Physics 307 Project 3

Due Tuesday, 2 October, by 5 PM

1. First we will study a DE that we can solve analytically so we can compute error as a function of stepsize.

   Consider a cooling cup of tea. When the water is removed from the heat, it is at 100°. Suppose that room temperature is 30°, and that $k = 0.1\text{min}^{-1}$.

   Newton's law of cooling says that the rate of temperature loss is proportional to the temperature difference between the cooling object and the ambient temperature T_a; that is,

$$\frac{\partial T}{\partial t} = -k(T - T_a) \tag{1}$$

   where k is a proportionality constant that depends on the materials involved and their surface area. The analytic solution to this DE is

$$T(t) = T_a + (T(0) - T_a)e^{-kt} \tag{2}$$

   Important note: The point of these methods – the Euler and RK2 algorithms – is that you can use them to solve differential equations where you do not know the analytic solution, or where one doesn't exist. In the previous assignment, you used the computer to calculate $\int_0^2 \sin(x)dx$. You didn't need to know that the analytic answer was $1 - \cos(2)$ to do this. Instead, the only thing you used this analytic solution for was to calculate the error in your numerical approximation.

   In this assignment, the same idea applies. You are not putting this into your Euler-algorithm solver; instead, you are letting the *computer* solve the differential equation, and comparing the answer to this to see how accurate your numeric solution is.

   For most of this class, we will be dealing with situations where there actually *is* no analytic solution – like the example from *Hidden Figures*, where Katherine points out that there is no analytic solution to the problem they are trying to solve.

   (a) Write a program that solves the differential equation above for the temperature T as a function of time using Euler's method. Plot this function vs. time for stepsizes dt of 1 minute, 10 seconds, and 1 second, along with the analytic solution. Does your numeric solution do a good job of capturing the behavior of the system?

   (b) Now, investigate the error in a rigorous way. Using a range of stepsizes dt, compute the temperature after five minutes and compute the error in each value. Make a log-log plot of the error vs. the stepsize. Is the scaling what you expect it to be?

   Note: You should actually solve the DE numerically using Euler's method; the exact solution given above is just so you can compute the error. This is the second-to-last thing we will study in this class that has an easy, obvious analytic solution!

2. Repeat the preceding problem with the second-order Runge-Kutta method. Comment on the accuracy of the Euler and RK2 methods.

3. Now, using RK2, answer the question "How long does it take the tea to cool to $T_f = 70°$?" Make a plot of the error in your result vs. stepsize and verify that you're seeing second-order scaling.

   Note that in order to achieve second-order accuracy, you will need to correct for "overshoot", as we will discuss in class.

   Briefly, if you discover that the temperature is T_1 (less than 70 degrees) at t_1, and you are trying to find the time t_f when the temperature was equal to T_f (70 degrees), in order to achieve second-order precision you need to interpolate to figure out at what time you actually hit 70 degrees. That time is given by

$$t_f = t_1 + \left(\left. \frac{\partial T}{\partial t} \right|_{t_1} \right)^{-1} (T_f - T_1) \tag{3}$$