MA Math Camp Exam 2020 Solutions

Instructions:

- This is a 48 hour take home exam.
- You may use any results covered in the class directly without proofs.
- All answers must be justified.
- You may only consult the slides and lecture material for this exam. **If any indication of cheating is suspected, 10 points will be deducted for every suspected answer copied from the internet or from your classmates.** No chance of explanation will be given.
- Please write your answers clearly. Points will be deducted for bad handwriting.

1. (**5 points**) Consider a function $f : \mathbb{R} \to \mathbb{R}$. Suppose f is increasing i.e. $x \geq y$ implies $f(x) \geq f(y)$. Prove that f is both quasiconvex and quasiconcave.

   Solution : Take any $x, y \in \mathbb{R}$. WLOG suppose $x \geq y$. Moreover $x \geq \lambda x + (1-\lambda)y \geq y \implies f(x) \geq f(\lambda x + (1-\lambda)y) \geq f(y)$ since f is increasing. Hence we have that :

   $$\max\{f(x), f(y)\} = f(x) \geq f(\lambda x + (1-\lambda)y) \geq f(y) = \min\{f(x), f(y)\}$$

   Hence f is quasiconcave and quasiconvex.

2. Consider the matrix $\boldsymbol{A} = \begin{pmatrix} 2 & 1 \\ -2 & 5 \end{pmatrix}$.

   (a) (**3 points**) Find the inverse of the matrix $\boldsymbol{A}$.

   Solution : The inverse is $A^{-1} = \begin{pmatrix} 5/12 & -1/12 \\ 1/6 & 1/6 \end{pmatrix}$

   (b) (**3 points**) Find the eigen values of this matrix

   Solution : The eigen values are $\lambda = 3, 4$

(c) (**3 points**) Find a set of linearly independent eigenvectors such that the Euclidean norm of each eigen-vector equals 1.

Solution : The set of linearly independent eigen vectors is :

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}$$

(d) (**3 points**) Find a matrix $\boldsymbol{P}$ such that $\boldsymbol{A} = \boldsymbol{PDP^{-1}}$

Solution : $P = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{2}} & \frac{2}{\sqrt{5}} \end{pmatrix}$ and $D = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$

(e) (**3 points**) Consider a sequence of matrices $\boldsymbol{A^1}, \boldsymbol{A^2}, \ldots, \boldsymbol{A^n}, \ldots$ where the n^{th} element of the sequence is the n^{th} power of $\boldsymbol{A}$. Does there exist a matrix $\boldsymbol{B}$ such that each of the elements of $\boldsymbol{A^n}$ converges to the corresponding element of $\boldsymbol{B}$ as $n \to \infty$

Solution : No. For any $n \in \mathbb{N}$ we have :

$$A^n = PD^nP^{-1} = P \begin{pmatrix} 3^n & 0 \\ 0 & 4^n \end{pmatrix} P^{-1}$$

which goes to infinity as $n \to \infty$

3. Are the following statements true or false? If you think it is true, provide a sketch of a proof. If you think it is false, provide a counterexample.

(a) (**5 points**) The intersection of 2 compact sets is a compact set.

Solution : True. For instance, using the sequential definition of compactness: Let K_1, K_2 be two compact sets of a metric space (X, d). Consider a sequence (x_n) of $K_1 \cap K_2$. Since (x_n) is a sequence of K_1 and K_1 is compact, it has a subsequence (x_{n_k}) that converges to $l \in K_1$. Since (x_{n_k}) is a sequence ofK_2 and K_2 is closed (because it is compact), $l \in K_2$. Hence (x_{n_k}) is a subsequence of (x_n) that converges in $K_1 \cap K_2$

(b) (**5 points**) The inverse image of a compact set by a continuous function is a compact set.

Solution : False. Consider $f(x) = 0 \ \forall x \in \mathbb{R}$. The set $\{0\}$ is compact, but $f^{-1}(\{0\}) = \mathbb{R}$ which is not compact.

(c) (**5 points**) Suppose $f : X \to Y$ is discontinuous at $x \in X$. Suppose $g : Y \to Z$ is a continuous function. Then $g \circ f : X \to Z$ is discontinous at x.

Solution : False. Consider $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$ and $g : \mathbb{R} \to \mathbb{R}$ as $g(x) = 1$. Thus $g \circ f(x) = 1 \ \forall x \in \mathbb{R}$ which is continuous.

4. On a non-empty set X, define the discrete metric d, as :

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

(a) (**5 points**) Verify that d is indeed a valid metric

Solution : Positive definiteness and symmetry are trivial. For triangle inequality, take $x, y, z \in X$. If $x = y$, then we have that $d(x, y) = 0 \leq d(x, z) + d(z, y)$. If $x \neq y$, then we must have that $d(x, y) = 1 \leq d(x, z) + d(z, y)$ since either $d(x, z) = 1$ or $d(z, y) = 1$.

(b) (**5 points**) Show that any subset of X is both open and closed

(i) Take any $S \subset X$. Consider any $x \in S$, then if we can show that it is an interior point, we are done. Now, by definition $B_1(x) = \{x\} \subset S$. Hence S is open.
(ii) To show that S is closed, we need to show that $S \supset S'$. We will show here that the set of limit points, $S' = \emptyset$. Take any $x \in X$. As before since, $B_1(x) = \{x\}$, we have that $B_1(x)/\{x\} \cap S = \emptyset$. Hence no point $x \in X$ is a limit point of S. Since $\emptyset \subset S$, we are done.

(c) (**5 points**) Show that a set S in X is compact if and only if it is finite.

Solution : $\implies$ Suppose S is compact. We know this implies that every open cover has a finite subcover. In particular consider the open cover $\{B_1(x)\}_{x \in S}$. Since S is compact, $\exists$ a finite set $\hat{S} \subset S$ s.t. $\{B_1(x)\}_{x \in \hat{S}}$ is also an open cover of S. But we know that :

$$\bigcup_{x \in \hat{S}} B_1(x) = \bigcup_{x \in \hat{S}} \{x\} = \hat{S} \supset S$$

This implies $\hat{S} = S$ which implies S is a finite set.
$\impliedby$ Suppose $S \subset X$ is finite. We need to show that it is compact. Take any open cover $\{E_\alpha\}_{\alpha \in A}$ of S. For each $x \in S$, $\exists \alpha_x \in A$ s.t. $x \in E_{\alpha_x}$. Then $\bigcup_{\alpha_x} E_{\alpha_x} \supset S$ by construction. Since S is finite, $\{E_{\alpha_x}\}_{x \in S}$ is a finite subcover.

S is compact iff it is finite.

$\Rightarrow$ Suppose ~~is~~ S is compact.

Take any seq $(x_n) \subset S$ & suppose there exists $(x_{n_k}) \to x \in S$

I'm going to argue that it is impossible for there to be infinitely many distinct terms.

Fix $\varepsilon \in (0,1) \Rightarrow \exists N_k$ s.t.

$$d(x_{n_k}, x) < \varepsilon \quad \forall n_k > N_k$$

$$\Rightarrow x_{n_k} = x \quad \forall n_k > N_k$$

No convergent sequence can have infinitely many distinct terms

$\Rightarrow$ set is finite.

$S = \{1, 2, 3\}$

$x_n = \{3, 2, 1, 1, 1 \, \ldots \ldots\} \to 1$

$\Leftarrow$ S is finite $\Rightarrow$ S is compact.

$x_n \subset S$ $\Rightarrow$ Some term of that set is repeated infinitely many times

$S = \{1, 2\}$

$x_n = \{1, 2, 2, 1, 2, \ldots\}$

$x_{n_k} = \{1, 1, 1, 1, \ldots\} \to 1$

$\Rightarrow$ S is compact.

$\tilde{p}$ $\qquad\qquad$ $\tilde{\omega}$

$$(\lambda p + (1-\lambda) p') \cdot x \leq \lambda w + (1-\lambda) w'$$

$$\lambda p \cdot x + (1-\lambda) p' \cdot x \leq \lambda w + (1-\lambda) w'$$

It must be that either

$$\boxed{\begin{array}{l} p \cdot x \leq w \\ \text{OR} \quad p' \cdot x \leq w' \end{array}}$$

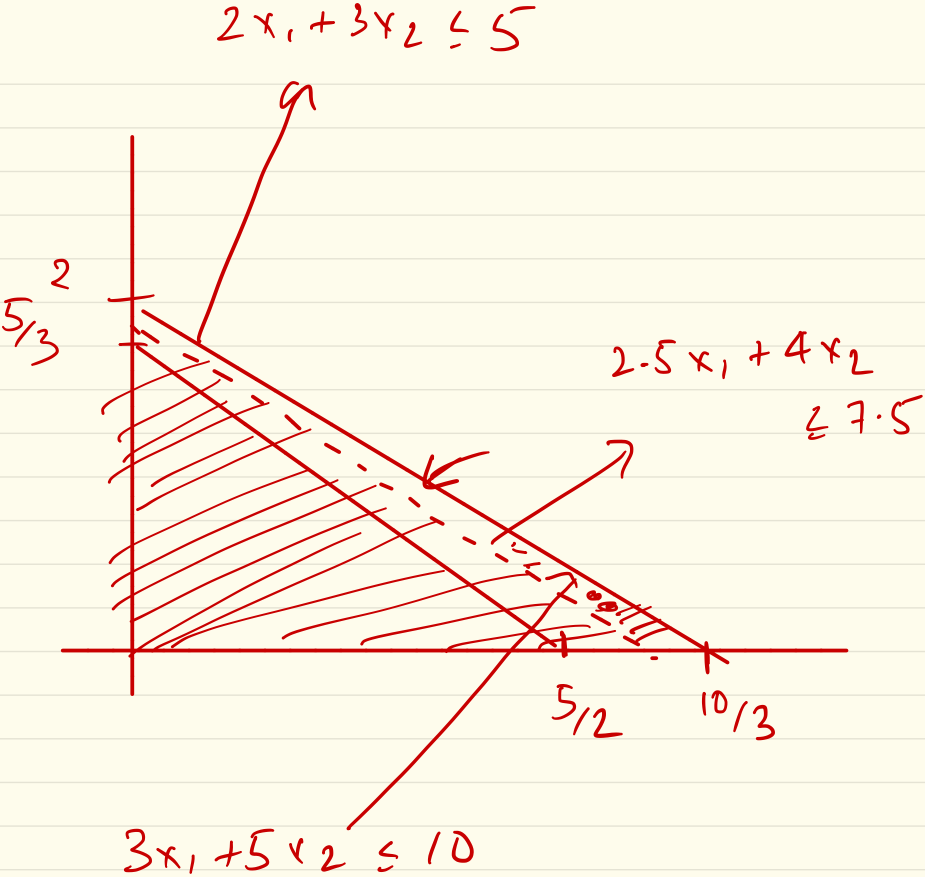
$$\underbrace{\lambda (p \cdot x - w)}_{< 0} + \underset{\text{OR}}{} (1-\lambda) \underbrace{(p' \cdot x - w')}_{< 0} \leq 0$$

Suppose $p \cdot x \leq w$

$$2x_1 + 3x_2 \leq 5 \qquad p = (2, 3)$$
$$\qquad\qquad\qquad\qquad w = 5$$
$$3x_1 + 5x_2 \leq 10 \qquad p' = (3, 5)$$
$$\qquad\qquad\qquad\qquad w' = 10$$

$$\lambda = 0.5$$

$$\tilde{p} = (2.5, 4) \qquad \tilde{w} = 7.5$$



$2x_1 + 3x_2 \leq 5$
2
$5/3$
$2.5x_1 + 4x_2 \leq 7.5$
$5/2$
$10/3$
$3x_1 + 5x_2 \leq 10$

f is Quasiconvex iff $f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\}$

5. (**10 points**) Consider the following maximization problem :

$$\max_{x \in \mathbb{R}^n} u(x)$$
$$s.t. \quad p \cdot x \leq w$$

Budget constraint $\sum p_i x_i \leq w$

where $x \in \mathbb{R}^n$, $p \in \mathbb{R}^n$ and $w \in \mathbb{R}$. (Note that $p \cdot x = \sum_{i=1}^n p_i x_i$). Suppose that the maximization problem has a solution for any (p, w) in some convex set $S \subset \mathbb{R}^n \times \mathbb{R}$. Show that the value function $v : S \to \mathbb{R}$ defined as $v(p, w) := \max_{x \in \mathbb{R}^n} \{u(x) \text{ s.t. } p \cdot x \leq w\}$ is quasiconvex.
(Hint : Kuhn Tucker is not required! Work with the definition of quasiconvexity)

Exogenous parameters

Solution : For any (p, w) and $(p', w') \in S$, we want to show $v(\lambda p + (1-\lambda)p', \lambda w + (1-\lambda)w') \leq \max\{v(p, w), v(p', w')\}$.

$\lambda(p, w) + (1-\lambda)(p', w')$

Notice that $(\underbrace{\lambda p + (1-\lambda)p'}_{\tilde{p}}) \cdot x \leq \underbrace{\lambda w + (1-\lambda)w'}_{\tilde{w}}$ implies either $p \cdot x \leq w$ or $p' \cdot x \leq w'$. Then any feasible point under $(\lambda p + (1-\lambda)p', \lambda w + (1-\lambda)w')$ must be feasible under either (p, w) or (p', w') and therefore either $v(\lambda p + (1-\lambda)p', \lambda w + (1-\lambda)w') \leq v(p, w)$ or $v(\lambda p + (1-\lambda)p', \lambda w + (1-\lambda)w') \leq v(p', w')$ and hence :

$$v(\lambda p + (1-\lambda)p', \lambda w + (1-\lambda)w') \leq \max\{v(p, w), v(p', w')\}$$

$\tilde{p} \cdot x \leq \tilde{w}$

$\tilde{p} = \lambda p + (1-\lambda)p'$

$\tilde{w} = \lambda w + (1-\lambda)w'$

Any bundle which can be bought under $\tilde{p}, \tilde{w}$ can also be bought under either (p, w) or (p', w').

6. Consider the function $f(x_1, x_2) = x_1^\alpha + x_2^\alpha$ defined for $x_1, x_2 \geq 0$ and $\alpha \in (0, 1]$. **Do notice that $\alpha = 1$ is included.** For a fixed α, consider the problem:

$$\max_{x_1, x_2} x_1^\alpha + x_2^\alpha$$
$$s.t. \; x_1 + x_2 = 1$$

(a) (**5 points**) Show that f is concave in (x_1, x_2) for a fixed α. (Hint : Use the result that a finite sum of concave functions is concave)

Solution : x^α is concave since $\alpha \in (0, 1]$. Hence f is concave since it is a finite sum of concave functions.

(b) (**5 points**) Does a solution exist for this maximization problem for all values of $\alpha \in (0, 1]$

Solution : The objective function is continuous and the constraint set $S = \{(x_1, x_2) \,|\, x_1 + x_2 = 1\}$ is closed and bounded and therefore compact. Hence a solution exists by Weierstrass' Theorem.

(c) (**15 points**) For a fixed α, find the solution to this maximization problem. (Hint : Consider separate cases for $\alpha < 1$ and $\alpha = 1$).

Solution : Consider the case $\alpha < 1$. Solving the FOCs of the Lagrangean, we get : $x_1 = x_2 = \frac{1}{2}$ is the only solution.

In the case $\alpha = 1$, there are infinite solutions since any x_1, x_2 such that $x_1 + x_2 = 1$ is a solution.

(d) (**5 points**) Do you need to check for the Second Order Conditions in this problem? Why or why not?

Solution : No. The objective function f is concave and the constraint is linear. Hence the Lagrangean is concave as well which is a sufficient condition for global maxima.

(e) (**5 points**) Find the value function $V^*(\alpha)$

Solution : $\boxed{V^*(\alpha) = 2\left(\frac{1}{2}\right)^\alpha}$

(f) (**5 points**) Verify that the Envelope theorem holds in this problem.

The Envelope theorem tells us that $V'(\alpha) = \mathcal{L}_\alpha(x_1^*(\alpha), x_2^*(\alpha), \alpha)$. Note that $\mathcal{L}_\alpha(x_1^*(\alpha), x_2^*(\alpha), \alpha)$ is the derivative of the Lagrangean with respect to α evaluated at $(x_1^*(\alpha), x_2^*(\alpha))$.

Solution : $V'(\alpha) = 2\ln\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^\alpha$ derivative wrt α
$\frac{\partial \mathcal{L}}{\partial \alpha}$ $\underline{L_2'(x,\alpha)} = \ln(x_1)\,x_1^\alpha + \ln(x_2)\,x_2^\alpha$.

This implies $L_2'(x^*(\alpha), \alpha) = 2\ln\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^\alpha$

Evaluated at $x = x^*(\alpha)$