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Project 5 | 10/17/18

Introduction

In the last project, the ability to solve differential equations using a computer was extended from only first-order differential equations to include second-order ones as well. This project focuses on finding a computational solution for planetary motion, especially that of the Earth, which is described by second-order differential equations in two dimensions:

$$\frac{dx}{dt} = v_x \qquad \frac{dy}{dt} = v_y$$

$$\frac{dv_x}{dt} = \frac{-GMx}{r^3} \qquad \frac{dv_y}{dt} = \frac{-GMy}{r^3}$$

In these equations, G is the gravitational constant and M is the mass of the sun. The solution will be tested in terms of its conformity to actual phenomena (such as how far the Earth travels in a year) and to the law of conservation of energy. Initially, the orbit of the Earth will be approximated as circular, but some adjustments will be made to observe other orbits and even to model one effect of general relativity.

Procedure

1. First, the units to be used are chosen such that they make the solution as simple as possible for a circular orbit. This means that distance is measured in astronomical units (AU) because 1 AU is equal to the radius of the Earth's orbit around the sun, mass is measured in solar masses, and time is measured in years because the period of the Earth's orbit is 1 year. Using these units, the product GM that appears in the equations is equal to $4\pi^2$. The Earth is then given an initial position of x equal to 1 AU and y equal to 0 AU relative to a Sun placed at the origin because that gives a radius of 1 AU. When the Earth is located at this point, its velocity, being tangential to the orbit, is all in the y-direction, and it is equal to 2π AU/yr since the Earth travels the circumference of a circle with a radius of 1 AU every year. See Figure 1 for an illustration of the initial conditions assigned to this simulation.

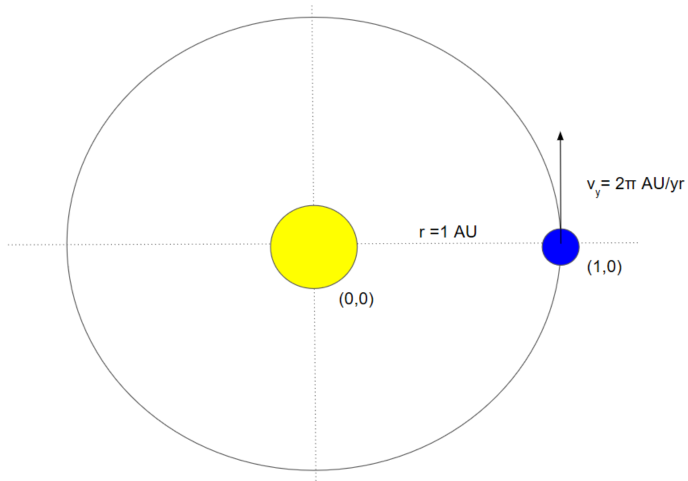

Figure 1: This image shows the initial conditions given to the Earth relative to the Sun at the beginning of the computer simulation used to model its orbit. The Cartesian axes have units of AU.

2. The numerical solution that describes Earth's orbit is computed after each of a series of time steps using the leapfrog second-order symplectic integrator. The x- and y-coordinates of Earth are printed at the beginning of each time step, and this output can be fed to anim to produce a simulation of planetary motion.
3. The simulation is tested for accuracy. First, the distance of the Earth from the Sun is monitored to determine whether it stays consistently at 1 AU as it should. Some small error can be seen, and it is examined more rigorously for different time steps by computing the distance as found by the computer after 200 years for each time step and calculating a percent error (($r_{measured}$-1)/1 * 100) that is plotted versus the step. This produces the plot shown in Figure 2. The error grows with the time step, as expected, but it remains very small. Even when the time step is 0.025 years, which is more than a week, the error is only 1%.

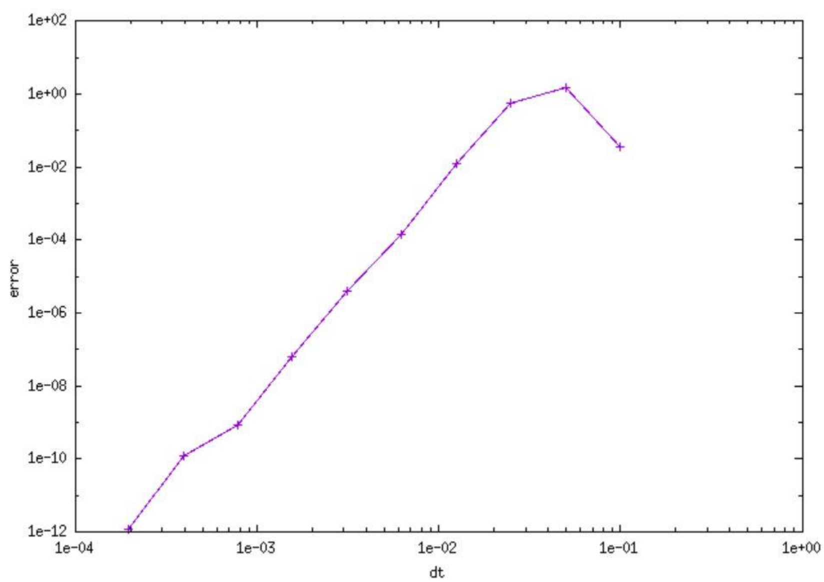

Figure 2: This graph shows the percent error in the computed distance of the Earth from the Sun after 200 years plotted versus the time step in years. It has logarithmic axes.

The next accuracy check performed is related to the period. If the simulation is perfect, the Earth should travel a full 2π radians in one year. To test the angle subtended by a year's orbit, the angle is found between the starting point of the Earth and its location after one year. This is done by forming a triangle with its vertices at the Sun and the Earth's initial and final positions. The side connecting the Sun to the Earth's initial position has a length of 1 AU. The length of the side connecting the Sun to the Earth's final position can be found by computing the radius using the final x- and y-values. The length of the last side can be found by adding the differences between the initial and final x- and y-values in quadrature. Knowing all three lengths, the angle can be found using the law of cosines. This angle is then subtracted from 2π to find the total angle subtended, whose absolute value is divided by 2π to find the fraction of a full orbit that the Earth travels in a year. This is repeated for different time steps and plotted as shown in Figure 3. For small time steps, the Earth completes very close to 100% of a full orbit. For larger time steps, it falls short of the expected 2π by several percentage points.

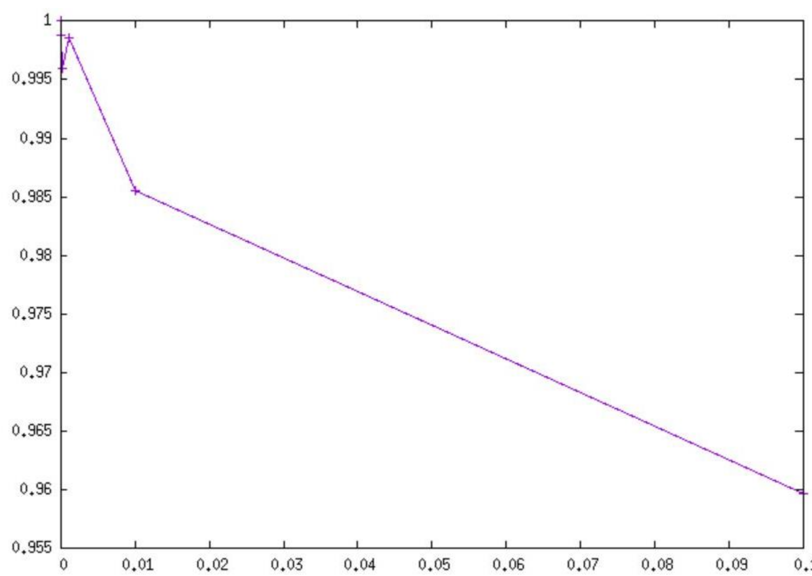

Figure 3: This plot shows the fraction of a full orbit traveled by the Earth in the computer simulation created for this project as function of the time step used. The x-axis is the time step in years and the y-axis is the unitless fraction of a full orbit traveled.

The final test applied to the simulation is that of energy conservation. The total energy for the Earth in orbit can be found as the sum of its kinetic energy and potential energy. The kinetic energy can be calculated using the familiar $\frac{1}{2}m(v_x{}^2 + v_y{}^2)$, where m is the mass of the Earth. The potential energy is equal to the work done to bring a planet from a defined radius out to infinity, calculated using the integral from r to infinity of the gravitational force, $\frac{GMm}{r^2}$, multiplied by the change in distance, dr. This yields $\frac{-GMm}{r}$ as the potential energy of the Earth orbiting the Sun. Therefore, the total energy is equal to

$\frac{1}{2}m(v_x{}^2 + v_y{}^2) - \frac{GMm}{r}$. Since the mass of the Earth in solar masses would be a messy number, a term called the specific orbital energy is defined as the total energy per unit mass, allowing the m term to drop out and leaving $E_{spec} = \frac{1}{2}(v_x{}^2 + v_y{}^2) - \frac{GM}{r}$. The specific orbital energy of the system is printed on the simulation, and it appears to just fluctuate. To analyze it more quantitatively, the specific orbital energy is computed initially and after 200 years for various time steps. For each step, the difference between the energies is taken, resulting in the plot in Figure 4. The energy gained by the system generally increases for increases in time step size. This is logical because, when the step size is too large, the simulation shows the Earth flying away from the Sun, indicating that is has gained energy due to computer error and escaped its orbit.

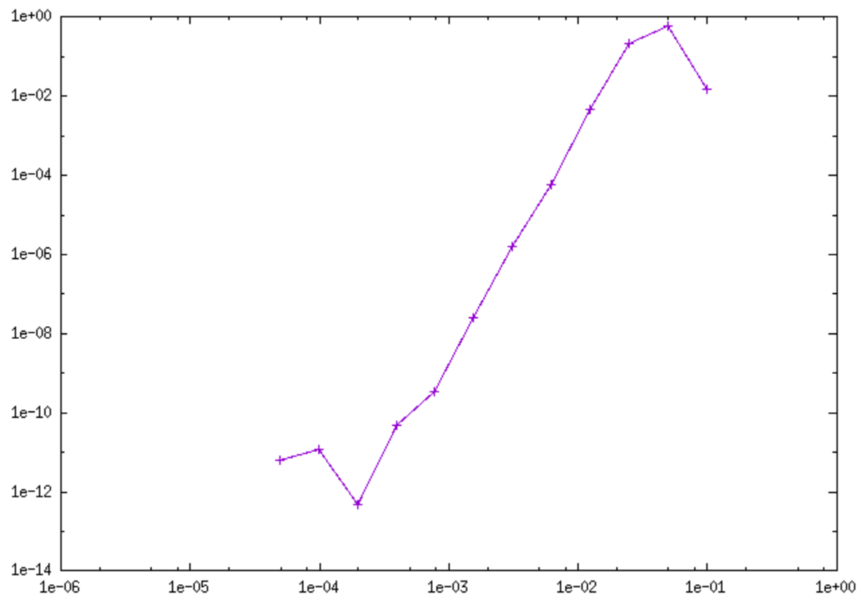

Figure 4: This plot shows the specific orbital energy gained by the Earth in the computer simulation after it completes 200 orbits as a function of the time step. The x-axis is the time step in years and the y-axis is the specific orbital energy gained in Joules per kilogram.

4. Next, the initial conditions of the system are altered to observe other orbits. Generally, changes in the initial condition lead to circular orbits, and energies of low negative magnitude are not sufficient to keep the Earth in its orbit. Specifically, a few situations are tested. When the initial energy is large and negative (done by shrinking the distance from the Earth to the Sun), Earth's orbit becomes elliptical. It seems to precess as it spirals into the Sun. When the initial energy is small and negative, the Earth flies away from the Sun. The specific orbital energy approaches smaller and smaller negative values and seems that it will do so infinitely without reaching zero because the energy still has a value after 500 years. When the initial energy is small and positive, the behavior is similar, but the specific orbital energy approaches zero much faster than when it is negative.
5. Using an elliptical orbit, a small change is made to Newton's law of gravity. The power on the radius term in the expression for force is changed by a little bit—in this case,

0.1—and the resulting motion is monitored. Under these conditions, the elliptical orbit precesses; that is, its major and minor axis are rotated a small amount each cycle. This effect actually happens to Mercury because of corrections to the laws of gravity that are supported by general relativity when the force of gravity is strong. For this simulation, the force of gravity was increased to cause an elliptical orbit in order to be able to observe this phenomenon.

6. Lastly, the simulation is tested when the Sun is located at a point other than the origin—specifically (0.2, 0.3). This, as expected, forms an elliptical orbit since the initial conditions are still those of a circular orbit but relative to a different point. The program no longer has the initial conditions appropriate for a circular orbit, although this could be rectified by adjusting the initial velocity to be tangential to the new initial radius between the Earth and the Sun.

Conclusion

While this project is specifically designed to model the (approximately) circular orbit of the Earth around the Sun, the technique used in solving the differential equations can be easily adjusted to suit different systems. It is proven here to be highly accurate, and, in the next project, it will even be extended to simulate binary systems where both bodies orbit. Therefore, the power of computational solutions is demonstrated once again.