Notes on pnege plasma

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Abstract

This note discusses the thermodynamics and weak interaction rates in a hot plasma consisting of protons (p), free neutrons (n), electron-positrons (e), gamma-rays (g), and alpha particles (a). Such "pnege plasma" exists (1) in the accretion disks around stellar-mass BHs or NSs at extremely high accretion rates of the order $10^{-3} M_\odot\,\mathrm{s}^{-1}$ or higher and (2) in the earliest few minutes of the Universe according to the theory of Big Bang Nucleosynthesis.

Thermodynamic Equilibrium

The plasma consists of baryons, electron-positron pairs, photons, and neutrinos. We only include neutrons n, protons p, and α (or ^{4}He) particles for baryons, and their mass fractions are denoted as X_n, X_p, and X_α. In the high temperature limit $T \gtrsim 5\,\mathrm{GK}$ (or $k_\mathrm{B}T \gtrsim m_\mathrm{e}c^2$), baryons are in nuclear statistical equilibrium (NSE, meaning all strong interactions are in balance with their inverse processes), so the abundance ratios for given density $\rho = 10^{10}\rho_{10}\,\mathrm{g\,cm^{-3}}$ and temperature $T = 10^{10}$ K are given by the Saha Equation (Shapiro & Teukolsky 1983)

$$X_p X_n = 1.25\times 10^2 X_\alpha^{1/2}\rho_{10}^{-3/2}T_{10}^{9/4}\exp(-16.4/T_{10}). \tag{1}$$

We ignore the small mass difference between neutrons and protons $\Delta = 1.294\,\mathrm{MeV}$ for the calculations of their number densities. The electron fraction Y_e is given by

$$Y_e \equiv X_p + X_\alpha/2. \tag{2}$$

Eqs. (1) and (2) can be combined with $X_p + X_n + X_\alpha = 1$ to solve for all three mass fractions for given Y_e, ρ, and T, which specify a certain thermodynamic state of the plasma.

The total pressure and energy density have three main contributions from baryons, radiation, and pairs,

$$P = P_\mathrm{b} + P_\gamma + P_+ + P_-,\ U = U_\mathrm{b} + U_\gamma + U_+ + U_-, \tag{3}$$

where we have ignored the contribution from neutrinos. Photons have Planck spectrum (or Bose-Einstein distribution), which means

$$P_\gamma = aT^4/3,\ U_\gamma = aT^4, \tag{4}$$

where a is the radiation energy density constant. Baryons are non-relativistic and non-degenerate and have Maxwell-Boltzmann distributions

$$f_\mathrm{MB}(\mathcal{E},\mu_i) = \exp\left(-\frac{\mathcal{E}-\mu_i}{\Theta}\right) = \exp\left(\frac{\mu_i - m_i/m_\mathrm{e}}{\Theta}\right)\exp\left(-\frac{m_\mathrm{e}}{2m_i\Theta}p^2\right),\ \Theta \equiv \frac{k_\mathrm{B}T}{m_\mathrm{e}c^2},\ i = n,\ p,\ \alpha, \tag{5}$$

where $\mathcal{E} = m_i/m_\mathrm{e} + m_\mathrm{e}p^2/(2m_i)$ and μ_i are the particle energy and chemical potential (both in units of $m_\mathrm{e}c^2$), and p is the particle momentum (in units of $m_\mathrm{e}c$). The number of particles per phase space volume of h^3 (h being the Planck constant) is given by gf/h^3, where $g = 2$ for Fermions of spin $1/2$ (protons, neutrons, electrons, positrons) and $g = 1$ for α particles of spin 0. Thus, the number density, pressure (or momentum flux), and energy density of baryons ($i = n,\ p,\ \alpha$) are given by

$$
\begin{aligned}
n_i &= \mathcal{V}^{-1} \int f_{\rm MB}(\mathcal{E}, \mu_i) p^2 {\rm d}p, \\
P_i &= \frac{m_{\rm e}c^2}{3\mathcal{V}} \int f_{\rm MB}(\mathcal{E}, \mu_i) \frac{p^4}{\mathcal{E}} {\rm d}p, \\
U_i &= \frac{m_{\rm e}c^2}{\mathcal{V}} \int f_{\rm MB}(\mathcal{E}, \mu_i) \mathcal{E} p^2 {\rm d}p, \\
\mathcal{V}^{-1} &\equiv \frac{4\pi g (m_{\rm e}c)^3}{h^3},
\end{aligned}
\tag{6}
$$

where the integrals are from 0 to $+\infty$. We obtain the well-known results

$$
\begin{aligned}
n_i &= \frac{\rho X_i}{m_i} = \frac{\sqrt{\pi}}{4\mathcal{V}} \left(\frac{2m_i\Theta}{m_{\rm e}}\right)^{3/2} \exp\left(\frac{\mu_i - m_i/m_{\rm e}}{\Theta}\right), \\
P_i &= n_i kT, \\
U_i &= \frac{3}{2} n_i kT + n_i m_i c^2,\ i = n,\ p,\ \alpha.
\end{aligned}
\tag{7}
$$

Note that we define the volume constant $\mathcal{V} \equiv h^3/(8\pi m_{\rm e}^3 c^3)$ based on $g = 2$, with only one caveat that, when calculating the number density of alpha particles, one must use $\mathcal{V}_\alpha = h^3/(4\pi m_{\rm e}^3 c^3)$ because $g_\alpha = 1$. The baryonic chemical potentials $\mu_i(\rho, T, Y_e)$ are obtained from the number density expression above, and it can be shown

$$
\mu_n - \mu_p = \Theta \ln(n_n/n_p) + Q = \Theta \ln(X_n/X_p) + Q,\ Q \equiv (m_n - m_p)/m_{\rm e} = 2.531. \tag{8}
$$

On the other hand, pairs may be relativistic and degenerate and obey Fermi-Dirac distributions

$$
f_{\rm FD}(\mathcal{E}, \mu_\pm) = \frac{1}{\exp[(\mathcal{E} - \mu_\pm)/\Theta] + 1},\ \Theta \equiv k_{\rm B}T/m_{\rm e}c^2, \tag{9}
$$

where $\mathcal{E}$ is the particle energy and $\mu_\pm$ is the chemical potential (both in units of $m_{\rm e}c^2$). The number density, pressure, and energy density are given by

$$
\begin{aligned}
n_\pm &= \mathcal{V}^{-1} \int f_{\rm FD}(\mathcal{E}, \mu_\pm) p^2 {\rm d}p, \\
P_\pm &= \frac{m_{\rm e}c^2}{3\mathcal{V}} \int f_{\rm FD}(\mathcal{E}, \mu_\pm) \frac{p^4}{\mathcal{E}} {\rm d}p, \\
U_\pm &= \frac{m_{\rm e}c^2}{\mathcal{V}} \int f_{\rm FD}(\mathcal{E}, \mu_\pm) \mathcal{E} p^2 {\rm d}p,
\end{aligned}
\tag{10}
$$

where the momentum (in units of $m_{\rm e}c$) is given by $p = (\mathcal{E}^2 - 1)^{1/2}$. Making use of the Fermi-Dirac integrals (Cox & Giuli 1968)

$$
F_k(\eta, \Theta) \equiv \int_0^\infty \frac{x^k (1 + 0.5\Theta x)^{1/2}}{\exp(x - \eta) + 1} {\rm d}x,\ \ \eta = (\mu - 1)/\Theta, \tag{11}
$$

the above quantities can be expressed as

$$
\begin{aligned}
n_{\pm} &= \frac{\sqrt{2}}{\mathcal{V}}\Theta^{3/2}\left(F_{\frac{1}{2}} + \Theta F_{\frac{3}{2}}\right), \\
P_{\pm} &= \frac{2^{3/2} m_{\mathrm{e}} c^2}{3\mathcal{V}}\Theta^{5/2}\left(F_{\frac{3}{2}} + \frac{\Theta}{2} F_{\frac{5}{2}}\right), \\
U_{\pm} &= n_{\pm} m_{\mathrm{e}} c^2 + \frac{\sqrt{2} m_{\mathrm{e}} c^2}{\mathcal{V}}\Theta^{5/2}(F_{\frac{3}{2}} + \Theta F_{\frac{5}{2}}),
\end{aligned} \tag{12}
$$

where the two arguments of the integral functions F_k are $\eta_{\pm} \equiv (\mu_{\pm} - 1)/\Theta$ and Θ. Since electrons and positrons are generated by photon annihilation, their chemical potentials satisfy

$$\mu_- + \mu_+ = 0. \tag{13}$$

In the following, we define $\mu \equiv \mu_- = -\mu_+$ and $\eta = (\mu - 1)/\Theta$ and hence

$$\eta = \eta_- = (\mu - 1)/\Theta, \;\; \eta_+ = (-\mu - 1)/\Theta. \tag{14}$$

From charge conservation

$$Y_e \frac{\rho}{m_{\mathrm{p}}} = n_- - n_+ = \frac{\sqrt{2}}{\mathcal{V}}\Theta^{3/2}\left[F_{\frac{1}{2}}(\eta_-, \Theta) - F_{\frac{1}{2}}(\eta_+, \Theta) + \Theta F_{\frac{3}{2}}(\eta_-, \Theta) - \Theta F_{\frac{3}{2}}(\eta_+, \Theta)\right], \tag{15}$$

we obtain the product $Y_e\rho$ as a function of $(\eta,\ \Theta)$. This is shown in Fig. 1 along with the total pressure of pairs.

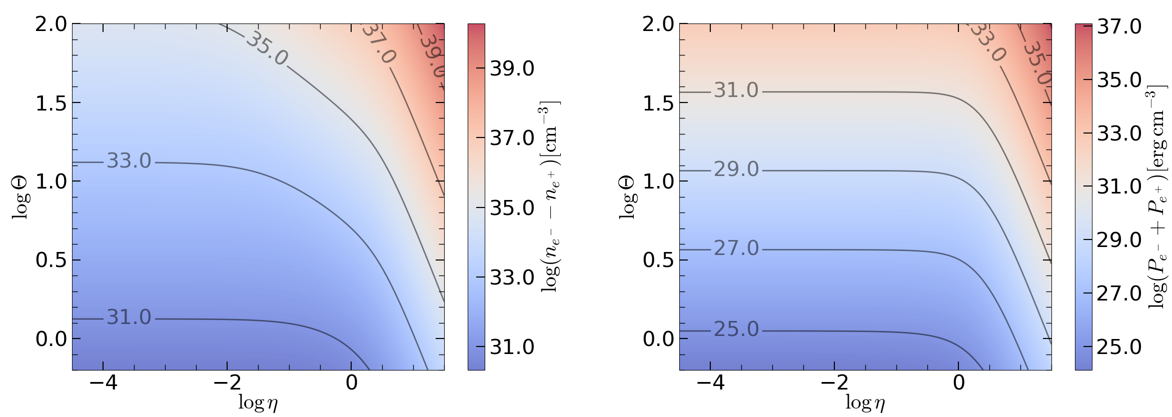

Fig. 1.— Net charge density and pressure of pairs, as functions of η (a dimensionless measurement of the degree of electron degeneracy) and Θ (dimensionless temperature).

Finally, the specific entropy of the whole mixture is given by the first law of thermodynamics

$$s = \frac{U + P - \sum_i \mu_i n_i m_{\mathrm{e}} c^2}{\rho T}, \;\; i = n,\ p,\ \alpha,\ e^-,\ e^+. \tag{16}$$

The entropies contributed by radiation and baryons are

$$s_\gamma = \frac{4aT^3}{3\rho}, \quad s_{\rm b} = \frac{5k_{\rm B}}{2m_{\rm p}}\left(X_p + X_n + \frac{X_\alpha}{4}\right) + k_{\rm B}\sum_{i=n,p,\alpha}\frac{X_i}{m_i}\ln\left[\sqrt{\frac{\pi}{2}}\frac{(m_i\Theta/m_{\rm e})^{3/2}}{n_i\mathcal{V}}\right]. \tag{17}$$

The entropy of pairs is

$$s_\pm = \frac{\sqrt{2}k_{\rm B}}{\rho\mathcal{V}}\Theta^{3/2}\sum_\pm\left[\frac{4}{3}\Theta F_{\frac{5}{2}} + \left(\frac{5}{3} - \eta_\pm\Theta\right)F_{\frac{3}{2}} - \eta_\pm F_{\frac{1}{2}}\right], \quad \eta_\pm \equiv (\mu_\pm - 1)/\Theta. \tag{18}$$

Weak Interactions

Neutrino emission/absorption affects the internal energy and the electron fraction Y_e of the plasma. Neutrino cooling is dominated by the URCA reactions

$$e^- + p \leftrightarrow n + \nu_e, \quad e^+ + n \leftrightarrow p + \bar{\nu}_e, \tag{19}$$

and the cooling rates (per unit mass) in the optically thin limit are given by (Shapiro & Teukolsky 1983)

$$\begin{aligned} q^-_{\nu,\rm thin} &= \Gamma\frac{X_p}{m_{\rm p}}m_{\rm e}c^2\int_0^\infty f_{\rm FD}(\mathcal{E}_-,\mu_-)\mathcal{E}_-\left(\mathcal{E}_-^2 - 1\right)^{1/2}\mathcal{E}_\nu^3{\rm d}\mathcal{E}_\nu \\ &= \frac{\sqrt{2}\Gamma X_p m_{\rm e}c^2}{m_{\rm p}}\Theta^{3/2}\times \\ &\left[\Theta^4\widetilde{F}_{\frac{9}{2}} - (3Q-4)\Theta^3\widetilde{F}_{\frac{7}{2}} + 3(Q-1)(Q-2)\Theta^2\widetilde{F}_{\frac{5}{2}} + (4-Q)(Q-1)^2\Theta\widetilde{F}_{\frac{3}{2}} - (Q-1)^3\widetilde{F}_{\frac{1}{2}}\right], \end{aligned} \tag{20}$$

$$\begin{aligned} q^-_{\bar{\nu},\rm thin} &= \Gamma\frac{X_n}{m_{\rm p}}m_{\rm e}c^2\int_{Q+1}^\infty f(\mathcal{E}_+,\mu_+)\mathcal{E}_+\left(\mathcal{E}_+^2 - 1\right)^{1/2}\mathcal{E}_{\bar{\nu}}^3{\rm d}\mathcal{E}_{\bar{\nu}} \\ &= \frac{\sqrt{2}\Gamma X_n m_{\rm e}c^2}{m_{\rm p}}\Theta^{3/2}\times \\ &\left[\Theta^4 F_{\frac{9}{2}} + (3Q+4)\Theta^3 F_{\frac{7}{2}} + 3(Q+1)(Q+2)\Theta^2 F_{\frac{5}{2}} + (4+Q)(Q+1)^2\Theta F_{\frac{3}{2}} + (Q+1)^3 F_{\frac{1}{2}}\right], \end{aligned} \tag{21}$$

where $\Gamma \simeq 6.93\times10^{-4}\,{\rm s}^{-1}$ is given by the life time of free neutron decay $(1.637\tau_n)^{-1}$ (and $\tau_n = 881.5\pm1.3$ s, Serebrov et al. 2018), $Q = (m_{\rm n} - m_{\rm p})/m_{\rm e} = 2.531$, and the energies of neutrinos and pairs are related as $\mathcal{E}_\nu = \mathcal{E}_- - Q$ and $\mathcal{E}_{\bar{\nu}} = \mathcal{E}_+ + Q$ are in units of $m_{\rm e}c^2$. The incomplete Fermi-Dirac integral are defined as

$$\widetilde{F}_k(\eta,\ \Theta) \equiv \int_{(Q-1)/\Theta}^\infty \frac{x^k(1+0.5\Theta x)^{1/2}}{\exp(x-\eta)+1}{\rm d}x. \tag{22}$$

The cooling rate due to pair annihilation $e^- + e^+ \rightarrow \nu + \bar{\nu}$ can be roughly estimated by (in the optically thin non-degenerate limit, Popham et al. 1999; Di Matteo et al. 2002)

$$q^-_{\rm annih} \sim (1.5\times10^{15}\,{\rm erg\,g^{-1}\,s^{-1}})\,\rho_{10}^{-1}T_{10}^9. \tag{23}$$

Similar to eqs. (20) and (21), the emission rates of neutrino number or pair capture rates (per nucleon) in the optically thin limit are

$$
\begin{aligned}
\lambda_{e^-p,\mathrm{thin}} &= \Gamma \int_0^\infty f_{\mathrm{FD}}(\mathcal{E}_-, \mu_-)\mathcal{E}_- \left(\mathcal{E}_-^2 - 1\right)^{1/2} \mathcal{E}_\nu^2 \mathrm{d}\mathcal{E}_\nu \\
&= \sqrt{2}\Gamma\Theta^{3/2}\left[\Theta^3 \widetilde{F}_{\frac{7}{2}} - (2Q-3)\Theta^2 \widetilde{F}_{\frac{5}{2}} + (Q-1)(Q-3)\Theta \widetilde{F}_{\frac{3}{2}} + (Q-1)^2 \widetilde{F}_{1/2}\right],
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
\lambda_{e^+n,\mathrm{thin}} &= \Gamma \int_{Q+1}^\infty f(\mathcal{E}_+, \mu_+)\mathcal{E}_+ \left(\mathcal{E}_+^2 - 1\right)^{1/2} \mathcal{E}_{\bar{\nu}}^2 \mathrm{d}\mathcal{E}_{\bar{\nu}} \\
&= \sqrt{2}\Gamma\Theta^{3/2}\left[\Theta^3 F_{\frac{7}{2}} + (2Q+3)\Theta^2 F_{\frac{5}{2}} + (Q+1)(Q+3)\Theta F_{\frac{3}{2}} + (Q+1)^2 F_{1/2}\right],
\end{aligned}
\tag{25}
$$

where the arguments of $\widetilde{F}_k$ are $(\eta_-,\ \Theta)$ and the arguments of F_k are $(\eta_+,\ \Theta)$. The pair capture rates are shown in Fig. 2.

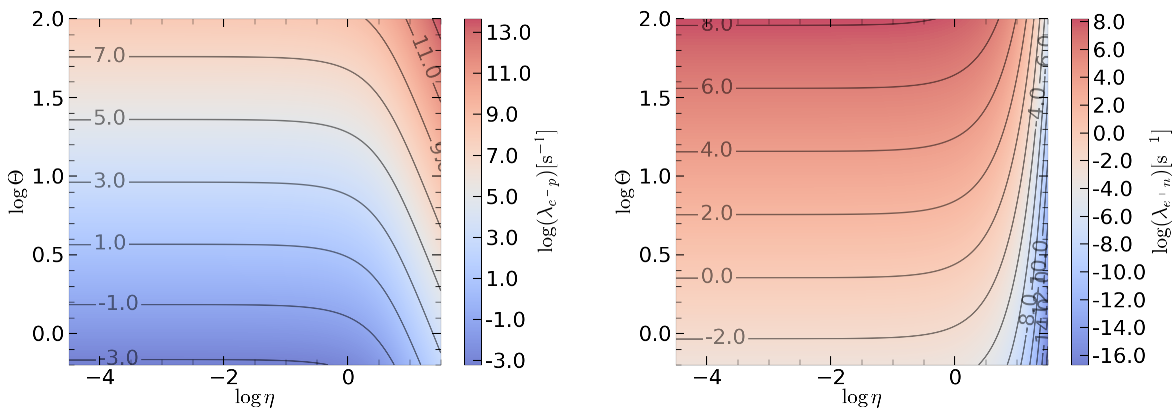

Fig. 2.— Pair capture rates per nucleon per second as functions of η (a dimensionless measurement of the degree of electron degeneracy) and Θ (dimensionless temperature). The plasma is assumed to be optically thin to neutrinos and anti-neutrinos. For a dense and relatively cold plasma such that electrons are highly degenerate ($\eta \gtrsim 1$), electron captures onto protons occur at a much higher rate than positron captures onto neutrons, so the system tends to evolve towards a lower Y_e (or more neutron rich).

As the neutrino/anti-neutrino optical depth of the entire system τ approaches (but is still below) unity, the cooling rates $q_{\nu/\bar{\nu},\mathrm{thin}}$ and pair capture rates $\lambda_{e^-p,\mathrm{thin}}$, $\lambda_{e^+n,\mathrm{thin}}$ are suppressed by a factor of $\mathrm{e}^{-\tau} \simeq (1+\tau)^{-1}$. In the extreme limit $\tau \gg 1$, since the absorption and scattering cross-sections are comparable, (anti-)neutrinos are efficiently thermalized into Fermi-Dirac distribution $f_{\mathrm{FD}}(\mathcal{E}_\nu, \mu_\nu)$ (eq. 9) for neutrino energy $\mathcal{E}_\nu$ and degeneracy μ_ν (both in units of $m_\mathrm{e}c^2$). Then the neutrino number density and energy density are given by

$$
\begin{aligned}
n_\nu &= \mathcal{V}^{-1} \int f_{\mathrm{FD}}(\mathcal{E}_\nu, \mu_\nu)\mathcal{E}_\nu^2 \mathrm{d}\mathcal{E}_\nu = \frac{\Theta^3}{\mathcal{V}} F_2(\eta_\nu, 0), \quad \eta_\nu \equiv \mu_\nu/\Theta, \\
U_\nu &= \frac{m_\mathrm{e}c^2}{\mathcal{V}} \int f_{\mathrm{FD}}(\mathcal{E}_\nu, \mu_\nu)\mathcal{E}_\nu^3 \mathrm{d}\mathcal{E}_\nu = \frac{m_\mathrm{e}c^2\Theta^4}{\mathcal{V}} F_3(\eta_\nu, 0),
\end{aligned}
\tag{26}
$$

where we have used ultra-relativistic neutrino momentum $p_\nu \approx \mathcal{E}_\nu$ and the neutrino pressure is given by $P_\nu = U_\nu/3$. The same expressions as eq. (26) with $\mathcal{E}_{\bar{\nu}}$ and $\mu_{\bar{\nu}}$ can be written for electron anti-neutrinos. The degeneracy parameter for neutrinos is given by detailed balance

$$\mu_\nu = \mu_{e^-} + \mu_p - \mu_n = \mu - \Theta \ln(X_n/X_p) - Q,\ \mu_{\bar{\nu}} = -\mu_\nu, \tag{27}$$

or

$$\eta_\nu \equiv \mu_\nu/\Theta = \eta - (Q-1)/\Theta - \ln(X_n/X_p),\ \eta_{\bar{\nu}} = -\eta_\nu, \tag{28}$$

where we have used eq. (8) for $\mu_n - \mu_p$. The energy flux escaping from the disk in the vertical direction depends on the Rosseland-mean opacity (Rybicki & Lightman 1979)

$$\begin{aligned} \kappa_{\mathrm{R},\nu}^{-1} &= \frac{1}{\Theta^4 F_3(\eta_\nu, 0)} \int_0^\infty \frac{1}{\kappa_{\mathrm{s},\nu} + \kappa_{\mathrm{a},\nu\mathrm{n}}} f_{\mathrm{FD}}(\mathcal{E}_\nu, \mu_\nu) \mathcal{E}_\nu^3 \mathrm{d}\mathcal{E}_\nu, \\ \kappa_{\mathrm{R},\bar{\nu}}^{-1} &= \frac{1}{\Theta^4 F_3(\eta_{\bar{\nu}}, 0)} \int_0^\infty \frac{1}{\kappa_{\mathrm{s},\bar{\nu}} + \kappa_{\mathrm{a},\bar{\nu}\mathrm{p}}} f_{\mathrm{FD}}(\mathcal{E}_{\bar{\nu}}, \mu_{\bar{\nu}}) \mathcal{E}_{\bar{\nu}}^3 \mathrm{d}\mathcal{E}_{\bar{\nu}}, \end{aligned} \tag{29}$$

and then the escaping energy fluxes from the system are (for a spatial thickness of H)

$$F_\nu \simeq \frac{U_\nu c}{\tau_\nu + 1},\ \ F_{\bar{\nu}} \simeq \frac{U_{\bar{\nu}} c}{\tau_{\bar{\nu}} + 1},\ \ \tau_\nu = \kappa_{\mathrm{R},\nu}\rho H,\ \ \tau_{\bar{\nu}} = \kappa_{\mathrm{R},\bar{\nu}}\rho H. \tag{30}$$

The scattering cross-sections are given by (a summary is provided by Burrows et al. 2006)

$$\sigma_{\mathrm{s,p}} = 0.30\,\sigma_0 \mathcal{E}_{\nu/\bar{\nu}}^2,\ \sigma_{\mathrm{s,n}} = 0.36\,\sigma_0 \mathcal{E}_{\nu/\bar{\nu}}^2,\ \sigma_{\mathrm{s},\alpha} = 0.21\,\sigma_0 \mathcal{E}_{\nu/\bar{\nu}}^2,\ \sigma_0 = 1.76 \times 10^{-44}\,\mathrm{cm}^{-2}, \tag{31}$$

We ignore (inelastic) neutrino scattering by pairs because the cross-sections are smaller than scattering by nucleons and scattering of neutrinos less energetic than the electron chemical potential is suppressed by degeneracy (Bruenn 1985). The absorption cross-sections are given by the following expressions with a few percent accuracy for neutrino energies up to 80 MeV (Bemporad et al. 2002; Strumia & Vissani 2003)

$$\sigma_{\mathrm{a},\nu\mathrm{n}} = 1.41\,\sigma_0 (1 - f_{e^-})(\mathcal{E}_\nu + Q)\left[(\mathcal{E}_\nu + Q)^2 - 1\right]^{1/2}, \tag{32}$$

$$\begin{aligned} \sigma_{\mathrm{a},\bar{\nu}\mathrm{p}} &= 1.48\, g_{\mathrm{fit}} \sigma_0 (1 - f_{e^+})(\mathcal{E}_{\bar{\nu}} - Q)\left[(\mathcal{E}_{\bar{\nu}} - Q)^2 - 1\right]^{1/2}, \\ g_{\mathrm{fit}}(y) &= y^{-0.07056 + 0.02018 \ln y - 0.001953 \ln^3 y},\ \ y = 0.511 \mathcal{E}_{\bar{\nu}}, \end{aligned} \tag{33}$$

where the Fermi blocking factors are given by

$$1 - f_{e^-} = 1 - \frac{1}{\exp\left[(\mathcal{E}_\nu + Q - \mu)/\Theta\right] + 1},\ \ 1 - f_{e^+} = 1 - \frac{1}{\exp\left[(\mathcal{E}_{\bar{\nu}} - Q + \mu)/\Theta\right] + 1}. \tag{34}$$

Knowing the escaping energy flux, the cooling rates per unit mass in the optically thick limit are given by

$$q^-_{\nu,\mathrm{thick}} = \frac{F_\nu}{\rho H} \simeq \frac{q_{\nu,\mathrm{max}}}{\tau_\nu + 1},\ q_{\nu,\mathrm{max}} = \frac{U_\nu c}{\rho H}, \tag{35}$$

$$q^-_{\bar{\nu},\mathrm{thick}} = \frac{F_{\bar{\nu}}}{\rho H} \simeq \frac{q_{\bar{\nu},\mathrm{max}}}{\tau_{\bar{\nu}} + 1},\ q_{\bar{\nu},\mathrm{max}} = \frac{U_{\bar{\nu}} c}{\rho H}. \tag{36}$$

For the escaping number flux of neutrinos, the equivalent Rosseland-mean opacities are

$$\begin{aligned}\widetilde{\kappa}_{\mathrm{R},\nu}^{-1} &= \frac{1}{\Theta^3 F_2(\eta_\nu,0)}\int_0^\infty \frac{1}{\kappa_{\mathrm{s},\nu}+\kappa_{\mathrm{a},\nu\mathrm{n}}} f_{\mathrm{FD}}(\mathcal{E}_\nu,\mu_\nu)\mathcal{E}_\nu^2 \mathrm{d}\mathcal{E}_\nu,\\ \widetilde{\kappa}_{\mathrm{R},\bar\nu}^{-1} &= \frac{1}{\Theta^3 F_2(\eta_{\bar\nu},0)}\int_0^\infty \frac{1}{\kappa_{\mathrm{s},\bar\nu}+\kappa_{\mathrm{a},\bar\nu\mathrm{p}}} f_{\mathrm{FD}}(\mathcal{E}_{\bar\nu},\mu_{\bar\nu})\mathcal{E}_{\bar\nu}^2 \mathrm{d}\mathcal{E}_{\bar\nu},\end{aligned} \tag{37}$$

and then the number fluxes are

$$\dot N_\nu \simeq \frac{n_\nu c}{\widetilde{\tau}_\nu+1},\quad \dot N_{\bar\nu} \simeq \frac{n_{\bar\nu} c}{\widetilde{\tau}_{\bar\nu}+1},\quad \widetilde{\tau}_\nu = \widetilde{\kappa}_\nu \rho H,\quad \widetilde{\tau}_{\bar\nu} = \widetilde{\kappa}_{\bar\nu}\rho H. \tag{38}$$

Then the emission rates of neutrino number or pair capture rates (per nucleon) in the optically thick limit are

$$\lambda_{e^-p,\mathrm{thick}} = \frac{\dot N_\nu m_\mathrm{p}}{X_p\rho H} \simeq \frac{\lambda_{e^-p,\max}}{\widetilde{\tau}_\nu+1},\ \lambda_{e^-p,\max} = \frac{n_\nu m_\mathrm{p} c}{X_p \rho H}, \tag{39}$$

$$\lambda_{e^+n,\mathrm{thick}} = \frac{\dot N_{\bar\nu} m_\mathrm{p}}{X_n\rho H} \simeq \frac{\lambda_{e^+n,\max}}{\widetilde{\tau}_{\bar\nu}+1},\ \lambda_{e^+n,\max} = \frac{n_{\bar\nu} m_\mathrm{p} c}{X_n \rho H}. \tag{40}$$

Since the opacities depends on neutrino energy in a complex way, eqs. (29) and (37) cannot be expressed in Fermi-Dirac integrals and are hence numerically integrated as follows

$$\begin{aligned}\kappa_{\mathrm{R},\nu}^{-1} &= \frac{m_\mathrm{p} K_\nu(Y_e,\eta_\nu,\Theta)}{\sigma_0\Theta^2 F_3(\eta_\nu,0)},\quad \kappa_{\mathrm{R},\bar\nu}^{-1} = \frac{m_\mathrm{p} K_{\bar\nu}(Y_e,\eta_{\bar\nu},\Theta)}{\sigma_0\Theta^2 F_3(\eta_{\bar\nu},0)},\\ \widetilde{\kappa}_{\mathrm{R},\nu}^{-1} &= \frac{m_\mathrm{p} \widetilde{K}_\nu(Y_e,\eta_{\bar\nu},\Theta)}{\sigma_0\Theta^2 F_2(\eta_\nu,0)},\quad \widetilde{\kappa}_{\mathrm{R},\bar\nu}^{-1} = \frac{m_\mathrm{p} \widetilde{K}_{\bar\nu}(Y_e,\eta_{\bar\nu},\Theta)}{\sigma_0\Theta^2 F_2(\eta_{\bar\nu},0)},\end{aligned} \tag{41}$$

$$\begin{aligned}K_\nu(Y_e,\eta_\nu,\Theta) = &\int_0^\infty \frac{x^3\mathrm{d}x}{\exp(x-\eta_\nu)+1}\times\\ &\left[\left(0.30X_p+0.36X_n+0.21X_\alpha\right)x^2 + 1.41X_n(1-f_{e^-})\left(x+\frac{Q}{\Theta}\right)\sqrt{\left(x+\frac{Q}{\Theta}\right)^2-\frac{1}{\Theta^2}}\right]^{-1},\\ K_{\bar\nu}(Y_e,\eta_{\bar\nu},\Theta) = &\int_0^\infty \frac{x^3\mathrm{d}x}{\exp(x-\eta_{\bar\nu})+1}\times\\ &\left[\left(0.30X_p+0.36X_n+0.21X_\alpha\right)x^2 + 1.48\,g_{\mathrm{fit}}X_p(1-f_{e^+})\left(x-\frac{Q}{\Theta}\right)\sqrt{\left(x-\frac{Q}{\Theta}\right)^2-\frac{1}{\Theta^2}}\right]^{-1},\end{aligned} \tag{42}$$

where $\eta_\nu \equiv \mu_\nu/\Theta$ and the term $((x-Q/\Theta)^2-\Theta^{-2})^{1/2}$ in $K_{\bar\nu}$ is only included when $\mathcal{E}_{\bar\nu} = \Theta x > Q+1$, because otherwise $\bar\nu p$ absorption is energetically prohibited (although $\bar\nu p$ and $\bar\nu n$ scatterings can always occur). The integral expressions for the number-flux opacities $\widetilde{K}_\nu$ and $\widetilde{K}_{\bar\nu}$ only differ from those in eq. (42) by a substitution $x^3 \rightarrow x^2$ in the integrands.

Finally, the two extreme limits of optically thin and thick cases can be smoothly connected by the following total cooling rate (per unit mass) and pair capture rates (per nucleon)

$$q_\nu^- = \frac{1}{\tau_\nu+1}\left(q_{\nu,\mathrm{thin}}^{-1}+q_{\nu,\max}^{-1}\right)^{-1},\ q_{\bar\nu}^- = \frac{1}{\tau_{\bar\nu}+1}\left(q_{\bar\nu,\mathrm{thin}}^{-1}+q_{\bar\nu,\max}^{-1}\right)^{-1}, \tag{43}$$

$$\lambda_{e^-p} = \frac{1}{\widetilde{\tau}_\nu + 1}\left(\lambda^{-1}_{e^-p,\mathrm{thin}} + \lambda^{-1}_{e^-p,\mathrm{max}}\right)^{-1},\ \lambda_{e^+n} = \frac{1}{\widetilde{\tau}_{\bar{\nu}} + 1}\left(\lambda^{-1}_{e^+n,\mathrm{thin}} + \lambda^{-1}_{e^+n,\mathrm{max}}\right)^{-1}. \tag{44}$$

Our prescription captures the same physics as in Di Matteo et al. (2002) and Chen & Beloborodov (2007) and is more accurate in that the energy flux and number flux are calculated separately based on different Rosseland-mean opacities.

Other than neutrino emission, the plasma may also be irradiated by external neutrino flux. External neutrino captures on free nucleons affect the plasma by changing the electron fraction Y_e and heating. The luminosity and mean energy of (anti-)neutrinos from the external neutrino source are denoted as L_ν (or $L_{\bar{\nu}}$) and $\langle \mathcal{E}_\nu \rangle$ (or $\langle \mathcal{E}_{\bar{\nu}} \rangle$), and the distance from the neutrino source is denoted as r.

In the optically thin limit, the heating rate per unit mass due to neutrino irradiation can be estimated by

$$q^+_{\nu\bar{\nu}} \simeq \frac{L_\nu \langle \sigma_{\mathrm{a},\nu\mathrm{n}} \rangle_q X_n}{4\pi r^2 m_\mathrm{p}} + \frac{L_{\bar{\nu}} \langle \sigma_{\mathrm{a},\bar{\nu}\mathrm{p}} \rangle_q X_p}{4\pi r^2 m_\mathrm{p}}, \tag{45}$$

where the averaged absorption cross-sections $\langle \sigma_{\mathrm{a},\nu\mathrm{n}} \rangle_q$ and $\langle \sigma_{\mathrm{a},\bar{\nu}\mathrm{p}} \rangle_q$ depend on the neutrino spectrum and are estimated later. On the other hand, the electron fraction of the outer disk can also be modified by neutrino irradiation and the rates of neutrino capture (per nucleon) are given by

$$\lambda_{\nu n} \simeq \frac{L_\nu \langle \sigma_{\mathrm{a},\nu\mathrm{n}} \rangle_\lambda}{4\pi r^2 \langle \mathcal{E}_\nu \rangle m_\mathrm{e} c^2},\ \ \lambda_{\bar{\nu} p} \simeq \frac{L_{\bar{\nu}} \langle \sigma_{\mathrm{a},\bar{\nu}\mathrm{p}} \rangle_\lambda}{4\pi r^2 \langle \mathcal{E}_{\bar{\nu}} \rangle m_\mathrm{e} c^2}, \tag{46}$$

where the averaged absorption cross-sections $\langle \sigma_{\mathrm{a},\nu\mathrm{n}} \rangle_\lambda$ and $\langle \sigma_{\mathrm{a},\bar{\nu}\mathrm{p}} \rangle_\lambda$ are estimated later.

Finally, with the rates for pair capture (eq. 44) and neutrino capture (eq. 46) in hand, we can calculate the time evolution of the electron fraction by

$$\frac{\mathrm{d}Y_e}{\mathrm{d}t} = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(X_n - X_p) = -(\lambda_{e^-p} + \lambda_{\bar{\nu}p})X_p + (\lambda_{e^+n} + \lambda_{\nu n})X_n. \tag{47}$$

Averaged absorption cross-sections for external neutrinos

For a given neutrino energy spectrum $\mathrm{d}\dot{N}/\mathrm{d}\mathcal{E}$, we define the k-th moments of the neutrino energy as

$$\langle \mathcal{E}^k \rangle = \dot{N}^{-1} \int \mathrm{d}\mathcal{E} \frac{\mathrm{d}\dot{N}}{\mathrm{d}\mathcal{E}} \mathcal{E}^k,\ \ \dot{N} = \int \mathrm{d}\mathcal{E} \frac{\mathrm{d}\dot{N}}{\mathrm{d}\mathcal{E}}, \tag{48}$$

where $\dot{N}$ is the total number flux (in $\mathrm{cm^{-2}\,s^{-1}}$). For a FD distribution $\mathrm{d}\dot{N}/\mathrm{d}\mathcal{E} \propto x^2/[\exp(x-\eta)+1]$ (where $x = \eta/\Theta$, $\eta = \mu/\Theta$), we can make use of the FD integrals $F_k(\eta, 0)$ to show that $\langle \mathcal{E}^k \rangle / \langle \mathcal{E} \rangle^k = F_{k+2} F_2^{k-1} / F_3^k$. For arbitrary degeneracy, these ratios are bounded by

$$\frac{\Gamma(k+3)}{2 \times 3^k} < \frac{\langle \mathcal{E}^k \rangle}{\langle \mathcal{E} \rangle^k} < \left(\frac{4}{3}\right)^k \frac{3}{k+3}. \tag{49}$$

In the following, we will need the above ratios for $k = 2, 3, 4$. Under the assumption that neutrinos are at most mildly degenerate ($|\eta_{\nu/\bar{\nu}}| \lesssim 1$), we take the following approximations $\langle \mathcal{E}^4 \rangle / \langle \mathcal{E} \rangle^4 \simeq 4$, $\langle \mathcal{E}^3 \rangle / \langle \mathcal{E} \rangle^3 \simeq 2$, and $\langle \mathcal{E}^2 \rangle / \langle \mathcal{E} \rangle^2 \simeq 1.3$.

The heating rates due to neutrino absorption are given by

$$q_\nu^+ = \frac{X_n m_e c^2}{m_p} \int d\mathcal{E}_\nu \frac{d\dot{N}_\nu}{d\mathcal{E}_\nu} (\mathcal{E}_\nu + Q) \sigma_{a,\nu n}(\mathcal{E}_\nu), \quad q_{\bar\nu}^+ = \frac{X_p m_e c^2}{m_p} \int d\mathcal{E}_{\bar\nu} \frac{d\dot{N}_{\bar\nu}}{d\mathcal{E}_{\bar\nu}} (\mathcal{E}_{\bar\nu} - Q) \sigma_{a,\bar\nu p}(\mathcal{E}_{\bar\nu}). \quad (50)$$

Note that the heat deposition due to each absorbed ν or $\bar\nu$ is $(\mathcal{E}_\nu + Q) m_e c^2$ or $(\mathcal{E}_{\bar\nu} - Q) m_e c^2$, respectively. The neutrino capture rates are given by

$$\lambda_{\nu n} = \int d\mathcal{E}_\nu \frac{d\dot{N}_\nu}{d\mathcal{E}_\nu} \sigma_{a,\nu n}(\mathcal{E}_\nu), \quad \lambda_{\bar\nu p} = \int d\mathcal{E}_{\bar\nu} \frac{d\dot{N}_{\bar\nu}}{d\mathcal{E}_{\bar\nu}} \sigma_{a,\bar\nu p}(\mathcal{E}_{\bar\nu}). \quad (51)$$

When Fermi blocking is unimportant (i.e. if the typical energies of electrons or positrons generated by neutrino capture are much greater than their Fermi energies in the plasma) and in the limit $\mathcal{E}_{\nu/\bar\nu} \gg Q$, the absorption cross-sections can be approximated by

$$\sigma_{a,\nu n} \simeq 1.41 \sigma_0 (\mathcal{E}_\nu + Q)^2, \quad \sigma_{a,\bar\nu p} \simeq 1.41 \sigma_0 (\mathcal{E}_{\bar\nu} - Q)^2 (1 - \chi \mathcal{E}_{\bar\nu}), \quad (52)$$

where $\chi \approx 5.8 m_e/m_p$ provides good agreement to the fitting formula of Strumia & Vissani (2003). We plug the cross-sections into the integrals in eqs. (50) and (51) and obtain the following averaged absorption cross-sections

$$\begin{aligned} \frac{\langle \sigma_{a,\nu n} \rangle_q}{1.41 \sigma_0 \langle \mathcal{E}_\nu \rangle} &\simeq \langle \mathcal{E}_\nu^3 \rangle + 3Q \langle \mathcal{E}_\nu^2 \rangle + 3Q^2 \langle \mathcal{E}_\nu \rangle + Q^3, \\ \frac{\langle \sigma_{a,\bar\nu p} \rangle_q}{1.41 \sigma_0 \langle \mathcal{E}_{\bar\nu} \rangle} &\simeq -\chi \langle \mathcal{E}_{\bar\nu}^4 \rangle + (1 + 3\chi Q) \langle \mathcal{E}_{\bar\nu}^3 \rangle - 3Q \langle \mathcal{E}_{\bar\nu}^2 \rangle + 3Q^2 \langle \mathcal{E}_{\bar\nu} \rangle - Q^3, \end{aligned} \quad (53)$$

$$\begin{aligned} \langle \sigma_{a,\nu n} \rangle_\lambda &\simeq 1.41 \sigma_0 \left(\langle \mathcal{E}_\nu^2 \rangle + 2Q \langle \mathcal{E}_\nu \rangle + Q^2 \right), \\ \langle \sigma_{a,\bar\nu p} \rangle_\lambda &\simeq 1.41 \sigma_0 \left[-\chi \langle \mathcal{E}_{\bar\nu}^3 \rangle + (1 + 2\chi Q) \langle \mathcal{E}_{\bar\nu}^2 \rangle - 2Q \langle \mathcal{E}_{\bar\nu} \rangle + Q^2 \right]. \end{aligned} \quad (54)$$

The above results can be combined with our estimates of the moments of the neutrino energy to calculate the heating rates and neutrino capture rates. Our approach is similar to that of Qian & Woosley (1996) except that we have included the additional correction term ($\chi \neq 0$) in $\sigma_{a,\bar\nu p}$ due to nucleon recoil and weak magnetism. In the absense of pair captures, the equilibrium electron fraction is given by

$$Y_e^\nu = \left(1 + \frac{\lambda_{\bar\nu p}}{\lambda_{\nu n}}\right)^{-1} = \left(1 + \frac{L_{\bar\nu}}{L_\nu} \frac{2\chi \langle \mathcal{E}_{\bar\nu} \rangle^2 + 1.3(1 + 2\chi Q) \langle \mathcal{E}_{\bar\nu} \rangle - 2Q + Q^2/\langle \mathcal{E}_{\bar\nu} \rangle}{1.3 \langle \mathcal{E}_\nu \rangle + 2Q + Q^2/\langle \mathcal{E}_\nu \rangle}\right)^{-1}. \quad (55)$$

On the other hand, if the typical energies of electrons or positrons generated by neutrino capture are much smaller than their Fermi energy in the plasma, then neutrino capture is strongly suppressed (or Fermi blocked). In this case, since the absorption cross-section depend on the neutrino energy in a complex way, and the detailed inner disk neutrino spectrum is required to calculate neutrino capture rate and heating rate in the outer disk. We make a rough estimate by multiplying the cross-sections in eqs. (53) and (54) by the following Fermi blocking factors based on the mean neutrino energies

$$1 - f_{e^-} \simeq 1 - \frac{1}{\exp\left[(\langle \mathcal{E}_\nu \rangle + Q - \mu)/\Theta\right] + 1}, \quad 1 - f_{e^+} \simeq 1 - \frac{1}{\exp\left[(\langle \mathcal{E}_{\bar\nu} \rangle - Q + \mu)/\Theta\right] + 1}. \quad (56)$$

Neutrino Eddington Luminosity

The Eddington luminosities of ν and $\bar{\nu}$ depends on their Planck-mean opacities, which are given by

$$\kappa_{\mathrm{P},\nu} = \frac{\sigma_0 \Theta^2 K_{\mathrm{P},\nu}(Y_e, \eta_\nu, \Theta)}{m_\mathrm{p} F_3(\eta_\nu, 0)}, \quad \kappa_{\mathrm{P},\bar{\nu}} = \frac{\sigma_0 \Theta^2 K_{\mathrm{P},\bar{\nu}}(Y_e, \eta_{\bar{\nu}}, \Theta)}{m_\mathrm{p} F_3(\eta_{\bar{\nu}}, 0)}, \tag{57}$$

$$\begin{aligned} K_{\mathrm{P},\nu}(Y_e, \eta_\nu, \Theta) = & \int_0^\infty \frac{x^3 \mathrm{d}x}{\exp(x - \eta_\nu) + 1} \times \\ & \left[\left(0.30 X_p + 0.36 X_n + 0.21 X_\alpha\right) x^2 + 1.41 X_n (1 - f_{e^-}) \left(x + \frac{Q}{\Theta}\right) \sqrt{\left(x + \frac{Q}{\Theta}\right)^2 - \frac{1}{\Theta^2}} \right], \\ K_{\mathrm{P},\bar{\nu}}(Y_e, \eta_{\bar{\nu}}, \Theta) = & \int_0^\infty \frac{x^3 \mathrm{d}x}{\exp(x - \eta_{\bar{\nu}}) + 1} \times \\ & \left[\left(0.30 X_p + 0.36 X_n + 0.21 X_\alpha\right) x^2 + 1.48\, g_\mathrm{fit} X_p (1 - f_{e^+}) \left(x - \frac{Q}{\Theta}\right) \sqrt{\left(x - \frac{Q}{\Theta}\right)^2 - \frac{1}{\Theta^2}} \right], \end{aligned} \tag{58}$$

where the absorption term $((x - Q/\Theta)^2 - \Theta^{-2})^{1/2}$ in $K_{\mathrm{P},\bar{\nu}}$ is only included when $\mathcal{E}_{\bar{\nu}} = \Theta x > Q + 1$ for energetic reason. In the advection-dominated regime $\dot{M} \gg 1\,\mathrm{M}_\odot\,\mathrm{s}^{-1}$, the disk temperature scales as $T \propto \dot{M}^{1/4}$, and because the opacity scales as $\kappa_\mathrm{P} \propto T^2$, the Eddington luminosity drops with accretion rate as $L_\mathrm{Edd} \propto \dot{M}^{-1/2}$. This property is unique to the neutrino-dominated accretion flow (NDAF), whereas for the accretion flows in black-hole X-ray binaries or active galactic nuclei (AGN), the Eddington luminosity is independent of the accretion rate (as a result of the constant electron scattering opacity).

Nuclear Reactions

At relatively high temperature $T \gtrsim 5\,\mathrm{GK}$, the abundance ratios among n, p, and α are given by NSE. The heating/cooling rate (per unit mass) from α particle formation/destruction is given by

$$q^+_\mathrm{nuc} = (6.8 \times 10^{18}\,\mathrm{erg\,g^{-1}}) \frac{\mathrm{d}X_\alpha}{\mathrm{d}t}. \tag{59}$$

Numerical Integrals

The Fermi-Dirac integrals (eq. 11) are numerically evaluated using the simple method by Natarajan & Kumar (1993), since there is no accurate analytical solution for arbitrary η and Θ.

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