Columbia MA Math Camp

Linearization and Log-linearization

Vinayak Iyer

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Motivation

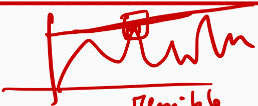
Max $c_1^{\alpha} c_2^{1-\alpha}$ s.t $p_1 c_1 + p_2 c_2 \le M$ $\Rightarrow$ $\boxed{c_1^* = c_1^*(p_1, p_2, M, \alpha)}$

$c_2^* = c_2^*(\quad)$

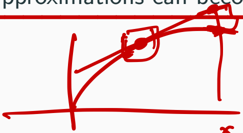
In many economic models we have a set of exogenous (given) parameters, from which we try to derive certain endogenous (or choice) variables

$\frac{dc_1^*}{dM}$

- One common exercise is to understand the impact of changes in the exogenous variables on the endogenous variables (this is called comparative statics)
- Difficult to do when the model is non-linear (one approach is the IFT, which we saw earlier)
- Another approach is to **linearize** the model, which we discuss here
- The benefit of this approach is its speed (can largely be done by hand, even for complicated models); the tradeoff is that linear approximations can become bad quickly if the model has a lot of curvature



Terrible.



Not bad

Linearization

Suppose we have a function $f(\theta) : \mathbb{R}^n \to \mathbb{R}$. We can generate a linear approximation of f around a point θ^* using a first-order Taylor series expansion:

$$f(\theta) - f(\theta^*) \approx f'(\theta^*)(\theta - \theta^*) = \sum_{i=1}^{n} \frac{\partial f}{\partial \theta_i}(\theta^*)(\theta_i - \theta_i^*)$$

$u \cdot v$ $u, v \in \mathbb{R}^n$

$\sum u_i v_i$

A useful bit of notation is $d\theta \equiv \theta - \theta^*$. Then we can restate our linearization result as:

$df(\theta) = f(\theta) - f(\theta^*)$

$$df(\theta) \approx f'(\theta^*)d\theta$$

$f(\theta) \approx f(\theta^*) + f'(\theta^*)(\theta - \theta^*)$ ← 1st order Taylor approx.

$\boxed{f(\theta)} - f(\theta^*) = \underbrace{f'(\theta^*)}_{1 \times n}\underbrace{(\theta - \theta^*)}_{n \times 1}$ → $\begin{pmatrix} \theta_1 - \theta_1^* \\ \theta_2 - \theta_2^* \\ \vdots \end{pmatrix}$

Linearization Example

Consider the equation $y = x^2$. To linearize this equation around (x^*, y^*), apply $\boxed{d}$ on both sides and use the formula from the previous slide:

$$\boxed{dy \approx 2x^* dx}$$

which shouldn't be too surprising

Total Derivative!

$U(x_1, x_2) =$

$$\underbrace{dU}_{\text{Total change in utility}} = \underbrace{\frac{\partial U}{\partial x_1} dx_1}_{\text{Total change due to change in } x_1} + \frac{\partial U}{\partial x_2} dx_2$$

Linearization: Rules for Speed

There are some rules that will help us linearize systems of equations quickly. Let α be a scalar, and x_i be scalar variables that we are expanding around x_i^*. Write $x = (x_1, ..., x_n)$.

- $d(x_1 + x_2) = dx_1 + dx_2$
- $d(\alpha x_1) = \alpha dx_1$
- $d(\alpha) = 0$
- $d(x_1 x_2) = x_1^* dx_2 + x_2^* dx_1$
- $d(x_1^n) = n x_1^{*n-1} dx$
- $d(f(x)) = f'(x^*)dx = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x^*)dx_i$ ← [Linearizing $f(x)$]

Linearization: Harder Example

Linearize the following system of equations:

$$
\begin{aligned}
x + 2y &= z \\
x^2 + y^2 + z^2 &= 5
\end{aligned}
$$

Apply our d operator to both equations. The rules above give:

$$
\begin{aligned}
dx + 2dy &= dz \\
x^* dx + y^* dy + z^* dz &= 0
\end{aligned}
$$

So if $(x^*, y^*, z^*) = (0, 1, 2)$, we can solve for dx and dy in terms of dz:

$$
\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} dz
$$

This implies

inflation changes

GDP changes

$$
\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} dz = \begin{pmatrix} 5 \\ -2 \end{pmatrix} dz
$$

$\frac{dx}{dz}$ & $\frac{dy}{dz}$

dz change in the interest rate

In macro, you will often encounter a complicated, non-linear system of dynamic equations.Linear approximations are not always good enough approximation.

(a) **Why** *log***?** - The different quantities we study in economics are measured in different units and hence it makes sense to talk about percentage deviations. So we transform all variables into deviation form and then linearize.

In other words, it frequently makes more sense to talk in terms of percent deviations as opposed to absolute deviations (e.g., the impact of a 2% change in GDP)

(b) **Linearize Around?** - Most natural point around which to consider linear approximations is the steady state of the variable concerned.

variable is not changing over time

$\frac{d}{dx} \log x \Rightarrow$ $d\log x = \boxed{\frac{dx}{x}}$

percentage change in x.

Log-linearization (continued)...

Expressing systems of equations in terms of percent deviations called **log-linearization**. Note that it is very similar to linearization.

- For a scalar x, define $\hat{x} = \frac{dx}{x^*}$: this is the percent change in x from x^*
- (This is called log-linearization because $\hat{x} = d\ln(x)$)
- For a function $f(x)$, take a Taylor series expansion in terms of $\hat{x}$:

$\hat{x} = d(\ln(x))$
$= \left[\frac{dx}{x^*}\right].$

$$\underbrace{f(x^* + dx) = f(x^* + x^*\hat{x})} \approx \underbrace{f(x^*) + f'(x^*)x^*}\hat{x}$$

Linear approx around x^*

Now notice the following:

$f(x^*+dx) \simeq f(x^*) + f'(x^*)x^*\hat{x}$
$\Rightarrow \frac{f(x^*+dx) - f(x^*)}{f(x^*)} \simeq f'(x^*)x^*$

$$\underbrace{\frac{f(x^* + dx) - f(x^*)}{f(x^*)}}_{\hat{f}} \approx \boxed{\underbrace{\frac{f'(x^*)x^*}{f(x^*)}}_{\text{elasticity}}}\hat{x}$$

$(x - x^*)$
$(\cancel{x^*} + x^*\hat{x} - \cancel{x^*})$

$f(x) = f(x - x^* + x^*) = f(x^* + dx)$
$= f(x^* + x^*\hat{x})$

Aim is to express all quantities in percentage deviation!

$$f(x^*+dx) \simeq f(x^*) + f'(x^*)\,x^*\hat{x}$$

$$\frac{f(x^*+dx)-f(x^*)}{f(x^*)} = \boxed{\frac{f'(x^*)}{f(x^*)}\,x^*}\,\hat{x}$$

$$\hat{f} = \boxed{\frac{df}{dx}\,\frac{x^*}{f(x^*)}}\,\hat{x}$$

Elasticity of f wrt x.

$$\hat{f} \simeq \varepsilon_f\,\hat{x}$$

$\hat{x}$ & $\hat{f}$

Log-Linearization: Simple Example

Consider the level curve $x_1 x_2 = 1$. Log-linearize this equation around the point x_1^*, x_2^*

$$dx_1 = x_1^* \hat{x}_1$$
$$dx_2 = x_2^* \hat{x}_2$$

$$\begin{aligned}
\widehat{x_1 x_2} &= \hat{1} \\
\frac{d(x_1 x_2)}{x_1^* x_2^*} &= \frac{d(1)}{1} \quad \text{(defn. of hat operator)} \\
\frac{x_1^* dx_2 + x_2^* dx_1}{x_1^* x_2^*} &= 0 \quad \text{(propreties of } d \text{ operator)} \\
\frac{x_1^* x_2^* \hat{x}_2 + x_2^* x_1^* \hat{x}_1}{x_1^* x_2^*} &= 0 \quad \text{(defn. of hat operator)} \\
\boxed{\hat{x}_1 + \hat{x}_2 = 0}
\end{aligned}$$

Linear in $\hat{x}_1$, $\hat{x}_2$

$$\hat{x}_1 = \frac{dx_1}{x_1^*} \qquad \widehat{x_1 x_2} = \frac{d(x_1 x_2)}{x_1^* x_2^*}$$

Log-linearization: Rules for speed

Similar to linearization, there are some basic rules that will let us log-linearize systems quickly. Let α be a scalar, and x_i scalar variables that we are expanding around x_i^*.

- $\widehat{x_1 x_2} = \hat{x}_1 + \hat{x}_2$
- $\widehat{x_i^n} = n\hat{x}_i$
- $\hat{\alpha} = 0$ for $\alpha \neq 0$
- $\widehat{\frac{x_1}{x_2}} = \hat{x}_1 - \hat{x}_2$
- $\widehat{x_1 + x_2} = \frac{x_1^*}{x_1^* + x_2^*}\hat{x}_1 + \frac{x_2^*}{x_1^* + x_2^*}\hat{x}_2$
- $\widehat{f(x_1, ..., x_n)} = \sum_{i=1}^n \epsilon_i^f(x^*)\hat{x}_i$, where $\epsilon_i^f(x^*)$ is the i-th elasticity of f at x^*:

$$\epsilon_i^f(x^*) = \frac{\frac{\partial f}{x_i}(x^*)x_i^*}{f(x^*)}$$

$d(\log x) = \frac{dx}{x} = \hat{x}$

$\hat{x} = \frac{dx}{x^*} \neq 0$

y= output
a = productivity
k = capital   l= labor

(a) Consider the production function : $y_t = a_t k_t^{\alpha} n_t^{1-\alpha}$. The log linearized production function is :

$$\hat{y}_t = \hat{a}_t + \alpha \hat{k}_t + (1-\alpha)\hat{n}_t$$

(b) Consider the Euler equation : $\left(\frac{c_{t+1}}{c_t}\right)^{\sigma} = \beta\left(1 + r_t\right)$. The log-linearized equation is :

$\sigma(\hat{c}_{t+1} - \hat{c}_t) = \hat{r}$ $\longrightarrow$

$$\hat{c}_{t+1} - \hat{c}_t = \frac{1}{\sigma}\hat{r}_t$$

$r^* = 0$

(c) Suppose $k_{t+1} = i_t + (1-\delta)\, k_t$. Then the log-linearized equation is:

$$\hat{k}_{t+1} = \frac{i^*}{k^*}\hat{i}_t + (1-\delta)\,\hat{k}_t$$

1+ r

Useful links here, here and here.

$$y_t = a_t k_t^{\alpha} l_t^{1-\alpha}$$

$$\hat{y}_t = \widehat{a_t k_t^{\alpha} l_t^{1-\alpha}}$$

$$a_t k_t^{\alpha} = z_t$$

$$l_t^{1-\alpha} = w_t$$

$$\hat{y}_t = \widehat{z_t w_t}$$

$$= \hat{z}_t + \hat{w}_t$$

$$= \widehat{a_t k_t^{\alpha}} + \widehat{l_t^{1-\alpha}}$$

$$= \hat{a}_t + \widehat{k_t^{\alpha}} + \widehat{l_t^{1-\alpha}}$$

$$= \hat{a}_t + \alpha \hat{k}_t + (1-\alpha)\hat{l}_t$$

Log-linearization: Harder Example

Log-linearize (1) around $(\lambda_t, \lambda_{t+1}, a_{t+1}, k_{t+1}, h_{t+1}) = (\lambda^*, \lambda^*, a^*, k^*, h^*)$.

$$\lambda_t = \beta\lambda_{t+1}\left(1-\delta+\alpha a_{t+1}k_{t+1}^{\alpha-1}h_{t+1}^{1-\alpha}\right) \tag{1}$$

First note $\beta(1-\delta+\alpha a^* k^{*\alpha-1}h^{*1-\alpha}) = 1$. Now keep applying the rules:

$$\begin{aligned}
\hat{\lambda}_t &= \hat{\beta} + \hat{\lambda}_{t+1} + \widehat{1-\delta+\alpha a_{t+1}k_{t+1}^{\alpha-1}h_{t+1}^{1-\alpha}} \text{ (product)} \\
&= \hat{\lambda}_{t+1} + \frac{d\left(1-\delta+\alpha a_{t+1}k_{t+1}^{\alpha-1}h_{t+1}^{1-\alpha}\right)}{1-\delta+\alpha a^* k^{*\alpha-1}h^{*1-\alpha}} \text{ (def. of hat)} \\
&= \hat{\lambda}_{t+1} + \beta\alpha d\left(a_{t+1}k_{t+1}^{\alpha-1}h_{t+1}^{1-\alpha}\right) \text{ (substitute } \beta\text{)} \\
&= \hat{\lambda}_{t+1} + \beta\alpha a^* k^{*\alpha-1}h^{*1-\alpha}\widehat{a_{t+1}k_{t+1}^{\alpha-1}h_{t+1}^{1-\alpha}} \text{ (def. of hat)} \\
&= \hat{\lambda}_{t+1} + s\left(\hat{a}_{t+1} + (\alpha-1)\hat{k}_{t+1} + (1-\alpha)\hat{h}_{t+1}\right) \text{ (product)}
\end{aligned}$$

where $s = \beta\alpha a^* k^{*\alpha-1}h^{*1-\alpha}$.

Log-linearization: Caveats

- Everyone has their preferred method for log-linearization - I'm sure you'll see different methods this year. Pick one that makes sense and you can do quickly.
- Note you can't log-linearize a quantity about 0: percent change isn't defined.
    - Most relevant for interest rates /Inflation
    - Most people just log-linearize using the gross interest rate $1 + r$
- Log-linearization for negative numbers doesn't make too much sense either. Either transform to absolute values or simply linearize

log(-negative) is undefined.