E1 244: Detection and Estimation Theory **(Due: 02/03/20)**

Assignment #1

Instructor: Sundeep Chepuri *Name:* Vineeth S, *SR No.:* 16543

PART A: Problem 1 Cramer-Rao Lower Bound (CRLB)

(Solution) From the range difference equations we have,

$$r_{ij} = d_i - d_j + n_{ij}$$

$$n_{ij} = w_i - w_j$$

We will use a set of non redundant range equations,

$$r_{12} = d_1 - d_2 + n_{12}$$

$$r_{23} = d_2 - d_3 + n_{23}$$

$$r_{34} = d_3 - d_4 + n_{34}$$

Concisely in vector notation we can write as,

$$\begin{bmatrix} r_{12} \\ r_{23} \\ r_{34} \end{bmatrix} = \begin{bmatrix} d_1 - d_2 \\ d_2 - d_3 \\ d_3 - d_4 \end{bmatrix} + \begin{bmatrix} n_{12} \\ n_{23} \\ n_{34} \end{bmatrix}$$

$$\underline{\mathbf{r}} = \underline{\mathbf{h}}(\underline{\mathbf{x}}) + \underline{\mathbf{n}}$$

$$\underline{\mathbf{r}} \sim \mathcal{N}(\underline{\mathbf{h}}(\underline{\mathbf{x}}), \mathbf{\Sigma_n})$$

With this we can write the likelihood and log-likelihood functions as,

$$\begin{aligned} P(\underline{\mathbf{r}};\underline{\mathbf{x}}) &= \frac{1}{(2\pi)^{3/2}|\sum_n|^{1/2}} exp(-\frac{1}{2}(\underline{\mathbf{r}} - \underline{\mathbf{h}}(\underline{\mathbf{x}}))^T \Sigma_n^{-1}(\underline{\mathbf{r}} - \underline{\mathbf{h}}(\underline{\mathbf{x}}))) \\ lnP(\underline{\mathbf{r}};\underline{\mathbf{x}}) &= K - \frac{1}{2}(\underline{\mathbf{r}} - \underline{\mathbf{h}}(\underline{\mathbf{x}}))^T \mathbf{\Sigma_n^{-1}}(\underline{\mathbf{r}} - \underline{\mathbf{h}}(\underline{\mathbf{x}})) \\ &= K - \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}(r_i - h_i(\underline{\mathbf{x}}))[\mathbf{\Sigma_n^{-1}}]_{ij}(r_j - h_j(\underline{\mathbf{x}})) \\ \frac{\partial}{\partial \underline{\mathbf{x}}} lnP(\underline{\mathbf{r}};\underline{\mathbf{x}}) &= \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}[\mathbf{\Sigma_n^{-1}}]_{ij}(r_i - h_i(\underline{\mathbf{x}}))\frac{\partial h_j(\underline{\mathbf{x}})}{\partial \underline{\mathbf{x}}} + \frac{\partial h_i(\underline{\mathbf{x}})}{\partial \underline{\mathbf{x}}}(r_j - h_j(\underline{\mathbf{x}})) \end{aligned}$$

We can find the elements of the Fischer Information Matrix one element at a time,

$$\frac{\partial^2}{\partial x^2} lnP(\underline{\mathbf{r}};\underline{\mathbf{x}}) = \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}[\mathbf{\Sigma_n^{-1}}]_{ij}((r_i - h_i(\underline{\mathbf{x}}))\frac{\partial^2 h_j(\underline{\mathbf{x}})}{\partial x^2} - \frac{\partial h_i(\underline{\mathbf{x}})}{\partial x}\frac{\partial h_j(\underline{\mathbf{x}})}{\partial x} + (r_j - h_j(\underline{\mathbf{x}}))\frac{\partial^2 h_i(\underline{\mathbf{x}})}{\partial x^2} - \frac{\partial h_j(\underline{\mathbf{x}})}{\partial x}\frac{\partial h_i(\underline{\mathbf{x}})}{\partial x})$$

$$\begin{aligned} [\mathbf{I}(\underline{\mathbf{x}})]_{11} &= -E[\frac{\partial^2}{\partial x^2} lnP(\underline{\mathbf{r}};\underline{\mathbf{x}})] \\ &= \sum_{i=1}^{3}\sum_{j=1}^{3}[\mathbf{\Sigma_n^{-1}}]_{ij}\frac{\partial h_i(\underline{\mathbf{x}})}{\partial x}\frac{\partial h_j(\underline{\mathbf{x}})}{\partial x} \\ &= \frac{\partial \underline{\mathbf{h}}(\underline{\mathbf{x}})}{\partial x}^T \mathbf{\Sigma_n^{-1}} \frac{\partial \underline{\mathbf{h}}(\underline{\mathbf{x}})}{\partial x} \end{aligned}$$

$$
\begin{aligned}
[\mathbf{I}(\underline{\mathbf{x}})]_{12} &= -E[\frac{\partial^2}{\partial x \partial y} lnP(\underline{\mathbf{r}};\underline{\mathbf{x}})] \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}[\mathbf{\Sigma_n^{-1}}]_{ij}\frac{\partial h_i(\underline{\mathbf{x}})}{\partial x}\frac{\partial h_j(\underline{\mathbf{x}})}{\partial y} \\
&= \frac{\partial \underline{\mathbf{h}}(\underline{\mathbf{x}})}{\partial x}^T \mathbf{\Sigma_n^{-1}} \frac{\partial \underline{\mathbf{h}}(\underline{\mathbf{x}})}{\partial y} \\
[\mathbf{I}(\underline{\mathbf{x}})]_{21} &= -E[\frac{\partial^2}{\partial y \partial x} lnP(\underline{\mathbf{r}};\underline{\mathbf{x}})] \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}[\mathbf{\Sigma_n^{-1}}]_{ij}\frac{\partial h_i(\underline{\mathbf{x}})}{\partial y}\frac{\partial h_j(\underline{\mathbf{x}})}{\partial x} \\
&= \frac{\partial \underline{\mathbf{h}}(\underline{\mathbf{x}})}{\partial y}^T \mathbf{\Sigma_n^{-1}} \frac{\partial \underline{\mathbf{h}}(\underline{\mathbf{x}})}{\partial x} \\
[\mathbf{I}(\underline{\mathbf{x}})]_{22} &= -E[\frac{\partial^2}{\partial y \partial y} lnP(\underline{\mathbf{r}};\underline{\mathbf{x}})] \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}[\mathbf{\Sigma_n^{-1}}]_{ij}\frac{\partial h_i(\underline{\mathbf{x}})}{\partial y}\frac{\partial h_j(\underline{\mathbf{x}})}{\partial y} \\
&= \frac{\partial \underline{\mathbf{h}}(\underline{\mathbf{x}})}{\partial y}^T \mathbf{\Sigma_n^{-1}} \frac{\partial \underline{\mathbf{h}}(\underline{\mathbf{x}})}{\partial y}
\end{aligned}
$$

where,

$$
\frac{\partial \underline{\mathbf{h}}(\underline{\mathbf{x}})}{\partial y} = \begin{bmatrix} \frac{\partial h_1(\underline{\mathbf{x}})}{\partial x} & \frac{\partial h_2(\underline{\mathbf{x}})}{\partial x} & \frac{\partial h_3(\underline{\mathbf{x}})}{\partial x} \end{bmatrix}
$$

$$
\begin{aligned}
\frac{\partial h_i(\underline{\mathbf{x}})}{\partial x} &= \frac{\partial}{\partial x}(d_i - d_{i+1}) \\
&= \frac{\partial}{\partial x}(||\mathbf{x}-\mathbf{x}_i||_2 - ||\mathbf{x}-\mathbf{x}_{i+1}||_2) \\
&= \frac{x - x_i}{||\mathbf{x}-\mathbf{x}_i||_2} - \frac{x - x_{i+1}}{||\mathbf{x}-\mathbf{x}_{i+1}||_2} \\
&= \frac{x-x_i}{d_i} - \frac{x-x_{i+1}}{d_{i+1}}
\end{aligned}
$$

Given an $\underline{\mathbf{x}}$, we can find the elements of Fischer Information Matrix and invert it to get the CRLB.

$$
\begin{aligned}
\mathbf{C}_{\underline{\hat{\mathbf{x}}}} &\geq [\mathbf{I^{-1}}(\underline{\mathbf{x}})] \\
Var[\hat{x}] &\geq [\mathbf{I^{-1}}(\underline{\mathbf{x}})]_{11}^{-1} \\
Var[\hat{y}] &\geq [\mathbf{I^{-1}}(\underline{\mathbf{x}})]_{22}
\end{aligned}
$$

PART A: Problem 2 Maximum Likelihood Estimator (MLE)

(Solution) We want obtain the maximum likelihood estimator (MLE) for $\underline{\mathbf{x}}$. The estimator that aximizes likelihood function also maximizes log-likelihood, since log function is a monotonically increasing function. We have our likelihood and log-likelihood functions from Problem 1 as,

$$\begin{aligned}
P(\underline{\mathbf{r}};\underline{\mathbf{x}}) &= \frac{1}{(2\pi)^{3/2}|\sum_n|^{1/2}}exp(-\frac{1}{2}(\underline{\mathbf{r}}-\underline{\mathbf{h}}(\underline{\mathbf{x}}))^T\Sigma_n^{-1}(\underline{\mathbf{r}}-\underline{\mathbf{h}}(\underline{\mathbf{x}}))) \\
lnP(\underline{\mathbf{r}};\underline{\mathbf{x}}) &= K-\frac{1}{2}(\underline{\mathbf{r}}-\underline{\mathbf{h}}(\underline{\mathbf{x}}))^T\mathbf{\Sigma_n^{-1}}(\underline{\mathbf{r}}-\underline{\mathbf{h}}(\underline{\mathbf{x}})) \\
\underline{\hat{\mathbf{x}}} &= \arg\max_{\underline{\mathbf{x}}} lnP(\underline{\mathbf{r}};\underline{\mathbf{x}}) \\
&= \arg\max_{\underline{\mathbf{x}}} K-\frac{1}{2}(\underline{\mathbf{r}}-\underline{\mathbf{h}}(\underline{\mathbf{x}}))^T\mathbf{\Sigma_n^{-1}}(\underline{\mathbf{r}}-\underline{\mathbf{h}}(\underline{\mathbf{x}})) \\
&= \arg\min_{\underline{\mathbf{x}}} \frac{1}{2}(\underline{\mathbf{r}}-\underline{\mathbf{h}}(\underline{\mathbf{x}}))^T\mathbf{\Sigma_n^{-1}}(\underline{\mathbf{r}}-\underline{\mathbf{h}}(\underline{\mathbf{x}})) \\
&= \arg\min_{\underline{\mathbf{x}}} J(\underline{\mathbf{x}})
\end{aligned}$$

We can proceed with this minimization as an optimization problem, using any of the available descent algorithms.We will use a simple iterative gradient descent algorithm with a fixed step size α such that $0<\alpha<\frac{2}{\lambda_{max}}$, where λ_{max} is the largest eigenvalue of $\mathbf{\Sigma_n^{-1}}$ (for which the gradient descent theoretically converges if objective function is Lipschitz).

$$\begin{aligned}
\underline{\hat{\mathbf{x}}}^{(k+1)} &= \underline{\hat{\mathbf{x}}}^{(k)}-\alpha\nabla J(\underline{\mathbf{x}}) \\
&= \underline{\hat{\mathbf{x}}}^{(k)}-\alpha\nabla\mathbf{h}(\underline{\mathbf{x}})^T\mathbf{\Sigma_n^{-1}}(\underline{\mathbf{r}}-\underline{\mathbf{h}}(\underline{\mathbf{x}}))
\end{aligned}$$

PART A: Problem 3 Best Linear Unbiased Estimator (BLUE) estimator for θ

(Solution) From the range difference equations we have,

$$r_{ij} = d_i - d_j + n_{ij}$$
$$r_{ij} + d_j = d_i + n_{ij}$$
$$r_{ij}^2 + 2r_{ij}dj + d_j^2 = d_i^2 + e_{ij}$$

where d_i and e_{ij} are given by,

$$d_i^2 = ||\mathbf{x} - \mathbf{x}_i||_2^2 = ||\mathbf{x}||^2 + ||\mathbf{x}_i||^2 - 2\mathbf{x}_i^T\mathbf{x}$$
$$e_{ij} = n_{ij}^2 = (w_i - w_j)^2$$

Substituting in range equation,

$$r_{ij}^2 = d_i^2 - d_j^2 - 2r_{ij}d_j + e_{ij}$$
$$r_{ij}^2 = (||\mathbf{x}||^2 + ||\mathbf{x}_i||^2 - 2\mathbf{x}_i^T\mathbf{x}) - (||\mathbf{x}||^2 + ||\mathbf{x}_j||^2 - 2\mathbf{x}_j^T\mathbf{x}) - 2r_{ij}d_j + e_{ij}$$
$$r_{ij}^2 = ||\mathbf{x}_i||^2 - ||\mathbf{x}_j||^2 - 2(\mathbf{x}_i - \mathbf{x}_j)^T\mathbf{x} - 2r_{ij}d_j + e_{ij}$$

Let $\gamma_{ij} = ||\mathbf{x}_i||^2 - ||\mathbf{x}_j||^2$,

$$r_{ij}^2 = \gamma_{ij} - 2(\mathbf{x}_i - \mathbf{x}_j)^T\mathbf{x} - 2r_{ij}d_j + e_{ij}$$

$$\begin{bmatrix} r_{12}^2 \\ r_{13}^2 \\ r_{14}^2 \\ r_{23}^2 \\ r_{24}^2 \\ r_{34}^2 \end{bmatrix} = \begin{bmatrix} -2(\mathbf{x}_1 - \mathbf{x}_2)^T & -2r_{12} & 0 & 0 \\ -2(\mathbf{x}_1 - \mathbf{x}_3)^T & 0 & -2r_{13} & 0 \\ -2(\mathbf{x}_1 - \mathbf{x}_4)^T & 0 & 0 & -2r_{14} \\ -2(\mathbf{x}_2 - \mathbf{x}_3)^T & 0 & -2r_{23} & 0 \\ -2(\mathbf{x}_2 - \mathbf{x}_4)^T & 0 & 0 & -2r_{24} \\ -2(\mathbf{x}_3 - \mathbf{x}_4)^T & 0 & 0 & -2r_{34} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} + \begin{bmatrix} e_{12} \\ e_{13} \\ e_{14} \\ e_{23} \\ e_{24} \\ e_{34} \end{bmatrix} + \begin{bmatrix} \gamma_{12} \\ \gamma_{13} \\ \gamma_{14} \\ \gamma_{23} \\ \gamma_{24} \\ \gamma_{34} \end{bmatrix}$$

$$\begin{bmatrix} r_{12}^2 \\ r_{13}^2 \\ r_{14}^2 \\ r_{23}^2 \\ r_{24}^2 \\ r_{34}^2 \end{bmatrix} = \begin{bmatrix} -2(x_1 - x_2)^T & -2(y_1 - y_2)^T & -2r_{12} & 0 & 0 \\ -2(x_1 - x_3)^T & -2(y_1 - y_3)^T & 0 & -2r_{13} & 0 \\ -2(x_1 - x_4)^T & -2(y_1 - y_4)^T & 0 & 0 & -2r_{14} \\ -2(x_2 - x_3)^T & -2(y_2 - y_3)^T & 0 & -2r_{23} & 0 \\ -2(x_2 - x_4)^T & -2(y_2 - y_4)^T & 0 & 0 & -2r_{24} \\ -2(x_3 - x_4)^T & -2(y_3 - y_4)^T & 0 & 0 & -2r_{34} \end{bmatrix} \begin{bmatrix} x \\ y \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} + \begin{bmatrix} e_{12} \\ e_{13} \\ e_{14} \\ e_{23} \\ e_{24} \\ e_{34} \end{bmatrix} + \begin{bmatrix} \gamma_{12} \\ \gamma_{13} \\ \gamma_{14} \\ \gamma_{23} \\ \gamma_{24} \\ \gamma_{34} \end{bmatrix}$$

Concisely in vector notation,

$$\underline{\mathbf{r}} = \mathbf{H}\underline{\theta} + \underline{\mathbf{e}} + \underline{\gamma}$$

where,

$$\underline{\mathbf{r}} = \begin{bmatrix} r_{12}^2 \\ r_{13}^2 \\ r_{14}^2 \\ r_{23}^2 \\ r_{24}^2 \\ r_{34}^2 \end{bmatrix}, \mathbf{H} = \begin{bmatrix} -2(x_1-x_2)^T & -2(y_1-y_2)^T & -2r_{12} & 0 & 0 \\ -2(x_1-x_3)^T & -2(y_1-y_3)^T & 0 & -2r_{13} & 0 \\ -2(x_1-x_4)^T & -2(y_1-y_4)^T & 0 & 0 & -2r_{14} \\ -2(x_2-x_3)^T & -2(y_2-y_3)^T & 0 & -2r_{23} & 0 \\ -2(x_2-x_4)^T & -2(y_2-y_4)^T & 0 & 0 & -2r_{24} \\ -2(x_3-x_4)^T & -2(y_3-y_4)^T & 0 & 0 & -2r_{34} \end{bmatrix}, \underline{\theta} = \begin{bmatrix} x \\ y \\ d_2 \\ d_3 \\ d_4 \end{bmatrix}, \underline{\mathbf{e}} \begin{bmatrix} e_{12} \\ e_{13} \\ e_{14} \\ e_{23} \\ e_{24} \\ e_{34} \end{bmatrix}, \underline{\gamma} \begin{bmatrix} \gamma_{12} \\ \gamma_{13} \\ \gamma_{14} \\ \gamma_{23} \\ \gamma_{24} \\ \gamma_{34} \end{bmatrix}$$

But here the mean of error $\underline{\mathbf{e}}$ is no longer $\underline{\mathbf{0}}$ and hence we should subtract the mean of the error from above observations to make error zero mean.

$$E[\underline{\mathbf{e}}] = \begin{bmatrix} E[e_{12}] \\ E[e_{13}] \\ E[e_{14}] \\ E[e_{23}] \\ E[e_{24}] \\ E[e_{34}] \end{bmatrix} = 2\sigma^2 \underline{\mathbf{1}}_{\mathbf{6x1}}$$

$$\begin{aligned} E[e_{ij}] &= E[(w_i - w_j)^2] \\ &= Var[w_i - w_j] + E[w_i - w_j] \\ &= Var[w_i] + Var[w_j] \\ &= 2\sigma^2 \end{aligned}$$

We can now have a model with zero mean as,

$$\begin{gathered} \underline{\mathbf{r}} - \underline{\gamma} - 2\sigma^2 \underline{\mathbf{1}}_{\mathbf{6x1}} = \mathbf{H}\underline{\theta} + \underline{\mathbf{e}} - 2\sigma^2 \underline{\mathbf{1}}_{\mathbf{6x1}} \\ \underline{\mathbf{r}'} = \mathbf{H}\underline{\theta} + \underline{\mathbf{e}'} \\ \underline{\mathbf{r}'} = \underline{\mathbf{r}} - \underline{\gamma} - 2\sigma^2 \underline{\mathbf{1}}_{\mathbf{6x1}} \\ \underline{\mathbf{e}'} = \underline{\mathbf{e}} - 2\sigma^2 \underline{\mathbf{1}}_{\mathbf{6x1}} \\ \underline{\mathbf{e}'} \sim \mathcal{N}(0, \mathbf{C}) \end{gathered}$$

where $\underline{\mathbf{C}}$ is the covariance matrix of $\underline{\mathbf{e}'}$ which is given by,

$$\begin{aligned} \mathbf{C} &= E[(\underline{\mathbf{e}'} - E[\underline{\mathbf{e}'}])(\underline{\mathbf{e}'} - E[\underline{\mathbf{e}'}])^T] \\ &= E[\underline{\mathbf{e}'}\underline{\mathbf{e}'}^T] \\ &= E[(\underline{\mathbf{e}} - 2\sigma^2 \underline{\mathbf{1}}_{\mathbf{6x1}})(\underline{\mathbf{e}} - 2\sigma^2 \underline{\mathbf{1}}_{\mathbf{6x1}})^T] \\ &= E[\underline{\mathbf{e}}\underline{\mathbf{e}}^T - \underline{\mathbf{e}}(2\sigma^2 \underline{\mathbf{1}}_{\mathbf{6x1}})^T - (2\sigma^2 \underline{\mathbf{1}}_{\mathbf{6x1}})]\underline{\mathbf{e}}^T + (2\sigma^2 \underline{\mathbf{1}}_{\mathbf{6x1}})(2\sigma^2 \underline{\mathbf{1}}_{\mathbf{6x1}})^T] \\ &= E[\underline{\mathbf{e}}\underline{\mathbf{e}}^T] - (2\sigma^2 \underline{\mathbf{1}}_{\mathbf{6x1}})(2\sigma^2 \underline{\mathbf{1}}_{\mathbf{6x1}})^T \end{aligned}$$

We can find element by element of $\mathbf{C}$ as,

$$\begin{aligned} [\mathbf{C}]_{11} &= E[e_{12}^2] - 4\sigma^4 \\ &= E[(w_1 - w_2)^2 (w_1 - w_2)^2] - 4\sigma^4 \\ &= E[(w_1^2 - 2w_1 w_2 + w_2^2)(w_1^2 - 2w_1 w_2 + w_2^2)] - 4\sigma^4 \end{aligned}$$

$$
\begin{aligned}
&= E[w_1^4 + w_2^4 + 6w_1^2 w_2^2] - 4\sigma^4 \\
&= 3\sigma^4 + 3\sigma^4 + 6\sigma^4 - 4\sigma^4 \\
&= 8\sigma^4 \\
[\mathbf{C}]_{12} &= E[e_{12}e_{13}] - 4\sigma^4 \\
&= E[(w_1 - w_2)^2(w_1 - w_3)^2] - 4\sigma^4 \\
&= E[(w_1^2 - 2w_1w_2 + w_2^2)(w_1^2 - 2w_1w_3 + w_3^2)] - 4\sigma^4 \\
&= E[w_1^4 + w_1^2w_3^2 + w_1^2w_2^2 + +w_2^2w_3^2] - 4\sigma^4 \\
&= 3\sigma^4 + \sigma^4 + \sigma^4 + \sigma^4 - 4\sigma^4 \\
&= 2\sigma^4 \\
[\mathbf{C}]_{13} &= E[e_{12}e_{14}] - 4\sigma^4 \\
&= 2\sigma^4 \\
[\mathbf{C}]_{14} &= E[e_{12}e_{23}] - 4\sigma^4 \\
&= 2\sigma^4 \\
[\mathbf{C}]_{15} &= E[e_{12}e_{24}] - 4\sigma^4 \\
&= 2\sigma^4 \\
[\mathbf{C}]_{16} &= E[e_{12}e_{34}] - 4\sigma^4 \\
&= E[(w_1 - w_2)^2(w_3 - w_4)^2] - 4\sigma^4 \\
&= E[(w_1^2 - 2w_1w_2 + w_2^2)(w_3^2 - 2w_3w_4 + w_4^2)] - 4\sigma^4 \\
&= E[w_1^2w_3^2 + w_1^2w_4^2 + w_2^2w_3^2 + +w_2^2w_4^2] - 4\sigma^4 \\
&= \sigma^4 + \sigma^4 + \sigma^4 + \sigma^4 - 4\sigma^4 \\
&= 0
\end{aligned}
$$

Similarly we compute the matrix $\mathbf{C}$ by identifying the pattern,

$$
\mathbf{C} = 2\sigma^2 \begin{bmatrix} 4 & 1 & 1 & 1 & 1 & 0 \\ 1 & 4 & 1 & 1 & 0 & 1 \\ 1 & 1 & 4 & 0 & 1 & 1 \\ 1 & 1 & 0 & 4 & 1 & 1 \\ 1 & 0 & 1 & 1 & 4 & 1 \\ 0 & 1 & 1 & 1 & 1 & 4 \end{bmatrix}
$$

Hence the best linear unbiased estimator (BLUE estimator),

$$
\underline{\hat{\theta}} = (\mathbf{H}^T\mathbf{C}^{-1}\mathbf{H})^{-1}\mathbf{H}^T\mathbf{C}^{-1}\underline{\mathbf{r'}}
$$