Research notes

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Contents

Chapter 1

Probability Theories

1.1 Slutsky's Theorem

Let $\{X_n\}$, $\{Y_n\}$ be sequences of random elements. If $X_n \xrightarrow{d} X$, $Y_n \xrightarrow{p} c$, where c is a constant, then

1. $X_n + Y_n \xrightarrow{d} X + c$;
2. $X_n Y_n \xrightarrow{d} cX$;
3. $Y_n^{-1} X_n \xrightarrow{d} c^{-1} X$.

Slustsky's theorem is very useful when proving convergence of some (complicated) random elements.

1.2 others

1. the intepretation of conditional expectation $E(\xi|\mathcal{F})$ as a version of the density $d(\xi \cdot P)/dP$ on the σ-field $\mathcal{F}$? (on Kallenberg p103).

2. **A general CLT**
   Let $\{X_{n,t}, n \geq 1, t = 1, \cdots, n\}$ be a triangular stochastic array,
   let $\{\mathbf{V}_{n,t}, -\infty < t < \infty, n \geq 1\}$ be a (possibly vector-valued) stochastic array,
   and $\mathcal{F}_{n,t-m}^{t+m} = \sigma(\mathbf{V}_{n,s}, t - m \leq s \leq t + m)$. Also, let $S_n = \sum_{t=1}^{n} X_{n,t}$.
   Suppose the following assumptions hold:

   (a) $X_{n,t}$ is $\mathcal{F}_{n,-\infty}^{t}/\mathcal{B}$-measurable, with $E(X_{n,t}) = 0$ and $E(S_n^2) = 1$.

   (b) $\exists \{c_{n,t} > 0\}$ s.t. $\sup_{n,t} \|X_{n,t}/c_{n,t}\|_r < \infty$ for some $r > 2$.

(c) $X_{n,t}$ is $L_2 - NED$ of size -1 on $\{\mathbf{V}_{n,t}\}$ w.r.t. $\{c_{n,t}\}$ specified above, and $\{\mathbf{V}_{n,t}\}$ is $\alpha - mixing$ of size $-(1+2\theta)r/(r-2)$, for some $\theta \in [0, 1/2)$.

(d) Let $b_n = [n^{1-\alpha}]$ and $r_n = [n/b_n]$ for some $\alpha \in (0, 1]$, and defining $M_{n,i} = \max_{(i-1)b_n < t \leq ib_n} c_{n,t}$ for $i = 1, \cdots, r_n$ and $M_{n,r_n+1} = \max_{r_n b_n < t \leq n}\{c_{n,t}\}$, the following conditions hold:

$$\max_{1 \leq i \leq r_n+1} M_{n,i} = o(b_n^{-1/2}),$$

$$\sum_{i=1}^{r_n+1} M_{n,i} = O(b_n^{\theta-1/2}),$$

where θ is given in (c), and

$$\sum_{i=1}^{r_n+1} M_{n,i}^2 = O(b_n^{-1}).$$

(e) $X_{n,t}$ is $L_2 - NED$ of size -1 on $\{\mathbf{V}_{n,t}\}$, which is $\alpha - mixing$ of size $-r/(r-2)$.

(f) Let $M_n = \max_{1 \leq t \leq n} c_{n,t}$, $\sup_n nM_n^2 < \infty$.

Then under assumptions $(a), (b), (c), (d)$, or $(a), (b), (e), (f)$

$$S_n \xrightarrow{D} N(0, 1).$$

To interpret this result, consider the case that $X_{n,t} = (Y_t - \mu_t)/s_n$, $s_n = E(\sum_{t=1}^n (Y_t - \mu_t))^2$, then we may set $c_{n,t} = (1 \vee \sigma_t)/s_n$, where $\sigma_t = Var(Y_t)$

3. The sum of a series

$$\sum_{j=1}^{k-1} jx^{j-1} = \frac{1 - kx^{k-1} + (k-1)x^k}{(1-x)^2}$$

$$\sum_{j=1}^{T-1} (T-j)x^{j-1} = \frac{T}{1-x} - \frac{1-x^T}{(1-x)^2}$$

4. Substochastic matrix
   A real matrix is called substochastic, if all of its elements lie in $[0, 1]$ and 1 is a upper bound for its row sums. Accordingly, a substochastic transition kernel $T(x, A)$ should be a kernel measure with $T(\omega, \Omega) \leq 1$, for all ω.

5. The indicator function I_C where C is open is not continuous! Since the preimage of the open set $(0-\epsilon, 0+\epsilon)$ is a closed set! But this function is lower semicontinuous, i.e., its value at any point is not larger than the lower limit of any series in its neighbourhood converging to itself.

   This is the reason why we need the Feller condition. ??

6. Sample paths and limit theorems of Markov chains.
   CLT for postive Harris chain with invariant probability π. positive Harris chain if Harris recurrent and positive.

7. intradaily prices — lognormal diffusion, whose coefficient $\leftarrow$ conditional variance equation

Chapter 2

Brownian Motion related

2010.08.13

Consider a Brownian motion

$$P(t),\ t \in [0, T]$$

with diffusion coefficient σ and drift μ:

$$dP_t = \mu dt + \sigma dW_t,$$

where W_t is a standard Wiener process.

The the joint p.d.f. of the highest value a, the lowest value b, and the close value r is given in (Lildholdt, 2002). also in Cox and Miller (the theory of stochastic process, P222).

Chapter 3

M- and Z- estimators

Theorem 3.0.1. *Let M_n be random functions and let M be a fixed function of θ s.t.*

- $\sup_{\theta\in\Theta} |M_n(\theta) - M(\theta)| \xrightarrow{P} 0,$
- $\forall\varepsilon, \sup_{\theta:d(\theta,\theta_0)>\varepsilon} M(\theta) < M(\theta_0),$

then

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Chapter 4

Random walks and Renewal Theory

Chapter 5

Applications in Finance

5.1 Option Picing

2010.08.13,10:37 P171, Applied Stochastic Processes by Lin Yuanlie, finished Ch5.1.

2010.08.13,15:55 Barrier option: an option with a payoff depending on the close value, but conditionally on the extrema lying in some region.

Assume that under some measure $\mathbb{Q}$, $S_T = S_0 \exp(\mu + cX)$, where X has cdf F. To compare it wit B-S model, some constraints are put on μ, c. Note that the variance of log-return in B-S model over length T is $\sigma^2 T$, where σ^2 is the annual volatility(?? precise need here), then $var(cX) = \sigma^2 T \Rightarrow c = \sqrt{\sigma^2 T / var(X)}$. Another constraint required of all option-pricing measures is the martingale constraint, this implies $e^{-rT} E_Q S_t = S_0 \Rightarrow e^{\mu - rT} E_Q \exp(cX) = 1 \Rightarrow e^{\mu - rT} m(c) = 1$, where $m(\cdot)$ is the moment generating function of X. Then $\mu = rT - \log(m(c))$.

The price of a call option with stike price K with mature date T will be priced with $e^{-rT} E_Q (S_T - K)^+$. To estimate its value, use $e^{-rT} \frac{1}{N} \sum_{i=1}^N (S_{T_i} - K)^+ = \frac{1}{N} \sum_{i=1}^N (\frac{S_0}{m(c)} e^{cX_i} - e^{-rT} K)^+$, where $X_i \sim_{iid} F$. To compare it with that of a normal distribution, suppose we can invert F, and the above $X_i = F^{-1}(U_i)$, the same U_i are transformed to normal random number by $\Phi^{-1}(U_i)$, then the difference is given by

$\frac{1}{N} \sum_{i=1}^N ((\frac{S_0}{m(c)} e^{cF^{-1}(U_i)} - e^{-rT} K)^+ - (\frac{S_0}{m(c)} e^{c\Phi^{-1}(U_i)} - e^{-rT} K)^+)$. In case that m and $var(X)$ is unknown, they can be simulated.

5.2 OHLC data

2010.08.12 Using OHLC data to analyze the dynamics of the stock returns. (Lildholdt, 2002) considered a GARCH model.

Consider a stochastic process X, let $X_i(t)$ be its value at time $i + f_i + t(1 - f_i)$, where $f_i \in [0, 1]$ and $t \in [0, 1]$. Assume that

- from $X(i)$ to $X(i + f_i)$, X unobservable except its values at the end-points,
- from $X(i + f_i)$ to $X(i + 1)$, that is, $X_i(t)$ for $t \in [0, 1]$ follow some general diffusions,
$$dX_i(t) = \mu_i dt + \nu_i dW_t,$$
$$d\sigma_t$$
a Brownian motion with drift μ_i and variance σ_i^2

$$\begin{aligned} r_i =& P_i(1) - P_i(0), \\ dP_i(t) =& \mu dt + \sigma_i dW(t),\ 0 \le t \le 1, \\ \sigma_i^2 =& \omega + \sum_j \alpha_j (r_{i-j} - \mu)^2 + \sum_j \beta_j \sigma_{i-j}^2, \\ a_i =& \sup_{t \in [0,1]} P_i(t) - P_i(0) \\ b_i =& P_i(0) - \inf_{t \in [0,1]} P_i(t) \end{aligned}$$

In this setting, the variance is determined by past squared closed value(the realized There are other ways to define the volatility process. For example, (Chou, 2006) considered a separate regression of the daily upper range and lower range.

We wish to establish a model which could handle with OPN, UPR, DNR, CLS, and VOL data. If we assume the price movement during a day is governed by a geometric Brownian motion. Note that this assumption is very common in the continuous-time stochastic process, but it is problematic to assume that the long-run price is a geometric Brownian motion, due to the change of information. However, it may be more plausible to assume that the price follows a GBM in a short time range, in which no apparent change of information occurs.

(UPR_i, DNR_i, CLS_i) can be characterized by the (μ_i, σ_i), we need only to specify the dymamics of

$$(OPN_i, \mu_i, \sigma_i, VOL_i) = (o_i, \mu_i, \sigma_i, v_i) \leftarrow ?(o_i, \mu_i, \log \sigma_i, v_i)$$

We would expect that the information before the trading of a day can be obsorbed in the movement of the the price in the day with some moises, which can have further influence with the trading in the sequent day.

How to compare the performance of two models?
Firstly, suppose the process is
Volume should have positive impact on the volatility. realized volatility

Chapter 6

Semimartingales and stochastic integrals

6.1 local martingale

6.1.1 intro.

A local martingale is a type of stochastic process satisfying the localized version of the martingale property.

6.1.2 defi.

Let $(\Omega, F, \mathcal{F}, P)$ be a filtered probability space, let $X : [0, \infty) \times \Omega) \to S$ be a $\mathcal{F}$-adapted s.p.. Then X is called an $\mathcal{F}$-local martingale if there exists a sequence of $\mathcal{F}$-stpping times $\tau_k : \Omega \to [0, \infty)$ s.t.

- $P(\tau_k \uparrow) = 1$, τ_k is a.s. strictly increasing;
- $P(\lim \tau_k = \infty) = 1$;
- the stopped process $1_{\tau_k > 0} X_t^{\tau_k} = 1_{\tau_k > 0} X_{t \wedge \tau_k}$ is a margingale for every k.

6.1.3 e.g.

Let W_t be the standard Wiener process and $T = \inf\{t : W_t = -1\}$, then $W_t^T = W_{t \wedge T}$ is a martingale,

6.2 semimartingales

X is a local martingale, if it is adapted, cadlag, and there exists a sequence of increasing stopping times, T_n s.t. $\lim T_n = \infty, a.s.$, and $\forall n$, $X^{T_n} 1\{T_n > 0\}$ is uniformly integrable martingale. Such T_n is called a fundamental sequence.

Topology generated by convergence of sequences.

An operator I_X, to be an integral, should be linear and s.t. some version of bounded convergence theorem.

X is a total semimartingale, if X is cadlag, adapted, I_X: $S_u \to L^0$ is continuous, where S_u is the set of simple predictable processes, and L^0 be the space of finite valued r.v. topologized by convergence in probability.

X is a semimartingale, if $\forall t \in [0, \infty)$, X^t, i.e., X stopped by t, is a total semimartingale.

The set of (total) semimartingales is a vector space.

Note that L^0 is dependent of the probability measure defined on it. If Q is a probability and is absolutely continuous w.r.t. P, and X is a P (total) semimartingale, then X is also a Q (total) semimartingale. (This is because Convergence in P-probability implies convergence in Q-probability, as $P(An) \to 0 \to P(fI\{An\}) \to 0)$, $\forall f$ integrable w.r.t. P.)

If X is a P_k semimartingale for each k, and let $P = \sum \lambda_k P_k$ with $\lambda_k >= 0$ and $\sum \lambda_k = 1$. Then X is a P semimartingale as well.

(Stricker's Theorem). Let X be a semimartingale for the filtration $\mathbb{F}$, $\mathbb{G}$ be a subfiltration of $\mathbb{F}$ and X is adapted to $\mathbb{G}$. Then X is also a $\mathbb{G}$ semimartingale. *Note for $\mathbb{G}$ its test simple predictable processes is a subset of that of $\mathbb{F}$.* Then we can always thrink a filtration and preserve the property of being a semimartingale as long as it is still adapted. Expanding the filtration, is a much delicate issue.

(Jacod's countable expansion) Let $\mathcal{A}$ be a collection of disjoint events in $\mathcal{F}$. Let $\mathcal{H}_t = \sigma\{\mathcal{F}_t, \mathcal{A}\}$. Then every $(\mathbb{F}, P)$ semimartingale is an $(\mathbb{H}, P)$ semimartingale. Particularly, this is true when $\mathcal{A}$ is a finite collections of events in $\mathcal{F}$.

Note if B is a Brownian motion for a filtration, then we are able to add, in a certain manner, an infinite number of future events to the filtration and B will no longer be a martingale but it will stay a semimartingale. This has interesting implications in theory of continuous trading (Duffie-Huang).

Being a semimartingale is a local property. A process X is stoped at $T-$ if $X^{T-} = X_t 1\{0 \leq t < T\} + X_{T-} 1\{t \geq T\}$, where $X_{0-} = 0$. If X is a cadlag, adapted process. Let (T_n) be a sequence of positive r.v. increasing to ∞ a.s., and (X^n) be a sequence of semimartingales s.t. $X^{T_n -} = (X^n)^{T_n -}$, then X is a semimartingale.*Note T_n may not be stopping times. We need to show X^t is a total semimartingale for every $t > 0$. Define $R_n = T_n 1\{T_n \leq t\} + \infty 1\{T_n > t\}$,* *then* $P(|I_{X^t}(H)| \geq c) \leq P(|I_{(X^n)^t}(H)| \geq c) + P(R_n < \infty)$. A corollary is that a process X s.t. there exists a sequence of stopping times (T_n) such that X^{T_n} or $X^{T_n}1\{T_n > 0\}$ is a semimartingale for each n, then X is a semimartingale.

Examples of semimartingales: adapted process with cadlag paths of finite variation on compacts (by definition), every L^2 martingale with cadlag paths. Cadlag, locally square integrable local martingale

Chapter 7

Financial Terminologies

7.1 spillover effect

Spillover effects are externalities of economic activity or processes upon those who are not directly involved in it. E.g.: the economic benefit of increased trade are the spillover effects anticipated in the formation of multilateral alliances of many of the regional nation states.

Analysis of comovement of stock trade may lead to prediction of stock price movement.

Jorgensen et al: model with a single dynamic factor and no idio syncratic components.

Wedel, Bockenholt and Wagner (2003): a static multivariate Poisson factor model for cross-sectional[1] analyses.

7.2 leverage effect

The leverage effect refers to the asymmetric behaviour of the stock price that the amplitude of relative price fluctuation tends to increase when the price drops.

[1]Cross-sectional data or cross section (of a study population) in statistics and econometrics is a type of one-dimensional data set. Cross-sectional data refers to data collected by observing many subjects (such as individuals, firms or countries/regions) at the same point of time, or without regard to differences in time. Analysis of cross-sectional data usually consists of comparing the differences among the subjects.

Chapter 8

Test for non-randomness

- Haigh χ^2 test, good, powerful
- Grenander-Rosenblatt test
- HOC high-order crossing test

Chapter 9

High-dimensional data analysis

Chapter 10

A point process with interactive marks

1. Epidemic type aftershock sequence (ETAS) model.

   Ground intensity function

   $$\lambda_g(t|\mathcal{F}_t) = \mu + A \sum_{i:t_i<t} e^{\alpha(M_i-M_0)}(1 + \frac{t-t_i}{c})^{-p}$$

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2.

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Chapter 11

Estimating theory

consider a family of sampling density $f(x|\theta)$ depending on parameters θ and the sampling space is $\mathfrak{X}$. The EM theory, based on the following inequality