Physics 307 Project 3

Due Tuesday, 3 October, at 11 AM

1. First we will study a DE that we can solve analytically so we can compute error as a function of stepsize.

   Newton's law of cooling says that the rate of temperature loss is proportional to the temperature difference between the cooling object and the ambient temperature T_a; that is,

   $$\frac{\partial T}{\partial t} = -k(T - T_a) \tag{1}$$

   where k is a proportionality constant that depends on the materials involved and their surface area. The analytic solution to this DE is

   $$T(t) = T_a + (T(0) - T_a)e^{-kt} \tag{2}$$

   Consider a cooling cup of tea. When the water is removed from the heat, it is at 100°. Suppose that room temperature is 30°, and that $k = 0.1\text{min}^{-1}$.

   (a) Write a program that solves the differential equation above for the temperature T as a function of time using Euler's method. Plot this function vs. time for stepsizes dt of 1 minute, 10 seconds, and 1 second, along with the analytic solution. Does your numeric solution do a good job of capturing the behavior of the system?

   (b) Now, investigate the error in a rigorous way. Using a range of stepsizes dt, compute the temperature after five minutes and compute the error in each value. Make a log-log plot of the error vs. the stepsize. Is the scaling what you expect it to be?

   Note: You should actually solve the DE numerically using Euler's method; the exact solution given above is just so you can compute the error. This is the second-to-last thing we will study in this class that has an easy, obvious analytic solution!

2. Repeat the preceding problem with the second-order Runge-Kutta method.

3. Now, using RK2, answer the question "How long does it take the tea to cool to $T_f = 70°$?" Make a plot of the error in your result vs. stepsize and verify that you're seeing second-order scaling.

   Note that in order to achieve second-order accuracy, you will need to correct for "overshoot", as we will discuss in class.

   Briefly, if you discover that the temperature is T_1 (greater than 70 degrees) at t_1, and T_2 (less than 70 degrees) at t_2, in order to achieve second-order precision you need to interpolate to figure out at what time you actually hit 70 degrees. That time is given by

$$t_1 + \frac{t_1 - t_2}{T_1 - T_2}(T_1 - T_f) \tag{3}$$

or alternatively

$$t_1 + \left(\left. \frac{\partial T}{\partial t} \right|_{t_1} \right)^{-1} (T_1 - T_f) \tag{4}$$

(I find the second more intuitive and easier to apply, since you already have been calculating $\frac{\partial T}{\partial t}$ many times already.)

4. Now we will solve a DE that we can't do with pen and paper. The equation of motion for a pendulum is

$$\frac{\partial^2 \theta}{\partial t^2} = -\frac{g}{L} \sin \theta \tag{5}$$

In mechanics class you solved this by taking the small-angle approximation $\sin \theta \approx \theta$; the equation then has a solution

$$\theta(t) = A \sin(\omega t + \phi) \tag{6}$$

where $\omega = \sqrt{\frac{g}{L}}$ giving a period $T = 2\pi\sqrt{\frac{L}{g}}$. This is valid only in the limit $\theta \to 0$.

This equation is very difficult to solve without making this approximation using pen and paper, but you have a computer!

Before Tuesday I'd like to see you at least code this using the Euler algorithm, and possibly the RK2 algorithm, using `anim` to animate your results. We will do more with the pendulum in Project 4.