Companion software for "Volker Ziemann, *Physics and Finance, Springer, 2021*" (https://link.springer.com/book/10.1007/978-3-030-63643-2)

Markowitz portfolio optimization (Section 3.2 and 3.3)

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This simulation illustrates Markowitz's theory of optimizing a portfolio of stocks by minimizing the variance of the random fluctuations of the portfolio. We assume that a finite amount of money is invested with ratios w_i and $\sum_i w_i = 1$ into the different stocks. This constraint is taken into account by a Lagrange multiplier λ_1. Likewise, the desired return $\rho = \sum_i w_i \langle r_i \rangle$ is treated as an additional constraint with the help of a second Lagrange multiplier λ_2. In Section 3.2 a portfolio with random stocks only is analyzed, while Section 3.3 treats a portfolio that also contains a risk-free asset.

In order to test the method, we first prepare arrays with the daily returns r_i of three stocks with $i = 1,2,3$ for a period of `N=200` trading days. The average returns are chosen to be 0.01, 0.02, and 0.03 with a random component of the same respective magnitude added. Note also that r_2 is correlated with r_1 and that r_3 with the other two stocks. With rng() we chose a seed for the random number generator in order to make the simulation random, yet reproducible.

```
clear all; close('all')
rng(1234)   % seed random number generator
N=200;      % number of trading days
r1=0.01*ones(N,1)+0.01*randn(N,1);
r2=0.02*ones(N,1)+0.02*randn(N,1)-0.3*r1;
r3=0.03*ones(N,1)+0.03*randn(N,1)+0.3*r1-0.5*r2;
```

Portfolio with risky assets only (Section 3.2)

For Equation 3.11 we need a column vector `ee=` $\vec{e}$ with three ones as entries, the averaged daily returns `ra=` $\langle r_i \rangle$ and the returns with teir average subtracted `rp=` $r_i - \langle r_i \rangle$. From the latter we calculate the covariance matrix C and its inverse.

```
ee=ones(3,1);
ra=[mean(r1); mean(r2) ; mean(r3)];   % average return
rp=[r1-mean(r1),r2-mean(r2),r3-mean(r3)]; % for the covariance matrix
C=rp'*rp/N  % covariance matrix
```

```
C = 3×3
    0.0001   -0.0000    0.0000
   -0.0000    0.0005   -0.0002
    0.0000   -0.0002    0.0012
```

```
CC=inv(C);  % and its inverse
```

Now we have all the ingredients to construct the matrix, here we call it A, that appears in Equation 3.16, and its inverse that appears in Equation 3.17.

```
A=[ra'*CC*ra , ee'*CC*ra ;     % matrix from eq. 3.16
```

```
    ra'*CC*ee , ee'*CC*ee]
```

```
A = 2×2
10^4 ×
    0.0003    0.0211
    0.0211    1.5975
```

```
AA=inv(A);                    % matrix from eq. 3.17
```

In the following loop we iterate over many different values of the desired return ρ, calculate the corresponding Lagrange multipliers from Equation 3.17 and the weights w_i from Equation 3.18, and store the value of ρ and the overall volatility of the portfolio $\sigma^2 = \vec{w}^T C \vec{w}$ where $\vec{w}$ is a column vector with the three weight w_i as entries adn σ^2 is defined just below Equation 3.18.

```
K=0;
for rho=0:0.0002:0.04;  %...loop over the desired returns
    lambda=AA*[rho;1];                    % eq. 3.17
    w=lambda(1)*CC*ra+lambda(2)*CC*ee;    % eq. 3.18
    K=K+1;
    rrho(K)=rho;
    sig(K)=sqrt(w'*C*w);    % definition of sigma
end
```

Finally we plot the portfolio return ρ versus the volatility σ as a black line and annottate the axes.

```
plot(sig,rrho,'k','LineWidth',2)
xlabel('Volatility \sigma')
ylabel('Portfolio return \rho')
```

Into the same figure we then plot the poiunt with the smallest volatility as a red asterisk.

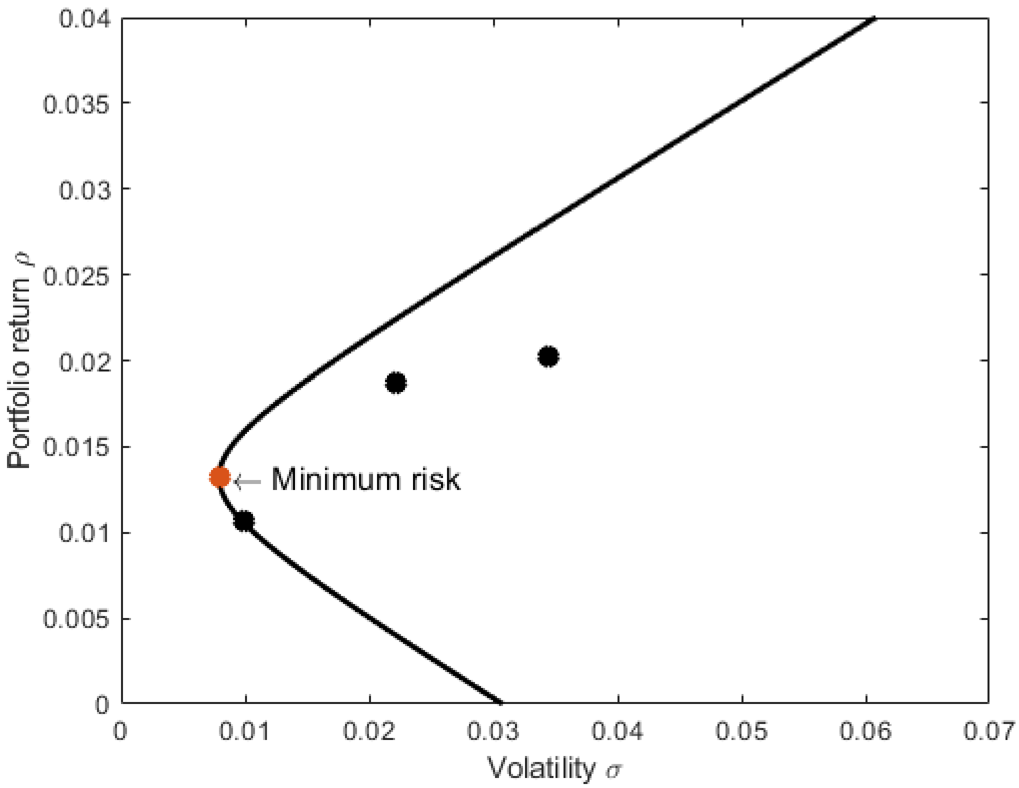
```
hold on
rhomin=-AA(1,2)/AA(1,1);          % eq.3.20
sigmin=sqrt(det(AA)/AA(1,1))      % eq.3.21
```

```
sigmin = 0.0079
```

```
plot(sigmin,rhomin,'*','LineWidth',6)
text(sigmin+0.001,rhomin,'\leftarrow Minimum risk','FontSize',12)
```

And then we place the volatilities and the average daily returns of the original stocks on the same plot.

```
plot(sqrt(C(1,1)),ra(1),'k*',sqrt(C(2,2)),ra(2),'k*',sqrt(C(3,3)),ra(3),'k*','LineWidth
```



We observe that the volatility of the minimum portfolio is actually slightly smaller than the smallest value if the individual volatilities--hedging and cancelling of correlated volatilities at work!

Portfolio with a risk-free asset (Section 3.3)

In the previous section, we inverted the covariance matrix C, but this is not possible if we have a risk-free asset in our portfolio, because one of the columns (and rows) of C contains zeros. So we have to considre this case separately here. The theory is outlined in Section 3.3. Here we assume that the risk-free return $r_f = 0.08$ and then calculate $\Delta r = \langle r \rangle - r_f \vec{e}$ (see top of page 22 in the book) and loop over the desired return ρ. Inside the loop we use Equation 3.28 to calculate the weights $\vec{w}$ for the different stocks and the resulting volatility.

```
r0=0.008;    % rate of return of the zero risk asset
deltar=ra-r0*ee;
K=0;
for rho=r0:0.0001:5*r0
    ww=(rho-r0)/(deltar'*CC*deltar)*CC*deltar;  % eq.5.22
    K=K+1;
    rrho2(K)=rho;
    sig2(K)=sqrt(ww'*C*ww);
end
```

Now we plot the return versus the volatility in the same plot as before, which is shown as the dashed red line, which is called the *Capital market line*, whose slope is referred to as the *Sharpe ratio*. See the discussion at the end of Section 3.3. We also place a red asterisk at the point at which the black line--the effcient frontier--and the red line meet; a portfolio with this mix of rsik-free and risky assets is called the tangent- or market portfolio. Please consult Sections 3.4 and 3.5 for further discussions.

```
plot(sig2,rrho2,'r--','LineWidth',2)
rhot=r0+(deltar'*CC*deltar)/(ee'*CC*deltar)
```

```
rhot = 0.0156
```

```
sigmat=sqrt((deltar'*CC*deltar)/(ee'*CC*deltar)^2)
```

```
sigmat = 0.0095
```

```
plot(sigmat,rhot,'r*','LineWidth',6)
text(sigmat+0.001,rhot-0.001,'\leftarrow Tangent/Market portfolio','FontSize',12)
text(0.035,0.027,'\leftarrow Efficient frontier','FontSize',12);
text(0.001,0.033,'Capital market line \rightarrow','FontSize',12)
```