

Columbia MA Math Camp

Real Analysis

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Motivation

- In real analysis (and also in micro) we often need to use concepts related to limits and convergence
- By saying that a sequence of objects converges to a limiting object, we mean, roughly speaking, that the sequence will get "as close as we want" to the limit.
- To be able to talk about how close 2 objects are, we need the concept of distance.
- Metric spaces are the general framework that capture the concept of distance, but we will focus on Euclidean metric spaces
 - Pretty much the only thing economists work with ✓
 - Easier to visualize ✓

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Defining Metric Spaces

Definition 1.1

Let X be a set, and $d : X \times X \rightarrow \mathbb{R}$ a function. We call d a metric on X if:

- Positive Definiteness : $d(x, y) \geq 0$ for all $x, y \in X$, and $d(x, y) = 0$ iff $x = y$
- Symmetry : $d(x, y) = d(y, x)$
- Triangle Inequality : $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$

A metric space (X, d) is a set X with a metric d defined on X



Example : A trivial example is the discrete metric :

$$X = \mathbb{R}$$

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}.$$

Does this satisfy the properties of a metric? We will however focus on Euclidean metric spaces.

Discrete Metric is actually a metric

Discrete metric : $d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$

$$d(x, y) = d(y, x)$$

$$= 1$$

if $x \neq y$

$= 0$ if $x = y$

Let us check that the discrete metric satisfies the 3 properties of a metric :

(a) By definition $d(x, y) = 0$ iff $x = y$

(b) Symmetric is also trivial

(c) Take $x, y, z \in X$. If $x = y$, then $d(x, y) = 0 \leq d(x, z) + d(z, y)$. If $x \neq y$, then we must have $d(x, z) = 1$ OR $d(z, y) = 1$. In either case we have :

$$d(x, y) = 1 \leq d(x, z) + d(z, y)$$

Take any $x, y, z \in X$.

WTS: $d(x, z) \leq d(x, y) + d(y, z)$

Case I: $x = y = z$

$$d(x, z) = 0 \leq d(x, y) + d(y, z)$$

$0 \quad 0$

$$0 \leq 0 \quad \cancel{+}$$

Case II: $x \neq y \neq z$

$$d(x, z) = 1 \quad 1 \leq 2$$

$$d(x, y) + d(y, z) = 2$$

Case III: $x \neq z$

$$d(x, z) = 1$$
$$[\underbrace{d(x, y)}_{\cdot} + \underbrace{d(y, z)}_{\cdot}] \rightarrow \text{at least } 1$$

Euclidean Distance

Focus on Euclidean Metric Spaces : (\mathbb{R}^n, d_2)

Definition 1.2

In \mathbb{R}^n , the Euclidean distance is the function $d : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^+$:

$$\rightarrow. [d(x, y) = \left(\sum_{i=1}^n (x_i - y_i)^2 \right)^{\frac{1}{2}}]$$

$x \in \mathbb{R}^n$
 $y \in \mathbb{R}^2$

Euclidean distance satisfies the 3 properties that we mentioned earlier:

- $d(x, y) \geq 0$, and $d(x, y) = 0$ iff $x = y$
- $d(x, y) = d(y, x)$
- $d(x, y) \leq d(x, z) + d(z, y)$

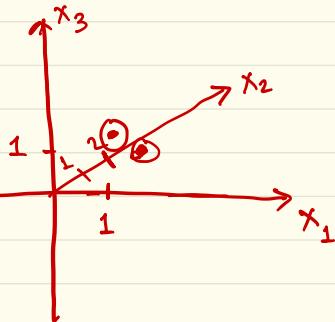
Let us prove the first two properties. Triangle Inequality is hard to show - requires the Cauchy Schwarz inequality.

$$3 \in \mathbb{R} \quad 1 \in \mathbb{R} \quad d(1, 3) = \left[(3-1)^2 \right]^{\frac{1}{2}} = |3-1| = |2|$$

$$x \in \mathbb{R}^3$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \in \mathbb{R}^3$$

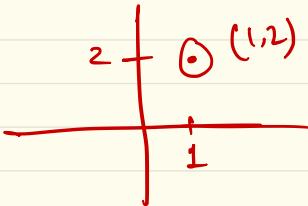


$$x \in \mathbb{R}^4$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$x \in \mathbb{R}^2$$

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \in \mathbb{R}^2$$



$$\mathbb{R}^3$$

Compute the distance between x & y where

$$x = \begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix}$$

$$y = \begin{pmatrix} 2 \\ 10 \\ 6 \end{pmatrix}$$

$$\left(\sum_{i=1}^n (x_i - y_i)^2 \right)^{\frac{1}{2}} = \left[\frac{(1-2)^2 + (5-10)^2 + (7-6)^2}{\sqrt{27}} \right]^{\frac{1}{2}} = 3\sqrt{3}$$

Euclidean metric proof

Fact 1

Prove that the Euclidean metric satisfies the first two properties of a metric

Proof.

(a) WTS that $d(x, y) = 0$ iff $x = y$ for any $x, y \in \mathbb{R}^n$.

(\Rightarrow) : If $d(x, y) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2} = 0$, this implies $\sum_{i=1}^n (x_i - y_i)^2 = 0$. Since $(x_i - y_i)^2 \geq 0$ for each i which means that $(x_i - y_i)^2 = 0 \forall i$ and thus $x_i = y_i \forall i$. Thus $x = y$

(\Leftarrow) : If $x = y$, we have $x_i = y_i$ for each i . Therefore

$$d(x, y) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2} = 0$$

(b) WTS that $d(x, y) = d(y, x)$. This follows from the fact that :

$$\sqrt{\sum_{i=1}^n (x_i - y_i)^2} = \sqrt{\sum_{i=1}^n (y_i - x_i)^2}$$

Bounded Sets

Example:- Consider \mathbb{R}^2



$$d(0, x) \leq M \in \mathbb{R}$$

$\forall x \in S$

replace with any constant

Definition 1.3

A subset S of \mathbb{R}^n is bounded if there exists $M \in \mathbb{R}$ such that for all $x \in S$, $d(0, x) \leq M$.

Note: We could have chosen any $a \in \mathbb{R}^n$ in the place of 0. Suppose S is bounded with respect to 0. Then for any $x \in S$ the triangle inequality tells us

$$d(a, x) \leq d(a, 0) + d(0, x) \leq d(a, 0) + M$$

Thus S is bounded with respect to a as well.

Trick Question :- Is \mathbb{R} bounded?

No! There does NOT exist $M \in \mathbb{R}$ s.t. $d(0, x) \leq M$ $\forall x \in \mathbb{R}$

Proof :- Suppose $\exists M \in \mathbb{R}$ s.t. $d(0, x) \leq M \quad \forall x \in \mathbb{R}$.
But consider any $x' \in \mathbb{R}$ s.t. $x' > M \Rightarrow d(0, x') = x' > M \therefore x' > M$
 $\Rightarrow \mathbb{R}$ is not bounded.

Least Upper Bounds

Example:-

$$[1, 2] \subset \mathbb{R}$$

Any number larger than 2
is an upper bound
for $[1, 2]$



8 2 is the LUB
for $[1, 2]$

Definition 1.4

Let $S \subseteq \mathbb{R}$. A number $M \in \mathbb{R}$ is an upper bound of S if $s \leq M$ for every $s \in S$.

If no $M' < M$ is an upper bound of S , then M is called the least upper bound or supremum of S .

We make one important assumption about the real numbers: every bounded set of real numbers has a least upper bound. This is called the least upper bound property.

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Sequences: Definition

Sequence is a fn $x: \mathbb{N} \rightarrow \mathbb{R}$ $x(1) = x_1$
 $x(2) = x_2$
 \vdots

Formally, a sequence in a set X is a function from \mathbb{N} to X . We denote x_n the image of n , and (x_n) the sequence.

Less formally, a sequence is an ordered collection of elements (x_0, x_1, x_2, \dots) . Many problems in math boil down to understanding the long-term behavior of some sequence.

We typically write sequences as a formula or by enumerating the first few terms.

- $(x_n) = (n)_{n=0}^{\infty} : (0, 1, 2, 3, \dots)$
- $(x_n) = (1)_{n=0}^{\infty} : (1, 1, 1, 1, \dots)$
- $(x_n) = \boxed{1 \left(\frac{1+(-1)^n}{2} \right)}_{n=0}^{\infty} : (1, 0, 1, 0, \dots)$
- $(x_n) = \left(\frac{1}{n+1} \right)_{n=0}^{\infty} : (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots)$

$$(x_n) = \langle x_1, x_2, x_3, \dots \rangle$$

$$x: \mathbb{N} \rightarrow X$$

 $x(n) = n$

$$(x_n) = \{x_1, \dots\}$$

is a set!!

You'll also see people write things like $x_n = n$.

$$x(n) = \left(\frac{1 + (-1)^n}{2} \right)$$

$$(x_n) = \langle 0, 1, 2, 3, \dots \rangle$$

Bounded Sequences

Example:- $x_n = \frac{(-1)^n}{n} \Leftarrow (-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \dots) \hookrightarrow M=1$

$x_n = \frac{(-1)^n}{n^2} \rightarrow (-1, 4, -9, 16, \dots) \leftarrow d(x_n, 0) \leq 1 \forall n$

$d(x_n, 0) \leq M$

In \mathbb{R}^n we can define properties of sequences that rely on the notion of distance.

Definition 2.1

A sequence (x_n) of $S \subseteq \mathbb{R}^n$ is bounded iff $\{x_0, \dots, x_n, \dots\}$ is a bounded subset of \mathbb{R}^n .

Are the sequences $x_n = n$ and $x_n = 1/n$ bounded or unbounded?

$$n \nless 0$$

- The sequence $x_n = n$ is not bounded. For any $M \in \mathbb{R}$, $d(x_n, 0) > M$ for $n > M$.
- The sequence $x_n = \frac{1}{n}$ is bounded: $d(x_n, 0) \leq 1$ for all n

$$(x_n) = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots \right\} \uparrow^{M=1} d(x_n, 0) \text{ keeps getting smaller as } n \text{ increases.}$$

$d(1, 0) = 1$

Limits

$x: \mathbb{N} \rightarrow S$

Definition 2.2

A sequence (x_n) of $S \subseteq \mathbb{R}$ converges to a limit $\ell \in S$ iff for all $\epsilon > 0$, there exists an integer $N(\epsilon)$ such that for all $n \geq N(\epsilon)$

$$d(x_n, \ell) < \epsilon$$

~~$d(x_n, \ell) = |x_n - \ell|$~~

$d(x_n, \ell) = |x_n - \ell|$
only in \mathbb{R} !!

We write $x_n \rightarrow \ell$ or $\lim_{n \rightarrow \infty} x_n = \ell$. If (x_n) does not converge, we say it diverges.

This definition is important, so let's unpack it a little:

- The sequence must eventually get and remain arbitrarily close to ℓ .
- N can be different for each ϵ .
- We require $\ell \in S$. Consider $(x_n) = (1, \frac{1}{2}, \frac{1}{3}, \dots)$. What happens if $S = \mathbb{R}$? If $S = (0, 2)$?
 - When $S = \mathbb{R}$, Does (x_n) have a limit? Yes, $\ell = 0$
 - When $S = (0, 2)$, Does (x_n) " " " ? No!
Actually diverges!!

$$\begin{array}{c}
 \begin{array}{ccccccc}
 & x_4 & x_3 & x_2 & x_1 & & \\
 \hline
 & | & | & | & | & & \\
 0 & \frac{1}{4} & \frac{1}{3} & \frac{1}{2} & & 1 & \\
 x_1 & x_2 & x_3 & x_4 & x_5 & \dots & \\
 x_n = \frac{1}{n} & = & \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots \right\}
 \end{array}
 \end{array}$$

Let's try & see how this converges to 0

ϵ	$N(\epsilon)$
0.5	3 }
0.05	21]
0.005	

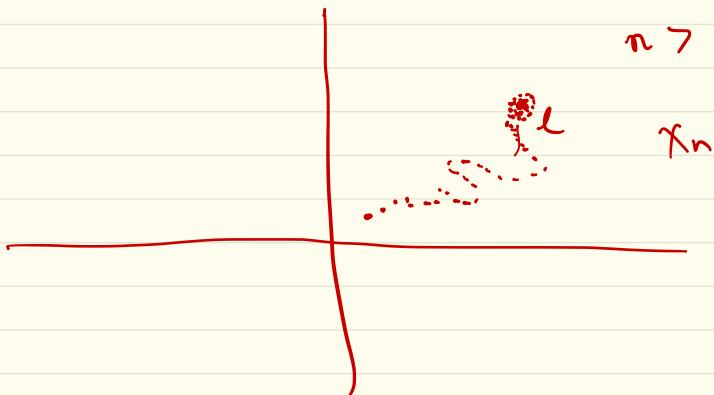
Give me $N(\epsilon)$
s.t.
 $d(x_n, 0) < \epsilon$
 $\forall n > N(\epsilon)$

$$d(x_3, 0) = d\left(\frac{1}{3}, 0\right) = \frac{1}{3}$$

$$d(x_4, 0) < 0.5$$

⋮

$$d(x_{21}, 0) = \frac{1}{21} < 0.05 = \frac{1}{20}$$

$$n > N(\epsilon)$$


Convergent sequences: Results

$$P \Rightarrow Q$$

Steps:

(1) Assume (x_n) converges to ℓ

Proposition 2.1

If a sequence (x_n) converges, then (x_n) is bounded.

Proof.

Assume $x_n \rightarrow \ell$. $\Rightarrow \forall \epsilon > 0, \exists N(\epsilon) \text{ s.t. } d(x_n, \ell) < \epsilon \forall n > N(\epsilon)$

WTS: (x_n) is bounded.

- Take $\epsilon = 1$

- ✓ • Since $x_n \rightarrow \ell$, by definition there exists $N \in \mathbb{N}$ such that for all $n \geq N$ $d(x_n, \ell) < 1$.

This proves that the sequence starting at N is bounded, but we need to deal with the first $N - 1$ terms.

- Define $M \equiv \max\{d(x_1, \ell), \dots, d(x_{N-1}, \ell), 1\}$

- For all n , $d(x_n, \ell) \leq M$

$d(x_1, \ell), d(x_2, \ell)$

\vdots
 $d(x_{N-1}, \ell)$

$\{x_1, x_2, x_3, \dots, x_N, x_{N+1}, \dots\}$

$d(x_1, \ell) \quad d(x_2, \ell) \quad \dots \quad d(x_{N-1}, \ell) \quad d(x_N, \ell) \leq 1$

$\Rightarrow (x_n)$ is a bounded set!

□

Limit (if exists) must be unique

P \Rightarrow Q

Proposition 2.2

The limit of a sequence (x_n) is unique provided it exists i.e. if $x_n \rightarrow x$ and $x_n \rightarrow x'$, then $x = x'$

Proof.

We prove this by contradiction. Suppose $x \neq x'$. Thus it must be that $d(x, x') > 0$.

Consider $\varepsilon = \frac{d(x, x')}{2}$.

Because $x_n \rightarrow x$, there exists N such that $d(x_n, x) < \varepsilon$ for any $n > N$. Similarly, because $x_n \rightarrow x'$, there exists N' such that $d(x_n, x') < \varepsilon$ for any $n > N'$. Take $\hat{n} = \max\{N, N'\} + 1$ so that $\hat{n} > N$ and $\hat{n} > N'$. Then $d(x_{\hat{n}}, x) < \varepsilon$ and $d(x_{\hat{n}}, x') < \varepsilon$. Thus it must be that :

$$d(x, x_{\hat{n}}) + d(x_{\hat{n}}, x') < 2\varepsilon = d(x, x')$$

which contradicts the triangle inequality of d . Thus we have reached a logical contradiction and therefore $x = x'$. □

Steps:

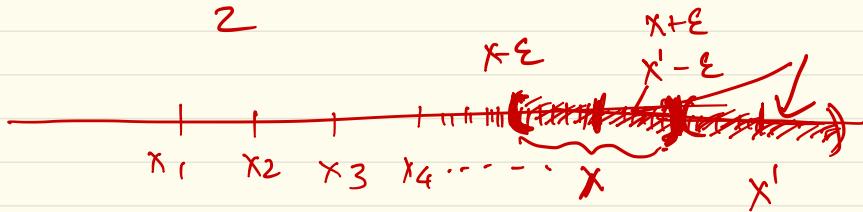
(1) Assuming the antecedent i.e. $x_n \rightarrow x$ AND $x_n \rightarrow x'$

Want to show:- $x = x'$

Proof by contradiction:-

(1) Assume the consequent is false :- $x \neq x'$
 $\Rightarrow d(x, x') > 0$

(2) $\epsilon = \frac{d(x, x')}{2}$



(3) Since $x_n \rightarrow x$, $\exists N_1$ s.t. $\forall n > N_1$
 $|d(x_n, x) < \epsilon|$

~~x~~

$x_n \rightarrow x' \Rightarrow \exists N_2$ s.t. $\forall n > N_2 \quad x_{101}$

$$d(x_n, x') < \epsilon$$

(4) Take $N = \max\{N_1, N_2\}$.

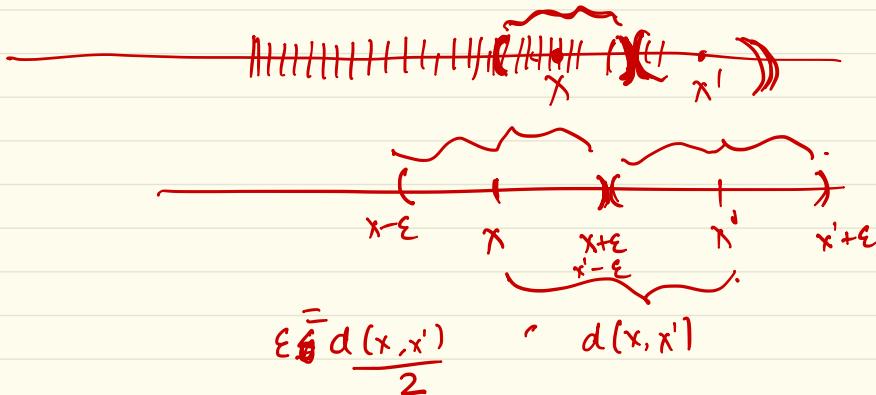
$\Rightarrow \forall n > N, \quad d(x_n, x) < \epsilon \text{ AND } d(x_n, x') < \epsilon$

$$d(x, x') \leq \underbrace{d(x, x_n)}_{\geq \varepsilon} + \underbrace{d(x_n, x')}_{\leq \varepsilon} \geq 2\varepsilon.$$

which contradicts the triangle inequality.

Hence our initial assumption is wrong & thus

$$x = x'$$



Order of Limits are preserved in sequences

$$d(x, y) = |x - y| \quad \text{in } \mathbb{R}. \quad \left. \begin{array}{l} \text{→ Euclidean metric.} \\ \end{array} \right.$$

Proposition 2.3

In the Euclidean metric space (\mathbb{R}, d) , suppose there are 2 convergent sequences $x_n \rightarrow x$ and $y_n \rightarrow y$. If $x_n \leq y_n \forall n$, then prove that $x \leq y$

Proof.

We do this by contradiction. Suppose $x > y$ and set $\varepsilon = \frac{x-y}{2}$. Since $x_n \rightarrow x$ and $y_n \rightarrow y$, then by definition, $\exists N_x$ and N_y such that $|x_n - x| < \varepsilon$ and $|y_n - y| < \varepsilon$ $\forall n > N_x$ and $n > N_y$. Take $\hat{n} > \max\{N_x, N_y\}$. Then we must have $|x_{\hat{n}} - x| < \varepsilon$ and $|y_{\hat{n}} - y| < \varepsilon$. Since $\varepsilon = \frac{x-y}{2}$, this implies $x - \varepsilon = y + \varepsilon$

$$x_{\hat{n}} > x - \varepsilon = y + \varepsilon > y_{\hat{n}}$$

which contradicts $x_n \leq y_n \forall n$

therefore:

Hence our initial assumption is wrong & $x \leq y$.

X If $x_n < y_n \forall n$ then $x < y$.

True or False?
False!!

Information :- $x_n \rightarrow x$ & $y_n \rightarrow y$.

$$(x_n) = \{x_1, x_2, \dots\}$$

$$(y_n) = \{y_1, y_2, \dots\}$$

$$\boxed{x_n \leq y_n \forall n}.$$

Step 1:- (Proof by contradiction) :-

• Contradiction
this.

Assume $x \neq y \Rightarrow x > y$



$$\underline{\text{Step 2}}: \epsilon = \frac{|x-y|}{2}$$

Step 3: $\exists N_x, N_y$ s.t. $d(x_n, x) = |x_n - x| < \epsilon$
 $\forall n > N_x$
AND

$$d(y_n, y) = |y_n - y| < \epsilon$$

$\forall n > N_y$

Step 4: $N = \max(N_x, N_y) \Rightarrow \forall n > N$

$$\begin{aligned} |x_n - x| &< \epsilon \quad \text{AND} \\ |y_n - y| &< \epsilon \end{aligned}$$

$$\boxed{|x_n - x| < \epsilon}$$

$$\begin{aligned} x_n - x &> -\epsilon \\ x_n - x &< \epsilon \end{aligned}$$

$$x_n > x - \epsilon = y + \epsilon > y_n.$$

$$\boxed{x_n > y_n} \quad \begin{aligned} &y_n - y < \epsilon \\ &\forall n > N \end{aligned}$$

$$|x_n - x| < \varepsilon$$

↓
 $x_n - x < \varepsilon$
 $x_n - x > -\varepsilon$

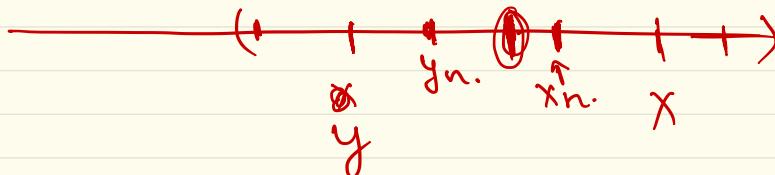
$$x_n - x > -\varepsilon$$

$$\Rightarrow x_n > x - \varepsilon$$

$$|y_n - y| < \varepsilon$$

$$\varepsilon = \frac{|x-y|}{2}$$

$$x - \varepsilon = y + \varepsilon$$



$$x_n > x - \varepsilon = y + \varepsilon > y_n$$

y_n

$x_n > y_n$

$$x_n = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \right\} \rightarrow 0$$

$$y_n = \left\{ 2, 1, \frac{1}{2}, \frac{1}{3}, \dots \right\} \rightarrow 0$$

$x_n < y_n \forall n$

Convergent sequences: results (cont.)

$$x^1 = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} - \text{first element of the sequence } x^k.$$

$x^k \in \mathbb{R}^n$

Here is a result you'll show on your problem set:

Proposition 2.4

A sequence $(x^k) = (x_1^k, \dots, x_n^k)$ of \mathbb{R}^n converges to a limit \underline{x} iff each component converges to the corresponding component of x in \mathbb{R} .

The result boils down to the fact that for all $j \in \{1, \dots, n\}$:

jth component of kth element in the sequence.

$$|x_j - x_j^k| \leq \left(\sum_{i=1}^n (x_j - x_j^i)^2 \right)^{\frac{1}{2}} \leq n \max_i |x_i - x_i^k|$$

jth component of limit

$$\text{When } n=1, \left(\sum_{i=1}^n (x_j - x_i^i)^2 \right)^{\frac{1}{2}} = |x_j - x_j^k| \stackrel{d(x_n, x)}{=} d(x_n, x)$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$$

$$(x_n) = \{x_1, x_2, x_3, \dots\} \quad x_i \in \mathbb{R}.$$

x_n could very well be a vector in \mathbb{R}^m .

$$x_n = \left\{ \begin{array}{c} 1 \quad 2 \quad \dots \\ \left(\begin{array}{c} x_1^1 \\ x_1^2 \\ \vdots \\ x_1^m \end{array} \right) + \left(\begin{array}{c} x_2^1 \\ x_2^2 \\ \vdots \\ x_2^m \end{array} \right) + \dots + \left(\begin{array}{c} x_j^1 \\ x_j^2 \\ \vdots \\ x_j^m \end{array} \right)^k \end{array} \right\}$$

$$x \in \mathbb{R}^m$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}$$

$$x_1^1, x_2^1, x_3^1, \dots \rightarrow x_1$$

$$\text{Example: } x_k \in \mathbb{R}^2$$

$$x_k = \left\{ \underbrace{\left(\begin{array}{c} \frac{1}{2} \\ 2 \end{array} \right)}, \underbrace{\left(\begin{array}{c} \frac{1}{2} \\ 1 \end{array} \right)}, \underbrace{\left(\begin{array}{c} \frac{1}{3} \\ \frac{1}{2} \end{array} \right)}, \dots \rightarrow \bar{x} \right\} \quad \bar{x} \in \mathbb{R}^2$$

$$x_k^1 = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \right\} \rightarrow x_1 \rightarrow \bar{x}_1 = \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix}$$

$$x_k^2 = \left\{ 2, 1, \frac{1}{2}, \dots \right\} \rightarrow x_2 \rightarrow \bar{x}_2 = \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix}$$

$\Rightarrow x^k \rightarrow x$. This means $\forall \epsilon > 0, \exists N$
 s.t. $d(x^n, x) \leq \epsilon \text{ if } n > N.$

$$\sqrt{\sum (x_i^k - x_i)^2} \rightarrow$$

WTS: $x_j^k \rightarrow x_j$. What we have to prove:-

$\forall \epsilon > 0, \exists N_i$ s.t. $d(x_i^k, x_i) \leq \epsilon \text{ if } k > N_i$

$$|x_i^k - x_i|$$

$$x^k = \left\langle x^1, x^2, x^3, \dots, x^k, \dots \right\rangle \quad x \in \mathbb{R}^n.$$

$$x^k = \begin{pmatrix} x_1^k \\ x_2^k \\ \vdots \\ x_j^k \\ \vdots \\ x_n^k \end{pmatrix} = \begin{pmatrix} (x_1^k - x_1)^2 \\ (x_2^k - x_2)^2 \\ \vdots \\ \vdots \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$|x_j^k - x_j| = |x_j -$$

Convergent sequences: results (cont.)

In general, working with the definition of convergence is cumbersome. There are some important results we'll use frequently

Proposition 2.5

Let (x_n) and (y_n) be sequences of \mathbb{R} . If $(x_n) \rightarrow x$ and $(y_n) \rightarrow y$:

- (a) $x_n + y_n \rightarrow x + y$
- (b) $x_n y_n \rightarrow xy$
- (c) $1/x_n \rightarrow 1/x$ if $x \neq 0$

$$\begin{aligned}x_n &= (x_1, x_2, x_3, \dots) \rightarrow x \\y_n &= (y_1, y_2, \dots) \rightarrow y \\ \Rightarrow x_n + y_n &= (x_1 + y_1, x_2 + y_2, \dots) \rightarrow x + y\end{aligned}$$

Proof.

We'll show the second one and the rest will probably be on your problem set. \square

Proof of Property (b)

Proposition 2.6

Let (x_n) and (y_n) be sequences of \mathbb{R} . If $(x_n) \rightarrow x$ and $(y_n) \rightarrow y$ then $x_n y_n \rightarrow xy$

Proof.

Take any $\varepsilon > 0$. I want to find N such that $|x_n y_n - xy| < \varepsilon$ for any $n > N$.

Because (y_n) is convergent, it is bounded i.e. there exists M such that $|y_n| < M \forall n$.

Because $x_n \rightarrow x$, there exists N_x s.t $|x_n - x| < \frac{\varepsilon}{2M}$. Again since $y_n \rightarrow y$, there exists N_y such that $|y_n - y| < \frac{\varepsilon}{2(|x|+1)}$.

$$|y_n - 0| < M$$

$$|y_n|$$

Let $N = \max\{N_x, N_y\}$ and I claim this is the N we need to find. This is because for any $n > N$,

$$\begin{aligned} |x_n y_n - xy| &= |(x_n - x)y_n + (y_n - y)x| \\ &\leq |x_n - x| |y_n| + |y_n - y| |x| \\ &< \frac{\varepsilon}{2M} \cdot M + \frac{\varepsilon}{2(|x|+1)} \cdot |x| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

$$\begin{aligned} |x_n - x| |y_n| &< \frac{\varepsilon}{2} \\ |y_n - y| |x| &< \frac{\varepsilon}{2} \\ \hline &< \varepsilon \end{aligned}$$

$$\tilde{\varepsilon} = \frac{\varepsilon}{2M}$$

$$|y_n| < M$$

□

$$|x+y| \leq |x| + |y| \quad |xy| = |x||y|$$

$$x_n y_n \rightarrow xy$$

WTS: $\forall \epsilon > 0, \exists N(\epsilon) \text{ s.t.}$

$$|x_n y_n - xy| < \epsilon \quad \forall n > N(\epsilon)$$

$$\begin{aligned} |x_n y_n - xy| &= |x_n y_n - x y_n + x y_n - xy| \\ &= |(x_n - x)y_n + (y_n - y)x| \\ &\leq |(x_n - x)y_n| + |(y_n - y)x| \\ &\quad (\text{By triangle inequality}) \\ &= \boxed{|x_n - x| |y_n| + |y_n - y| |x|} < \epsilon \end{aligned}$$

$$\frac{\epsilon}{2(|x|+1)} \text{ is a constant}$$

Subsequences

Definition 2.3

Let (x_n) be a sequence. A subsequence of (x_n) is a sequence (x_{n_k}) where $n_1 < n_2 < \dots$ is an increasing sequence of indices.

- (x_2, x_3, x_5, \dots) is a subsequence of (x_n) is a valid subsequence.
- (x_4, x_3, x_2, \dots) is not (the terms are out of order). NOT VALID!

Proposition 2.7

A sequence (x_n) converges to a limit ℓ iff all its subsequences converge to the same limit ℓ .

Proof.

(\Rightarrow) Assume $(x_n) \rightarrow \ell$ and consider a subsequence (x_{n_k}) of (x_n) . Fix $\epsilon > 0$. There exists $N \in \mathbb{N}$ such that for all $n \geq N$, $d(x_n, \ell) < \epsilon$. Take K such that $n_K \geq N$. Then for all $k \geq K$, $n_k \geq n_K \geq N$, so $d(x_{n_k}, \ell) < \epsilon$.

(\Leftarrow) Since (x_n) is a subsequence of itself, this implication is immediate. □

\Leftarrow

If all subsequences of a sequence (x_n) converge to l , then the sequence $(x_n) \rightarrow l$

Proof:-

\Rightarrow If a sequence $(x_n) \rightarrow l$, then
all of its subsequences converge to l .

Proof:-

WTS: Any subsequence $(x_{n_k}) \rightarrow l$.

WTS: $\forall \epsilon > 0, \exists N_K$ s.t. $d(x_{n_k}, l) < \epsilon$
 $\forall n > N_k$

Info:- ~~x_n~~ $(x_n) \rightarrow l$.

$\Rightarrow \exists \underline{N}$ s.t. $\forall n > N, d(x_n, l) < \epsilon$

choose an index n_k s.t. $\boxed{n_k > N}$

$\Rightarrow \forall n_k > n_k > N \Rightarrow d(x_{n_k}, l) < \epsilon$

$$x_n = \{x_1, x_2, x_3, x_4, x_5\}$$

$$x_{n_k} = \{x_2, x_5, x_6, \dots\}$$

$$N = 100$$

$$d(x_n, l) < \varepsilon \quad \forall n > 100$$

$$x_{101}, x_{102}, \dots$$

$$(x_{n_k}) = \{x_{n_1}, x_{n_2}, x_{n_3}, x_{n_4}\}$$

$$x_{n_1} = x_{10}$$

$$x_{n_2} = x_{15}$$

$$x_{n_3} = \cancel{x_{101}}$$

$$n_k > 100$$

$$n_4, n_5, n_6, \dots \quad d(x_{n_k}, l) < \varepsilon -$$

~~$n_k > n_{100}$~~

Subsequences (cont.)

Proposition 2.8

Bolzano-Weierstrass Theorem Every bounded sequence of real numbers has a convergent subsequence.

Proof.

To do this, we will first prove :

- **Monotone Convergence Theorem** : Every increasing (decreasing) and bounded from above (below) real sequence is convergent in $(\mathbb{R}, \|\cdot\|)$ ✓
- Every sequence in \mathbb{R} has a monotone subsequence. ✓

Combining these 2 results, we get that the Bolzano-Weierstrass Theorem holds. □

-
- (1) *Result 2 : Argue that every sequence has a monotone subsequence.*
- (2) *Use that monotone subsequence & Result 1 (MCT) to finish proof*

Monotone Convergence Theorem

Proposition 2.9

Monotone Convergence Theorem Every increasing (decreasing) and bounded from above (below) real sequence is convergent in (\mathbb{R}, d)

Proof.

Take any increasing and bounded real sequence (x_n) . By the least upper bound property of \mathbb{R} , it has a l.u.b.

Let $x := \sup x_1, x_2, \dots$. We want to show $x_n \rightarrow x$. Take any $\varepsilon > 0$. Want to find N s.t. $d(x_n, x) < \varepsilon \forall n > N$. Since x is the lub $\Rightarrow x - \varepsilon$ is not an upper bound and therefore $\exists N$ s.t. $x_N > x - \varepsilon$. Therefore $\forall n > N$, it must be that :

$$x \geq x_n \geq x_N > x - \varepsilon$$

$$x > x_n > x_N > x - \varepsilon.$$

and therefore $|x_n - x| < \varepsilon$

The proof for a decreasing and bounded from below sequence (x_n) is completed by applying the proof above to $(-x_n)$. □

$2 \notin (0, 2)$ $(0, 2)$ Is there a maximum? No!
2 is the supremum.

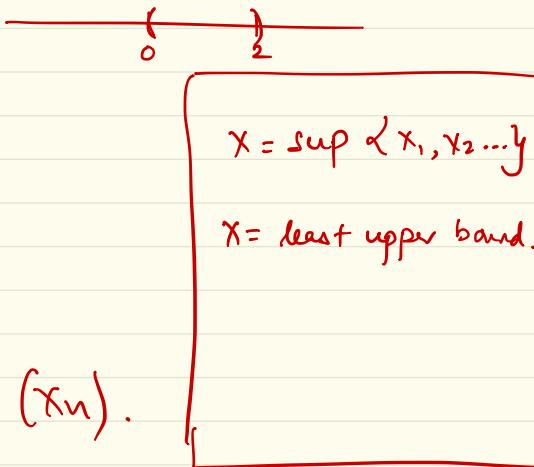
$$x_n = \{1, 2, 3, 4, 5, 6, 7, \dots\} \checkmark$$

Increasing: $x_{n+1} > x_n$

$$x_6 < x_5$$

$$1 < 5$$

x : is the lub of (x_n) .



$$\Rightarrow \exists N. \quad x_N > x - \varepsilon$$

(x_n) is an increasing sequence.

$$\Rightarrow x_n > x - \varepsilon \quad \forall n > N$$

$$\Rightarrow |x_n - x| < \varepsilon \quad \forall n > N.$$

Every sequence in \mathbb{R} has a monotone subsequence

$$(x_n) = \{ 1, 2, 3, 5, 4, 1, 1, 1, \dots \}$$

↓
dominant?
No!
 $x_1 > x_m \forall m \geq 1$
 $x_4 = 5 > x_m \forall m \geq n$

Proposition 2.10

Every sequence in \mathbb{R} has a monotone subsequence

Proof.

Take any sequence (x_n) in \mathbb{R} . Call the term x_n a dominant term if $x_n > x_m$ for all $m > n$.

- **Case 1 :** (x_n) has infinitely many dominant terms. Then these dominant terms constitute a decreasing sequence Example: $(x_n) = (\underbrace{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots})$
- **Case 2 :** (x_n) has finitely many dominant terms. Let x_N be the last dominant term. Let $n_1 = N + 1$. So x_{n_1} is not a dominant term. This means there exists $n_2 > n_1$ s.t. $x_{n_2} \geq x_{n_1}$. Again x_{n_2} is not dominant which implies there exists n_3 st $x_{n_3} \geq x_{n_2}$ and so on. We thus obtain a strictly increasing sequence weakly.
- **Case 3 :** (x_n) has no dominant terms. We can let $n_1 = 1$ and then we're back in the previous case.

Example: $x_n = \{ 1, 2, 3, 5, 4, 1, 2, 3, 1, 2, 3, 1, 2, 3, \dots \}$

$x_5 = 4$ is the last dominant term. $n_1 = 6$, x_6

□

$$x_{n_k} = \{ 1, 2, 3, 3, \dots \}$$

x_6 x_2

Cauchy Sequences

- When a sequence converges, its terms become closer and closer to the limit as n increases. As a by-product, the terms themselves become closer and closer together.
- The notion of a Cauchy sequence weakens the requirement of convergence by only requiring that the terms get closer together
 - Without necessarily converging to a limit.

Cauchy Sequences (contd)

$$a_n = \sum_{k=1}^n \frac{1}{k^2}$$

$$a_1 = 1$$

$$a_2 = 1 + \frac{1}{4}$$

$$a_3 = 1 + \frac{1}{4} + \frac{1}{9}$$

- How do we actually know when a sequence converges? Do we have to calculate the limit?

$$a_n = \sum_{k=1}^n \frac{1}{k^2}$$

- Turns out the right notion is that of **Cauchy sequences**
- In many situations, we will show that Cauchy sequences are the same as convergent sequences which will help us show that a sequence is convergent without specifying the limit

Cauchy Sequences (cont.)

Definition 2.4

A sequence (x_n) is **Cauchy** iff:

Convergent seq:-
 $\forall \epsilon > 0, \exists N(\epsilon) \in \mathbb{N} \text{ such that } \forall n, m \geq N(\epsilon), d(x_n, x_m) < \epsilon$

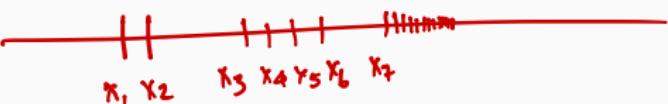
$$\forall \epsilon > 0, \exists N(\epsilon) \in \mathbb{N} \text{ such that } \boxed{\forall n, m \geq N(\epsilon), d(x_n, x_m) < \epsilon}$$

- Not enough for subsequent terms to be close: **all terms after a certain point must be close.**
- Let's show (x_n) on the previous slide is Cauchy. Fix $\epsilon > 0$, and take $N > 1/\epsilon$:

$$x_n = \sum_{k=1}^n \frac{1}{k^2}$$
$$x_m = \sum_{k=1}^m \frac{1}{k^2}$$

$$\begin{aligned} d(x_n, x_m) &= \sum_{k=n+1}^m \frac{1}{k^2} &< \sum_{k=n+1}^m \frac{1}{k(k-1)} \\ &= \sum_{k=n+1}^m \frac{1}{k-1} - \frac{1}{k} \\ &= \frac{1}{n} - \frac{1}{m} < \epsilon \end{aligned}$$

$$\begin{aligned} \Rightarrow \epsilon &> \frac{1}{N} \\ \Rightarrow n &> N \\ \Rightarrow \frac{1}{n} &< \epsilon \\ \Rightarrow \frac{1}{n} - \frac{1}{m} &< \epsilon \end{aligned}$$



$$x_n = \sum_{k=1}^n \frac{1}{k^2}$$

$$x_m = \sum_{k=1}^m \frac{1}{k^2}$$

Suppose:-
 $m > n$

$$d(x_n, x_m) = \left| \sum_{k=1}^m \frac{1}{k^2} - \left(\sum_{k=1}^n \frac{1}{k^2} \right) \right|$$

$$= \sum_{k=n+1}^m \frac{1}{k^2} \approx \int_1^\infty \frac{1}{x^2} dx$$

$$< \sum_{k=n+1}^m \frac{1}{k(k-1)} \cdot k < \epsilon$$

$$= \sum_{k=n+1}^m \left(\frac{1}{k-1} - \frac{1}{k} \right)$$

$$= \left(\frac{1}{n} - \cancel{\frac{1}{n+1}} + \cancel{\frac{1}{n+1}} - \cancel{\frac{1}{n+2}} \dots \right.$$

$$\left. + \cancel{\frac{1}{m-1}} - \frac{1}{m} \right)$$

$$= \left[\left(\frac{1}{n} - \frac{1}{m} \right) \right] \leq \epsilon$$

Cauchy sequences are bounded

$$d(x_n, \ell) < M$$

Proposition 2.11

If (x_n) is Cauchy, it is bounded

Proof.

As in a proof before, let us take $\varepsilon = 1$. Since (x_n) is Cauchy, there exists N such that $d(x_n, x_m) < 1$ for all $n, m \geq N$. In particular, $d(x_n, x_N) < 1$, so $\{x_n | n \geq N\}$ is bounded.

$$d(x_n, x_N) \leq 1 \quad \forall n \geq N$$

Since $\{x_n | n < N\}$ is also bounded, the whole sequence is bounded. \square

Question: Is the reverse implication true? Counter-example?

Is it true that
if x_n is bounded
it is Cauchy?

$$M = \max \{d(x_1, x_N), d(x_2, x_N), \dots, 1\}$$

Counter Example: $x_n = \{0, 1, 0, 1, 0, 1, \dots\}$ This is bounded!
 $M = 2 \quad d(x_n, 0) \leq 2 \quad \forall x_n$. But not Cauchy!!
Any $\varepsilon < 1 \Rightarrow \nexists N$ s.t. $d(x_n, x_m) < \varepsilon$

Convergence of a subsequence implies convergence

Proposition 2.12

If a subsequence of a Cauchy sequence converges to ℓ , the sequence itself converges to ℓ .

Proof.

Let (x_n) be Cauchy, and suppose $x_{n_k} \rightarrow \ell$. Fix $\epsilon > 0$. Since (x_n) is Cauchy, there exists an N such that

$$d(x_n, x_m) < \epsilon/2, \forall n, m \geq N$$

Since $(x_{n_k}) \rightarrow \ell$, there exists K such that

$$d(x_{n_k}, \ell) < \epsilon/2, \forall k \geq K$$

Take k such that $N_k \geq \max(N, n_K)$. Then for all $n \geq N_k$:

$$d(x_n, \ell) \leq d(x_n, x_{n_k}) + d(x_{n_k}, \ell) < \epsilon/2 + \epsilon/2 = \underline{\epsilon}$$

Thus $x_n \rightarrow \ell$.

$$d(x_n, \ell) < \epsilon - \text{Want to show.}$$

$$\begin{aligned} d(x_n, \ell) &< d(x_n, x_{n_k}) + d(x_{n_k}, \ell) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \underline{\epsilon}. \end{aligned}$$

□

Info :-

(x_n) is Cauchy.
 $\exists (x_{n_k}) \rightarrow l$

WTS: $(x_n) \rightarrow l$

$\forall \epsilon > 0, \quad \exists N \text{ s.t. } d(x_{n_l}, l) < \epsilon \quad \forall n > N$

Do Cauchy sequences converge?

Theorem 2.1

In \mathbb{R}^n a sequence converges iff it is Cauchy.

Proof.

(\Rightarrow) Suppose $x_n \rightarrow l$. Let $\epsilon > 0$. There exists N such that for all $n \geq N$ $d(x_n, l) < \epsilon/2$.

Let $n, m \geq N$. By the triangle inequality,

$$d(x_n, x_m) \leq d(x_n, l) + d(l, x_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$\underbrace{\quad}_{n, m > N} \quad \underbrace{\quad}_{l}$

$\Rightarrow d(x_n, x_m) < \epsilon$
 $\Rightarrow x_n$ is Cauchy.

(\Leftarrow) Since Cauchy sequences are bounded, by Prop. 2.8, (x_n) has a convergent subsequence. From the result above, (x_n) converges. \square

Definition 2.5

Spaces in which Cauchy sequences converge are called **complete metric spaces**

Can you give an example of a Cauchy sequence which does not converge? (**Hint:** Impossible in \mathbb{R} .)

Do Cauchy sequences converge?

Theorem 2.1

In \mathbb{R}^n a sequence converges iff it is Cauchy.

Proof.

(\Rightarrow) Suppose $x_n \rightarrow \ell$. Let $\epsilon > 0$. There exists N such that for all $n \geq N$, $d(x_n, \ell) < \epsilon/2$. Let $n, m \geq N$. By the triangle inequality,

$$d(x_n, x_m) \leq d(x_n, \ell) + d(\ell, x_m) < \epsilon/2 + \epsilon/2 = \epsilon$$

(\Leftarrow) Since Cauchy sequences are bounded, by Prop. 2.8, (x_n) has a convergent subsequence. From the result above, (x_n) converges. □

Definition 2.5

Spaces in which Cauchy sequences converge are called **complete metric spaces**

Can you give an example of a Cauchy sequence which does not converge? (**Hint:** Impossible in \mathbb{R} .)

Consider the rational numbers \mathbb{Q} and the sequence given by :

$a_n \in \mathbb{Q}$
But $\ell \notin \mathbb{Q}$

$$a_n = \left(1 + \frac{1}{n}\right)^n$$
$$\lim_{n \rightarrow \infty} a_n = e$$

$a_n \rightarrow e$
 a_n is Cauchy. But
doesn't converge within \mathbb{Q}

$e = 2.718\cdots$

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Open and Closed Sets

- Most of economics deals with sets: budget sets, indifference curves, production sets, the set of unbiased estimators, etc. It's useful to have well-defined language to describe the structure of sets.
- In addition, many common concepts and results (limits, continuity, existence of maxima, etc.) rely on the notion of open and closed sets.
- As a silly example consider the sets $S_1 = [0, 1]$ and $S_2 = (0, 1)$. The points 0 and 1 seem special: they are on the “edge” of both sets. The difference between S_1 and S_2 is whether they contain these points on the edge (turns out to be an important distinction!)

Open Balls

Definition 3.1

In \mathbb{R}^n , an open ball with center a and radius $r > 0$ is the subset of all points at a distance less than r from a :

$$B(a, r) = \{x \in \mathbb{R}^n \mid d(x, a) < r\}$$

$a \in \mathbb{R}^n$

Two things reqd
to define an open
ball :-
(1) $a \rightarrow$ center
(2) $r \rightarrow$ radius

Note: Even though I'm writing $d(x, a)$, I mean the Euclidean metric.

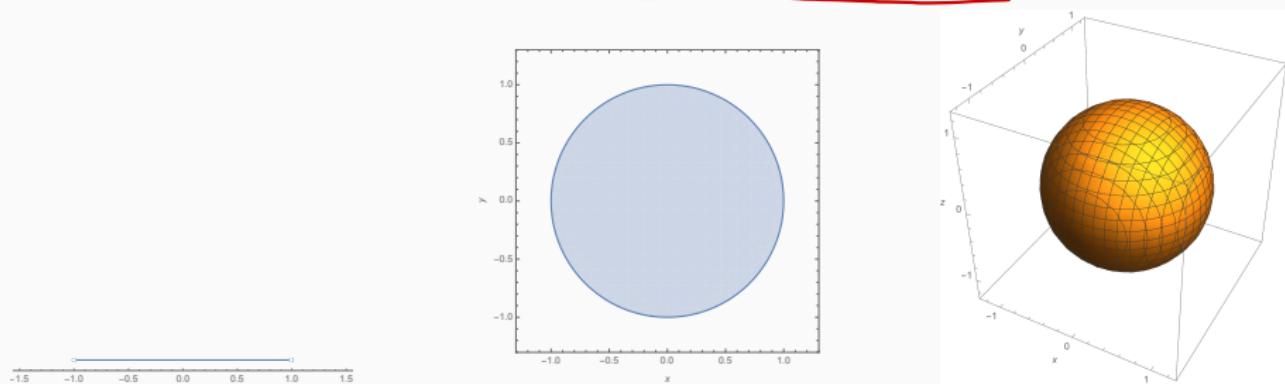


Figure 1: Open balls in \mathbb{R}, \mathbb{R}^2 , and \mathbb{R}^3

Euclidean metric :- $d(x, y) = \left(\sum_{i=1}^n (x_i - y_i)^2 \right)^{\frac{1}{2}}$

Open Balls

$$B(0, 1)$$

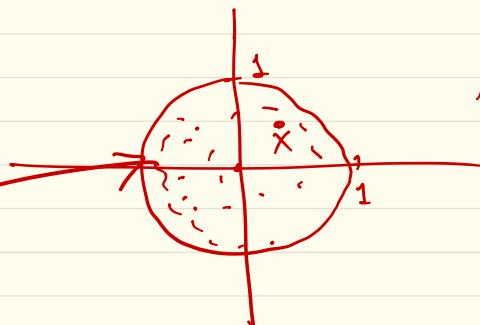
$0 \downarrow$

$$\mathbb{R}^1$$



Open ball is an interval on the real line.

$$\mathbb{R}^2$$



$$\boxed{d(0, x) < 1}$$

$B(0, 1)$ is a circle

$$\begin{aligned} B(0, 1) &= \{x \in \mathbb{R}^2 \mid d(x, 0) < 1\} \\ &= \{x \in \mathbb{R}^2 \mid ((x_1 - 0)^2 + (x_2 - 0)^2)^{\frac{1}{2}} < 1\} \end{aligned}$$

$$= \{x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < 1\}$$

Open Sets: Definition



Example:



x is an interior point because
 $\exists r > 0$ s.t. $B(x, r) \subseteq (-1, 1)$

Definition 3.2

Let S be a subset of \mathbb{R}^n

- A point $a \in \mathbb{R}^n$ is an **interior point** of S if $\exists r > 0$ such that $B(a, r) \subseteq S$.
- The **interior** of S , noted $\text{int}(S)$ is the set of all its interior points.
- A subset S is an **open set** if all its points are interior points of S , i.e. if $\text{int}(S) = S$.
Because an interior point of S always belongs to S , $\text{int}(S) \subseteq S$.
To show a set is open, take an arbitrary point $s \in S$ and find a radius such that $B(s, r) \subseteq S$.
 $\text{AND } S \subseteq \text{int}(S)$

(1) $S = (0, 1)$. This is an open set. Why?

~~if $x \in S$~~ x is outside $[0, 1]$.

Take any $x \in (0, 1)$, you can find

$r > 0$ s.t. $B(x, r) \subseteq (0, 1)$. All points are

(2) $S = [0, 1]$. Is the point 0 an interior point? No!
You can't find $r > 0$ s.t. $B(0, r) \subseteq [0, 1]$.

Open Sets: Results

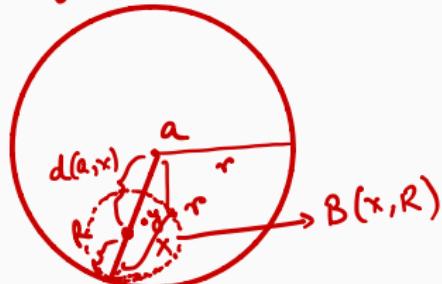
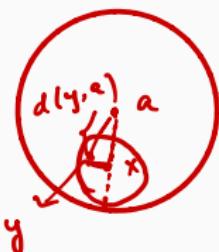
Need to show $d(y, a) < r \Rightarrow y \in B(a, r)$

Proposition 3.1

Any open ball is an open set.

Proof.

Terminology is not a proof!



Consider an open ball $B(a, r)$. We will show every point is interior.

- Let $x \in B(a, r)$. We need to find a radius $R > 0$ such that $B(x, R) \subseteq B(a, r)$
- Define $R = r - d(a, x) > 0$
- Using the triangle inequality, for any $y \in B(x, R)$:

$$d(y, a) \leq d(y, x) + d(x, a) < r.$$

\downarrow \downarrow
 $< R$ $r - R$

How do we verify that
 $B(x, R) \subseteq B(a, r)$
 $y \in B(x, R)$ \square
 $\Rightarrow y \in B(a, r)$

To show!!

$$\Leftrightarrow d(a, y) < r$$

Open Sets: Results (cont.)

$$\{S_i\} = \{S_1, S_2, \dots\}$$

Proposition 3.2

- Any union (possibly infinite) of open sets is an open set.
- Any finite intersection of open sets is an open set.

Proof.

We will show the first result.

- Let $\{S_i\}$ be a family of open sets and $S \equiv \bigcup_i S_i$. Need to show $S = \text{int}(S)$.
- For any $x \in S$, $x \in S_i$ for some i .
- Since S_i is open, there exists $r > 0$ such that $B(x, r) \subseteq S_i$.
- But $S_i \subseteq S$, so $B(x, r) \subseteq S$, so $x \in \text{int}(S)$.

$$B(x, r) \subseteq S_i \subseteq S \Rightarrow B(x, r) \subseteq S \\ \Rightarrow x \in \text{int}(S)$$

$\text{int}(S) \subseteq S$ AND

$S \subseteq \text{int}(S)$

↓
Need to show

we showed this.

□

Closed Sets: Definition

Example :-

Qn :- Does $1 \in (-1, 1)$
Is 1 a closure point?

Yes! Because $\forall r > 0$, $B(1, r) \cap S \neq \emptyset$



$$S = (-1, 1)$$

Qn: Is 2 a closure point of S ?

No! Take any $r < 1$
 $\text{So } B(2, r) \cap S = \emptyset$

Definition 3.3

Let S be a subset of \mathbb{R}^n .

- A point $a \in \mathbb{R}^n$ is a **closure point** of S iff for any radius $r > 0$, the open ball $B(a, r)$ around a contains some point of S . In other words, for all $r > 0$, $B(a, r) \cap S \neq \emptyset$
- The **closure** of a subset S , $\text{cl}(S)$, is the set of all its closure points.
- A subset S is a **closed set** iff $\text{cl}(S) = S$.

A point that belongs to S is always a closure point of S , so $S \subseteq \text{cl}(S)$.

So to show a set is closed, we need to show that

$$\text{cl}(S) \subseteq S.$$

Links between open and closed sets

$$S = [0, 1)$$



$$(int(S))^c = cl(S^c)$$

Proposition 3.3

- The complement of $int(S)$ is the closure of S^c : $(int(S))^c = cl(S^c)$.
- The complement of $cl(S)$ is the interior of S^c : $(cl(S))^c = int(S^c)$.

Proof.

We'll show the first result. The definition of $int(S)$ is:

$$a \in int(S) \Leftrightarrow \exists r > 0 : B(a, r) \subseteq S$$

Take the negation of each side of the equivalence:

$$a \in (int(S))^c \Leftrightarrow \forall r > 0 : B(a, r) \cap S^c \neq \emptyset$$

But the right-hand side is precisely the definition of belonging to $cl(S^c)$!

□

Links between open and closed sets (cont.)

An important theorem follows directly from the previous result

Theorem 3.1

A set S is open (closed) iff its complement is closed (open).

Proof.

Applying the previous result, we see

$$\begin{aligned} S \text{ is open} &\Leftrightarrow \underline{\text{int}(S)} = S \text{ (defn)} & \underline{\text{int}(S)}^c &= \underline{S^c} \\ &\Leftrightarrow \underline{\text{cl}(S^c)} = S^c \text{ (previous result)} & S^c &= \underline{\text{cl}(S^c)} \\ &\Leftrightarrow \underline{S^c} \text{ is closed (defn)} & \end{aligned}$$

$\text{cl}(S^c)$
 $\text{cl}(S^c)^c = S^c$
 $S^c = \text{cl}(S^c)$



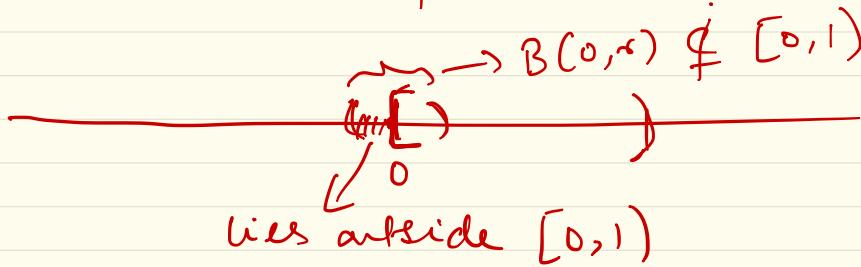
A similar proof shows S is closed iff S^c is open. □

Qn 1: Consider the metric space (\mathbb{R}, d) .

Is the set $S = [0, 1)$ open or closed?

Does $0 \in S$? Yes!

Is 0 an interior point? No!



So S is not open. You can S is not closed.

(Because 1 is a closure point but $\notin S$)

Qn 2: Consider the metric space (\mathbb{R}_+, d) .

Is the set $S = [0, 1)$ open or closed?



Qn 3: Suppose the metric space is

$$([0,1], d)$$

Is $[0,1]$ open or closed?

Does $1 \in$ to metric space?

S
It is both open & closed.

- ∅ (1) All points are interior points!
- (2) It contains ALL its closure points!

Closed Sets: Results

Proposition 3.4

- Any intersection (possibly infinite) of closed sets is a closed set.
- Any finite union of closed sets is a closed set.

Not infinite!!

Proof.

We'll show the first result:

- For a collection of closed sets $\{S_i\}$, $\cap_i S_i = (\cup_i S^c)^c$ by Morgan's laws.
- Since S_i is closed, S_i^c is open.
- By Proposition 3.2, $\cup_i S_i^c$ is open.
- Therefore $(\cup_i S_i^c)^c$ is closed, finishing the proof.



Finding the closure: sequential characterization

Proposition 3.5

Let $S \subseteq \mathbb{R}^n$. A point x is in $\text{cl}(S)$ iff there exists a sequence of elements of S that converges (in \mathbb{R}^n) to x .

Proof. Antecedent

(\Rightarrow) Assume $x \in \text{cl}(S)$. We will construct a sequence of S that converges to x .

- For each $n \in N$ take $x_n \in S \cap B(x, 1/n) \neq \emptyset$ (we can do this because x is a closure point of S)
- To check convergence, fix $\epsilon > 0$. For any $n \geq 1/\epsilon$, $d(x_n, x) < 1/n \leq \epsilon$.

$$x_n \in B\left(x, \frac{1}{n}\right)$$

(\Leftarrow) Assume there exists a sequence x_n of S converging to x .

- WTS: x is a closure point.
- Let $r > 0$. We need to show $B(x, r) \cap S \neq \emptyset$.
 - Since $x_n \rightarrow x$, there exists an N such that for all $n \geq N$, $x_n \in B(x, r)$. Since $x_n \in S$, we see $B(x, r) \cap S \neq \emptyset$

$$d(x_n, x) < r$$

$$B(x, r) = \{x_n \mid d(x_n, x) < r\}$$

□

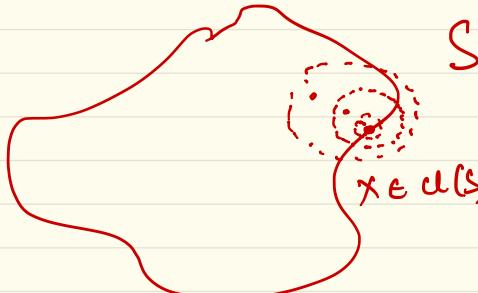
\Rightarrow Assume $x \in \text{cl}(S)$

WTS: \exists a sequence $(x_n) \subseteq S$ s.t. $\underline{x_n \rightarrow x}$

$\cancel{x \in \text{cl}(S)} \Rightarrow \forall \epsilon > 0, B(x, \epsilon) \cap S \neq \emptyset$

$\{x_1, x_2, x_3, \dots\} \rightarrow x$

Construction:- $\forall n \in \mathbb{N}, B(x, \frac{1}{n}) \cap S \neq \emptyset$
 \Rightarrow choose any point x_n in $B(x, \frac{1}{n}) \cap S$



Step 1: $B(x, 1)$. Choose x_1
 $B(x, \frac{1}{2})$. Choose x_2
⋮

$\forall n.$
 $\frac{1}{n} \rightarrow 0$

Finding the closure: discussion

- { 1) Closed set contains all its closure points i.e. $\text{cl}(S) = S$
- 2) If $x_n \rightarrow x$, then $x \in \text{cl}(S)$

The previous result is quite useful since it gives us another approach for determining whether a set is closed.

Corollary 3.1

A set S is closed iff the limits of all convergent sequences of S belong to S .

Example: Show the set $S = [a, \infty)$ is closed.

- Approach: Take a convergent sequence of S , $x_n \rightarrow x$. If $x \in S$, then S is closed.
- Since $x_n \geq a$ for all n , from our previous results we know $x \geq a$, so $x \in S$.

$$x_n \rightarrow x, y_n \rightarrow y$$

$$x_n > y_n \forall n$$

$$\Rightarrow x > y$$

$$a_n = a \quad \forall n$$

$$\Rightarrow a_n \rightarrow a$$

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Compact sets

Definition 4.1

A set $S \subseteq \mathbb{R}^n$ is **compact** iff every sequence of S has a convergent subsequence (in S).

Examples:

- \mathbb{R} ? not compact $x_n = \{1, 2, 3, 4, 5, 6, \dots\} \quad (x_n) \subseteq \mathbb{R}$
 - $[0, 1]$? compact
 - $(0, 1]$? not compact $x_n = \{\frac{1}{n}\} \quad \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$

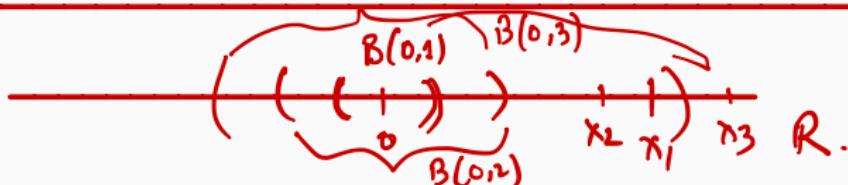
Proposition 4.1

If S is a compact subset of \mathbb{R}^n , S is bounded.

Proof.

We show the contrapositive. Assume S is not bounded. For each n , let $x_n \in B(0, n)^c$.

This sequence has no bounded subsequence, so it has no convergent subsequence.



Compact sets (cont.)

Steps:-

- (1) Assume S is compact
- (2) WTS:- S is closed.

If $x \in cl(S) \Rightarrow x \in S$.

Proposition 4.2

If S is a compact subset of \mathbb{R}^n , S is closed

Proof.

Let $x \in cl(S)$; we want to show $x \in S$.

- Since $x \in cl(S)$, it is the limit of a sequence $(x_n) \in S^{\mathbb{N}}$.
- Since S is compact, (x_n) has a subsequence (x_{n_k}) that converges to a limit $l \in S$.
- But (x_{n_k}) is a subsequence of the converging sequence (x_n) , so l is necessarily equal to x

Thus $x = l \in S$, so S is closed.

$$l \in S \Rightarrow x \in S$$

(1) Because $x \in cl(S)$
 $\Rightarrow \exists x_n \rightarrow x$

(2) S is compact, for x_n
 $\exists x_{n_k} \rightarrow l \in S$

Compact sets (cont.)

S is compact $\Rightarrow S$ is closed & bounded.

The two previous results have an important implication:

Corollary 4.1

Let S be a subset of \mathbb{R} . If S is compact, it has a maximal element.

Proof.

- Since S is compact, it is bounded, meaning it has a least upper bound s . ie $x \in S$ s.t $x \geq s$ $\forall x' \in S$
- By way of contradiction, suppose $s \notin S$. Then $s \in S^c$.
- Since S is closed, S^c is open, so there exists some $\epsilon > 0$ such that $B(s, \epsilon) \subseteq S^c$.
- This implies $s - \epsilon$ is an upper bound for S , contradicting the fact that s is the least upper bound.

Therefore we must have $s \in S$.



$S = (0, 1)$ has a supremum / lub which is $1 \notin S$
But no maximal element.

How can we tell whether a set is compact?

S is compact $\Rightarrow S$ is closed + bounded.
 \Leftarrow

We've seen that closed and bounded are necessary conditions. Are they sufficient as well?

Theorem 4.1

(Heine-Borel) Let $S \subseteq \mathbb{R}^n$. Then S is compact iff it is closed and bounded.

\Rightarrow already showed previously

Proof.

For simplicity we'll show this in \mathbb{R} ; the proof extends easily.

\Leftarrow

Assume S is closed and bounded, and let (x_n) be a sequence of S . By Proposition 2.8, (x_n) has a convergent subsequent, $x_{n_k} \rightarrow \ell$. Since S is closed, $\ell \in S$. Therefore S is compact. \square

WTS: S is closed + bounded $\Rightarrow S$ is compact.

S is closed.

Steps: (1) Take any $(x_n) \subseteq S \Rightarrow (x_n)$ is bounded

\Rightarrow if x_{n_k} is convergent (to some point)
 $\Rightarrow x_{n_k} \rightarrow x \in S$ because

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Limits of functions

$$x_n = \langle x_1, x_2, \dots \rangle$$
$$f(x_n) = \langle f(x_1), f(x_2), \dots \rangle$$

- A sequence (x_n) converges to ℓ if as n gets closer and closer to ∞ , x_n gets closer and closer to ℓ .
- Analogously, we say a function f converges to a limit ℓ at a point x_0 if as x gets closer and closer to x_0 , $f(x)$ gets closer and closer to ℓ .

Definition 5.1

Let $D \subseteq \mathbb{R}^n$ and let $f : D \rightarrow \mathbb{R}^m$ with $x_0 \in cl(D)$. We say $\lim_{x \rightarrow x_0} f(x) = \ell$ ("the limit as x approaches x_0 of $f(x)$ is ℓ ") if for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$0 < d(x, x_0) < \delta \Rightarrow d(f(x), \ell) < \epsilon$$

Example: $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = 3x$. What is $\lim_{x \rightarrow 0} f(x)$?

If x lies in a δ -ball around $x_0 \Rightarrow f(x)$ lies in an ϵ -ball around ℓ

Limit: Discussion

- Another way to put the ϵ, δ criteria is:

$$f(B(x_0, \delta) \setminus \{x_0\}) \subseteq B(\ell, \epsilon)$$

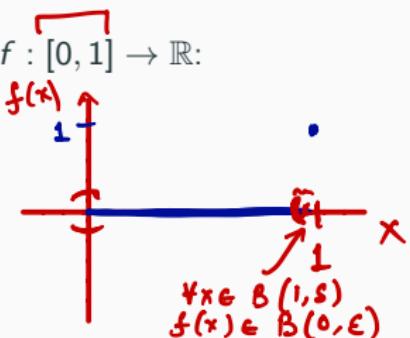
Consider the δ -ball around x_0 . All points in that (bearing x_0) are mapped into the ϵ -ball around ℓ .

(Be careful: there are two distances in this equation!)

- What happens when $x = x_0$ is not relevant. Consider $f : [0, 1] \rightarrow \mathbb{R}$:

$$f(x) = \begin{cases} 0 & x < 1 \\ 1 & x = 1 \end{cases}$$

What is $\lim_{x \rightarrow 1} f(x)$? $= 0$



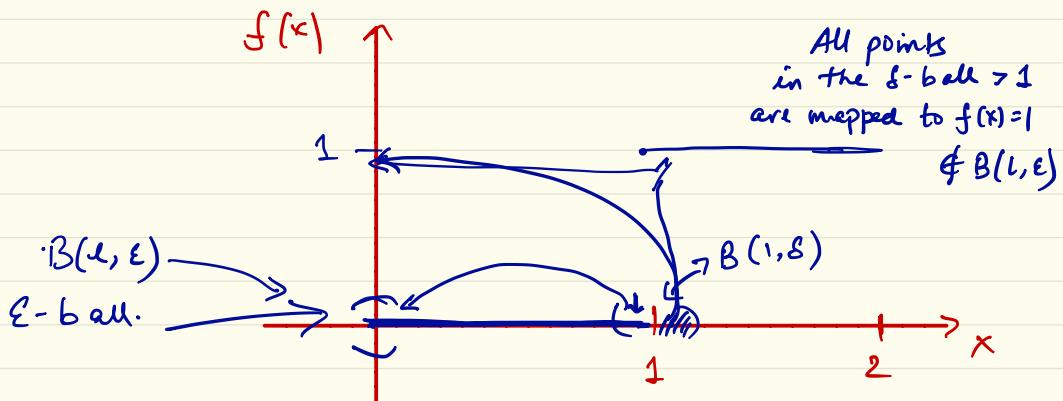
- We require $x_0 \in cl(D)$. For a function defined on $(0, 1)$ it makes sense to talk about its limit as x approaches 1. What about its limit as x approaches 2?

I claim 0 is the limit. I need to show $\forall \epsilon > 0, \exists \delta > 0$ s.t.
 $f(B(1, \delta) \setminus \{1\}) \subseteq B(0, \epsilon)$

$$f: [0, 2] \rightarrow \mathbb{R}$$

$$\begin{aligned} f(x) &= 0 && \text{if } x < 1 \\ &= 1 && \text{if } x \geq 1 \end{aligned}$$

Claim:- There does NOT exist any limit of f at $x=1$



Is $l=0$ a limit of $f(x)$ as $x \rightarrow 1$?

Take any ϵ -ball around $l=0$. If any δ -ball s.t. if $x \in B(1, \delta) \Rightarrow f(x) \in B(l, \epsilon)$

Continuity

Continuous functions are functions that preserve limits. Remember this!

Definition 5.2

Let $x_0 \in D$. Then f is continuous at x_0 iff $\lim_{x \rightarrow x_0} f(x) = f(x_0)$. That is, for all $\epsilon > 0$ there exists a $\delta > 0$ such that

$$x \in B(x_0, \delta) \Rightarrow f(x) \in B(f(x_0), \epsilon)$$

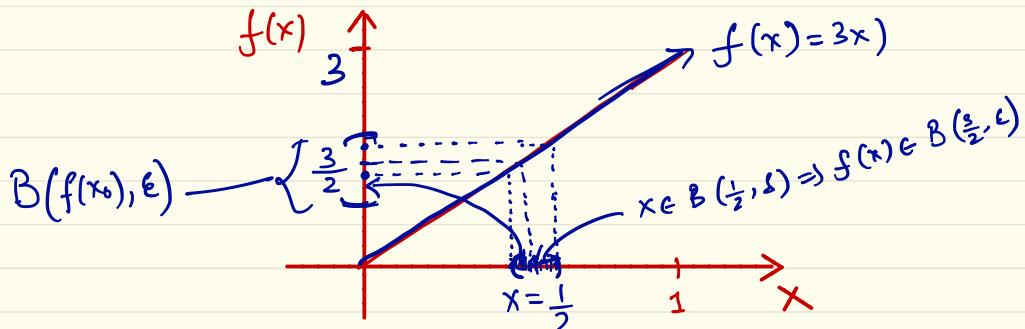
We say f is continuous if it is continuous at all points of its domain.

Most functions that we use in economics are continuous:

- Polynomials (x, x^2 , etc.)
- Exponential functions ($2^x, e^x$, etc.)
- The log function is continuous for $x > 0$
- Addition, multiplication, and composition of continuous functions
- Division of continuous functions, so long as the denominator $\neq 0$

Examples :- $f : [0, 1] \rightarrow \mathbb{R}$

(1) $f(x) = 3x$



Is $f(x)$ continuous at $x = \frac{1}{2}$?

$$f\left(\frac{1}{2}\right) = \frac{3}{2} .$$

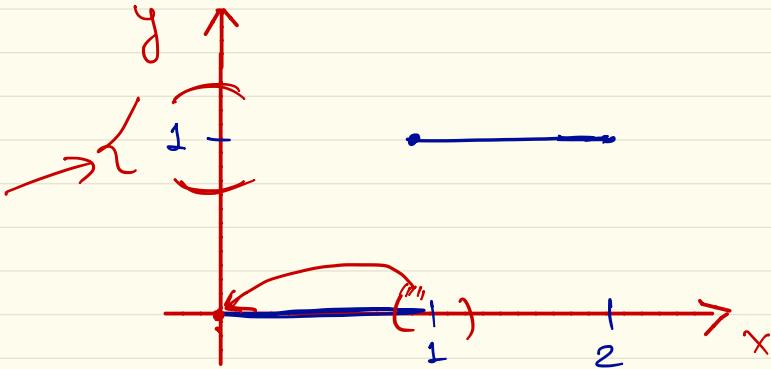
Steps :- (1) Take any ϵ -ball around $f(x_0)$

(2) Construct a δ -ball around $x_0 = \frac{1}{2}$

s.t. all points in the δ -ball are mapped into the ϵ -ball

Example :- $f: [0, 2] \rightarrow \mathbb{R}$.

$$\begin{aligned}f(x) &= 0 && \text{if } x < 1 \\&= 1 && \text{if } x \geq 1\end{aligned}$$



Is $f(x)$ continuous at $x = 1$?

No! : $f(1) = 1$. Consider any ϵ -ball around $f(1)$ such that $0 \in B(1, \epsilon)$.

Take any δ -ball around $x = 1$, it points to the left of 1 which are mapped to $f(x) = 0 \notin B(1, \epsilon)$.

Sequential characterization of continuity

If f is continuous at x , then $x_n \rightarrow x \Rightarrow (f(x_1), f(x_2), \dots) \rightarrow f(x)$

Proposition 5.1

A function f is continuous at x_0 iff for all $(x_n) \rightarrow x_0$, $(f(x_n)) \rightarrow f(x_0)$.

Proof.

(\Rightarrow). Suppose f is continuous at x_0 , and let (x_n) be a sequence of D that converges to x_0 . We want to show $f(x_n) \rightarrow f(x_0)$.

- Fix $\epsilon > 0$. Since f is continuous, there exists $\delta > 0$ such that $d(f(x_0), f(x_n)) < \epsilon$ whenever $x \in B(x_0, \delta)$.
- Since $x_n \rightarrow x_0$, there exists N such that $d(x_n, x_0) < \delta$ for all $n \geq N$.
- This implies $d(f(x_n), f(x_0)) < \epsilon$ for all $n \geq N$, so $f(x_n) \rightarrow f(x_0)$

(\Leftarrow) Suppose f is not continuous at x_0 . There exists ϵ such that for every $\delta > 0$, we can find $x \in B(x_0, \delta)$ with $f(x) \notin B(f(x_0), \epsilon)$. Construct (x_n) by taking $\delta = 1/n$ for $n = 1, 2, \dots$. We have $x_n \rightarrow x_0$, but $f(x_n) \not\rightarrow f(x_0)$. □

Open/closed set characterization of continuity

Does not mean that if S is open $\Rightarrow f(S)$ is open $\forall s \in S$

Theorem 5.1

- f is continuous iff the inverse image by f of any open set is open $\Leftrightarrow \boxed{f^{-1}(V)}$ is open
- f is continuous iff the inverse image by f of any closed set is closed

Proof.

(\Rightarrow) Let $f : D \rightarrow \mathbb{R}^m$ be continuous, $V \subseteq \mathbb{R}^m$ open. WTS $f^{-1}(V)$ open

Codomain.

- Let $x_0 \in f^{-1}(V)$; by definition, $f(x_0) \in V$
- Since V is open, there exists $\epsilon > 0$ such that $B(f(x_0), \epsilon) \subseteq V$. From defn of open set.
- Since f is continuous, there exists a $\delta > 0$ such that $f(B(x_0, \delta)) \subseteq B(f(x_0), \epsilon) \subseteq V$.
- Therefore $B(x_0, \delta) \subseteq f^{-1}(V)$, so $f^{-1}(V)$ is open

(\Leftarrow) : Try it yourself.

all points in the δ -ball around x_0 are mapped into ϵ -ball around $f(x_0)$ \square

Result for closed sets follows your problem set: $(f^{-1}(S))^c = f^{-1}(S^c)$

$f(x_0)$

Continuous functions preserve compact sets

- The inverse image of a closed set by a continuous function is a closed set. **True.**
- Is the image of a closed set by a continuous function a closed set? **False!!**
- Is the image of a bounded set by a continuous function bounded? **False!!**

(Find counterexample)

Theorem 5.2

Let K be a compact subset of D and $f : D \rightarrow \mathbb{R}^m$ a continuous function. Then $f(K)$ is compact.

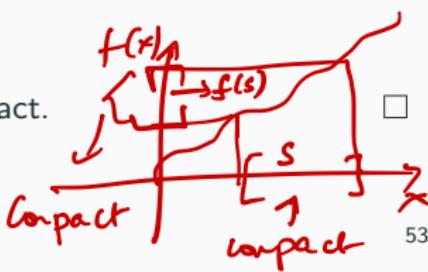
VERY IMPORTANT! A cts fn maps compact sets into compact sets.

Proof.

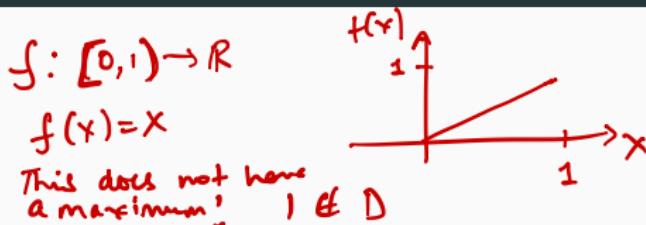
Let (y_n) be a sequence of $f(K)$.

- By definition there exists $x_n \in K$ such that $y_n = f(x_n)$.
- Since K is compact, (x_n) has a convergent subsequence $x_{n_k} \rightarrow l \in K$.
- Since f is continuous, $y_{n_k} \equiv f(x_{n_k}) \rightarrow f(l) \in f(K)$.

Thus $f(K)$ has a convergent subsequence, so $f(K)$ is compact. □



Weierstrass Theorem



A central application of the previous theorem is the existence of a maximizer for a continuous, real-valued function over a compact domain.

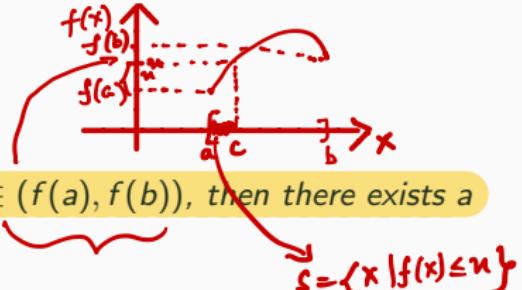
Theorem 5.3

(Weierstrass) Let $f : D \rightarrow \mathbb{R}$. If f is continuous and D is compact, then f attains a maximum and a minimum.

Proof.

By the previous theorem, $f(D)$ is a compact subset of \mathbb{R} . By Theorem 4.1 (Heine-Borel), $f(D)$ has a maximum element. □

Intermediate Value Theorem



Proposition 5.2

Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function. If $u \in (f(a), f(b))$, then there exists a $c \in (a, b)$ such that $f(c) = u$.

- If $f(a) = f(b)$ the theorem says nothing. Assume $f(a) < f(b)$
- Let $f(a) < u < f(b)$ and define $S = \{x \in [a, b] | f(x) \leq u\}$.
- S is non-empty and bounded above, so $c \equiv \sup S$ exists
- By continuity, $f(c) \leq u$. Suppose $f(c) < u$.
- Let $T = f^{-1}((f(a), u))$. Since f is continuous, T is open. By assumption, $c \in T$.
- However, c is not an interior point of T , a contradiction. Therefore $f(c) = u$.

WTS: $f(c) = u$

Questions:-

- (1) Qn 2 of Real Analysis
 (2) Qn 4 of Real Analysis

Qn 4 Is the sequence $a_n = \sum_{k=1}^n \frac{1}{2^k}$ Cauchy?

WTS: $\forall \epsilon > 0, \exists N$ s.t. $\forall m, n \geq N$

$$d(x_n, x_m) < \epsilon$$

Prof:- fix $\epsilon > 0$,

$$d(x_n, x_m) = \left| \sum_{k=1}^n \frac{1}{2^k} - \sum_{k=1}^m \frac{1}{2^k} \right|$$

Suppose $m > n$

$$= \sum_{k=n+1}^m \frac{1}{2^k}$$

$$= \left(\frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \dots \right)$$

$$= \frac{1}{2^n} \left[\frac{1}{2} + \frac{1}{2^2} + \dots \right]$$

$$= \frac{1}{2^n} \sum_{k=1}^{m-n} \frac{1}{2^k}$$

Want to find $N \in \mathbb{N}, n > N$ $d(x, x_m) < \varepsilon$

$$d(x_n, x_m) = \left| \frac{1}{2^n} \right| + \sum_{k=1}^{m-n} \frac{1}{2^k}$$

if n is large enough, then $\frac{1}{2^n} < \varepsilon$

$$< \frac{1}{2^n} + \sum_{k=1}^{\infty} \frac{1}{2^k}$$

We know $\sum_{k=1}^{\infty} \frac{1}{2^k} = 1$

$$= \frac{1}{2^n}$$

choose N s.t. $\frac{1}{2^N} < \varepsilon$ & we are done!

$$\Rightarrow \forall m, n > N \Rightarrow d(x_n, x_m) < \frac{1}{2^n} < \varepsilon$$

Qn 2: (x_n) is a seq in $\overset{\downarrow}{(\mathbb{R}^k, d)}$ which converges to $x \in \mathbb{R}^k$



(x_n^i) converges to x^i in \mathbb{R} $\forall i \in 1, 2, \dots k$

$$(x_n) = \left\{ x_1, x_2, \dots \right\} \rightarrow x \Rightarrow \begin{pmatrix} x^1 \\ \vdots \\ x^k \end{pmatrix}$$

$$= \left\{ \begin{pmatrix} x_1^1 \\ x_1^2 \\ \vdots \\ x_1^k \end{pmatrix}, \begin{pmatrix} x_2^1 \\ x_2^2 \\ \vdots \\ x_2^k \end{pmatrix}, \dots \right\}$$

\Rightarrow Info:- $x_n \rightarrow x \in \mathbb{R}^k$

$$\left\{ x_1^1, x_2^1, x_3^1, \dots \right\} \rightarrow x^1$$

\Rightarrow WTS. $x_n^i \rightarrow x^i \quad \forall i=1,2,\dots,k.$

$\forall \epsilon > 0$, we need to find N s.t.

$$\boxed{d(x_n^i, x^i) < \epsilon \quad \forall n > N}$$

Proof:- Take any $\epsilon > 0$.

Because $x_n \rightarrow x \Rightarrow \exists N$ s.t.

$$\boxed{d(x_n, x) < \epsilon \quad \forall n > N}$$

$\forall n > N$

$$d(x_n^i, x^i) = \sqrt{\sum_{i=1}^k (x_n^i - x^i)^2} \leq \sqrt{\sum_{i=1}^k (x_n^i - x^i)^2} = d(x_n, x) < \epsilon$$

We know $x_n^i \rightarrow x^i \forall i$
 WTS: $\exists x_n \rightarrow x \in \mathbb{R}^k$
 Fix $\epsilon > 0$
 we need N s.t.
 $d(x_n, x) < \epsilon$ $\forall n > N$.

$\exists N_i$ s.t $\forall n > N_i$,
 $|x_n^i - x^i| < \frac{\epsilon}{\sqrt{k}}$ $\tilde{\epsilon}$
 $N = \max \{N_1, \dots, N_k\}$

Now, $\forall n > N$,

$$\begin{aligned}
 d(x_n, x) &= \sqrt{\sum_{i=1}^k (x_n^i - x^i)^2} \\
 &< \sqrt{\sum_{i=1}^k \left(\frac{\epsilon}{\sqrt{k}}\right)^2} \\
 &= \sqrt{k \frac{\epsilon^2}{k}} \\
 &= \sqrt{k} \frac{\epsilon^2}{\sqrt{k}}
 \end{aligned}$$

$$\Rightarrow d(x_n, x) \stackrel{?}{=} \epsilon$$

Fixed Points

If $x=1$ is a fixed point
 $\Rightarrow f(1)=1$

$$f(x) = x^2$$

$x=1$ is a F.P. of
 $f(x) = x^2$

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$. A point x is a **fixed point** of f if $f(x) = x$

- Common equation in econ (game theory, dynamic programming)
- General setup: can express roots in terms of fixed points

$$g(x) = 0 \Leftrightarrow \underbrace{g(x) + x}_{{}=f(x)} = x$$

Question: how do we find fixed points? For a certain class of functions, we can compute them easily

↓
Contractions

Contractions

$f: \mathbb{R} \rightarrow \mathbb{R}$. $f(x) = 2x + 3$. Can you check if this is a contraction or not.

A function $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a **contraction** if there exists a $k \in [0, 1)$ such that for all $x, y \in \mathbb{R}^n$:

$$d(f(x), f(y)) \leq kd(x, y)$$

modulus of the contraction

RHS: $kd(x, y)$
 $< d(x, y)$

Example: $f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = \alpha x + \beta$. Then:

$$\begin{aligned} d(f(x), f(y)) &= |\alpha x + \beta - (\alpha y + \beta)| \\ &= |\alpha| |x - y| \\ &= |\alpha| d(x, y) \end{aligned}$$

Thus f is contraction iff $|\alpha| < 1$.

$d(f(x), f(y)) = \left(\frac{1}{2} \cdot \frac{1}{xy}\right) d(x, y)$

Ans

$$f(x^*) = x^*$$

$$f: [\sqrt{2}, \infty) \rightarrow [\sqrt{2}, \infty)$$

$$\begin{aligned} k &= \frac{1}{2} \\ x, y &> \sqrt{2} \end{aligned}$$

$$f(x) = 2x + 3 \rightarrow \text{Not a contraction!}$$

Take 2 points x & y .

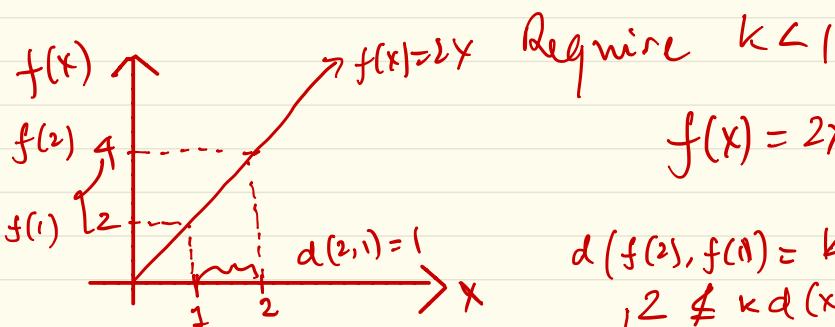
$$f(x) = 2x + 3$$

$$f(y) = 2y + 3$$

$$\begin{aligned} d(f(x), f(y)) &= |f(x) - f(y)| \quad \left(\begin{array}{l} \text{since } f \text{ is} \\ \text{a real} \\ \text{valued fn} \end{array} \right) \\ &= |2x + 3 - 2y - 3| \\ &= 2|x - y| \\ &= 2d(x, y) \end{aligned}$$

$$d(f(x), f(y)) = \underbrace{2d(x, y)}$$

$$d(f(x), f(y)) \leq k d(x, y)$$



Contractions are continuous

WTS: f is continuous. $\Rightarrow \forall \epsilon > 0, \exists \delta > 0$ s.t. $x \in B(x, \delta) \Rightarrow f(x) \in B(f(x), \epsilon)$

Assume f is a contraction.

- Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a contraction with modulus k and fix $x \in \mathbb{R}^n$; we will show f is continuous at x .

$$\text{choose } \delta = \frac{\epsilon}{k}$$

$$d(f(x), f(y))$$

- Fix $\epsilon > 0$. By definition, for any $y \in B_{\epsilon/k}(x)$ we have

$$f(y) \in B_\epsilon(f(x))$$

$$\leq k d(x, y)$$

$$\text{Because } y \in B(x, \frac{\epsilon}{k})$$

$$\Rightarrow d(f(x), f(y)) < \epsilon$$

- Thus f is continuous at x (we took $\delta = \epsilon/k$)

- (We didn't need the fact that $k < 1$. Any function satisfying $d(f(x), f(y)) < kd(x, y)$ for some $k \in \mathbb{R}_+$ is continuous. These are called

Lipschitz functions.)

special cases

Contractions are Lipschitz functions

$$\hat{\beta}_{kx_1} = \frac{(x'x)^{-1}x'y}{k}$$

$x = n \times k$ data matrix
 $y = n \times 1$ data of others

$$\hat{\beta}_1 = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2}$$

Contraction Mapping Theorem

$f: X \rightarrow X$

Banach

\exists only 1 x^* s.t. $f(x^*) = x^*$

Theorem 5.4

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a contraction. Then f has a unique fixed point, x^* . Moreover, given any x_0 , the sequence $x_{n+1} = f(x_n)$ converges to x^* .

$x_0, f(x_0), f(f(x_0))$

$f^2(x_0)$
→ show
existence of x^*

- Show x_n is Cauchy, and therefore converges to some x^*
- By continuity, $f(x_n) \rightarrow f(x^*)$. However, $f(x_n) = x_{n+1} \rightarrow x^*$. Thus x^* is a fixed point of f
- Show uniqueness, suppose x_1 and x_2 are fixed points. Then

$$d(x_1, x_2) = d(f(x_1), f(x_2)) \leq kd(x_1, x_2)$$

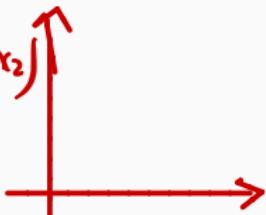
Therefore $d(x_1, x_2) = 0$

$$f(x_1) = x_1$$

$$f(x_2) = x_2$$

$$d(x_1, x_2) < d(x_1, x_2)$$

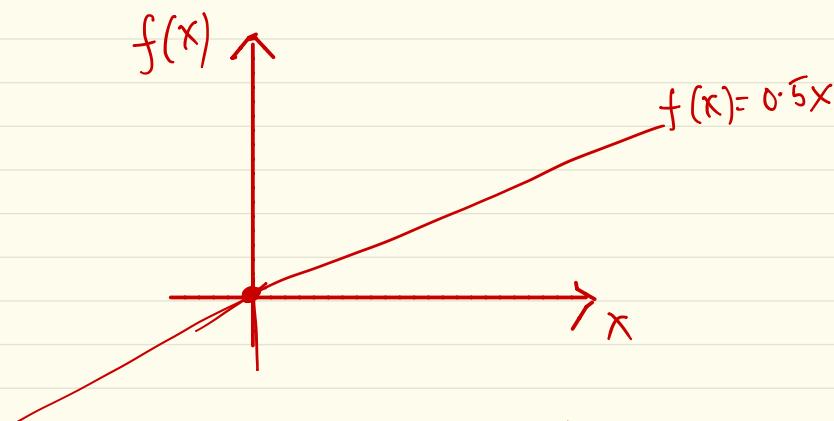
as $k < 1$



$f(x) = 0.5x$
 \therefore
is a contraction

$$f(x) = 0.5x$$

Find a point x^* s.t. $f(x^*) = \underbrace{x^*}_{x^*}$



$$\boxed{0.5x^* = x^*} \Rightarrow x^* = 0 \quad \text{Unique Fixed Point}$$



Start with $x_0 = 1$

$$x_{n+1} = f(x_n)$$

$$(x_n) = \{x_0, x_1, \dots\}$$

$$= f^2(f(x_{n-1}))$$

$$= \{x_0, f(x_0), f^2(x_0), f^3(x_0), \dots\};$$

$$= \{1, 0.5, 0.25, 0.125, \dots\}$$

$$\begin{aligned} & \xrightarrow{x^* = 0} \\ & x_{n+1} = f^n(x_0) \end{aligned}$$

$$x_{n+1} = f(x_n) = f(f(f(x_{n-1}))) = f^2(x_{n-1})$$

$$\underbrace{x_n = f(x_{n-1})}_{\vdots}$$

$$x_{n-1} = f(x_{n-2})$$

$$= f^3(x_{n-2})$$

\vdots

$$x_{n+1} = f^n(x_0)$$

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Metric Spaces

A brief recap of metric spaces

Let X be a set, and $d : X \times X \rightarrow \mathbb{R}$ a function. We call d a **metric** on X if:

- $d(x, y) \geq 0$ for all $x, y \in X$, and $d(x, y) = 0$ iff $x = y$
- $d(x, y) = d(y, x)$
- $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$

These are properties which are more general than the Euclidean distance that we have seen

A brief recap of metric spaces (cont.)

$(\mathbb{R}^n, d) \rightarrow \text{Euclidean Metric Space}$

(X, d)

- The definitions of results we've developed for \mathbb{R}^n largely carry over to general metric spaces
- One special property of \mathbb{R} we used was the LUB property. This was important for the proofs of:
 - A bounded sequence of \mathbb{R}^n has a convergent subsequence
 - A sequence of \mathbb{R}^n converges iff it is Cauchy, which we used to prove the Contraction Mapping Theorem
 - A compact subset of \mathbb{R} has a maximal element
 - A set $S \subseteq \mathbb{R}^n$ is compact iff it is closed and bounded
 - The intermediate value theorem
- These results don't hold in every metric space, but in some metric spaces similar results hold
 - The contraction mapping theorem holds in any complete metric space

Examples of other metric spaces?

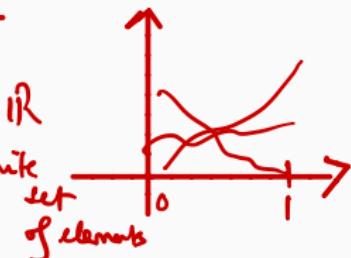
Function Space (X, d) $d: X \times X \rightarrow \mathbb{R}$ (\mathbb{R}, d)

- Let X denote the set of bounded functions from $[0, 1]$ to \mathbb{R} .
- Define the function $d : X \times X \rightarrow \mathbb{R}$ as follows:

sup metric

$$d(f, g) = \sup_{x \in [0, 1]} |f(x) - g(x)| \quad \in \mathbb{R}$$

- Is d a distance? To show triangle inequality, let $f, g, h \in X$:



$$\begin{aligned} d(f, g) &= \sup_{x \in [0, 1]} |f(x) - g(x)| \\ &= \sup_{x \in [0, 1]} |f(x) - h(x) + h(x) - g(x)| \\ &\leq \sup_{x \in [0, 1]} (|f(x) - h(x)| + |h(x) - g(x)|) \quad \text{Add & subtract } h(x) \\ &\leq \sup_{x \in [0, 1]} |f(x) - h(x)| + \sup_{x \in [0, 1]} |h(x) - g(x)| \quad (|x+y| \leq |x| + |y|) \\ &= d(f, h) + d(h, g) \end{aligned}$$

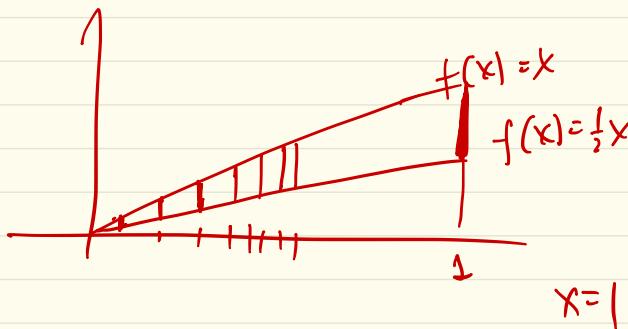
WTS: $d(f, g) \leq d(f, h) + d(h, g)$

$$f(x) = x \quad g(x) = \frac{1}{2}x$$

$$d(f, g) = \sup_{x \in [0, 1]} |f(x) - g(x)|$$

$$= \sup_{x \in [0, 1]} |x - \frac{1}{2}x| = \sup_{x \in [0, 1]} \frac{1}{2}|x|$$

$$\sup \left[d(0, 0.00\ldots 1), d(0, 0\ldots 2), \dots, d(0, \frac{1}{2}) \right]$$



Function spaces

More common
for function
spaces!!

- The metric on the previous slide is called the sup metric. There are other intuitive notions of distance (e.g. the “average distance” between f and g), but those are a little tricky to work with (the $d(x, y) = 0 \Leftrightarrow x = y$ property breaks down)
- It turns out that spaces of bounded functions with the sup metric are complete (Cauchy sequences converge)
- Therefore the contraction mapping theorem holds for spaces of bounded functions! This is a powerful result for dynamic programming, which we'll return to later in the semester