# Physics 307 Homework 3

## Due Tuesday, 3 October, at 11 AM

1. First we will study a DE that we can solve analytically so we can compute error as a function of stepsize.

   Newton's law of cooling says that the rate of temperature loss is proportional to the temperature difference between the cooling object and the ambient temperature $T_a$; that is,

   $$\frac{\partial T}{\partial t} = -k(T - T_a) \tag{1}$$

   where $k$ is a proportionality constant that depends on the materials involved and their surface area. The analytic solution to this DE is

   $$T(t) = T_a + (T(0) - T_a)e^{-kt} \tag{2}$$

   Consider a cooling cup of tea. When the water is removed from the heat, it is at 100°. Suppose that room temperature is 30°, and that $k = 0.1\text{min}^{-1}$.

   (a) Write a program that solves the differential equation above for the temperature $T$ as a function of time using Euler's method. Plot this function vs. time for stepsizes $dt$ of 1 minute, 10 seconds, and 1 second, along with the analytic solution. Does your numeric solution do a good job of capturing the behavior of the system?

   (b) Now, investigate the error in a rigorous way. Compute the temperature after five minutes for a range of stepsizes $dt$, and make a log-log plot of the error vs. the stepsize. Is the scaling what you expect it to be?

   Note: You should actually solve the DE numerically using Euler's method; the exact solution given above is just so you can compute the error. This is the second-to-last thing we will study in this class that has an easy, obvious analytic solution!

2. Repeat the preceding problem with the second-order Runge-Kutta method.

3. Now, using RK2, answer the question "How long does it take the tea to cool to $T_f = 70°$?" Make a plot of the error in your result vs. stepsize and verify that you're seeing second-order scaling. Note that in order to achieve second-order accuracy, you will need to correct for "overshoot", as we will discuss in class.

   Briefly, if you discover that the temperature is $T_1$ (greater than 70 degrees) at $t_1$, and $T_2$ (less than 70 degrees) at $t_2$, in order to achieve second-order precision you need to interpolate to figure out what time you actually hit 70 degrees at. That time is given by

$$t_1 + \frac{t_1 - t_2}{T_1 - T_2}(T_1 - T_f) \tag{3}$$

or alternatively

$$t_1 + \left( \left. \frac{\partial T}{\partial t} \right|_{t_1} \right)^{-1} (T_1 - T_f) \tag{4}$$

(I find the second more intuitive and easier to apply, since you already have been calculating $\frac{\partial T}{\partial t}$ many times already.)

4. Now we will solve a DE that we can't do with pen and paper. The equation of motion for a pendulum is

$$\frac{\partial^2 \theta}{\partial t^2} = -\frac{g}{L} \sin \theta \tag{5}$$

In mechanics class you solved this by taking the small-angle approximation $\sin \theta \approx \theta$; the equation then has a solution

$$\theta(t) = A \sin(\omega t + \phi) \tag{6}$$

where $\omega = \sqrt{\frac{g}{L}}$ giving a period $T = 2\pi\sqrt{\frac{L}{g}}$. This is valid only in the limit $\theta \to 0$.

This equation is very difficult to solve without making this approximation using pen and paper, but you have a computer!

(a) Without making the small-angle approximation, write a computer program that solves Newton's law (rotational form) to compute the oscillation of a swinging pendulum using the RK2 algorithm. (It may be helpful to code Euler as a first step.)

(b) Animate your pendulum using `anim`.

(c) Modify your program to determine the period of the pendulum and print it out. You can determine when a period has elapsed by looking for sign changes in $\omega$.

(d) Suppose a pendulum clock keeps accurate time when $\theta_{\text{max}} = 5°$. How many seconds will it gain or lose per day if it is swinging at an angle of $\theta_{\text{max}} = 20°$?

(e) Now, make a plot of the fractional deviation in the period, defined as $\Delta = \frac{T - T_0}{T_0}$. Here $T_0$ is the small-angle-limiting period, $T_0 = 2\pi\sqrt{\frac{L}{g}}$. Calculate this for a range of $\theta_{\text{max}}$ from $10^{-5}$ to 2 (radians). Make a log-log plot of $\Delta$ vs. $\theta_{\text{max}}$. Based on what you know about the power series expansion of $\sin(x)$, comment on its appearance. Is it what you expect?

(f) **Graduate students and ambitious undergraduates:** Looking at the animation from the previous, you'll notice that the pendulum still oscillates essentially sinusoidally, but with a frequency that now depends on the amplitude.

This information still doesn't let us get an exact solution to the differential equation

$$\ddot{\theta} = -k^2 \sin\theta$$

where $k^2 = \sqrt{g/L}$ for simplicity.

However, for small amplitude, we can obtain an approximate solution using perturbation theory. You can do this by expanding everything out to next-to-next-to-leading-order (written NNLO) in power series in $\theta$ – both the DE and your guess for its solution.

This means that the differential equation becomes

$$\ddot{\theta} = -k^2 \left(\theta - \frac{1}{6}\theta\right).$$

What about its solution? You might guess that it's still sinusoidal (which you can write as a complex exponential for simplicity), but now the frequency is also a power series in the amplitude $A$.

and your guess at its solution can be written

$$\theta(t) = A \exp\left[i\left(\omega_0 + A\omega_1 + A^2\omega_2\right)t\right]$$