Physics 307 Homework 5
Due Tuesday, 24 October, at 11 AM

In this homework assignment, you will simulate the Earth going around the Sun, and then generalize your work to other orbits.

To review the discussion in class: For an orbit in the x-y plane, you have four dynamical variables x, y, v_x, and v_y. The four differential equations that govern them are

$$\dot{x} = v_x \tag{1}$$
$$\dot{y} = v_y \tag{2}$$
$$\dot{v}_x = -\frac{GMx}{r^3} \tag{3}$$
$$\dot{v}_y = -\frac{GMy}{r^3} \tag{4}$$

where M is the mass of the Sun.

1. First we will pretend that the Earth is in an exactly circular orbit; this is approximately the case. If we measure distance in astronomical units and time in years,
   - what initial conditions for the Earth's position and velocity vectors will give a circular orbit with the correct radius, and
   - what is the numerical value of GM?

   (Hint: Remember your freshman physics; what is v^2/r? I know some of you took or taught Physics 211 with me; treat it as a 211 problem!)

2. Simulate the Earth going around the Sun. (Show one of us your animation in class.) Do you get a circular orbit with radius 1 AU and a period of 1 year with the initial conditions you found earlier?

3. Above, you chose specific initial conditions to produce a circular orbit. However, not all orbits are circles. Change the initial velocity and position of the Earth. (Note that if the total energy becomes positive, the Earth will escape the Sun's gravity. If your planet is flying away from the Sun, test this first!) What shape do these orbits take? At what part of the planet's orbit does it move faster? Spend at least half an hour playing around with different initial conditions and seeing what happens, and write about that in your report.

4. The total energy per unit mass ("specific orbital energy") is given by the sum of the potential and kinetic energy

$$E_{\text{spec}} = \frac{1}{2}(v_x^2 + v_y^2) - \frac{GM}{r}. \tag{5}$$

   Even without an exact analytic solution for orbits (in general), we can look at departures from conservation of energy as another way to gauge the error in the simulation. Plot the total specific energy over a period of many years. What's happening? Does your leapfrog solver do a "good job" of conserving energy?

5. Now, let's investigate something fancy!

   It turns out that, when you're very close to a star, the force law from gravity isn't quite $\vec{F} = \frac{GMm}{r^2}(-\hat{r})$. Newton's law of universal gravity is just an approximation, valid in regions of relatively weak gravity. In regions of stronger gravity, there are corrections due to general relativity.

   When gravity gets very strong, the behavior of curved spacetime gets extremely complicated, and results in things like gravitational waves. Computer simulations were crucial to the LIGO work, since they involved solving the Einstein equation for general relativity to figure out what sort of gravitational waves two spiraling black holes ought to emit as they collide.

   However, we're not going to do anything quite *that* fancy. As you know from the pendulum project, if you have a system that's quite close to an easy one (pendulum swinging at small amplitudes), its behavior is a small correction to the behavior of the easy one. In our case, this means that the gravitational force law is modified to $\vec{F} = \frac{GMm}{r^{2+\epsilon}}(-\hat{r})$, where ϵ is a small value.

   Try this! Change the power to something close to 2, and see how your simulation behaves.[1]

6. Consider what happens if you move the Sun away from the origin. Suppose you put the Sun at coordinates (0.2, 0.3). Change your code to simulate orbits around a star not at the origin. This is a stepping-stone to the next problem; you may omit it if you want.

7. Finally, change your code to simulate a binary star system. Spend some time playing with it to see what kinds of animations you can generate.

[1] It turns out that there is some history here! The orbit of Mercury is observed to precess around the Sun like this very slightly. Some of that precession is due to the influence of Jupiter's gravity, but if you add in Jupiter's gravity you still don't get the right value for the precession of Mercury: it's off by 1/900 of a degree per century! (Astronomers can be very precise people when need be!)

Before Einstein's theory of general relativity this was a mystery. However, Einstein realized that the correction to the law of gravity from general relativity might explain this; sure enough, it does, and this is one of the proofs of GR Einstein demonstrates in his 1916 paper.

8. **Graduate students and ambitious undergrads:**

   Once you know how to simulate two planets, it shouldn't be a challenge to simulate N planets. You can do this using a programming construct called an *array*; I will post the notes on arrays next week, or you can ask me.

   I'll give you the specific assignment next week as I figure out what it is, but the more pressing task is just coding this. It'll take three nested `for` loops:

   - One looping over timesteps
   - One looping over planets feeling forces
   - One looping over planets applying forces

   ... since every planet pulls on every other planet.