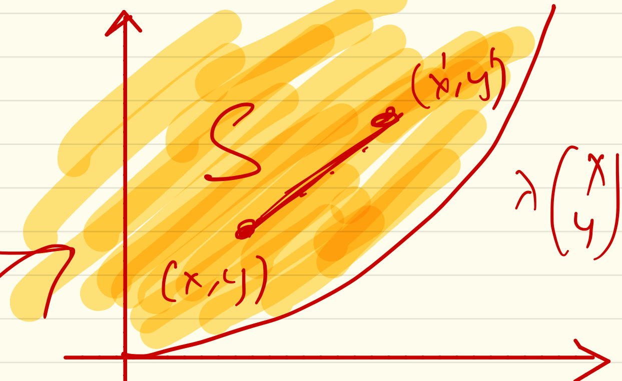


A straight line (weakly) convex & concave!

Concave

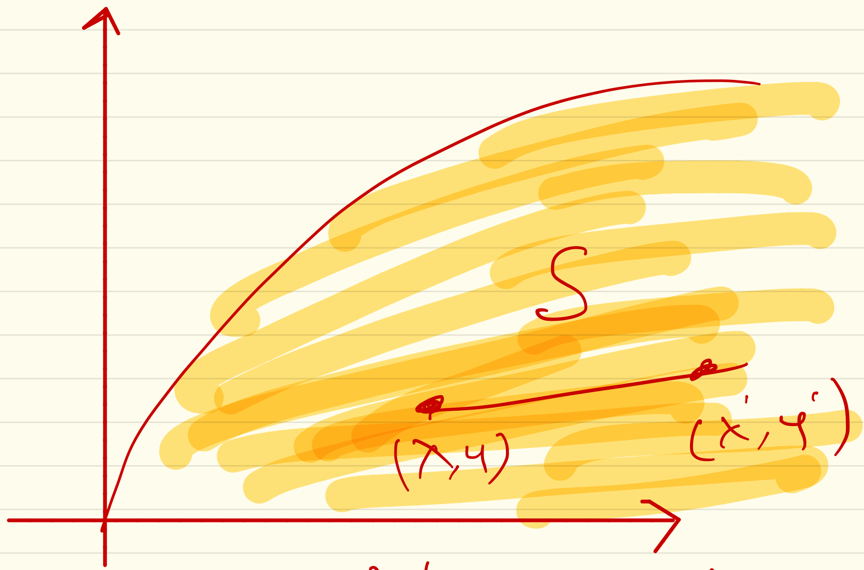


We know this is convex



$$\lambda\begin{pmatrix}x\\y\end{pmatrix}+(1-\lambda)\begin{pmatrix}x'\\y'\end{pmatrix} \in S$$

$S = \{(x,y) \in \mathbb{R}\times\mathbb{R} \mid y \geq f(x)\}$. What is S?



$$S = \{(x,y) \in \mathbb{R}^2 \mid y \leq f(x)\}$$

is a convex set.

Convex Functions: Properties

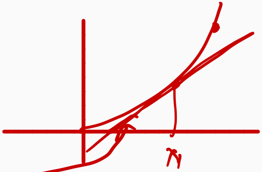
Convex functions have a whole host of nice properties - people write books on convex analysis. Some include:

- If f and g are **convex (concave)**, $f + g$ is **convex (concave)**
- If f is **convex** **(concave)** and g is **convex (concave) and increasing**, then $f \circ g$ is **convex (concave)**

$f(g(x))$

Some properties are a little surprising at first glance :

- Convex functions are **continuous**
- Convex functions are **differentiable** almost everywhere



Definition 1.3
Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable. Then

- *f is* convex *iff for all $x_1, x_2 \in \mathbb{R}^n$:*

$$f(x_2) \geq \overbrace{f(x_1) + f'(x_1)(x_2 - x_1)}$$

Linear Approximation of f at x_1

- *f is* strictly convex *iff for all $x_1 \neq x_2$:*

$$f(x_2) > f(x_1) + f'(x_1)(x_2 - x_1)$$

Convex functions sit above their tangent lines. The analogous result holds for concave functions (just flip the inequality)

Concave fns sit below their tangent lines!

matrix of second derivatives

Definition 1.4
Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^2 function. Then

- *f is convex (concave) iff its Hessian is positive (negative) semi-definite for all x*
- *If the Hessian is positive (negative) definite for all x, then f is strictly convex (concave)*

(Proof intuition) : Use a second-order Taylor series expansion

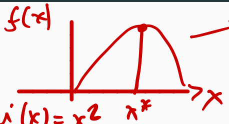
$$f(x) \approx f(a) + f'(a)(x-a) + \frac{1}{2}(x-a)^T H(a)(x-a)$$

If $H(a)$ is positive definite, f will sit above its tangent approximation.

$A \, R \, B \iff u(A) \geqslant u(B)$

5 4

$u(x) = x$ represents the same preferences as $u'(x) = x^2$



→ Concave

max f(x)

- In micro, we think of preferences that are represented by a utility function: x **is preferred to** y if $u(x) \geq u(y)$
- This is an *ordinal notion*: if $f(\cdot)$ is an **increasing** function, then $f(u(x)) > f(u(y))$, so $f \circ g$ represents the same preferences

Increasing

- However, convexity is not an ordinal notion. Let $u(x) = x^2$ and $f(x) = \log x$. Then u is *convex* and f an *increasing transformation*, but $f(u(x)) = 2 \log x$ is **concave, not convex**
- We will develop a notion of **quasiconcavity (quasiconvexity)** that will be preserved by increasing transformations

Quasiconcave functions

Not convex!

Not concave!

Quasiconvex

but not Q. concave.

Definition 1.5

Let $f : \mathbb{R}^n \to \mathbb{R}$. We say f is quasiconvex if the lower level set

$$S_\alpha \equiv \{x | f(x) \leq \alpha\}$$

is convex for every value α. If the upper level sets

$$U_\alpha \equiv \{x | f(x) \geq \alpha\}$$

is convex for every α, then f is quasiconcave

$S_\alpha = \{x \mid f(x) \le \alpha\}$

You require that S_α is a convex set.

$\Rightarrow \forall x, x' \in S_\alpha$

$\lambda x + (1-\lambda) x' \in S_\alpha$

Quasiconcave

but not

Q. convex

What is S_α?

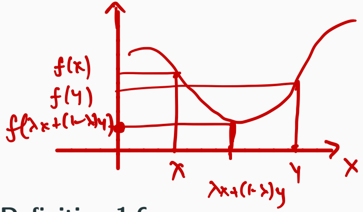
$U_\alpha = \{x \mid f(x) \geq \alpha\}$.

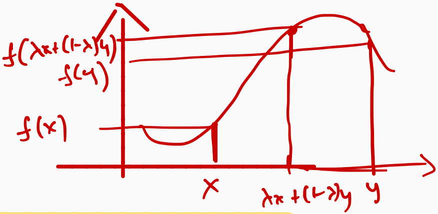
Quasiconvex :-

We say f is quasiconvex if $\forall \alpha \in \mathbb{R}$ the set S_α

$$S_\alpha = \{ x \mid f(x) \leq \alpha \}$$

is a convex set.





Definition 1.6
A function $f : \mathbb{R}^n \to \mathbb{R}$ *is* quasiconvex *iff for all* $x, y \in \mathbb{R}^n$ *and* $\lambda \in [0, 1]$,

$$f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\}$$

If the inequality is strict for $x \neq y$ *and* $\lambda \in (0, 1)$, f *is* strictly quasiconvex

For quasiconcavity, *we have* $f(\lambda x + (1-\lambda)y) \geq \min\{f(x), f(y)\}$

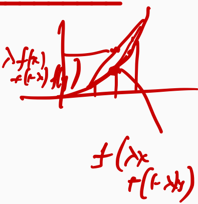
Qn :- Prove that if $f : \mathbb{R} \to \mathbb{R}$ and is an increasing function, then f is quasiconcave AND quasiconvex

Quasiconcave funtions : Properties

- **Convexity** $\implies$ **Quasiconvexity** : Suppose f is convex. Then for all $x, y \in \mathbb{R}^n$ and $\lambda \in [0,1]$:

$$\begin{aligned} f(\lambda x + (1-\lambda)y) &\leq \lambda f(x) + (1-\lambda) f(y) \\ &\leq \max\{f(x), f(y)\} \end{aligned}$$

$\lambda f(x) + (1-\lambda) f(y)$
$\leq \lambda \max\{f(x), f(y)\} + (1-\lambda)\max\{f(x), f(y)\}$
$= \max\{f(x), f(y)\}$



so f is **quasiconvex**. (Similar argument for Quasiconcavity)

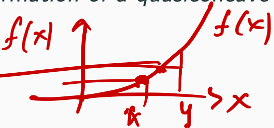
- **Increasing transformation of quasiconvex function is quasiconvex :**

  Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is quasiconvex and $g : \mathbb{R} \to \mathbb{R}$ is an increasing function. Then for all $x, y \in \mathbb{R}^n$ and $\lambda \in [0,1]$:

$$\begin{aligned} g(f(\lambda x + (1-\lambda)y)) &\leq g(\max\{f(x), f(y)\}) \\ &= \max\{g(f(x)), g(f(y))\} \end{aligned}$$

So $g \circ f$ is **quasiconvex**. Similarly, an *increasing transformation of a quasiconcave function is quasiconcave.*

Quasiconcave but not concave.



WTS: If f is Q. convex & g is an increasing fn $\Rightarrow$ $g(f(x))$ is Q. convex

Q. convexity :-

$S_\alpha = \{x \mid f(x) \leq \alpha\}$ is convex for all $\alpha \in \mathbb{R}$

$$S_\alpha = \{x \mid \overbrace{f(x)}^{f(x)} \leq \alpha\} \Leftrightarrow \{x \mid \overbrace{g(f(x))}^{g(f(x))} \leq g(\alpha)\}$$

$$f(x) \leq \alpha \Rightarrow g(f(x)) \leq g(\alpha)$$

as g is increasing

Quasiconcave functions (cont.)

Proposition 1.2
Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^1 function. Then f is quasiconcave iff
$f(y) \geq f(x) \Rightarrow f'(x)(y-x) \geq 0$.

Proof.
($\Rightarrow$) Suppose f is quasiconcave. Let $x, y \in \mathbb{R}^n$ such that $f(y) \geq f(x)$, and $\lambda \in (0,1)$.

$$f((1-\lambda)x + \lambda y) \geq f(x)$$

Rearranging gives

$$\frac{f(x + \lambda(y-x)) - f(x)}{\lambda} \geq 0$$

Taking $\lambda \to 0$ gives $f'_{y-x}(x) \geq 0$. So $f'(x)(y-x) \geq 0$. $\square$

Does this proposition guarantee existence of a global maximum?

$\boxed{\underline{\lambda x + (1-\lambda)y}}$

Proposition 1.3
A strictly quasiconcave function can have at most one global maximum.

Assume D is convex

Proof : Suppose there are 2 maximizers x and y. If $x \neq y$ are both maximizers, then $f(x) = f(y)$. → since both are maximizers.

However, $f(\lambda x + (1-\lambda)y) > f(x) = f(y)$ by the definition of strict quasiconcavity which contradicts that x and y are maximizers. since $\lambda x + (1-\lambda) y$ gives a higher value contradicting that x & y are max

Since f is strictly Q. concave ⇒

$\forall x \neq y \quad f(\lambda x + (1-\lambda)y) > \min \{f(x), f(y)\}$

$\therefore f(\underline{\lambda x + (1-\lambda)y}) > f(x) = f(y)$