Task 1A, List04

Step 0: Introduction to the task

We are tasked with finding a specific example of two random variables X and Y such that they are dependent, but their correlation is 0.

Step 1: My variables

- X is a random variable uniformly distributed on the interval $[-1, 1]$. The probability density function (pdf) of this function is $\frac{1}{2}$.
- Let's also define another random variable dependent on the first one: Y as $Y = X^2$.

Step 2: How to prove that it works?

Dependency

Quite obvious, one has other in it's equation so they are dependent.

Lack of Correlation

The correlation coefficient is given by:

$$\rho(X, Y) = \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}}.$$

We are happy (our variables satisfy task requirements) if $\mathrm{Cov}(X, Y) = 0$.

Step 3: Proving it works - Showing that Cov$(X, Y) = 0$

The covariance is given by:

$$\mathrm{Cov}(X, Y) = E[XY] - E[X]E[Y].$$

Let's start by computing $E[X]$ and $E[Y]$

$$E[X] = \int_{-1}^{1} x \cdot \frac{1}{2}\, dx.$$

which is clearly equal to 0. (Skipping $E[Y]$ as $E[X]$ will nullify it anyways).

Since $E[X] = 0$, we have:

$$\mathrm{Cov}(X, Y) = E[XY].$$

Computing $E[XY]$:

$$E[XY] = \int_{-1}^{1} x \cdot x^2 \cdot \frac{1}{2}\, dx = \int_{-1}^{1} \frac{x^3}{2}\, dx.$$

$$E[XY] = \frac{1}{2}\left(\frac{1}{4} - \frac{1}{4}\right) = 0.$$

Which gives us desired result:

$$\mathrm{Cov}(X, Y) = 0.$$

Step 4: Summary

I have shown that X and Y have 0 correlation despite being dependent.