DVCP: Directionally-Varying Change Points Model

Statistical Model

$$Y_i = \mathbf{x}\left(\mathbf{s}_i\right)^{\mathrm{T}} \boldsymbol{\beta} + \theta h(\|\mathbf{s}_i - \mathbf{p}\|)\mathbf{1}(\|\mathbf{s}_i - \mathbf{p}\| \leq \lambda \exp\left[\eta\left\{a_p\left(\mathbf{s}_i\right)\right\}\right]) + \epsilon\left(\mathbf{s}_i\right), \ i = 1, ..., n; \tag{1}$$

$$\epsilon\left(\mathbf{s}_i\right) | \sigma^2_{\epsilon} \overset{\text{iid}}{\sim} \mathrm{N}\left(0, \sigma^2_{\epsilon}\right);$$

$$\boldsymbol{\eta} = \left\{\eta\left(\mathrm{z}_1\right), \ldots, \eta\left(\mathrm{z}_m\right)\right\}^{\mathrm{T}} | \sigma^2_{\eta}, \phi_{\eta} \sim \mathrm{MVN}\left\{\mathbf{0}_m, \sigma^2_{\eta} \Sigma\left(\phi_{\eta}\right)\right\},$$
$$\Sigma\left(\phi_{\eta}\right)_{ij} = \exp\left[-\phi_{\eta} \min\left\{|\mathrm{z}_i - \mathrm{z}_j|, \ 2\pi - \max\left(\mathrm{z}_i, \mathrm{z}_j\right) + \min\left(\mathrm{z}_i, \mathrm{z}_j\right)\right\}\right]$$

- $\mathbf{p}$: Location of the point source;
- $a_p\left(\mathbf{s}_i\right)$: Angle of separation between $\mathbf{s}_i$ and $\mathbf{p}$ in degrees (with respect to the imaginary horizontal line extending through $\mathbf{p}$), measured counterclockwise;
- m: Number of unique angles.

$h\left(||\mathbf{s}_i - \mathbf{p}||\right)$ Options

- Indicator (h_model = 0): $h\left(\|\mathbf{s}_i - \mathbf{p}\|\right) = 1$ for all i;
- Linear (h_model = 1): $h\left(\|\mathbf{s}_i - \mathbf{p}\|\right) = \max\left\{\|\mathbf{s}_i - \mathbf{p}\|; i = 1, \ldots, n\right\} - \|\mathbf{s}_i - \mathbf{p}\|$;
- Exponential (h_model = 2): $h\left(\|\mathbf{s}_i - \mathbf{p}\|\right) = \exp\left\{-\|\mathbf{s}_i - \mathbf{p}\|\right\}$;
- Gaussian (h_model = 3): $h\left(\|\mathbf{s}_i - \mathbf{p}\|\right) = \exp\left\{-\|\mathbf{s}_i - \mathbf{p}\|^2\right\}$;
- Spherical (h_model = 4): $h\left(\|\mathbf{s}_i - \mathbf{p}\|; \lambda, \eta\left\{a_p\left(\mathbf{s}_i\right)\right\}\right) = 1 - 1.5\left(\frac{\|\mathbf{s}_i - \mathbf{p}\|}{\lambda \exp\left[\eta\left\{a_p\left(\mathbf{s}_i\right)\right\}\right]}\right) + 0.5\left(\frac{\|\mathbf{s}_i - \mathbf{p}\|}{\lambda \exp\left[\eta\left\{a_p\left(\mathbf{s}_i\right)\right\}\right]}\right)^3$;
  - Note: The spherical option includes a directionally-varying rate of decay within the change point buffer.

Prior Information

$\theta, \beta_j \overset{\text{iid}}{\sim} \mathrm{N}\left(0, \sigma^2_{\beta}\right), \ j = 1, ..., p_{\mathrm{x}}$;

- p_{x}: Length of $\mathbf{x}\left(\mathbf{s}_i\right)$ vector (same for all i);
- Default setting: $\sigma^2_{\beta} = 10,000$.

$\lambda \sim \mathrm{Beta}\left(\alpha_{\lambda}, \beta_{\lambda}\right)$

- Default setting: $\alpha_{\lambda} = \beta_{\lambda} = 1$

$\sigma^2_{\epsilon}, \sigma^2_{\eta} \overset{\text{iid}}{\sim} \text{Inverse Gamma}\left(\alpha_{\sigma^2}, \beta_{\sigma^2}\right)$

- Default setting: $\alpha_{\sigma^2} = \beta_{\sigma^2} = 0.01$

$\phi_{\eta} \overset{\text{iid}}{\sim} \mathrm{Gamma}\left(\alpha_{\phi_{\eta}}, \beta_{\phi_{\eta}}\right)$;

- Default setting: $\alpha_{\phi_{\eta}} = 1, \ \beta_{\phi_{\eta}} = 1$.

Default Initial Values

- $\beta_j = 0$ for all j;
- $\theta = 1$;
- $\lambda = 0.01$;
- $\sigma^2_\epsilon = \sigma^2_\eta = 1$;
- $\phi_\eta = 1$.

Additional Notes

- The Gaussian predictive process approximation is used during model fitting where the user must input the $k << m$ approximation angles.

Alternate Likelihood

$$Y(\mathbf{s}_i) | p(\mathbf{s}_i) \overset{\text{ind}}{\sim} \text{Binomical}\left\{c(\mathbf{s}_i), p(\mathbf{s}_i)\right\}, \ i = 1, ..., n.$$

$$\text{logit}\left\{p(\mathbf{s}_i)\right\} = \mathbf{x}(\mathbf{s}_i)^{\mathrm{T}} \boldsymbol{\beta} + \theta h(\|\mathbf{s}_i - \mathbf{p}\|)\mathbf{1}(\|\mathbf{s}_i - \mathbf{p}\| \leq \lambda \exp\left[\eta\left\{a_p(\mathbf{s}_i)\right\}\right])$$

- likelihood_indicator = 0: Binomial;
- likelihood_indicator = 1: Gaussian.