Notes on numerical solution of differential equations

As we discussed in class, a great many physics phenomena are described by (systems of) differential equations, so we need a way to solve them numerically. The most common scenario is the *time evolution* of some phenomena: the motions of objects, the evolution of electromagnetic fields, the propagation of heat, and so on.

In general, these situations take the form of a set of coupled second-order differential equations. However, such a situation can be reduced to a set of first-order DE's. So, in these notes I will build up the general case in three steps:

- A single first-order DE: the cooling cup of tea
- A single second-order DE: the swinging pendulum
- Many second-order DE's: celestial motion

1 Euler's method for a single first-order DE

Throughout I will use time as the independent variable, since this is usually the case we care about. However, everything works the same if some other coordinate is the independent variable.

A first-order differential equation has the form

$$\frac{\partial y}{\partial t} = f(y,t) \tag{1}$$

where y is a generic dependent variable and $f(y,t)$ gives its rate of change over time as a function of its value (and, sometimes, the time). A specific example is Newton's law of cooling

$$\frac{\partial T}{\partial t} = f(t) = -k(T - T_a) \tag{2}$$

The Euler method consists of doing the simplest thing you can think of: you approximate the value of the function at the end of each timestep by its value at the beginning, plus the rate of change times the length. In symbols,

$$y(t+h) = y(t) + hf\left[y\left(t\right)\right]; \tag{3}$$

that is, the value after a timestep is equal to the value before the step, plus the length of the step times the rate of change. Since we need a value of y to evaluate the rate of change, we use the value at the beginning of the interval $y(t)$. For instance, for Newton's law of cooling, $f(T) = -k(T - T_a)$, and Euler's algorithm is

$$T(t+h) = T(t) - hk(T - T_a) \tag{4}$$

2 The RK2 algorithm

The RK2 algorithm is analogous to the midpoint rule. Here, we would like to use the function's slope *halfway through the interval*, rather than at the beginning. We'd like to do something like this:

$$y(t+h) = y(t) + hf\left[y\left(t+\frac{h}{2}\right)\right]; \tag{5}$$

The only trouble is that we don't know $y\left(t+\frac{h}{2}\right)$ – in the Newton's law of cooling example, the temperature halfway through the interval. However, we can estimate it using the Euler method:

$$y(t+\frac{h}{2}) = y(t) + \frac{h}{2}f\left[y\left(t\right)\right]; \tag{6}$$

This gives the following procedure for RK2:

1. Take an Euler-method half step to estimate the value of the variable y at $t + h/2$
2. Use that value $y(t+h/2)$ to compute the slope over the whole interval, and take a full step using that slope

In symbols, we have

$$y\left(t+\frac{h}{2}\right) = y(t) + \frac{h}{2}f\left[y\left(t\right)\right] \tag{7}$$

$$y(t+h) = y(t) + hf\left[y\left(t+\frac{h}{2}\right)\right] \tag{8}$$

3 Euler's method for a single second-order DE

A second-order differential equation can be reduced to two first-order DE's. In general, given the second-order differential equation

$$\frac{\partial^2 y}{\partial t^2} = f(y) \tag{9}$$

we can separate it into two *first-order DE's*:

$$\frac{\partial y}{\partial t} = \dot{y} \tag{10}$$

$$\frac{\partial \dot{y}}{\partial t} = f(y, \dot{y}) \tag{11}$$

at the cost of introducing an extra *dynamical variable* $\dot{y}$. Dynamical variables are simply those variables that the simulation is evolving through time – for instance, positions and velocities.

Then you do a straightforward Euler update for each:

$$
\begin{aligned}
y(t+h) =& y(t) + h\dot{y}(t) && (12) \\
\dot{y}(t+h) =& \dot{y}(t) + hf\left[y(t), \dot{y}(t)\right] && (13)
\end{aligned}
$$

As an example, for the pendulum we have

$$
\begin{aligned}
\frac{\partial \theta}{\partial t} =& \omega && (14) \\
\frac{\partial \omega}{\partial t} =& -g/L \sin\theta && (15)
\end{aligned}
$$

and the Euler update looks like

$$
\begin{aligned}
\theta_{new} =& \theta_{old} + h\omega_{old} && (16) \\
\omega_{new} =& \omega_{old} + h\frac{g}{L}\sin\theta_{old} && (17) \\
& && (18)
\end{aligned}
$$

where I have used some new notation that should be easy to figure out.

4 RK2 for a single second-order DE

The approach here is just the same. The only trick is that you must take *both* half-steps together, since you need both y and $\dot{y}$ evaluated at the halfway point to take the full step. The algorithm can be written

$$
\begin{aligned}
y\left(t+h/2\right) =& y(t) + \frac{h}{2}\dot{y}(t) && (19) \\
\dot{y}\left(t+h/2\right) =& \dot{y}(t) + \frac{h}{2}f\left[y(t), \dot{y}(t)\right] && (20) \\
y(t+h) =& y(t) + h\dot{y}(t+h/2) && (21) \\
\dot{y}(t+h) =& \dot{y}(t) + hf\left[y\left(t+h/2\right), \dot{y}\left(t+h/2\right)\right] && (22)
\end{aligned}
$$

For the pendulum in particular, our y is the angle θ and $\dot{y}$ is the angular velocity ω. Thus we have:

$$\theta_{1/2} = \theta_{old} + \frac{h}{2}\omega_{old} \tag{23}$$

$$\omega_{1/2} = \omega_{old} - \frac{h}{2}\frac{g}{L}\sin\theta_{old} \tag{24}$$

$$\theta_{new} = \theta_{old} + h\omega_{1/2} \tag{25}$$

$$\omega_{new} = \omega_{old} - h\frac{g}{L}\sin\theta_{1/2} \tag{26}$$

$$\tag{27}$$

5 Euler's method for a system of many DE's

In general, a large system of DE's with N dynamical variables (which may be a combination of velocities and positions, as in the previous example, or may be something else) may be written

$$\frac{\partial y_1}{\partial t} = f_1(y_1, y_2, y_3, y_4 ...)\frac{\partial y_2}{\partial t} = f_2(y_1, y_2, y_3, y_4 ...)\frac{\partial y_3}{\partial t} = f_3(y_1, y_2, y_3, y_4 ...)\frac{\partial y_4}{\partial t} = f_4(y_1, y_2, y_3, y_4 ...), \tag{28}$$

where each rate of change f_i depends on *all* of the dynamical variables y_i. In compact vector form, we can represent the vector of dynamical variables as $\vec{y}$ and their rates of change as $\vec{f}(\vec{y})$. This is a vector function (it gives us a rate of change for each y_i) which has a vector parameter (it needs the value of all y_i's). Then the collection of DE's can be written simply as

$$\frac{\partial \vec{y}}{\partial t} = \vec{f}(\vec{y}) \tag{29}$$

and the Euler algorithm becomes

$$\vec{y}(t + h) = \vec{y}(t) + h\vec{f}\left[\vec{y}\,(t)\right]; \tag{30}$$

Note that all I did here is put vector signs over everything. It is easiest to understand how these algorithms work if you think of your collection of dynamical variables as a single object, and perform any operations on all of them at once. For HW4, for instance, you have four of them.

6 RK2 for a system of many DE's

You should already be able to guess how this goes: this is the same as RK2 for a single variable, except now we think about updating an entire *set* (which I've written as $\vec{y}$) of dynamical variables. Some of these may be position variables (in which case their rate of change is a corresponding velocity variable), and some may be velocity variables (in which case their rates of change are given by Newton's second law), but it doesn't matter; the procedure is just "do a half-step for everything to get the values at the midpoint, and then use those values to do a full step".

We can write this in vector form as follows:

$$\vec{y}(t+\frac{h}{2}) =\vec{y}(t)+\frac{h}{2}\vec{f}\left[\vec{y}\left(t\right)\right] \tag{31}$$

$$\vec{y}(t+h) =\vec{y}(t)+h\vec{f}\left[\vec{y}\left(t+\frac{h}{2}\right)\right] \tag{32}$$

where I have just scribbled vector signs above all the y's and h's that appeared in the one-equation RK2. It really is the same – you just need to update a set of variables, rather than just one.

7 Orbits around a fixed star

Since planets orbit in a plane, we can restrict ourselves to the $x-y$ plane. This means we need four dynamical variables: $x, y, v_x, and v_y$. Let's put the Sun at the origin; then our variables tell us the position and velocity of the planet. We are going to have the set of DE's

$$\dot{x} =v_x \tag{33}$$

$$\dot{y} =v_y \tag{34}$$

$$\dot{v}_x =a_x \tag{35}$$

$$\dot{v}_y =a_y \tag{36}$$

$$\tag{37}$$

Now we just need to figure out what a_x and a_y are.

Newton's law of gravity tells us

$$\vec{F}_g = (-\hat{r})\frac{GMm}{r^2} \tag{38}$$

where $\hat{r}$ is a unit vector pointing from the Sun to the planet, M is the mass of the Sun, and m is the mass of the planet. To compute the acceleration due to this force, we can use Newton's second law, which gives us

$$m\vec{a} = (-\hat{r})\frac{GMm}{r^2} \rightarrow \vec{a} = (-\hat{r})\frac{GM}{r^2} \tag{39}$$

Since we will be working in Cartesian coordinates, we need to find the xi and $y-$ components of $\vec{a}$. This is done readily with the $\hat{r}-$trick. Note that

$$\hat{r} = \frac{\vec{r}}{|r|}. \tag{40}$$

This makes sense: $\hat{r}$ is a vector in the direction of r, but with length 1, so we just take $\vec{r}$ and divide it by its length $|r|$. Going forward I will drop the absolute value signs: $r \equiv |r| = \sqrt{x^2 + y^2}$. Now we convert everything into x and y components:

$$\vec{r}_x = x \tag{41}$$
$$\vec{r}_y = y \tag{42}$$
$$\hat{r}_x = \frac{x}{r} \tag{43}$$
$$\hat{r}_y = \frac{y}{r} \tag{44}$$
$$r = \sqrt{x^2 + y^2} \tag{45}$$

We can substitute this into Newton's law of gravity to get the x and y coordinates of the acceleration, which are what we want:

$$\vec{a} = (-\hat{r})\frac{GM}{r^2} \tag{46}$$
$$\vec{a} = (-\vec{r})\frac{GM}{r^3} \tag{47}$$
$$a_x = \frac{-GMx}{r^3} \tag{48}$$
$$a_y = \frac{-GMy}{r^3} \tag{49}$$

where $r = \sqrt{x^2 + y^2}$. This gives us what we need: four DE's that give the rates of change of our four dynamical variables:

$$\dot{x} = v_x \tag{50}$$
$$\dot{y} = v_y \tag{51}$$
$$\dot{v}_x = \frac{-GMx}{(x^2 + y^2)^{\frac{3}{2}}} \tag{52}$$
$$\dot{v}_y = \frac{-GMy}{(x^2 + y^2)^{\frac{3}{2}}} \tag{53}$$
$$\tag{54}$$