

## Assignment #1

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## PART A: Problem 1 Cramer-Rao Lower Bound (CRLB)

**(Solution)** From the range difference equations we have,

$$r_{ij} = d_i - d_j + n_{ij}$$

$$n_{ij} = w_i - w_j$$

We will use a set of non redundant range equations,

$$r_{12} = d_1 - d_2 + n_{12}$$

$$r_{23} = d_2 - d_3 + n_{23}$$

$$r_{34} = d_3 - d_4 + n_{34}$$

Concisely in vector notation we can write as,

$$\begin{bmatrix} r_{12} \\ r_{23} \\ r_{34} \end{bmatrix} = \begin{bmatrix} d_1 - d_2 \\ d_2 - d_3 \\ d_3 - d_4 \end{bmatrix} + \begin{bmatrix} n_{12} \\ n_{23} \\ n_{34} \end{bmatrix}$$

$$\underline{\mathbf{r}} = \underline{\mathbf{h}}(\underline{\mathbf{x}}) + \underline{\mathbf{n}}$$

$$\underline{\mathbf{r}} \sim \mathcal{N}(\underline{\mathbf{h}}(\underline{\mathbf{x}}), \underline{\Sigma}_{\mathbf{n}})$$

where  $\underline{\Sigma}_{\mathbf{n}}$  is the noise covariance matrix, given by,

$$\underline{\mathbf{n}} = \begin{bmatrix} n_{12} \\ n_{23} \\ n_{34} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}$$

$$\underline{\mathbf{n}} = \underline{\mathbf{A}}\underline{\mathbf{w}}$$

$$\underline{\Sigma}_{\mathbf{n}} = E[\underline{\mathbf{n}}\underline{\mathbf{n}}^T]$$

$$= E[\underline{\mathbf{A}}\underline{\mathbf{w}}\underline{\mathbf{w}}^T \underline{\mathbf{A}}]$$

$$= \underline{\mathbf{A}}E[\underline{\mathbf{w}}\underline{\mathbf{w}}^T] \underline{\mathbf{A}}^T$$

$$= \underline{\mathbf{A}}\underline{\Sigma}_{\mathbf{w}} \underline{\mathbf{A}}^T$$

With this we can write the likelihood and log-likelihood functions as,

$$P(\underline{\mathbf{r}}; \underline{\mathbf{x}}) = \frac{1}{(2\pi)^{3/2} |\underline{\Sigma}_{\mathbf{n}}|^{1/2}} \exp\left(-\frac{1}{2}(\underline{\mathbf{r}} - \underline{\mathbf{h}}(\underline{\mathbf{x}}))^T \underline{\Sigma}_{\mathbf{n}}^{-1} (\underline{\mathbf{r}} - \underline{\mathbf{h}}(\underline{\mathbf{x}}))\right)$$

$$\ln P(\underline{\mathbf{r}}; \underline{\mathbf{x}}) = K - \frac{1}{2}(\underline{\mathbf{r}} - \underline{\mathbf{h}}(\underline{\mathbf{x}}))^T \underline{\Sigma}_{\mathbf{n}}^{-1} (\underline{\mathbf{r}} - \underline{\mathbf{h}}(\underline{\mathbf{x}}))$$

$$\begin{aligned}
&= K - \frac{1}{2} \sum_{i=1}^3 \sum_{j=1}^3 (r_i - h_i(\underline{\mathbf{x}})) [\boldsymbol{\Sigma}_{\mathbf{n}}^{-1}]_{ij} (r_j - h_j(\underline{\mathbf{x}})) \\
\frac{\partial}{\partial \underline{\mathbf{x}}} \ln P(\underline{\mathbf{r}}; \underline{\mathbf{x}}) &= \frac{1}{2} \sum_{i=1}^3 \sum_{j=1}^3 [\boldsymbol{\Sigma}_{\mathbf{n}}^{-1}]_{ij} (r_i - h_i(\underline{\mathbf{x}})) \frac{\partial h_j(\underline{\mathbf{x}})}{\partial \underline{\mathbf{x}}} + \frac{\partial h_i(\underline{\mathbf{x}})}{\partial \underline{\mathbf{x}}} (r_j - h_j(\underline{\mathbf{x}}))
\end{aligned}$$

We can find the elements of the Fischer Information Matrix one element at a time,

$$\frac{\partial^2}{\partial x^2} \ln P(\underline{\mathbf{r}}; \underline{\mathbf{x}}) = \frac{1}{2} \sum_{i=1}^3 \sum_{j=1}^3 [\boldsymbol{\Sigma}_{\mathbf{n}}^{-1}]_{ij} ((r_i - h_i(\underline{\mathbf{x}})) \frac{\partial^2 h_j(\underline{\mathbf{x}})}{\partial x^2} - \frac{\partial h_i(\underline{\mathbf{x}})}{\partial x} \frac{\partial h_j(\underline{\mathbf{x}})}{\partial x} + (r_j - h_j(\underline{\mathbf{x}})) \frac{\partial^2 h_i(\underline{\mathbf{x}})}{\partial x^2} - \frac{\partial h_j(\underline{\mathbf{x}})}{\partial x} \frac{\partial h_i(\underline{\mathbf{x}})}{\partial x})$$

$$\begin{aligned}
[\mathbf{I}(\underline{\mathbf{x}})]_{11} &= -E\left[\frac{\partial^2}{\partial x^2} \ln P(\underline{\mathbf{r}}; \underline{\mathbf{x}})\right] \\
&= \sum_{i=1}^3 \sum_{j=1}^3 [\boldsymbol{\Sigma}_{\mathbf{n}}^{-1}]_{ij} \frac{\partial h_i(\underline{\mathbf{x}})}{\partial x} \frac{\partial h_j(\underline{\mathbf{x}})}{\partial x} \\
&= \frac{\partial \underline{\mathbf{h}}(\underline{\mathbf{x}})}{\partial x}^T \boldsymbol{\Sigma}_{\mathbf{n}}^{-1} \frac{\partial \underline{\mathbf{h}}(\underline{\mathbf{x}})}{\partial x}
\end{aligned}$$

$$\begin{aligned}
[\mathbf{I}(\underline{\mathbf{x}})]_{12} &= -E\left[\frac{\partial^2}{\partial x \partial y} \ln P(\underline{\mathbf{r}}; \underline{\mathbf{x}})\right] \\
&= \sum_{i=1}^3 \sum_{j=1}^3 [\boldsymbol{\Sigma}_{\mathbf{n}}^{-1}]_{ij} \frac{\partial h_i(\underline{\mathbf{x}})}{\partial x} \frac{\partial h_j(\underline{\mathbf{x}})}{\partial y} \\
&= \frac{\partial \underline{\mathbf{h}}(\underline{\mathbf{x}})}{\partial x}^T \boldsymbol{\Sigma}_{\mathbf{n}}^{-1} \frac{\partial \underline{\mathbf{h}}(\underline{\mathbf{x}})}{\partial y}
\end{aligned}$$

$$\begin{aligned}
[\mathbf{I}(\underline{\mathbf{x}})]_{21} &= -E\left[\frac{\partial^2}{\partial y \partial x} \ln P(\underline{\mathbf{r}}; \underline{\mathbf{x}})\right] \\
&= \sum_{i=1}^3 \sum_{j=1}^3 [\boldsymbol{\Sigma}_{\mathbf{n}}^{-1}]_{ij} \frac{\partial h_i(\underline{\mathbf{x}})}{\partial y} \frac{\partial h_j(\underline{\mathbf{x}})}{\partial x} \\
&= \frac{\partial \underline{\mathbf{h}}(\underline{\mathbf{x}})}{\partial y}^T \boldsymbol{\Sigma}_{\mathbf{n}}^{-1} \frac{\partial \underline{\mathbf{h}}(\underline{\mathbf{x}})}{\partial x}
\end{aligned}$$

$$\begin{aligned}
[\mathbf{I}(\underline{\mathbf{x}})]_{22} &= -E\left[\frac{\partial^2}{\partial y \partial y} \ln P(\underline{\mathbf{r}}; \underline{\mathbf{x}})\right] \\
&= \sum_{i=1}^3 \sum_{j=1}^3 [\boldsymbol{\Sigma}_{\mathbf{n}}^{-1}]_{ij} \frac{\partial h_i(\underline{\mathbf{x}})}{\partial y} \frac{\partial h_j(\underline{\mathbf{x}})}{\partial y} \\
&= \frac{\partial \underline{\mathbf{h}}(\underline{\mathbf{x}})}{\partial y}^T \boldsymbol{\Sigma}_{\mathbf{n}}^{-1} \frac{\partial \underline{\mathbf{h}}(\underline{\mathbf{x}})}{\partial y}
\end{aligned}$$

where,

$$\frac{\partial \underline{\mathbf{h}}(\underline{\mathbf{x}})}{\partial y} = \begin{bmatrix} \frac{\partial h_1(\underline{\mathbf{x}})}{\partial y} & \frac{\partial h_2(\underline{\mathbf{x}})}{\partial y} & \frac{\partial h_3(\underline{\mathbf{x}})}{\partial y} \end{bmatrix}$$

$$\begin{aligned}
\frac{\partial h_i(\underline{\mathbf{x}})}{\partial x} &= \frac{\partial}{\partial x} (d_i - d_{i+1}) \\
&= \frac{\partial}{\partial x} (\|\mathbf{x} - \mathbf{x}_i\|_2 - \|\mathbf{x} - \mathbf{x}_{i+1}\|_2) \\
&= \frac{x - x_i}{\|\mathbf{x} - \mathbf{x}_i\|_2} - \frac{x - x_{i+1}}{\|\mathbf{x} - \mathbf{x}_{i+1}\|_2} \\
&= \frac{x - x_i}{d_i} - \frac{x - x_{i+1}}{d_{i+1}}
\end{aligned}$$

Given an  $\underline{\mathbf{x}}$ , we can find the elements of Fischer Information Matrix and invert it to get the CRLB.

$$\mathbf{C}_{\hat{\mathbf{x}}} \geq [\mathbf{I}^{-1}(\underline{\mathbf{x}})]$$

$$Var[\hat{x}] \geq [\mathbf{I}^{-1}(\underline{\mathbf{x}})]_{11}$$

$$Var[\hat{y}] \geq [\mathbf{I}^{-1}(\underline{\mathbf{x}})]_{22}$$

**PART A: Problem 2 Maximum Likelihood Estimator (MLE)**

**(Solution)** We want obtain the maximum likelihood estimator (MLE) for  $\underline{\mathbf{x}}$ . The estimator that aximizes likelihood function also maximizes log-likelihood, since log function is a monotonically increasing function.

We have our likelihood and log-likelihood functions from Problem 1 as,

$$\begin{aligned}
 P(\underline{\mathbf{r}}; \underline{\mathbf{x}}) &= \frac{1}{(2\pi)^{3/2} |\Sigma_n|^{1/2}} \exp\left(-\frac{1}{2}(\underline{\mathbf{r}} - \underline{\mathbf{h}}(\underline{\mathbf{x}}))^T \Sigma_n^{-1} (\underline{\mathbf{r}} - \underline{\mathbf{h}}(\underline{\mathbf{x}}))\right) \\
 \ln P(\underline{\mathbf{r}}; \underline{\mathbf{x}}) &= K - \frac{1}{2}(\underline{\mathbf{r}} - \underline{\mathbf{h}}(\underline{\mathbf{x}}))^T \Sigma_n^{-1} (\underline{\mathbf{r}} - \underline{\mathbf{h}}(\underline{\mathbf{x}})) \\
 \hat{\underline{\mathbf{x}}} &= \arg \max_{\underline{\mathbf{x}}} \ln P(\underline{\mathbf{r}}; \underline{\mathbf{x}}) \\
 &= \arg \max_{\underline{\mathbf{x}}} K - \frac{1}{2}(\underline{\mathbf{r}} - \underline{\mathbf{h}}(\underline{\mathbf{x}}))^T \Sigma_n^{-1} (\underline{\mathbf{r}} - \underline{\mathbf{h}}(\underline{\mathbf{x}})) \\
 &= \arg \min_{\underline{\mathbf{x}}} \frac{1}{2}(\underline{\mathbf{r}} - \underline{\mathbf{h}}(\underline{\mathbf{x}}))^T \Sigma_n^{-1} (\underline{\mathbf{r}} - \underline{\mathbf{h}}(\underline{\mathbf{x}})) \\
 &= \arg \min_{\underline{\mathbf{x}}} J(\underline{\mathbf{x}})
 \end{aligned}$$

We can proceed with this minimization as an optimization problem, using any of the available descent algorithms. We will use a simple iterative gradient descent algorithm with a fixed step size  $\alpha$  such that  $0 < \alpha < \frac{2}{\lambda_{max}}$ , where  $\lambda_{max}$  is the largest eigenvalue of  $\Sigma_n^{-1}$  (for which the gradient descent theoretically converges if objective function is Lipschitz).

$$\begin{aligned}
 \hat{\underline{\mathbf{x}}}^{(k+1)} &= \hat{\underline{\mathbf{x}}}^{(k)} - \alpha \nabla J(\underline{\mathbf{x}}) \\
 &= \hat{\underline{\mathbf{x}}}^{(k)} + \alpha \nabla \underline{\mathbf{h}}(\underline{\mathbf{x}})^T \Sigma_n^{-1} (\underline{\mathbf{r}} - \underline{\mathbf{h}}(\underline{\mathbf{x}}))
 \end{aligned}$$

where,

$$\nabla \underline{\mathbf{h}}(\underline{\mathbf{x}}) = \frac{\partial \underline{\mathbf{h}}(\underline{\mathbf{x}})}{\partial \underline{\mathbf{y}}} = \begin{bmatrix} \frac{\partial h_1(\underline{\mathbf{x}})}{\partial x} & \frac{\partial h_2(\underline{\mathbf{x}})}{\partial x} & \frac{\partial h_3(\underline{\mathbf{x}})}{\partial x} \end{bmatrix}$$

$$\begin{aligned}
 \frac{\partial h_i(\underline{\mathbf{x}})}{\partial x} &= \frac{\partial}{\partial x} (d_i - d_{i+1}) \\
 &= \frac{\partial}{\partial x} (\|\mathbf{x} - \mathbf{x}_i\|_2 - \|\mathbf{x} - \mathbf{x}_{i+1}\|_2) \\
 &= \frac{x - x_i}{\|\mathbf{x} - \mathbf{x}_i\|_2} - \frac{x - x_{i+1}}{\|\mathbf{x} - \mathbf{x}_{i+1}\|_2} \\
 &= \frac{x - x_i}{d_i} - \frac{x - x_{i+1}}{d_{i+1}}
 \end{aligned}$$

**PART A: Problem 3 Best Linear Unbiased Estimator (BLUE) estimator for  $\theta$** 

**(Solution)** From the range difference equations we have,

$$\begin{aligned} r_{ij} &= d_i - d_j + n_{ij} \\ r_{ij} + d_j &= d_i + n_{ij} \\ r_{ij}^2 + 2r_{ij}d_j + d_j^2 &= d_i^2 + e_{ij} \end{aligned}$$

where  $d_i$  and  $e_{ij}$  are given by,

$$\begin{aligned} d_i^2 &= \|\mathbf{x} - \mathbf{x}_i\|_2^2 = \|\mathbf{x}\|^2 + \|\mathbf{x}_i\|^2 - 2\mathbf{x}_i^T \mathbf{x} \\ e_{ij} &= n_{ij}^2 = (w_i - w_j)^2 \end{aligned}$$

Substituting in range equation,

$$\begin{aligned} r_{ij}^2 &= d_i^2 - d_j^2 - 2r_{ij}d_j + e_{ij} \\ r_{ij}^2 &= (\|\mathbf{x}\|^2 + \|\mathbf{x}_i\|^2 - 2\mathbf{x}_i^T \mathbf{x}) - (\|\mathbf{x}\|^2 + \|\mathbf{x}_j\|^2 - 2\mathbf{x}_j^T \mathbf{x}) - 2r_{ij}d_j + e_{ij} \\ r_{ij}^2 &= \|\mathbf{x}_i\|^2 - \|\mathbf{x}_j\|^2 - 2(\mathbf{x}_i - \mathbf{x}_j)^T \mathbf{x} - 2r_{ij}d_j + e_{ij} \end{aligned}$$

Let  $\gamma_{ij} = \|\mathbf{x}_i\|^2 - \|\mathbf{x}_j\|^2$ ,

$$r_{ij}^2 = \gamma_{ij} - 2(\mathbf{x}_i - \mathbf{x}_j)^T \mathbf{x} - 2r_{ij}d_j + e_{ij}$$

$$\begin{aligned} \begin{bmatrix} r_{12}^2 \\ r_{13}^2 \\ r_{14}^2 \\ r_{23}^2 \\ r_{24}^2 \\ r_{34}^2 \end{bmatrix} &= \begin{bmatrix} -2(\mathbf{x}_1 - \mathbf{x}_2)^T & -2r_{12} & 0 & 0 \\ -2(\mathbf{x}_1 - \mathbf{x}_3)^T & 0 & -2r_{13} & 0 \\ -2(\mathbf{x}_1 - \mathbf{x}_4)^T & 0 & 0 & -2r_{14} \\ -2(\mathbf{x}_2 - \mathbf{x}_3)^T & 0 & -2r_{23} & 0 \\ -2(\mathbf{x}_2 - \mathbf{x}_4)^T & 0 & 0 & -2r_{24} \\ -2(\mathbf{x}_3 - \mathbf{x}_4)^T & 0 & 0 & -2r_{34} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} + \begin{bmatrix} e_{12} \\ e_{13} \\ e_{14} \\ e_{23} \\ e_{24} \\ e_{34} \end{bmatrix} + \begin{bmatrix} \gamma_{12} \\ \gamma_{13} \\ \gamma_{14} \\ \gamma_{23} \\ \gamma_{24} \\ \gamma_{34} \end{bmatrix} \\ \begin{bmatrix} r_{12}^2 \\ r_{13}^2 \\ r_{14}^2 \\ r_{23}^2 \\ r_{24}^2 \\ r_{34}^2 \end{bmatrix} &= \begin{bmatrix} -2(x_1 - x_2)^T & -2(y_1 - y_2)^T & -2r_{12} & 0 & 0 \\ -2(x_1 - x_3)^T & -2(y_1 - y_3)^T & 0 & -2r_{13} & 0 \\ -2(x_1 - x_4)^T & -2(y_1 - y_4)^T & 0 & 0 & -2r_{14} \\ -2(x_2 - x_3)^T & -2(y_2 - y_3)^T & 0 & -2r_{23} & 0 \\ -2(x_2 - x_4)^T & -2(y_2 - y_4)^T & 0 & 0 & -2r_{24} \\ -2(x_3 - x_4)^T & -2(y_3 - y_4)^T & 0 & 0 & -2r_{34} \end{bmatrix} \begin{bmatrix} x \\ y \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} + \begin{bmatrix} e_{12} \\ e_{13} \\ e_{14} \\ e_{23} \\ e_{24} \\ e_{34} \end{bmatrix} + \begin{bmatrix} \gamma_{12} \\ \gamma_{13} \\ \gamma_{14} \\ \gamma_{23} \\ \gamma_{24} \\ \gamma_{34} \end{bmatrix} \end{aligned}$$

Concisely in vector notation,

$$\mathbf{r} = \mathbf{H}\theta + \mathbf{e} + \boldsymbol{\gamma}$$

where,

$$\underline{\mathbf{r}} = \begin{bmatrix} r_{12}^2 \\ r_{13}^2 \\ r_{14}^2 \\ r_{23}^2 \\ r_{24}^2 \\ r_{34}^2 \end{bmatrix}, \mathbf{H} = \begin{bmatrix} -2(x_1 - x_2)^T & -2(y_1 - y_2)^T & -2r_{12} & 0 & 0 \\ -2(x_1 - x_3)^T & -2(y_1 - y_3)^T & 0 & -2r_{13} & 0 \\ -2(x_1 - x_4)^T & -2(y_1 - y_4)^T & 0 & 0 & -2r_{14} \\ -2(x_2 - x_3)^T & -2(y_2 - y_3)^T & 0 & -2r_{23} & 0 \\ -2(x_2 - x_4)^T & -2(y_2 - y_4)^T & 0 & 0 & -2r_{24} \\ -2(x_3 - x_4)^T & -2(y_3 - y_4)^T & 0 & 0 & -2r_{34} \end{bmatrix}, \underline{\theta} = \begin{bmatrix} x \\ y \\ d_2 \\ d_3 \\ d_4 \end{bmatrix}, \underline{\mathbf{e}} = \begin{bmatrix} e_{12} \\ e_{13} \\ e_{14} \\ e_{23} \\ e_{24} \\ e_{34} \end{bmatrix}, \underline{\gamma} = \begin{bmatrix} \gamma_{12} \\ \gamma_{13} \\ \gamma_{14} \\ \gamma_{23} \\ \gamma_{24} \\ \gamma_{34} \end{bmatrix}$$

But here the mean of error  $\underline{\mathbf{e}}$  is no longer  $\underline{\mathbf{0}}$  and hence we should subtract the mean of the error from above observations to make error zero mean.

$$E[\underline{\mathbf{e}}] = \begin{bmatrix} E[e_{12}] \\ E[e_{13}] \\ E[e_{14}] \\ E[e_{23}] \\ E[e_{24}] \\ E[e_{34}] \end{bmatrix} = 2\sigma^2 \mathbf{1}_{6 \times 1}$$

$$\begin{aligned} E[e_{ij}] &= E[(w_i - w_j)^2] \\ &= \text{Var}[w_i - w_j] + E[w_i - w_j] \\ &= \text{Var}[w_i] + \text{Var}[w_j] \\ &= 2\sigma^2 \end{aligned}$$

We can now have a model with zero mean as,

$$\begin{aligned} \underline{\mathbf{r}} - \underline{\gamma} - 2\sigma^2 \mathbf{1}_{6 \times 1} &= \mathbf{H}\underline{\theta} + \underline{\mathbf{e}} - 2\sigma^2 \mathbf{1}_{6 \times 1} \\ \underline{\mathbf{r}}' &= \mathbf{H}\underline{\theta} + \underline{\mathbf{e}}' \\ \underline{\mathbf{r}}' &= \underline{\mathbf{r}} - \underline{\gamma} - 2\sigma^2 \mathbf{1}_{6 \times 1} \\ \underline{\mathbf{e}}' &= \underline{\mathbf{e}} - 2\sigma^2 \mathbf{1}_{6 \times 1} \\ \underline{\mathbf{e}}' &\sim \mathcal{N}(0, \mathbf{C}) \end{aligned}$$

where  $\mathbf{C}$  is the covariance matrix of  $\underline{\mathbf{e}}'$  which is given by,

$$\begin{aligned} \mathbf{C} &= E[(\underline{\mathbf{e}}' - E[\underline{\mathbf{e}}']) (\underline{\mathbf{e}}' - E[\underline{\mathbf{e}}'])^T] \\ &= E[\underline{\mathbf{e}}' \underline{\mathbf{e}}'^T] \\ &= E[(\underline{\mathbf{e}} - 2\sigma^2 \mathbf{1}_{6 \times 1})(\underline{\mathbf{e}} - 2\sigma^2 \mathbf{1}_{6 \times 1})^T] \\ &= E[\underline{\mathbf{e}} \underline{\mathbf{e}}^T - \underline{\mathbf{e}}(2\sigma^2 \mathbf{1}_{6 \times 1})^T - (2\sigma^2 \mathbf{1}_{6 \times 1}) \underline{\mathbf{e}}^T + (2\sigma^2 \mathbf{1}_{6 \times 1})(2\sigma^2 \mathbf{1}_{6 \times 1})^T] \\ &= E[\underline{\mathbf{e}} \underline{\mathbf{e}}^T] - (2\sigma^2 \mathbf{1}_{6 \times 1})(2\sigma^2 \mathbf{1}_{6 \times 1})^T \end{aligned}$$

We can find element by element of  $\mathbf{C}$  as,

$$\begin{aligned} [\mathbf{C}]_{11} &= E[e_{12}^2] - 4\sigma^4 \\ &= E[(w_1 - w_2)^2(w_1 - w_2)^2] - 4\sigma^4 \\ &= E[(w_1^2 - 2w_1w_2 + w_2^2)(w_1^2 - 2w_1w_2 + w_2^2)] - 4\sigma^4 \end{aligned}$$

$$\begin{aligned}
&= E[w_1^4 + w_2^4 + 6w_1^2w_2^2] - 4\sigma^4 \\
&= 3\sigma^4 + 3\sigma^4 + 6\sigma^4 - 4\sigma^4 \\
&= 8\sigma^4
\end{aligned}$$

$$\begin{aligned}
[\mathbf{C}]_{12} &= E[e_{12}e_{13}] - 4\sigma^4 \\
&= E[(w_1 - w_2)^2(w_1 - w_3)^2] - 4\sigma^4 \\
&= E[(w_1^2 - 2w_1w_2 + w_2^2)(w_1^2 - 2w_1w_3 + w_3^2)] - 4\sigma^4 \\
&= E[w_1^4 + w_1^2w_3^2 + w_1^2w_2^2 + w_2^2w_3^2] - 4\sigma^4 \\
&= 3\sigma^4 + \sigma^4 + \sigma^4 + \sigma^4 - 4\sigma^4 \\
&= 2\sigma^4
\end{aligned}$$

$$\begin{aligned}
[\mathbf{C}]_{13} &= E[e_{12}e_{14}] - 4\sigma^4 \\
&= 2\sigma^4
\end{aligned}$$

$$\begin{aligned}
[\mathbf{C}]_{14} &= E[e_{12}e_{23}] - 4\sigma^4 \\
&= 2\sigma^4
\end{aligned}$$

$$\begin{aligned}
[\mathbf{C}]_{15} &= E[e_{12}e_{24}] - 4\sigma^4 \\
&= 2\sigma^4
\end{aligned}$$

$$\begin{aligned}
[\mathbf{C}]_{16} &= E[e_{12}e_{34}] - 4\sigma^4 \\
&= E[(w_1 - w_2)^2(w_3 - w_4)^2] - 4\sigma^4 \\
&= E[(w_1^2 - 2w_1w_2 + w_2^2)(w_3^2 - 2w_3w_4 + w_4^2)] - 4\sigma^4 \\
&= E[w_1^2w_3^2 + w_1^2w_4^2 + w_2^2w_3^2 + w_2^2w_4^2] - 4\sigma^4 \\
&= \sigma^4 + \sigma^4 + \sigma^4 + \sigma^4 - 4\sigma^4 \\
&= 0
\end{aligned}$$

Similarly we compute the matrix  $\mathbf{C}$  by identifying the pattern,

$$\mathbf{C} = 2\sigma^2 \begin{bmatrix} 4 & 1 & 1 & 1 & 1 & 0 \\ 1 & 4 & 1 & 1 & 0 & 1 \\ 1 & 1 & 4 & 0 & 1 & 1 \\ 1 & 1 & 0 & 4 & 1 & 1 \\ 1 & 0 & 1 & 1 & 4 & 1 \\ 0 & 1 & 1 & 1 & 1 & 4 \end{bmatrix}$$

Hence the best linear unbiased estimator (BLUE estimator),

$$\hat{\underline{\theta}} = (\mathbf{H}^T \mathbf{C}^{-1} \mathbf{H})^{-1} \mathbf{H}^T \mathbf{C}^{-1} \underline{\mathbf{r}}'$$