Columbia MA Math Camp

Optimization

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[a] Material adapted from notes by David Thompson and Xingye Wu

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Unconstrained Optimization $\max f(x)$

Equality Constrained Optimization $\max f(x)$ s.t. $g(x) = c$

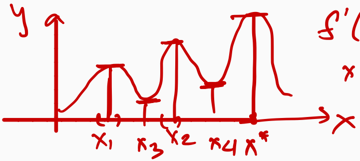
$\max x^{\frac{1}{2}} y^{\frac{1}{2}}$ s.t. $x + y = 1$

$\underbrace{\text{Inequality Constrained Opt}}_{\text{May not cover.}}$ $\max f(x)$ s.t. $g(x) \leq c$

(Kuhn Tucker Theorem)

Some Definitions

x_1 & x_2 are local maxima



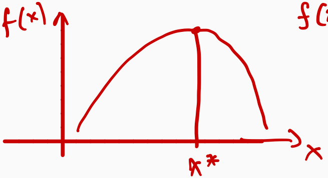
$f'(x) = 0$

$x = x_1, x_2$ or x^*, x_3 or x_4

Let $\mathcal{D} \subseteq \mathbb{R}^n$ and $f : \mathcal{D} \to \mathbb{R}$.

- A point c is a **maximum point** or **global maximum** of f if $f(c) \geq f(x)$ for all $x \in \mathcal{D}$
- A point c is a **local maximum** of f if there exists an $\epsilon > 0$ such that $f(c) \geq f(x)$ for all $x \in B_\epsilon(c)$
- If f is differentiable, a point such that $f'(c) = 0$ is a **critical point** of f

Maximum and minimum points are also called **extreme points, extremum** and **optimal points**



$f(x^*) \geq f(x) \ \forall x \in D$

& cts & differentiable

Suppose f is concave & $\mathcal{D}$ is a closed set & bounded

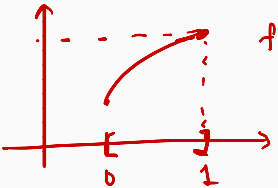
Is it necessary that at the maximizer, $x^* \in \mathcal{D}$

$f'(x^*) = 0$?

Proposition 1.1
Let $\mathcal{D} \subseteq \mathbb{R}^n$ be an open set, and $f : \mathcal{D} \to \mathbb{R}$ a differentiable function. If f has a local extreme point *at x, then $f'(x) = 0$* $\Longleftrightarrow$. $\longrightarrow$ This condition isolates extreme points!

Notes : This could mean that x is either a maximum or minimum or neither

$f : [0,1] \to \mathbb{R}$.



$f'(1) \neq 0$

Twice continuously differentiable function!

Proposition 1.2
Let $\mathcal{D}$ be an open set of $\mathbb{R}^n$ and $f : \mathcal{D} \to \mathbb{R}$ a C^2 function.

- *If f has a* local maximum (minimum) *at x, the Hessian of f at x is* negative (positive) semi-definite
- *If $f'(x) = 0$ and $H(x)$ is* negative (positive) definite, *then x is a* strict local maximum (minimum)

(Proof of second result): Since $H(x)$ is negative definite, there exists $\epsilon > 0$ such that $H(\zeta)$ is negative definite for all $\zeta \in B_\epsilon(x)$. Using the exact form of Taylor's theorem, for any $y \in B_\epsilon(x)$:

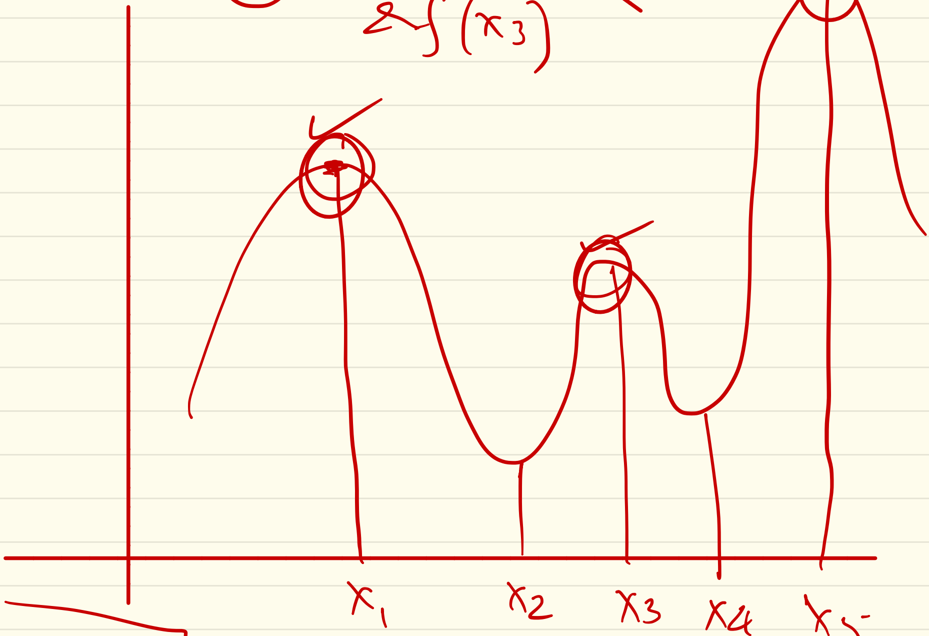
$$\begin{aligned} f(y) &= f(x) + f'(x)(y-x) + \frac{1}{2}(y-x)^T H(\zeta)(y-x) \\ &< f(x) \end{aligned}$$

Maximum:- $f'(x) = 0$ & $f''(x) \leq 0$

Hessian negative semi-definite.

Global maxima

$f(x_5) > f(x_1)$ & $f(x_3)$

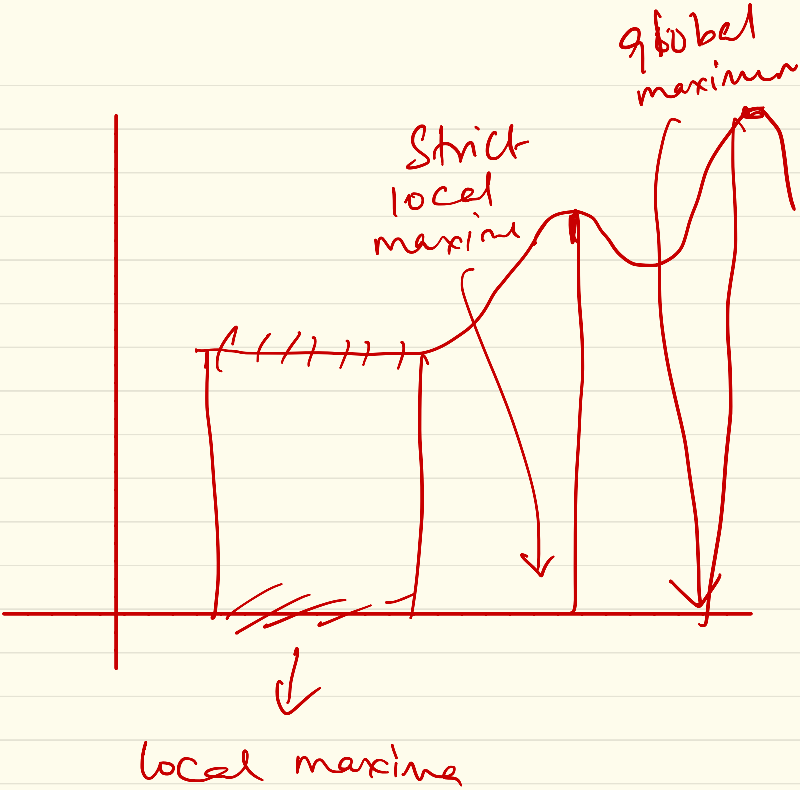


$\boxed{f'(x) = 0}$ $\leftarrow$ $\{x_1, x_2, x_3, x_4, x_5\}$ Critical Points.

$f''(x) < 0 \leftarrow \{x_1, x_3, x_5\}$

✓ $f(x_1), f(x_3), f(x_5)$

To find which point is the global maxima.


Strict local maxima
global maximum
local maxima

Summarizing

maxima / minima

- Critical points are a **necessary (not sufficient)** condition for extrema
- The second derivative can give us **local sufficient conditions**

To this point we've only discussed maximization over an open set

- Maxima need not exist on an open set (e.g. $f(x) = x$ on $(0, 1)$)
- If you're maximizing over a closed set S, you can decompose it as $S = int(S) \cup Boundary(S)$. Need to check the boundary

max $f(x)$

$x \in (0,1)$

$f(x) = x^{\frac{1}{2}}$

maximizer does not exist !

General recipe to maximize a function :

- Find all critical points. If there are many, $\boxed{\text{SOC}}$ can help filter
- Find the critical point with the largest value
- Check if the function takes on a higher value along the boundary

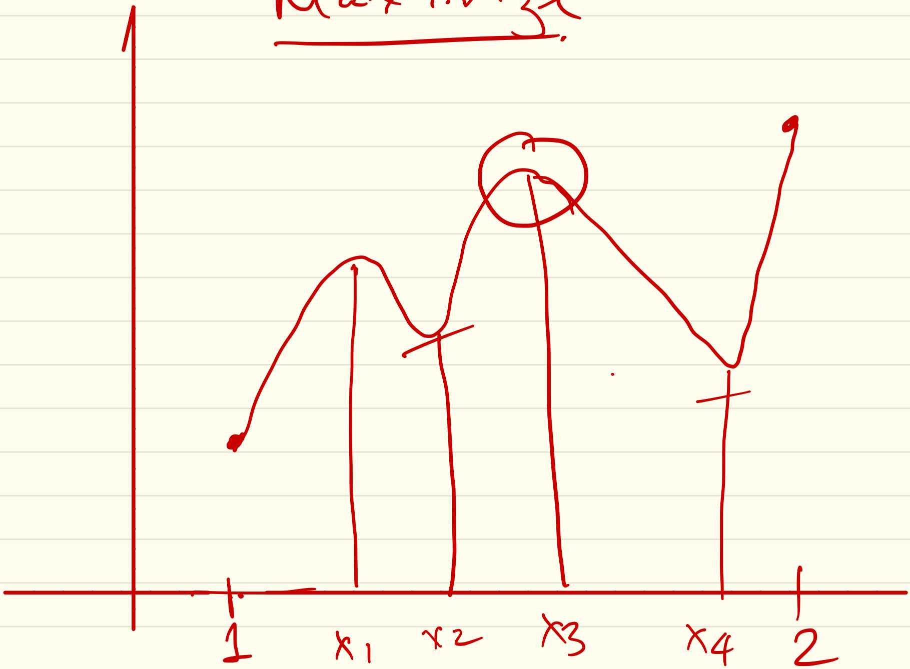
$f'(x) = 0$

filter out the minima !

If $f'(x)=0$ & $f''(x)<0 \Rightarrow x$ is local max

$f : [1, 2] \to \mathbb{R}$.

Maximize



(1)(a) find all critical points.

$\to \{x_1, x_2, x_3, x_4\}$

(b) Filter out based on SOC

$\to \{x_1, x_3\}$

critical pt

(2) $f(x_1)$ & $f(x_3)$ $\to$ $f(x_3) > f(x_1)$

(3) Check $f(1)$, $f(2)$ & compare to $f(x_3)$

$\longrightarrow$ 2 is the maxima

because

$f(2) \geq f(x) \quad \forall\, x \in [1,2]$

$$\max \; x^{\frac{1}{2}}$$

$x \in (0,1)$ — Not compact

$\Rightarrow$ max not exists necessarily.

$$f'(x) = 0 = \left[\frac{1}{2} x^{-\frac{1}{2}}\right]$$

$$x = 0 \notin (0,1)$$

Why is it not guaranteed in this simple problem that a max exists?

$x^* \in D \qquad x^* \in (0,1)$

<u>Weierstrass</u>: If f is cts & D is compact $\Rightarrow$ max exists.

$\not\Rightarrow \quad \neg P \Rightarrow \neg Q$.

Sufficient Conditions for Global Extrema

For convex functions, optimization is dramatically simpler, as evidenced by the following proposition :

Proposition 1.3
Let $\mathcal{D}$ be a convex open set of $\mathbb{R}^n$ and $f : \mathcal{D} \to \mathbb{R}$ be a convex function. Then :

- *The set of minimizers of f is* convex
- *If f is* strictly convex, *it has* at most one minimizer
- *Any* local minimum *of f is a* global minimum
- *If f is* differentiable, *then x is a* global minimum *of f iff $f'(x) = 0$*

$f(x) = x^2$

$D = (0,1)$

X

The same results hold for concave functions, replacing "minimizers" with "maximizers"

Note :

- Note that we already proved the second result for quasiconvex functions.

Convex fn :- $f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y)$

(1) Set of minimizers is a convex set

$\forall\, x_1^*, x_2^* \in S_{minimizers}$

$\lambda x_1^* + (1-\lambda) x_2^* \in S_{min}$,

$\Rightarrow f(\lambda x_1^* + (1-\lambda) x_2^*) \le f(x) \quad \forall x \in D$

$f(x_1^*) = f(x_2^*)$

$$\Rightarrow f(\lambda x_1^* + (1-\lambda) x_2^*) \le f(x_1^*) = f(x_2^*) \le f(x) \quad \forall x \in D.$$

$\Rightarrow \lambda x_1^* + (1-\lambda) x_2^* \in S_{minimizers}.$

$$f(\lambda x + (1-\lambda) y) < \lambda f(x) + (1-\lambda) f(y)$$

f is strictly convex $\Rightarrow$ At most 1 minimizer.

<u>Pf:</u> Suppose there are 2 minimizers x_1^* & x_2^*. $f(x_1^*) = f(x_2^*)$

$$f(\underbrace{\lambda x_1^* + (1-\lambda) x_2^*}) < \lambda f(x_1^*) + (1-\lambda) f(x_2^*)$$

$$= f(x_1^*)$$

which contradicts x_1^* & x_2^* being minimizers!

Hence minimizer (if exists) is unique.

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Constrained Optimization Problems

- In economics, it's more common to maximize an objective function subject to some constraints
- For example, in consumer theory, you will see problems of this form:

$$\max_{c_1,c_2} \log c_1 + \alpha \log c_2 \;\text{ s.t. }\; p_1 c_1 + p_2 c_2 = M$$

- One approach to these problems is to put the constraints in the objective. For instance, in the above example

$$c_1 = \frac{M - p_2 c_2}{p_1}$$

  So we could do the unconstrained maximization problem:

$$\max_{c_2} \log\left(\frac{M - p_2 c_2}{p_1}\right) + \alpha \log c_2$$

[Do More Examples]

A more formal statement

Theorem 2.1
Let $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}^k$ be C^1 functions, and consider the program:

$$\max_{x \in \mathbb{R}^n} f(x) \text{ s.t. } g(x) = 0$$

If x^* is a local maximum and x^* satisfies the constraint qualification ; $rank(g'(x^*)) = k$, then there exist k Lagrange multipliers $\lambda = (\lambda_1, ..., \lambda_k)^T \in \mathbb{R}^k$ *such that the first-order condition holds:*

$$f'(x^*) + \lambda^T g'(x^*) = 0$$

It is common to talk about the **Lagrangian** of a system:

$$\mathcal{L}(x, \lambda) = f(x) + \lambda^T g(x)$$

The first-order conditions wrt x and λ give us the critical point of the Lagrangian.

A sketch of Lagrange's Theorem in two variables

- Write $x = (x_1, x_2)$. Let $x^* = (x_1^*, x_2^*)$ be a local maximum of f subject to g
- By the IFT, we can write $x_2 = h(x_1)$, with $h'(x_1) = -\frac{g_1(x_1,x_2)}{g_2(x_1,x_2)}$
- We now do unconstrained optimization of $f(x_1, h(x_1))$. The FOC is

$$f_1(x^*) + f_2(x^*)h'(x_1^*) = 0$$

- Define $\lambda = -\frac{f_2(x^*)}{g_2(x^*)}$. Then

$$\begin{aligned} f_1 + \lambda g_1(x^*) &= 0 \\ f_2 + \lambda g_2(x^*) &= 0 \end{aligned}$$

The general case is similar, just with more cumbersome matrix notation.

Comments on Lagrange's Theorem

- The Lagrange condition is a necessary condition.
- As with unconstrained optimization, there are second order conditions that let you check whether a critical point of the Lagrangian is a local maximum or minimum (see FMEA Section 3.4 for details)
- In order to get sufficient conditions for global maxima along the constraint, we need additional structure
- If the constraint qualification fails, the theorem says nothing. So you need to check points where the CQ fails separately

An example of a sufficient condition

Proposition 2.1
Let $f : \mathbb{R}^n \to \mathbb{R}$ be strictly quasiconcave and consider the program

$$\max_x f(x) \text{ s.t. } Ax = b \tag{1}$$

where A is an $m \times n$ matrix with $m < n$. If (x^*, λ^*) is a critical point of the Lagrangian and $f'(x^*) \neq 0$, then x^* solves (1).

Proof.
The FOC of the Lagrangian implies $f'(x^*) + \lambda^T A = 0$. Suppose there were an $\hat{x}$ such that $A\hat{x} = b$ and $f(\hat{x}) > f(x^*)$. Since f is strictly quasiconcave:

$$\begin{aligned} 0 &< f'(x^*)(\hat{x} - x^*) \\ &= -\lambda^T A(\hat{x} - x^*) \\ &= 0 \end{aligned}$$

a contradiction. □

Note : $f(x)$ being strictly quasiconcave and the constraint being linear ensures that the Lagrangian is strictly quasiconcave as well (which we have shown before implies a unique maximizer)

Interpretation of the multipliers

Define the **"value function"** V as follows

$$V(b) = \max_x f(x) \text{ s.t. } g(x) = b$$

Form the Lagrangian :

$$\mathcal{L}(x, \lambda, b) = f(x) + \lambda^T (b - g(x))$$

Write the solution of this problem as $x^*(b), \lambda^*(b)$. Then

$$V(b) = \mathcal{L}(x^*(b), \lambda^*(b), b)$$

Interpretation of the multipliers (cont.)

Using the chain rule, we have :

$$V'(b) = \frac{\partial \mathcal{L}}{\partial x}\frac{dx^*}{db} + \frac{\partial \mathcal{L}}{\partial \lambda}\frac{d\lambda^*}{db} + \frac{\partial \mathcal{L}}{\partial b}$$

However, we know $\frac{\partial \mathcal{L}}{\partial x}$ and $\frac{\partial \mathcal{L}}{\partial \lambda}$ are 0 at x^*, λ^*, so we have

$$V'(b) = \lambda^T$$

Interpretation: Lagrange multipliers measure the marginal value of loosening a constraint by 1 unit