Indian Institute of Science
Department of Electrical Communications Engineering

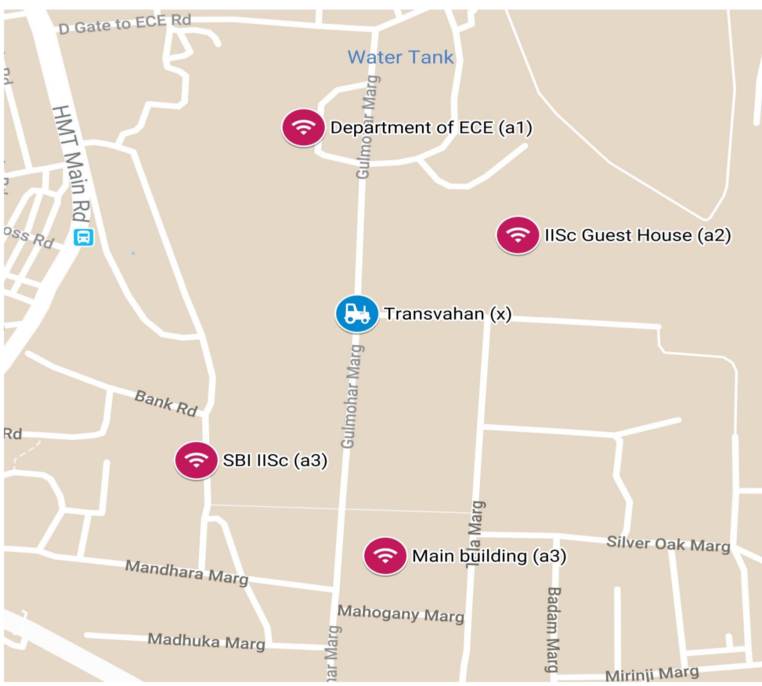
E1 244: Detection and Estimation

January-April 2020

Homework 1 (deadline 29 Feb. 5pm)

This homework consists of two parts: (a) Developing different estimators to localize Transvahan - the e-vehicle on IISc Campus using measurements from receivers at four different locations in IISc and (b) implementing and evaluating the performance of the estimators that you have derived. Make a short report using LaTeX containing the required explanations, answers, plots, and Matlab/Python scripts, and turn it in by the deadline using Microsoft Teams. Only PDF files will be evaluated.

TDOA localization of IISc Transvahan



Let $\mathbf{x} = [x, y]^{\mathrm{T}}$ be the location of the Transvahan (e-vehicle), and call $\mathbf{x}_i = [x_i, y_i]^{\mathrm{T}}$ for $i = 1, 2, 3, 4$ the known locations of the reference receivers (sometimes also referred to as anchors). The anchors are located at the Department of ECE, IISc guest house, SBI, and the main building as illustrated in the picture above. The Transvahan emits a beacon at time $t = T_0$ and each anchor measures the time-of-arrival of the beacon signal that we denote as t_i for $i = 1, 2, 3, 4$. Based on these time-of-arrival measurements, we would like to find the location of the Transvahan.

The time-of-arrival measurements are modeled by

$$t_i = T_0 + d_i/c + w_i', \qquad i = 1, 2, 3, 4,$$

where w_i' are the measurement noises, c denotes the propagation speed and

$$d_i = \|\mathbf{x} - \mathbf{x}_i\|_2 = \sqrt{(x - x_i)^2 + (y - y_i)^2}$$

is the distance between the Transvahan and the receiver at the ith anchor location. By multiplying by the propagation speed, we get noisy range measurements

$$r_i = cT_0 + d_i + w_i, \qquad i = 1, 2, 3, 4, \tag{1}$$

where w_i is white Gaussian noise (WGN) with zero mean and variance σ^2. Then the covariance matrix of the noise $\mathbf{w} = [w_1, w_2, w_3, w_4]^T$ will be $\mathbf{C} = E\{\mathbf{w}\mathbf{w}^T\} = \sigma^2\mathbf{I}$.

Since we might not always know T_0, we will eliminate T_0 by working with the range diferences

$$r_{ij} = d_i - d_j + n_{ij}, \qquad i, j = 1, 2, 3, 4, i < j \tag{2}$$

where $n_{ij} = w_i - w_j$.

Our aim is to estimate from the available measurements, r_{ij} for $i, j = 1, 2, 3, 4, i < j$ the location $\mathbf{x}$ of the Transvahan. Here, each range measurement is a hyperbolic curve in the 2-D plane of possible locations $\mathbf{x}$, and the combination of measurements allows us to find the intersection of these curves. This is called *multilateration.*

Part A: Derivation and modeling

1. Compute the Cramér-Rao lower bound for $\mathbf{x}$ based on the measurement model in (2). Note that n_{ij} is no more WGN.

2. Derive the maximum likelihood (ML) estimator for $\mathbf{x}$ and give the update equations to iteratively solve the ML optimization problem.

3. Since the ML estimator requires solving a system of nonlinear equations, we can transform (2) to a linear system, linear in $[\mathbf{x}, d_j]^{\mathrm{T}}$ as

   $$r_{ij}^2 + d_j^2 + 2r_{ij}d_j = d_i^2 + e_{ij}, \qquad i, j = 1, 2, 3, 4, i < j \tag{3}$$

   with $d_i^2 = \|\mathbf{x} - \mathbf{x}_i\|_2^2 = \|\mathbf{x}\|^2 + \|\mathbf{x}_i\|^2 - 2\mathbf{x}_i^{\mathrm{T}}\mathbf{x}$. Although the above linearization will make the noise depend on the unknown noiseless range, we will approximate it as $e_{ij} = n_{ij}^2 = (w_i - w_j)^2$.

   Express compactly using the matrix-vector notation the above equations as a linear model with vector parameter $\boldsymbol{\theta} = [\mathbf{x}, d_2, d_3, d_4]^{\mathrm{T}}$ and derive the best linear unbiased estimator (BLUE) for $\boldsymbol{\theta}$.

   Note: e_{ij} will not have zero mean anymore, but since the mean will be know you should subtract the mean of e_ij from the above measurements. You will have to compute the covariance matrix of the noises in (3) to use in the BLUE though.

Part B: Implementation

Unzip the .zip file to load the dataset. The readme text file has the description of the variables. The dataset contains 1000 realizations of the range measurements as in (1) with $T_0 = 0$ for 10 different values of σ^2.

1. Make Matlab subroutine to implement the MLE that you derived in part A

   ```
   function x = mle_tdoa(noisy_distances(:,1,1),anchor_location,x_0)
   ```

   where `x_0` is the initial guess; initialize with $[0, 0]^{\mathrm{T}}$. Plot the estimated location of the Transvahan on the image of the campus (similar to the one of the first page) provided in the dataset.

2. Make Matlab subroutine to implement the BLUE that you derived in part A

   ```
   function x = blue_tdoa(noisy_distances(:,1,1),anchor_location)
   ```

   Overlay the estimated location of the Transvahan on the previous image to compare the location estimates of MLE and BLUE.

3. Plot the mean squared error, $\|\mathbf{x} - \hat{\mathbf{x}}\|_2^2$, averaged over 1000 realizations of MLE and BLUE and compare it to the Cramer-Rao lower bound that you derived in the previous section for 10 different values of σ^2 provided. $\hat{\mathbf{x}}$ denotes the estimated locations obtained either with MLE or BLUE.