

Columbia MA Math Camp

Differential Calculus

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Derivative

Continuous fns $\not\rightarrow$ Differentiable

The fundamental concept in calculus is that of the **derivative**, which is simply a rate of change. Consider a function $f : \mathbb{R} \rightarrow \mathbb{R}$. The quantity

$$\frac{f(x_0 + h) - f(x_0)}{h}$$

tells us the average rate of change of f between x_0 and $x_0 + h$.

The big idea with derivatives is simply that we let h go to 0.

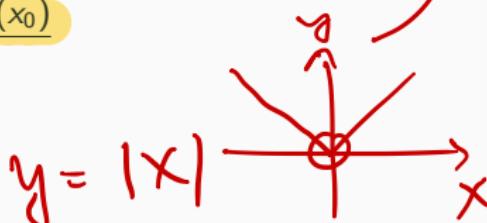
Definition 1.1

Let $A \subset \mathbb{R}$. A function $f : A \rightarrow \mathbb{R}$ is said to be differentiable at x_0 iff the limit :

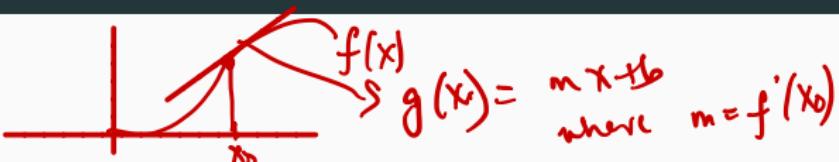
$$f'(x_0) \equiv \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

exists.

We define the derivative of f at x_0 as $f'(x_0)$



Derivative as an approximation



There's another interpretation of the derivative: it is the best linear approximation of a function.

- Suppose you wanted to approximate $f(x)$ by a linear function $g(x) = mx + b$ around the point x_0
- A good approximation should have the following properties:
 - The functions should agree at x_0 : $g(x_0) = f(x_0)$. So $g(x_0 + h) = f(x_0) + mh$
 - The per-unit error should be small near x_0 :

$$\lim_{h \rightarrow 0} \frac{f(x_0 + h) - g(x_0 + h)}{h} = 0$$

when $h=0$
 $\Rightarrow g(x_0) = f(x_0)$

These properties combined imply $m = f'(x_0)$

Common derivatives and rules

Common functions :

- $\frac{d}{dx} c = 0$
- $\frac{d}{dx} x^n = nx^{n-1}$
- $\frac{d}{dx} e^x = e^x$
- $\frac{d}{dx} \log x = \frac{1}{x}$



Differentiate a function
 $f(x)$ wrt x .

Combining derivatives :

- $\frac{d}{dx} (f(x) + g(x)) = \underline{f'(x) + g'(x)}$
- $\frac{d}{dx} (\alpha f(x)) = \alpha f'(x)$ α is a constant
- $\frac{d}{dx} (f(x)g(x)) = \underline{f'(x)g(x) + f(x)g'(x)}$ (product rule)
- $\frac{d}{dx} (f(g(x))) = f'(g(x))g'(x)$ (chain rule)



Quotient Rule :-

$$\frac{d}{dx} \cdot \left(\frac{f(x)}{g(x)} \right) = \frac{f'g - gf'}{(g(x))^2}$$

$$f' = f'(x)$$

Chain Rule example

- Consider a consumer whose utility u is only directly dependent on consumption c :
 $u = u(c)$
- However, consumption depends on the consumer's wealth: $c = c(w)$
- Therefore $u = u(c(w))$. We can capture the dependencies in a graph

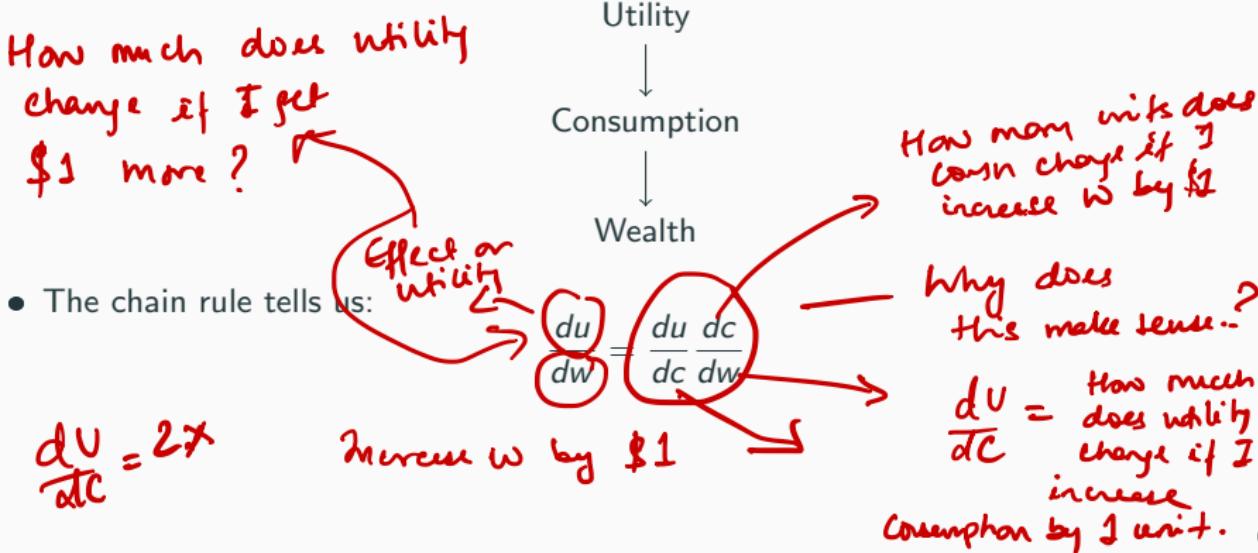


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The Implicit Function Theorem

Aside: Multivariable Functions

Utility:-

$$u(x_1, x_2, \dots, x_n) : \mathbb{R}^n \rightarrow \mathbb{R}$$

- We will be working with functions from \mathbb{R}^n to \mathbb{R} a lot
- For functions of two variables, one common tool we'll use is **level curves**. This is the graph of the equation $f(x, y) = c$
- Example: let $a \in \mathbb{R}^2$ and consider the function $f(x, y) = a_1x + a_2y$. The graph of this function is called a **hyperplane**

$$f: \mathbb{R}^2 \rightarrow \mathbb{R}$$

$$z = f(x, y)$$

$$\text{fix } z = c \\ f(x, y) = c$$

Give me an example of this!



Indifference Curve!

Notion of Linear Approximation

In the single-variable setting, $f'(x)$ was the term in the **best linear approximation** of f

$$f(y) \approx f(x) + f'(y)(y - x)$$



— Linear function
 $g(x)$

where “best” meant the relative error goes to 0. That is, $f'(x)$ is the value m such that

$$\lim_{y \rightarrow x} \frac{f(y) - (f(x) + m(y - x))}{y - x} = 0$$

Approximation

Notion of Linear Approximation (cont.)

$$f: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \quad f(x_1, x_2) = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix}$$

If $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$, then a *linear approximation* of f is:

$$f(y) = f(x) + \underbrace{A(y-x)}_{\text{an } m \times 1 \text{ vector}} \quad \begin{matrix} y \in \mathbb{R}^n (\text{nx1}) \\ f(y) \in \mathbb{R}^m \text{ vector} \\ \text{is an } m \times 1 \text{ vector} \end{matrix}$$

where A is a $m \times n$ matrix. Thus we will define the **derivative of f at x** or the **Jacobian of f at x** as the matrix A such that

$$\lim_{y \rightarrow x} \frac{f(y) - (f(x) + A(y-x))}{\|y-x\|} \rightarrow 0$$

A is the derivative of f at x

$A = f'(x)$ $m \times n$ matrix

Partial Derivative

Let us introduce the notion of a very useful concept of a partial derivative :

Definition 2.1

For a function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, it's partial derivative of the ith coordinate wrt to the jth argument at $x \in \mathbb{R}^n$ is :

$$\frac{\partial f_i}{\partial x_j}(x) := \frac{d}{dt} f_i(x + te_j)|_{t=0}$$

What's the partial derivative of $f(x) = x_1^2 + x_1x_2$?

$$f : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

$$\text{we want } \frac{\partial f_i(x)}{\partial x_j}$$

$$f_i(x)$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$f: \mathbb{R}^2 \rightarrow \mathbb{R}$$

$$f(x_1, x_2) = \cancel{x_1^2} + \underbrace{x_1 x_2}_{\cdot}$$

$$\frac{\partial f}{\partial x_1} = ? \quad 2x_1 + x_2 \quad \left(\begin{array}{l} \text{Assume} \\ x_2 \text{ is} \\ \text{constant} \end{array} \right)$$

$$\frac{\partial f}{\partial x_2} = x_1$$

$$f: \mathbb{R}^2 \rightarrow \mathbb{R}^2.$$

$$f(x_1, \textcircled{x}_2) = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix}$$

$$\frac{\partial f_1(x_1, x_2)}{\partial x_2} = \frac{d}{dt} (f_1(x_1, x_2 + t))_{t=0}$$

$$x + te_j = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + t \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_j + t \\ \vdots \\ x_n \end{pmatrix}$$

Properties of the derivative

$$f'(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \dots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

m x n matrix
 $x \in \mathbb{R}^n$

$f(x) = \begin{pmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{pmatrix}$

Properties :

- If $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, then $f'(x)$ is an $m \times n$ matrix also known as the **Jacobian**.
- If f is real-valued, the column vector $f'(x)^T$ is called the **gradient** of f or sometimes denoted as $\nabla f(x)$
- The derivative is **linear**. If $f, g : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $\alpha \in \mathbb{R}$:
 - $(f + g)'(x) = f'(x) + g'(x)$
 - $(\alpha f)'(x) = \alpha f'(x)$
- If f is differentiable at x , f is continuous at x

$f'(x) = 1 \times n \text{ vector } \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right)$

$n \times 1 \text{ vector}$
 $(f'(x)^T) = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right)$

$\nabla f(x) = 0$

$$\begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

The Chain Rule

- Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $g : \mathbb{R}^m \rightarrow \mathbb{R}^k$.
- Define $h : \mathbb{R}^n \rightarrow \mathbb{R}^k$ by $h(x) = g(f(x))$.
 $\Leftrightarrow v(\omega) = n(\pi(\omega))$
- If f is differentiable at x and g is differentiable at $f(x)$, then
$$h'(x) = g'(f(x))f'(x)$$

$$\begin{aligned}x(\omega) &: \mathbb{R} \rightarrow \mathbb{R}^2 \\u &: \mathbb{R}^2 \rightarrow \mathbb{R} \\v &: \mathbb{R} \rightarrow \mathbb{R}\end{aligned}$$

- (Heuristic proof) two linear approximations:

$$\begin{aligned}h(y) &= g(f(y)) \\&\approx g[f(x) + f'(x)(y - x)] \\&\approx g(f(x)) + g'(f(x))f'(x)(y - x) \\&= h(x) + \underbrace{g'(f(x))f'(x)}_{h'(x)}(y - x)\end{aligned}$$

- Full proof in FMEA, page 96

What does $f'(x)$ look like?

- Consider the partial derivative wrt x_i i.e. :

$$\frac{\partial f(x)}{\partial x_i} = \text{i-th column of } f'(x)$$

- This allows us to calculate the Jacobian

$$f'(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

1st column of
 $f'(x)$

$$\left(\begin{array}{c} \frac{\partial f_1}{\partial x_1} \\ \vdots \\ \frac{\partial f_m}{\partial x_1} \end{array} \right)$$



The Chain Rule: Interpretation

Returning to the chain rule, consider the following example:

$$x: \mathbb{R} \rightarrow \mathbb{R}^2$$
$$u(x): \mathbb{R}^2 \rightarrow \mathbb{R}$$
$$v(w) = \mathbb{R} \rightarrow \mathbb{R}$$

- Utility u depends on consumption c and hours worked h
- However c and h depend on the going wage w . Define $x(w) : \mathbb{R} \rightarrow \mathbb{R}^2$ by
$$x(w) = (c(w), h(w)).$$
- Define $v(w) = u(x(w))$. The chain rule says $\underline{v'(w) = u'(x(w))x'(w)}$:

$$v(w) = u(x(w))$$

$$\begin{aligned} v'(w) &= \left(\frac{\partial u}{\partial c} \quad \frac{\partial u}{\partial h} \right) \begin{pmatrix} \frac{\partial c}{\partial w} \\ \frac{\partial h}{\partial w} \end{pmatrix} \\ &= \frac{\partial u}{\partial c} \frac{\partial c}{\partial w} + \frac{\partial u}{\partial h} \frac{\partial h}{\partial w} \end{aligned}$$

$$u'(x(w)) = \left(\frac{\partial u}{\partial c} \quad \frac{\partial u}{\partial h} \right)$$
$$\mathbb{R}^2 \rightarrow \mathbb{R}$$

$$x: \mathbb{R} \rightarrow \mathbb{R}^2$$
$$x'(w) = \left(\frac{\partial x_1}{\partial w} \quad \frac{\partial x_2}{\partial w} \right)$$

How much does utility change if
 w increases by \$1.

$$u(c(w), h(w))$$

$$\frac{\partial u}{\partial w} = \underbrace{\frac{\partial u}{\partial c} \frac{\partial c}{\partial w}}_{\text{How much utility changes due to a \$1 wealth increase through the consumption channel!}} + \underbrace{\frac{\partial u}{\partial h} \frac{\partial h}{\partial w}}_{\text{Effect on utility of wealth through the hours worked channel!}}$$

Some common derivatives

Being comfortable taking vector derivatives in one step can save you a **lot** of algebra (especially in econometrics). You should know these identities by heart :

- Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ with $f(x) = Ax$ where A is an $m \times n$ matrix:



$$f'(x) = A$$

- Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ with $f(x) = x'Ax$ where A is an $n \times n$ matrix:

$$f'(x) = x'(A + A')$$

If A is symmetric, $f'(x) = 2x'A$

\downarrow A' refers to
 x transpose A transpose

- If $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$ and $h(x) = f(x)g(x)$, then

$$\beta = \begin{pmatrix} \beta_0 \\ \vdots \\ \beta_n \end{pmatrix}$$

$$h'(x) = f'(x)g(x) + f(x)g'(x)$$

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 \dots + \beta_n x_n + \epsilon$$

OLS: $(\beta)^{\text{min}} \| y - x\beta \| \leftarrow$ derivative wrt β vector

$$f : \mathbb{R} \rightarrow \mathbb{R}$$
$$f(x) = cx$$

$$f'(x) = c$$

generalization
of

$$f(x) = ax^2$$
$$f'(x) = 2ax$$

$$f: \mathbb{R}^n \rightarrow \mathbb{R}$$

$f'(x)$
 $1 \times n$ vector

$$f(x) = x^T A x$$

$$f'(x) = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right)$$

$$f(x) = x^T A x = \sum_i \sum_j a_{ij} x_i x_j$$

$$\frac{\partial f}{\partial x_1} = \frac{\partial}{\partial x_1} \left(a_{11} x_1^2 + a_{12} x_1 x_2 + a_{13} x_1 x_3 + \dots + a_{1n} x_1 x_n \right) + \dots$$

$$= 2a_{11}x_1 + 2a_{12}x_2 + \dots + 2a_{1n}x_n$$

$$\frac{\partial f}{\partial x_2} = 2a_{22}x_2 + 2a_{21}x_1 + \dots + 2a_{2n}x_n$$

$1 \times n$

$$f'(x) = 2x^T A_{n \times n}$$

$$= 2 \begin{pmatrix} x_1 & \dots & x_n \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \dots \\ \vdots & \ddots & \ddots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

$$= 2x_1 a_{11} + 2x_2 a_{21} + \dots + 2x_n a_{n1}$$

Some technical concerns

If f is differentiable \Rightarrow partial derivatives exist & Jacobian is matrix of the partials

- We have seen that if f is differentiable, its partials exist and the Jacobian is just the matrix of partial derivatives.
- What if we only know that the partials exist? Is that enough for differentiability?
- Sadly, the answer is no (idea: function could behave nicely along the axes, but misbehave along other directions)
- However, if the partials are also continuous, then the derivative exists. Almost every function we work with in economics will have continuous partial derivatives

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Higher Derivatives: Single Variable

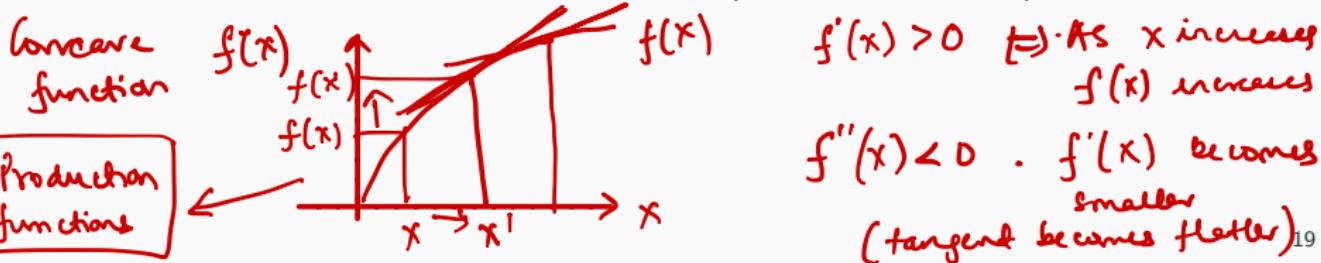
For a function $f : \mathbb{R} \rightarrow \mathbb{R}$, the **second derivative** of f at x is the derivative of f' at x

$$f''(x) = \lim_{h \rightarrow 0} \frac{f'(x+h) - f'(x)}{h}$$

The second derivative measures the *change in the slope per unit change in x* :

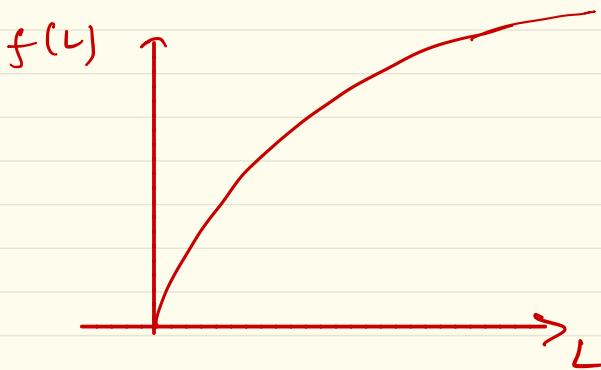
- $f''(x) > 0$ means the derivative is (locally) increasing in x
- $f''(x) < 0$ means the derivative is (locally) decreasing in x

Can keep going to third, fourth derivatives, etc. (not commonly used)



Production function

$$f(L) = \sqrt{L} \quad L = \text{labor}$$

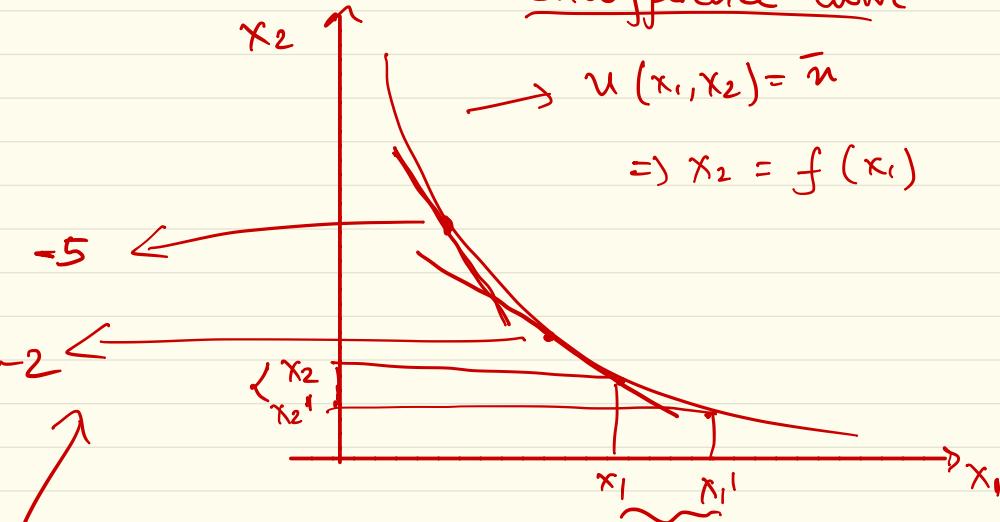


Properties :-

- (1) $f'(L) > 0$: Adding labor increases output
- (2) $f''(L) < 0$: Each additional unit of labor adds lesser output than the previous units,

(Law of Diminishing Returns)

Indifference Curve



Call this indifference curve:

- (1) $f'(x_1) < 0$: More of $x_1 \Rightarrow$ less of x_2 to maintain same level
- (2) $f''(x_1) > 0$: Law of diminishing of utility returns is at play here

Taylor Series: Single Variable

$$f(x_0) = h(x_0) = a_0$$

$$f: \mathbb{R} \rightarrow \mathbb{R}$$

of order n

Suppose you want to approximate f by a polynomial around the point x_0

$$h(x) \equiv a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n$$

Linear Quadratic

Two intuitive criteria for a "good" approximation are:

✓ $h(x_0) = f(x_0)$, which implies $a_0 = f(x_0)$

$$f(x) \approx h(x)$$

- The first n derivatives of h should match those of f at x_0

Differentiating repeatedly gives $h^k(x_0) = k!a_k$. Thus the Taylor series expansion of order n of f around x_0 is

approximately equal $f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2 + \dots + \frac{1}{n!}f^n(x_0)(x - x_0)^n$

$$(1) \quad f'(x_0) = h'(x_0)$$

$$(2) \quad f''(x_0) = h''(x_0)$$

$$\vdots \quad \vdots \quad \vdots$$
$$(3) \quad f^n(x_0) = h^n(x_0)$$

$$h(x) = f(x_0) + a_1(x - x_0) + \dots + a_n(x - x_0)^n$$

$$h'(x) = a_1 + 2a_2(x - x_0) + \dots + na_n(x - x_0)^{n-1}$$

$$h'(x_0) = a_1$$

$$h^2(x) = 2a_2 + 6a_3(x-x_0) + \dots + \frac{n(n-1)a_n}{(x-x_0)^{n-2}}$$

$$h^2(x_0) = 2a_2$$

$$h^3(x_0) = 6a_3 = 1 \times 2 \times 3 \times a_3$$

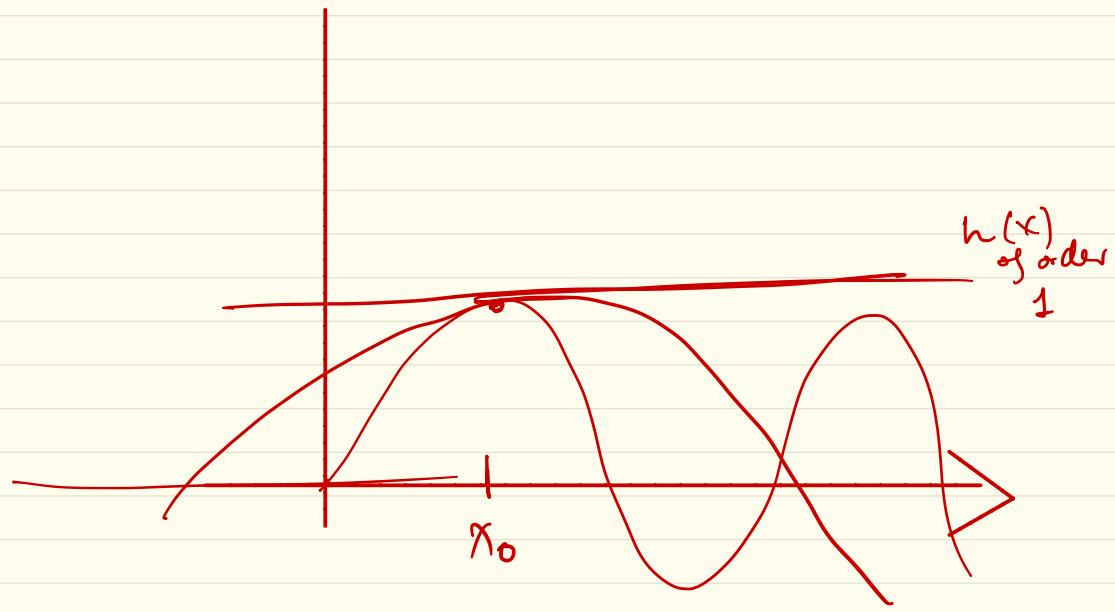
$$h^k(x_0) = 1 \times 2 \times 3 \times \dots \times k \times a_k$$

$$= k! a_k$$

$$k! = 1 \times 2 \times 3 \times 4 \times 5 \times \dots \times k$$

$$f^k(x_0) = h^k(x_0) = k! a_k$$

$$\Rightarrow a_k = \frac{f^k(x_0)}{k!}$$



$$f(x) = \sin(x)$$

$$f(x) = x^2 e^x$$

$$\boxed{f'(x) = 2xe^x + x^2e^x}$$

$$f'(x) \Big|_{x=x_0}$$

$$f'(x_0) \equiv f'(x) \Big|_{x=x_0}$$

$$f'(1) = \frac{d}{dx} f(1) \quad X$$

$$= \frac{d}{dx} (1) = 0 \quad X$$

$$f'(1) = [2xe^x + x^2e^x]_{x=1} \quad \checkmark$$

$$= 2e + e = 3e$$

Taylor Series: Exact Form of Remainder

Approximating f with a k^{th} order polynomial

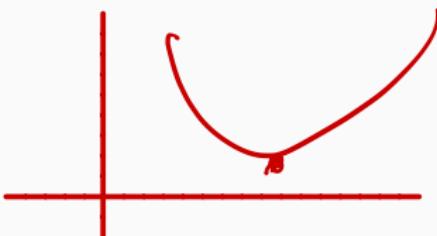
Proposition 3.1

Let f be $k + 1$ times differentiable on $[a, x]$. Then

$$f(x) = f(a) + f'(a)(x - a) + \dots + \frac{f^k(a)}{k!}(x - a)^k + \frac{f^{k+1}(\zeta)}{(k+1)!}(x - a)^{k+1}$$

for some $\zeta \in (a, x)$.

Taylor Series: Single Variable (cont.)



- Taylor series are useful tools for many proofs in econometrics
- Approximation methods are used frequently in economics to help simplify nonlinear equations
- Accuracy of the approximation depends on distance from x_0

[Show picture]

Second Derivations: Multiple Variables

$f: \mathbb{R}^n \rightarrow \mathbb{R}$: $f'(x) = \left(\frac{\partial f}{\partial x_1} \frac{\partial f}{\partial x_2} \cdots \frac{\partial f}{\partial x_n} \right)$ $1 \times n$ vector

$f(x) \in \mathbb{R}$, $x \in \mathbb{R}^n$. Taking derivative of a scalar w.r.t to a vector

- For a function $f: \mathbb{R}^n \rightarrow \mathbb{R}$, we define second derivatives similarly, as the derivative of ∇f .

$$f'(x) = \nabla f(x) \quad \text{if } f: \mathbb{R}^n \rightarrow \mathbb{R}$$

- Evidently, the second derivative of f is an $n \times n$ matrix, called the **Hessian** of f at x
- The form of the Hessian is

$$H(x) = \begin{pmatrix} \frac{\partial}{\partial x_1} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial}{\partial x_n} \frac{\partial f}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial x_1} \frac{\partial f}{\partial x_n} & \cdots & \frac{\partial}{\partial x_n} \frac{\partial f}{\partial x_n} \end{pmatrix} \quad \text{— Hessian matrix!}$$

$$f''(x) = \frac{d}{dx} (f'(x)) \quad \left[\text{Taking a derivative of w.r.t to } n \times 1 \text{ vector} \right] \quad \begin{matrix} 1 \times n \text{ vector} \\ n \times n \text{ matrix} \end{matrix}$$

$$f: \mathbb{R}^n \rightarrow \mathbb{R}^m, f(x) \text{ } m \times n \text{ matrix}$$

Schwarz's Theorem

Worried about remembering the order of differentiation for the Hessian? In most cases,

there's no need:

2nd derivative is continuous.

Theorem 3.1

(Schwarz) If f is *twice continuously differentiable*, the Hessian matrix is symmetric.

$$\boxed{\frac{\partial}{\partial x_i} \frac{\partial f}{\partial x_j} = \frac{\partial}{\partial x_j} \frac{\partial f}{\partial x_i}} \Rightarrow \text{H is symmetric}$$

Taylor Series for Multivariable Functions

- We will consider second-order expansions, as the notation is cumbersome beyond that point (and expansions beyond two orders are rare in economics).
- Let's take $f : \mathbb{R}^2 \rightarrow \mathbb{R}$. Suppose we want to approximate f around (x_1^*, x_2^*) with a second-order polynomial

$$h(x_1, x_2) = a_0 + a_1(x_1 - x_1^*) + a_2(x_2 - x_2^*) + a_{11}(x_1 - x_1^*)^2 + a_{12}(x_1 - x_1^*)(x_2 - x_2^*) + a_{22}(x_2 - x_2^*)^2$$

- We again require that $h(x_0, y_0) = f(x_0, y_0)$ and that all first and second order derivatives of h match f at x_1^*, x_2^*

$f : \mathbb{R} \rightarrow \mathbb{R}$.

$$h(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2$$

Taylor Series for Multivariable Functions (cont.)

$$a_1 = \frac{\partial f(x_1^*, x_2^*)}{\partial x_1}$$

$$a_{11} = \frac{1}{2} \frac{\partial^2 f(x_1^*, x_2^*)}{\partial x_1^2}$$

- Differentiating and matching terms gives

$$h(x_1, x_2) = f(x_1^*, x_2^*) + \frac{\partial f(x_1^*, x_2^*)}{\partial x_1}(x_1 - x_1^*) + \frac{\partial f(x_1^*, x_2^*)}{\partial x_2}(x_2 - x_2^*)$$

$$\begin{aligned} &+ \frac{1}{2} \frac{\partial^2 f(x_1^*, x_2^*)}{\partial x_1^2} (x_1 - x_1^*)^2 + \frac{\partial^2 f(x_1^*, x_2^*)}{\partial x_1 \partial x_2} (x_1 - x_1^*)(x_2 - x_2^*) \\ &+ \frac{1}{2} \frac{\partial^2 f(x_1^*, x_2^*)}{\partial x_2^2} (x_2 - x_2^*)^2 \end{aligned}$$

$$a_{12} = \frac{\partial^2 f(x_1^*, x_2^*)}{\partial x_1 \partial x_2}$$

- Much cleaner to write in matrix form:

$f: \mathbb{R}^n \rightarrow \mathbb{R}$ \Rightarrow $|x|$ $h(x) = f(x^*) + f'(x^*)(x - x^*) + \frac{1}{2} (x - x^*)^T H(x^*) (x - x^*)$ Quadratic form

- This formula holds for functions with more than 2 variables, and is known as the second order Taylor series expansion of f around x^*

$$f'(x^*) = \left(\frac{\partial f}{\partial x_1} \quad \frac{\partial f}{\partial x_2} \right) \Big|_{x_1^*, x_2^*} \quad (x - x^*) = \begin{pmatrix} x_1 - x_1^* \\ x_2 - x_2^* \end{pmatrix}$$

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Motivation

Exogenous v/s Endogenous Variables

$$\max_{\substack{c_1, c_2}} c_1^\alpha c_2^{1-\alpha} \quad \text{s.t. } p_1 c_1 + p_2 c_2 \leq M$$

$c_1^* = c_1(M, p_1)$
 $c_2^* = c_2(M, p_2)$

Exogenous: M, α, p_1, p_2
Endogenous: c_1, c_2

- Many economic analyses introduce equations of the form $f(x, y) = 0$, where x is a vector of "exogenous" variables and y a vector of "endogenous" variables
- We are frequently interested in understanding the impact of x on y , namely $y'(x)$
- However, the equations are complicated and it may not be possible to solve explicitly for $y(x)$ in order to take derivatives
- The implicit function theorem gives us a way to do such comparative statics even in the absence of a closed form solution

[Endogenous vars in terms
of exogenous variables]

$\frac{\partial c_1}{\partial p_2}$ / $\boxed{\frac{dc_1}{dM}}$ - Effect of an exogenous on endogenous variable.

$\frac{dc_1}{dM}$. This requires $c_1 = c_1(M, \dots)$

$c_1 = \frac{\partial M}{P_1} \Rightarrow \frac{\partial c_1}{\partial M} = \frac{1}{P_1}$

IFT: Two Variables

1 exogenous + 1 endogenous

Given x , I can find
y s.t. $f(x, y) = 0$

- Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$
- Assume for every x there is a unique y that satisfies $f(x, y) = 0$. Write $y = y(x)$
- Differentiate the expression $f(x, y(x)) = 0$ with respect to x and apply the chain rule:

$$f_x(x, y(x)) + f_y(x, y(x))y'(x) = 0$$

- So long as $f_y(x, y(x)) \neq 0$, we can solve for $y'(x)$:

Goal $y'(x)$

Exo Endogenous

$$f(x, y(x)) = 0$$

$$y'(x) = -\frac{f_x(x, y(x))}{f_y(x, y(x))}$$

without explicitly defining $y(x)$.

$$\left| \begin{array}{l} f(x, y(x)) = 0 \\ f_x(x, y(x)) + \frac{\partial f(x, y(x))}{\partial y} \cdot y'(x) \end{array} \right.$$

Diff wrt x

IFT: Many variables

n exogenous variables *m endogenous variables*

- Let $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m$

- Assume for every x there is a unique y that satisfies $f(x, y) = 0$. Write $y = y(x)$
- Now differentiate the expression $f(x, y(x)) = 0$ with respect to x and apply the chain rule:

$$f_x(x, y(x)) + f_y(x, y(x))y'(x) = 0$$

*matrix
d*

- So long as $f_y(x, y(x))$ is invertible, we have

$$y'(x) = - \left(\begin{matrix} f_y(x, y(x))^{-1} \\ \text{m} \times m \end{matrix} \right) \begin{matrix} f_x(x, y(x)) \\ \text{m} \times n \end{matrix}$$

$\frac{d}{dx} f(\)$
 $f(x, y) \in \mathbb{R}^m$
 $x \in \mathbb{R}^n$

- So long as f is continuously differentiable and $\det(f_y(x, y(x))) \neq 0$, the above formula is correct. See FMEA page 84 for a full statement.

$$y'(x) : \begin{matrix} y \in \mathbb{R}^m \\ x \in \mathbb{R}^n \end{matrix}$$

Taking derivative of
a $m \times 1$ vector w.r.t.
 $n \times 1$ vector

IFT: Supply and Demand Example

*Oil prices
Govt policies
Macroeconomic shocks*



- Let $\theta \in \mathbb{R}^n$ be a vector of variables that affect supply and demand.
- Market clearing implies

for all θ

$$\underline{Q^s(\theta, p) = Q^d(\theta, p)}$$

supply = demand

- To put it into our format:

$$f(x, y) = 0$$

$$\underline{\underbrace{Q^s(\theta, p) - Q^d(\theta, p)}_{f(\theta, p): \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}} = 0}$$

Diff wrt θ

- This implicitly defines p as a function of θ . Differentiating gives:

$$Q_\theta^s(\theta, p(\theta)) + Q_p^s(\theta, p(\theta))p'(\theta) = Q_\theta^d(\theta, p(\theta)) + Q_p^d(\theta, p(\theta))p'(\theta)$$

Have not explicitly told you how p depends on θ .

$p = p(\theta)$?

$$\frac{dp}{d\theta_1}$$

Endogenous: $P \in \mathbb{R}$
Exogenous: $\theta \in \mathbb{R}^n$

x = exogenous var
(θ)

y = endogenous var
(p)

$$\underbrace{Q^s(\theta, p(\theta)) - Q^d(\theta, p(\theta))}_{\text{Diff wrt } \theta} = 0$$

$$\Rightarrow Q_\theta^s(\theta, p(\theta)) + Q_P^s(\theta, p(\theta)) \underbrace{p'(\theta)}_{- Q_\theta^d(\theta, p(\theta)) - Q_P^d(\theta, p(\theta)) \underbrace{p'(\theta)}_{= 0}}$$

$$\Rightarrow p'(\theta) = \frac{Q_\theta^d(\theta, p(\theta)) - Q_\theta^s(\theta, p(\theta))}{Q_P^d(\theta, p(\theta)) - Q_P^s(\theta, p(\theta))}$$

$\theta \in \mathbb{R}$

$n \times 1$ vector

$$p'(\theta) = \left(\begin{array}{c} \frac{\partial P}{\partial \theta_1} \\ \vdots \\ \frac{\partial P}{\partial \theta_n} \end{array} \right)$$

IFT: Supply and Demand Example (cont.)

If $\theta \in \mathbb{R}$ \Rightarrow

$\frac{\text{How supply increases as price increases}}{Q_p^s(p, p(\theta)) > 0}$

$Q_p^d(p, p(\theta)) < 0$

- Solving for $p'(\theta)$ gives:

$$p'(\theta) = \frac{Q_\theta^d - Q_\theta^s}{Q_p^s - Q_p^d} > 0$$

- The denominator is positive
- Therefore the sign of $p'(\theta)$ depends on the sign of $Q_\theta^d - Q_\theta^s$
- If demand reacts more strongly than supply to changes in θ , price increases

$$Q_\theta^d > Q_\theta^s \Rightarrow p'(\theta) > 0$$

