

# PHYSICS 307 HOMEWORK 3

## Due Tuesday, 3 October, at 11 AM

1. First we will study a DE that we can solve analytically so we can compute error as a function of stepsize.

Newton's law of cooling says that the rate of temperature loss is proportional to the temperature difference between the cooling object and the ambient temperature  $T_a$ ; that is,

$$\frac{\partial T}{\partial t} = -k(T - T_a) \quad (1)$$

where  $k$  is a proportionality constant that depends on the materials involved and their surface area. The analytic solution to this DE is

$$T(t) = T_a + (T(0) - T_a)e^{-kt} \quad (2)$$

Consider a cooling cup of tea. When the water is removed from the heat, it is at  $100^\circ$ . Suppose that room temperature is  $30^\circ$ , and that  $k = 0.1\text{min}^{-1}$ .

- (a) Write a program that solves the differential equation above for the temperature  $T$  as a function of time using Euler's method. Plot this function vs. time for stepsizes  $dt$  of 1 minute, 10 seconds, and 1 second, along with the analytic solution. Does your numeric solution do a good job of capturing the behavior of the system?
- (b) Now, investigate the error in a rigorous way. Compute the temperature after five minutes for a range of stepsizes  $dt$ , and make a log-log plot of the error vs. the stepsize. Is the scaling what you expect it to be?

Note: You should actually solve the DE numerically using Euler's method; the exact solution given above is just so you can compute the error. This is the second-to-last thing we will study in this class that has an easy, obvious analytic solution!

2. Repeat the preceding problem with the second-order Runge-Kutta method.
3. Now, using RK2, answer the question "How long does it take the tea to cool to  $T_f = 70^\circ$ ?" Make a plot of the error in your result vs. stepsize and verify that you're seeing second-order scaling. Note that in order to achieve second-order accuracy, you will need to correct for "overshoot", as we will discuss in class.

Briefly, if you discover that the temperature is  $T_1$  (greater than 70 degrees) at  $t_1$ , and  $T_2$  (less than 70 degrees) at  $t_2$ , in order to achieve second-order precision you need to interpolate to figure out what time you actually hit 70 degrees at. That time is given by

$$t_1 + \frac{t_1 - t_2}{T_1 - T_2}(T_1 - T_f) \quad (3)$$

or alternatively

$$t_1 + \left( \frac{\partial T}{\partial t} \Big|_{t_1} \right)^{-1} (T_1 - T_f) \quad (4)$$

(I find the second more intuitive and easier to apply, since you already have been calculating  $\frac{\partial T}{\partial t}$  many times already.)

4. Now we will solve a DE that we can't do with pen and paper. The equation of motion for a pendulum is

$$\frac{\partial^2 \theta}{\partial t^2} = -\frac{g}{L} \sin \theta \quad (5)$$

In mechanics class you solved this by taking the small-angle approximation  $\sin \theta \approx \theta$ ; the equation then has a solution

$$\theta(t) = A \sin(\omega t + \phi) \quad (6)$$

where  $\omega = \sqrt{\frac{g}{L}}$  giving a period  $T = 2\pi\sqrt{\frac{L}{g}}$ . This is valid only in the limit  $\theta \rightarrow 0$ .

This equation is very difficult to solve without making this approximation using pen and paper, but you have a computer!

- (a) Without making the small-angle approximation, write a computer program that solves Newton's law (rotational form) to compute the oscillation of a swinging pendulum using the RK2 algorithm. (It may be helpful to code Euler as a first step.)
- (b) Animate your pendulum using `anim`.
- (c) Modify your program to determine the period of the pendulum and print it out. You can determine when a period has elapsed by looking for sign changes in  $\omega$ .
- (d) Suppose a pendulum clock keeps accurate time when  $\theta_{\max} = 5^\circ$ . How many seconds will it gain or lose per day if it is swinging at an angle of  $\theta_{\max} = 20^\circ$ ?
- (e) Now, make a plot of the fractional deviation in the period, defined as  $\Delta = \frac{T - T_0}{T_0}$ . Here  $T_0$  is the small-angle-limiting period,  $T_0 = 2\pi\sqrt{\frac{L}{g}}$ . Calculate this for a range of  $\theta_{\max}$  from  $10^{-5}$  to 2 (radians). Make a log-log plot of  $\Delta$  vs.  $\theta_{\max}$ . Based on what you know about the power series expansion of  $\sin(x)$ , comment on its appearance. Is it what you expect?