Reangularity, Semangularity, and Ideality

By

Dr. Michael S. Slocum
Vice President of Innovation and Design
MichaelS@BMGi.com

Catherine Lundberg
Ontro, Inc.

Katrina Walter
Ontro, Inc.

Abstract:

The calculation and use of Ideality to drive the evolution of a technological system can be a subjective process. The identification and definition as an element as useful or harmful can be convoluted and non-repeatable when calculated by different members of the design team. Also, there is not standardization of this technique that would yield consistency across product domains. Axiomatic Design was employed to create a robust disposable self-heating chafing dish for use by consumers. This paper compares the standard combustible fuel disposable chafing system with a self-heating disposable chafing system in order to demonstrate the key elements of the Axiomatic Design Process. This demonstration evolves into the use of Reangularity and Semangularity (metrics of the degree of coupling in a system between design elements) to apply a rigorous standard to the calculation and effectiveness of Ideality. The results of this effort are considered to be the primary intent of this paper.

A Brief Review of Axiomatic Design and its' Application to Ideality

Axiomatic Design is a system that uses design axioms and corollaries to govern the design process. The two main axioms govern dependencies between functional requirements (FR's) and design parameters (DP's) and the total information content of a system. The FR-DP dependencies should be maintained on a one-to-one basis with no cross-correlation between FR's and DP's (independence). When cross-correlations occur this is called "coupling". The degree of coupling is an indication of the state of the design with the desire that coupling either not exist or exist at a very minimal level. The structure and mapping of the elements in Axiomatic Design is well established, therefore, it is possible to consistently apply this technique to a system. This consistency lends itself to application in the Ideality element of TRIZ. The useful functions of a system would be categorized with the inclusion of a DP for each element thereby supplying the information of the numerator of the Ideality equation. Those elements that are indicative of coupling would be included in the denominator with the costs associated with each DP:

$$\begin{Bmatrix} FR_1 \\ FR_2 \\ FR_3 \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{Bmatrix} DP_1 \\ DP_2 \\ DP_3 \end{Bmatrix}$$

$$Ideality = \frac{\sum F_u}{\sum \left(F_h + \text{cost}\right)} = \frac{\left(A_{11} + A_{22} + A_{33}\right)}{\left(A_{12} + A_{13} + A_{21} + A_{23} + A_{31} + A_{32}\right) + \text{cost}\left(\mathrm{DP}_1 + DP_2 + DP_3\right)}$$

The coupled elements may be substituted with the inverse of their associated reangularities (R) in order to normalize the harmful effect content:

$$Ideality = \frac{\sum F_u}{\sum \left(F_h + \text{cost}\right)} = \frac{\left(A_{11} + A_{22} + A_{33}\right)}{\left(\frac{1}{R_{12}} + \frac{1}{R_{13}} + \frac{1}{R_{21}} + \frac{1}{R_{23}} + \frac{1}{R_{31}} + \frac{1}{R_{32}}\right) + \text{cost}\left(\mathrm{DP}_1 + DP_2 + DP_3\right)}$$

TRIZ offers algorithms for the creative resolution of problems, the evolution of a technological system, and the ideal final result concept but there is no rigorous application of axioms to the correspondence of requirements and parameters. This core feature of Axiomatic Design when incorporated with the creative elements of TRIZ will enable the practitioner to increase the idealness of the their system.

Axiomatic Design Axioms

1. *The Independence Axiom* – Maintain the independence of FRs
2. *The Information Axiom* – Minimize the information content of the design

a. The information content (IC) is calculated by:

$$IC = \log\left(\frac{\text{total \# of observations}}{\text{total \# of acceptable observations}}\right)$$

Design Matrix

Standard Combustible Fuel Disposable Chafing System

Functional Requirement	**Design Parameter**
1.0 Hold Food	DP1.1 Aluminum food pan(s) DP1.2 Frame to hold food pan(s)
2.0 Maintains food 2.1 $\bar{T} \geq T_{i+30\min}$ 2.2 t ≥ 2 hours	DP2.1 Combustible Fuel Source DP2.2 Water Pan
3.0 Inexpensive Remove as constraint (RAC) RAC RAC	DP3.1 Aluminum (standard) food pan(s) DP3.2 Fuel (standard) is inexpensive DP3.3 Frame design is simple DP3.4 Assembly is basic DP3.5 Low weight system (reduces shipping cost)
4.0 Portability RAC	DP4.1 Same as DP3.5 DP4.2 Handles for carrying
5.0 Manufacturability RAC	DP5.1 Most assembly is pick and place DP5.2 Use of standard (off the shelf) items
6.0 Easy to use RAC	DP6.1 3-step activation

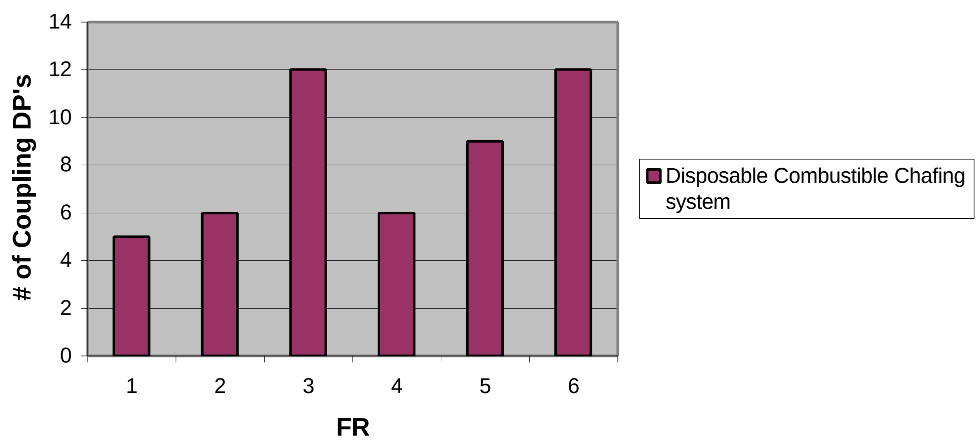
Design Matrix to identify coupling within the design. Utilizing this equation:

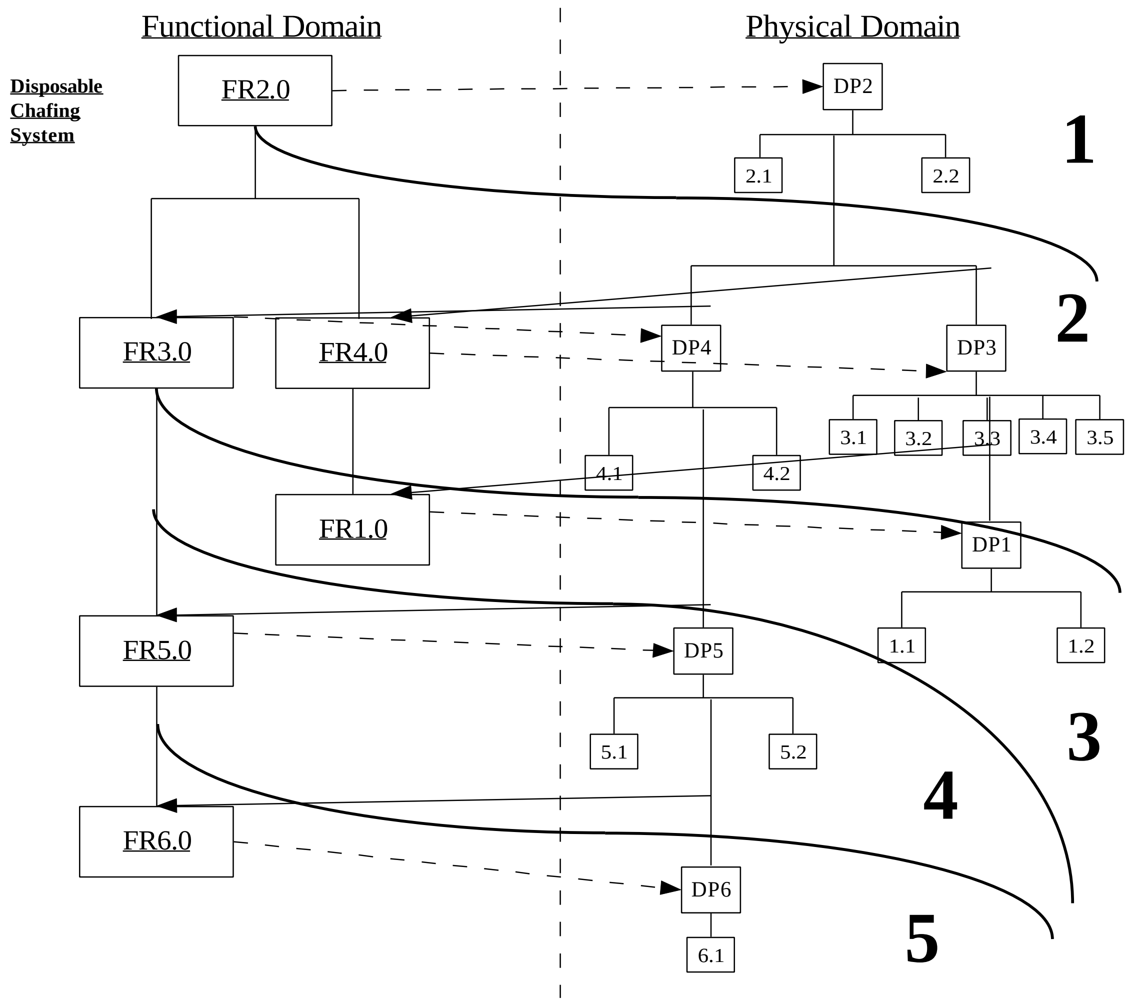
$\{FR\} = [\underline{A}]\{DP\} \quad FR_i = \sum_j A_{ij} DP_j$ Equation 1.0

Design Matrix for a Disposable Combustible Chafing System

DP/FR	FR1.0	FR2.1	FR2.2	FR3.0	FR4.0	FR5.0	FR6.0
DP1.1	X	X	X	X	X	X	X
DP1.2	X	O	O	X	X	X	X
DP2.1	0	X	X	X	X	O	X
DP2.2	O	X	X	X	O	O	X
DP3.1	X	X	X	X	O	X	O
DP3.2	O	O	O	X	O	O	O
DP3.3	X	O	O	X	O	X	X
DP3.4	O	O	O	X	O	X	X
DP3.5	O	O	O	X	X	X	X
DP4.1	O	O	O	X	X	X	X
DP4.2	O	O	O	O	X	O	X
DP5.1	O	O	O	X	O	X	O
DP5.2	X	O	O	X	O	X	X
DP6.1	O	O	O	O	O	O	X

Graphical Representation of Coupling by FR Group





- **This flowchart outlines the design progress**
- **The DP's from the previous FR feeds into the next FR's**
- **Analyze the design at each of the 5 levels.**

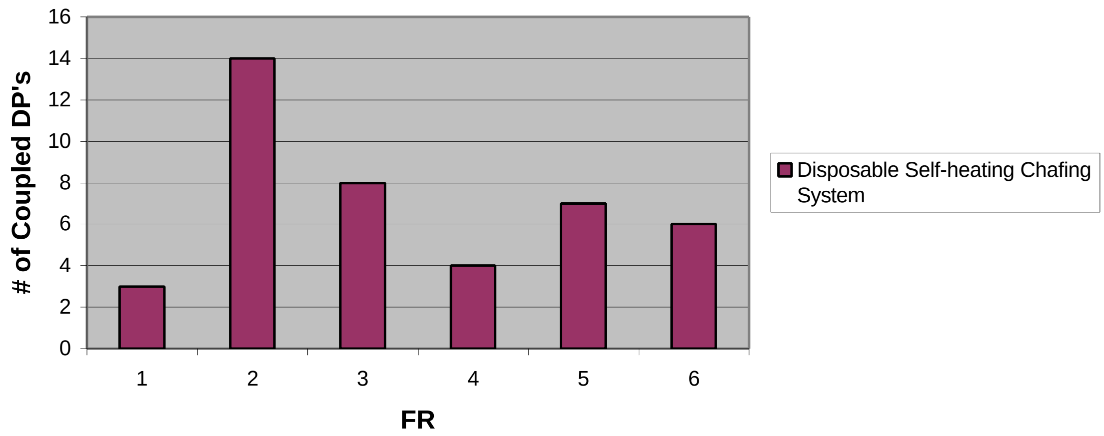
Self-heating Disposable Chafing System

Functional Requirement	Design Parameter
1.0 Hold Food	DP1.1 Aluminum food pan(s) DP1.2 Supported by SCRB
2.0 Maintains food 2.1 $\overline{T} \geq T_{i+30\min}$ 2.2 t ≥ 2 hours	DP2.1 Exothermic Chemical Reaction DP2.2 Surface contact between food tray and SCRB DP2.3 Insulation internal to SCRB DP2.4 Food pan lid
3.0 Inexpensive RAC	DP3.1 Materials are inexpensive DP3.2 Food pans are off the shelf
4.0 Portability	DP4.1 Outer tray has handles DP4.2 Fully integrated system
5.0 Manufacturability RAC	DP5.1 Utilizing vacuum form and injection molding DP5.2 Basic mfg/assembly process
6.0 Easy to use RAC	DP6.1 2-step activation

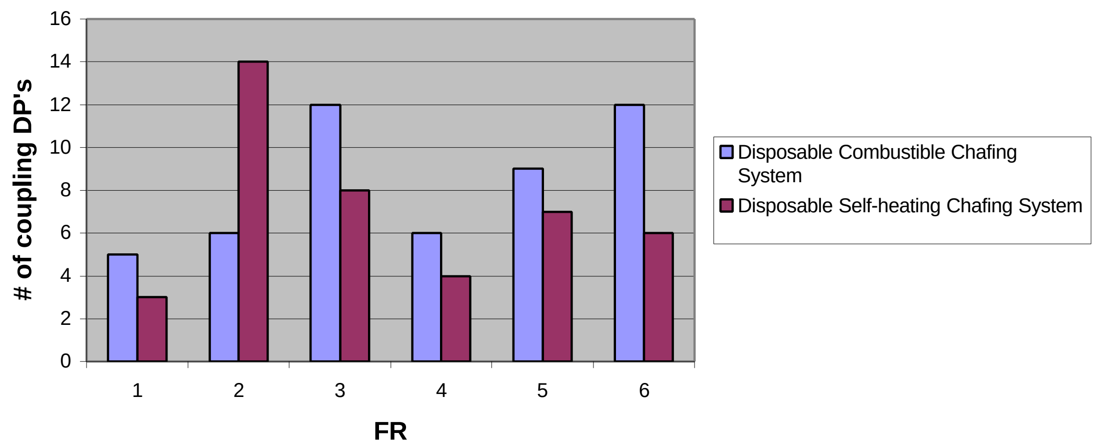
Design Matrix for the Self-heating Disposable Chafing System

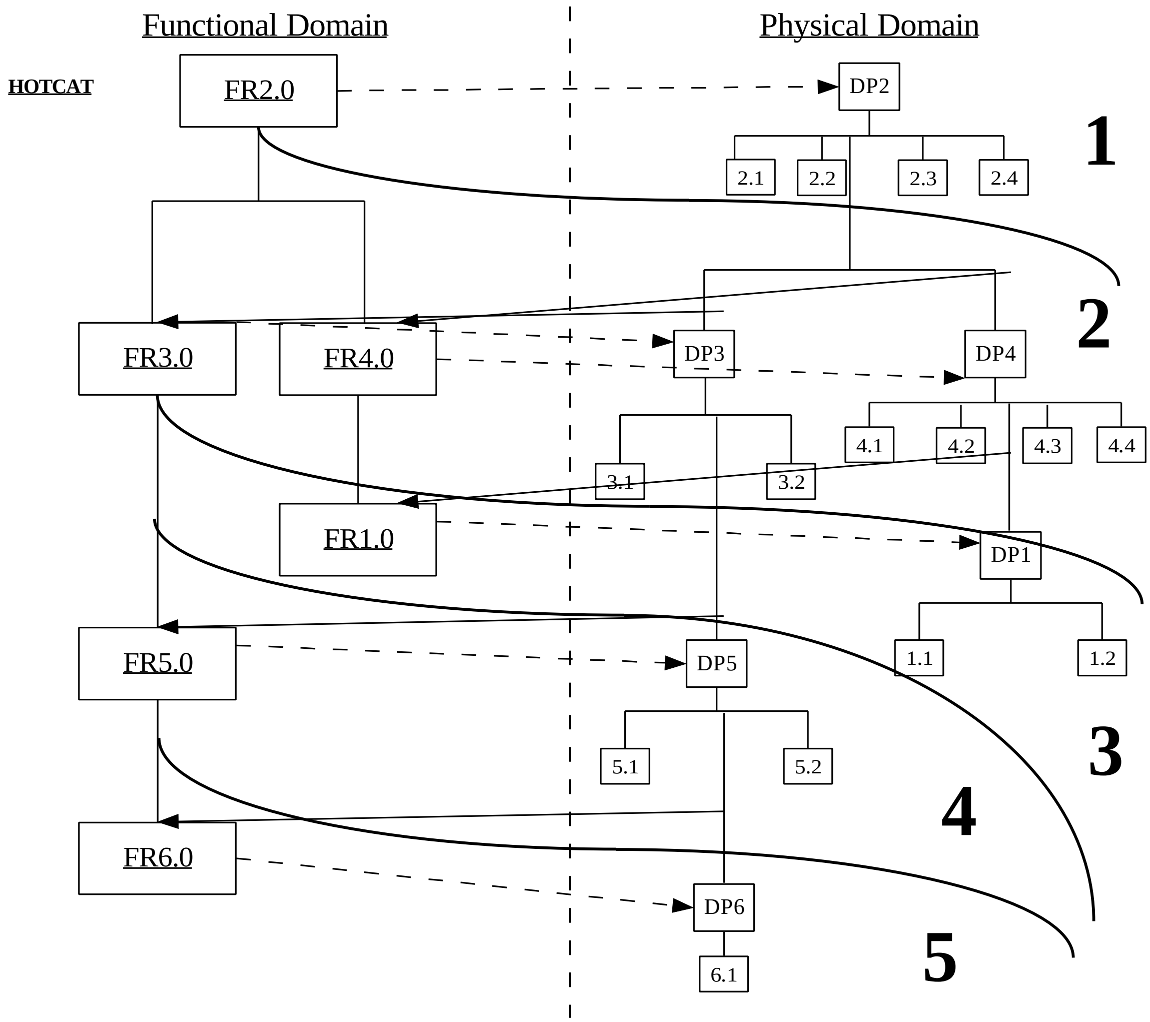
DP/FR	FR1.0	FR2.1	FR2.2	FR3.0	FR4.0	FR5.0	FR6.0
DP1.1	X	X	X	X	O	X	O
DP1.2	X	X	X	O	O	X	O
DP2.1	O	X	X	O	X	O	X
DP2.2	O	X	X	O	O	O	O
DP2.3	O	X	X	X	O	X	O
DP2.4	O	X	X	X	X	X	X
DP3.1	O	O	O	X	O	O	O
DP3.2	X	O	O	X	O	X	X
DP4.1	O	O	O	X	X	O	X
DP4.2	O	X	X	O	X	O	X
DP5.1	O	O	O	X	O	X	O
DP5.2	O	O	O	X	O	X	O
DP6.1	O	O	O	O	O	O	X

Graphical Representation of Coupling by FR Group



Graphical Representation of Coupling DP's by FR Group




Functional Domain
Physical Domain
HOTCAT
FR2.0
DP2
1
2.1
2.2
2.3
2.4
2
FR3.0
FR4.0
DP3
DP4
4.1
4.2
4.3
4.4
3.1
3.2
FR1.0
DP1
FR5.0
DP5
1.1
1.2
3
5.1
5.2
4
FR6.0
DP6
5
6.1

Information Content (Partial)

The partial information content was calculated utilizing the above described equation and data from Buffet Test #1 and #2. The rating was considered acceptable if the food was rated between 2 and 4 (rating scale was 1-5, cold to hot).

For the Standard Combustible System

$$IC_{Partial} = \log\left(\frac{187}{136}\right) = 0.138$$

For the Self-heating Disposable Chafing System

$$IC_{Partial} = \log\left(\frac{182}{164}\right) = 0.045$$

Removing FR's 3.0, 5.0, and 6.0

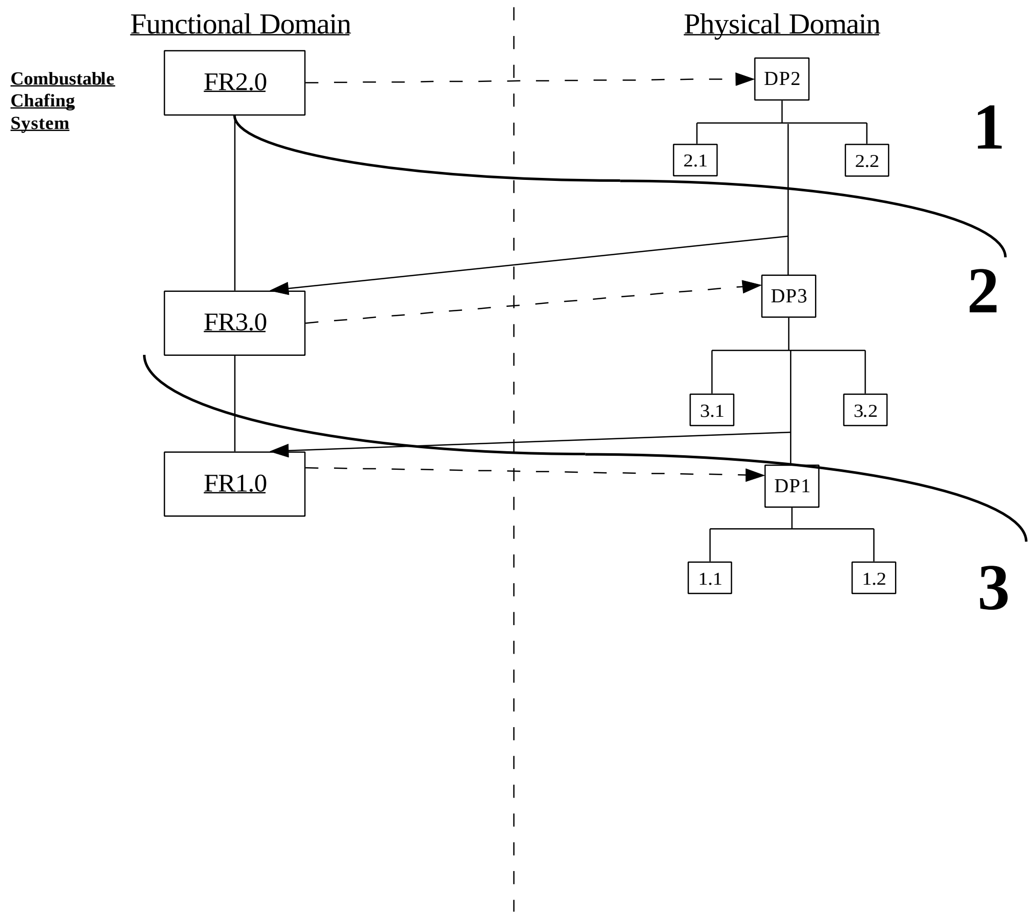
FR's 3.0, 5.0, and 6.0 could be considered system constraints instead of FR's. The above analysis is repeated after the removal of these FR's. The *new* FR's are as follows:

Disposable Combustible Fuel Chafing System

Functional Requirement	Design Parameter
1.0 Hold Food	DP1.1 Aluminum food pan(s) DP1.2 Frame to hold food pan(s)
2.0 Maintains food 2.1 $\bar{T} \geq T_{i+30\min}$ 2.2 t ≥ 2 hours	DP2.1 Combustible Fuel Source DP2.2 Water Pan
3.0 Portability	DP3.1 Low weight of system DP3.2 Handles for carrying

Design Matrix for a Disposable Combustible Chafing System

DP/FR	FR1.0	FR2.1	FR2.2	FR3.0
DP1.1	X	X	X	X
DP1.2	X	O	O	X
DP2.1	O	X	X	X
DP2.2	O	X	X	O
DP3.1	O	O	O	X
DP3.2	O	O	O	X

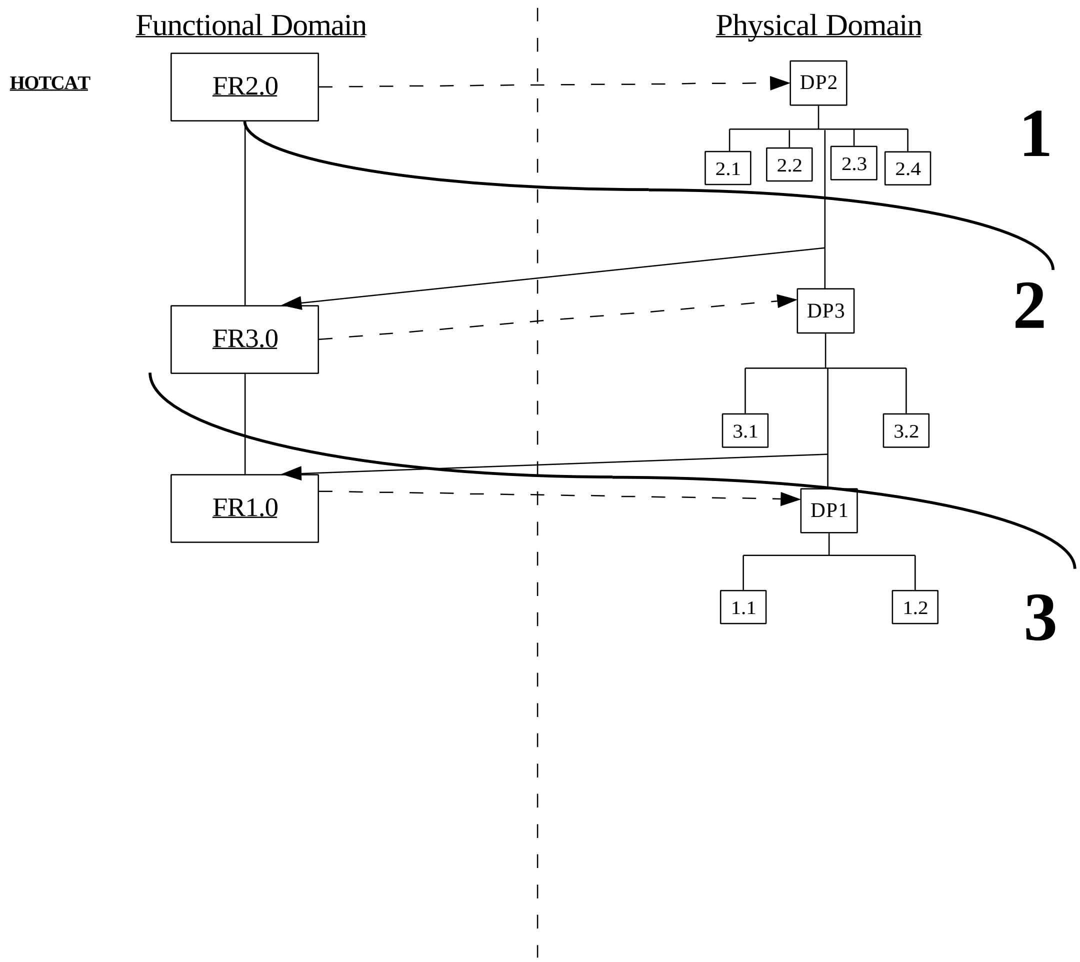

Functional Domain
Physical Domain
Combustable
Chafing
System
FR2.0
DP2
2.1
2.2
1
2
FR3.0
DP3
3.1
3.2
FR1.0
DP1
1.1
1.2
3

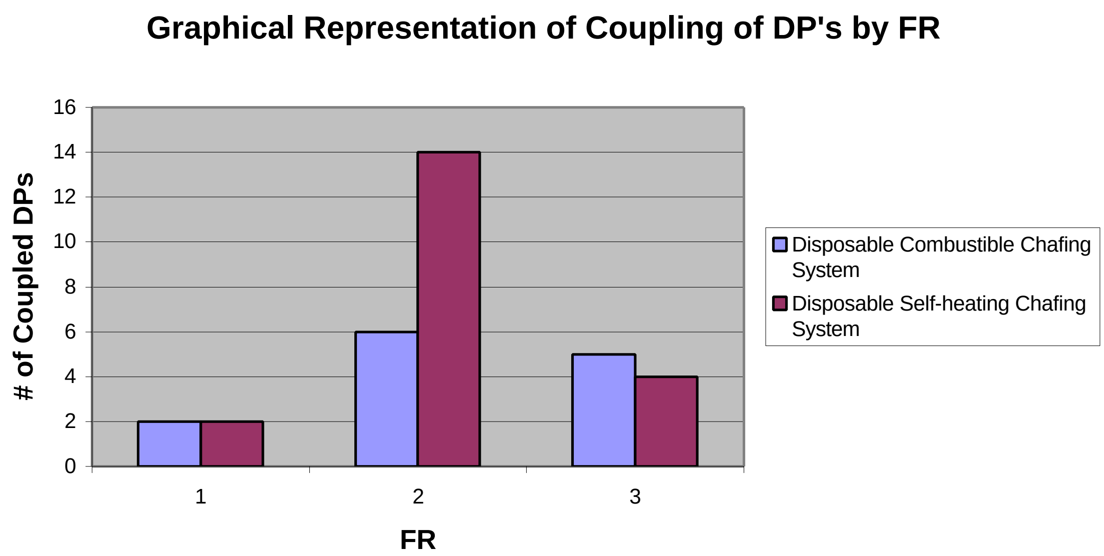
Self-heating Disposable Chafing System

Functional Requirement	Design Parameter	Status	Action
1.0 Hold Food	DP1.1 Aluminum food pan(s) DP1.2 Supported by SCRB	 Redundant	
2.0 Maintains food 2.1 $\overline{T} \geq T_{i+30\min}$ 2.2 t ≥ 2 hours	DP2.1 Exothermic Chemical Reaction DP2.2 Surface contact between food tray and SCRB DP2.3 Insulation internal to SCRB DP2.4 Food pan lid	 Redundant Redundant	 Eliminate insulation Eliminate lid
3.0 Portability	DP3.1 Outer tray has handles DP3.2 Fully integrated system	 Redundant	

Design Matrix for the Self-heating Disposable Chafing System

DP/FR	FR1.0	FR2.1	FR2.2	FR3.0
DP1.1	X	X	X	O
DP1.2	X	X	X	O
DP2.1	O	X	X	X
DP2.2	O	X	X	O
DP2.3	O	X	X	O
DP2.4	O	X	X	X
DP3.1	O	O	O	X
DP3.2	O	X	X	X


Functional Domain
Physical Domain
HOTCAT
FR2.0
DP2
2.1
2.2
2.3
2.4
1
2
FR3.0
DP3
3.1
3.2
FR1.0
DP1
1.1
1.2
3


Graphical Representation of Coupling of DP's by FR
# of Coupled DPs
0
2
4
6
8
10
12
14
16
1
2
3
FR
Disposable Combustible Chafing System
Disposable Self-heating Chafing System

Our design has FR<DP’s. This means our design is redundant or coupled. This means it is over designed and probably costs too much. To reduce the design to a simpler design we will:

1. Delete the lid (DP 2.4)
2. Injection mold the polypropylene SCRB which may reduce or eliminate the insulation (DP2.3)

<u>Quantitative Methods for Determining Coupling - *Reangularity and Semangularity*</u>

Semangularity is a value that, when equal to unity, 1, then the design is uncoupled providing the reangularity is also equal to unity, 1. Semangularity, *S*, is defined by the following:

$$S = \prod_{j=1}^{n} \frac{\left| A_{jj} \right|}{\left(\sum_{k=1}^{n} A_{kj}^{2} \right)^{1/2}}$$

Reangularity, *R* is a value that has an inverse relationship to coupling (i.e. as reangularity decreases, coupling increases). If R is 0, the design is completely coupled. If R *and* S are unity, 1 then the design is uncoupled. Reangularity is defined by the following:

$$R = \sin q = \left(1 - \cos^{2} q\right)^{1/2}$$

For the n[th] dimensional case:

$$R = \prod_{\substack{i=1,n-1 \\ j=1,n}} \left(1 - \frac{\sum_{k=1}^{n} \Delta_{ki} \Delta_{kj}}{\left(\sum_{k=1}^{n} \Delta_{ki}^{2} \right) \left(\sum_{k=1}^{n} \Delta_{kj}^{2} \right)} \right)^{1/2}$$

This information combined with the design equation 1.0 and corresponding matrix to create isotherms utilizing the following the methodology outlined in the following 2-dimensional example:

$[FR] = [A][DP]$ Equation 1

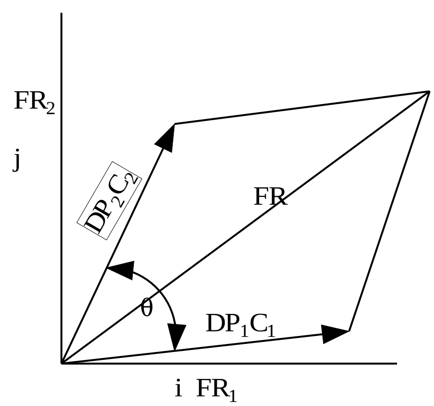
For the 2 dimensional case:

$$\begin{Bmatrix} \mathrm{FR}_1 \\ \mathrm{FR}_2 \end{Bmatrix} = \begin{bmatrix} A_{11} A_{12} \\ A_{21} A_{22} \end{bmatrix} \begin{Bmatrix} DP_1 \\ DP_2 \end{Bmatrix} = \begin{Bmatrix} A_{11} \\ A_{12} \end{Bmatrix} DP_1 + \begin{Bmatrix} A_{21} \\ A_{22} \end{Bmatrix} DP_2 \quad \text{OR}$$

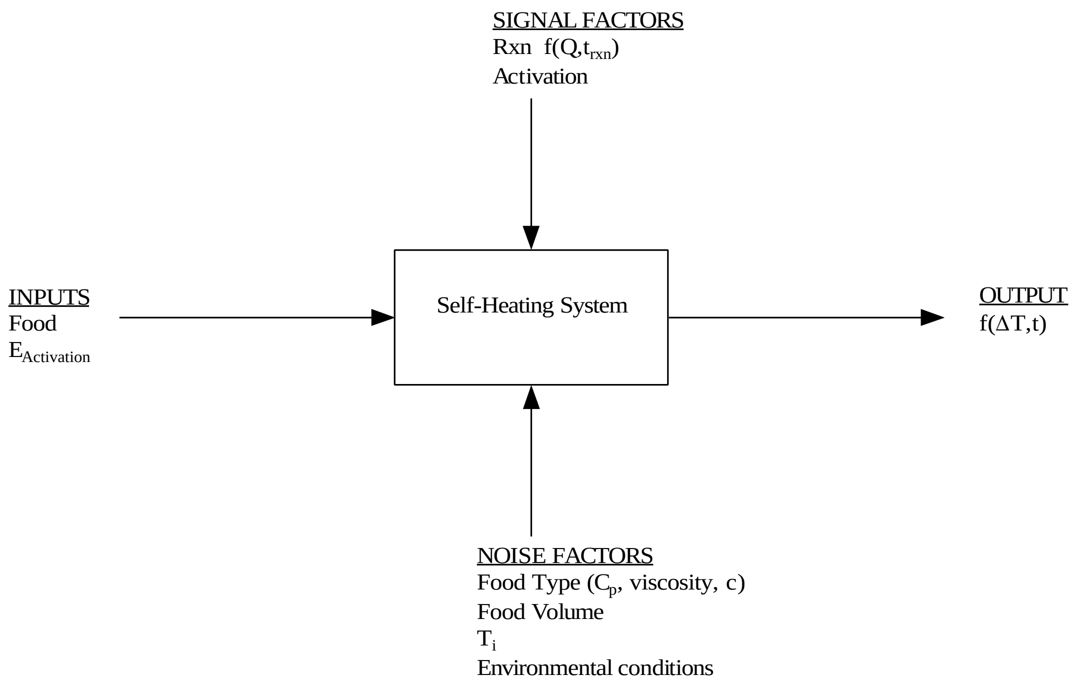
$$FR = C_1 DP_1 + C_2 DP_2 = FR_{1j} + FR_{2j}$$

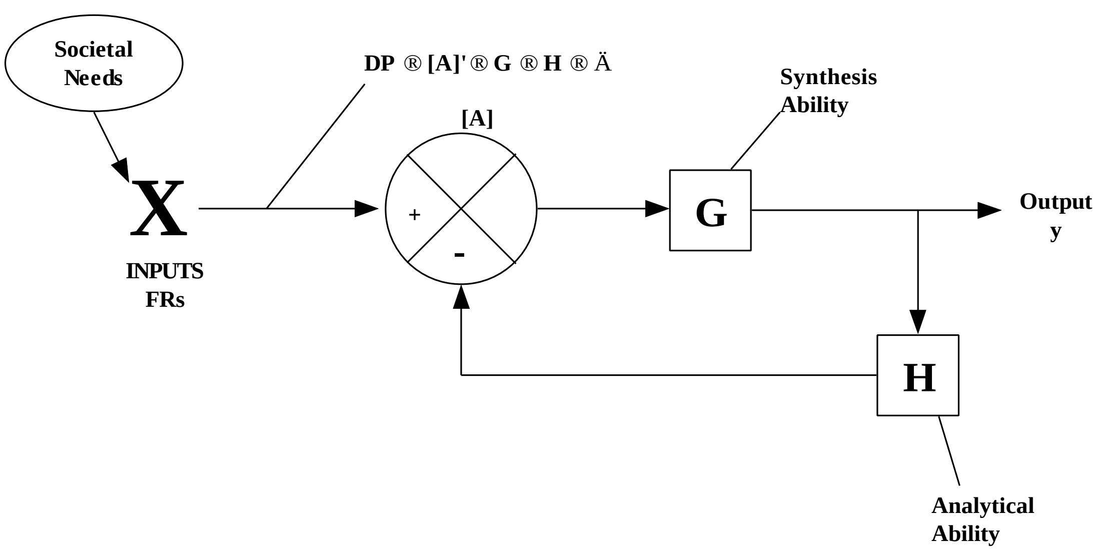
Vector C_k are: $\mathrm{C}_1 = \begin{Bmatrix} A_{11} \\ A_{12} \end{Bmatrix}$; $C_2 = \begin{Bmatrix} A_{21} \\ A_{22} \end{Bmatrix}$

$$\cos q = \frac{C_1 \bullet C_2}{|C_1| \bullet |C_2|}$$



p-diagram





$$\frac{y}{x} = \frac{G}{1+GH} \approx \frac{G}{GH};$$

H^{-1} for GH>>1

FR=$\sum^{h}\left[\left(l + l' + l'' + \ldots + l^{h}\right)\right] = DP$; where l is each iteration of the [$\otimes \rightarrow$ G $\rightarrow$ H] cycle

Robustness

$$s^2 = \text{var}\,iance = \frac{1}{n-1}\sum_{p=1}^{n}\left(FR_j^p - \overline{\overline{FR}}_i\right)^2$$

$$\overline{\overline{FR}}_i = m = \frac{1}{n}\sum_{p=1}^{n} FR_i^p$$

Taguchi

$SN = 10\log\left(\dfrac{m^2}{s^2}\right)$; the higher the better

Decibles, dB

The FR's are dimensionless because a log normalization of the terms is performed.

Bias - Aiming for a 0 value

$$b = \left(FR_i\right)_o - FR = \left(FR_i\right)_0 - m$$

Functional Requirements

Functional Requirements	Design Parameters
FR 1.0 Holds Food	DP 1.1 Aluminum Food Pan
FR 2.0 Maintain Food Temperature	DP 2.1 Exothermic Chemical Reaction (SCRB)
FR 3.0 Portability	DP3.1 Fully integrated system with handles
Constraints	
C.1 Inexpensive	
C.2 Manufacturable	
C.3 Easy to Use	

Design Matrix

$$\{FR\} = [A]\{DP\}$$

$$\begin{Bmatrix} FR1.0 \\ FR2.0 \\ FR3.0 \end{Bmatrix} = \begin{bmatrix} A_{11} & 0 & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & A_{33} \end{bmatrix} \begin{Bmatrix} DP1.1 \\ DP2.1 \\ DP3.1 \end{Bmatrix} \text{ Coupled Design}$$

$FR_1 = 2L$ $\quad$ $DP_{1.1} = (l * w * h) = 2L$

$FR_2 \geq 60°C$ $\quad$ $DP_{2.1} = 168,000 cal$

$FR_3 \leq 5500 g$ $\quad$ $DP_{3.1} = \left(M_t = \frac{2}{9} b^2 h S_v \right) (h > b);$

where, S_v = vertical shear=$S_{vMax} = \frac{3}{2}\frac{V}{bh}$

$\therefore$

$A_{11} = \{2L = l * w * h\}$

$A_{21} = 33 \frac{mL}{°C \Delta T}$

$A_{22} = 2800 \frac{cal}{°C \Delta T}$

$A_{33} = \{M_t > 5500 g\}$

Isograms

$$FR_{3,1} = C_3 DP_3 + C_1 DP_1$$

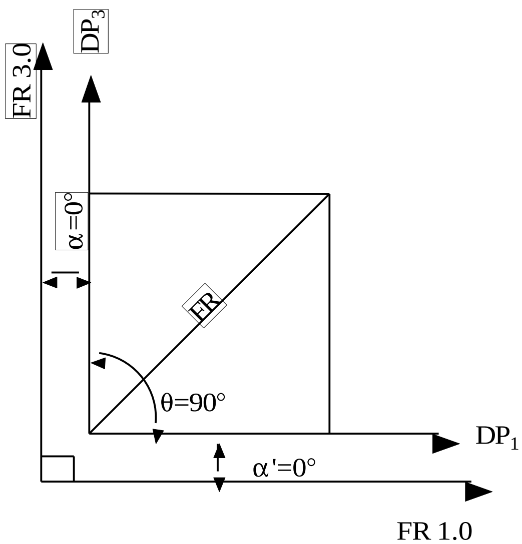
$$C_1 = \begin{Bmatrix} A_{11} \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 2L = lwh \\ 0 \\ 0 \end{Bmatrix} \qquad C_3 = \begin{Bmatrix} 0 \\ 0 \\ A_{33} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ M_t > 5500g \end{Bmatrix}$$

$$\cos\theta = \frac{C_1 \bullet C_3}{|C_1| \bullet |C_3|} = \frac{(A_{11} * 0) + (0 * 0) + (0 * A_{33})}{(A_{11}^2 + 0^2 + 0^2)^{1/2} * (0^2 + 0^2 + A_{33}^2)^{1/2}} = \frac{0}{A_{11} * A_{33}} = 0$$

$$\theta = 90°$$

$$\alpha = \tan^{-1}\left(\frac{-A_{13}}{A_{33}}\right) = \tan^{-1}\left(\frac{-0}{A_{33}}\right) = 0°$$

$$\alpha' = \tan^{-1}\left(\frac{A_{31}}{A_{11}}\right) = \tan^{-1}\left(\frac{0}{A_{11}}\right) = 0°$$



$$FR_{2,1} = C_2 DP_2 + C_1 DP_1$$

$$C_1 = \begin{Bmatrix} A_{11} \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 2L = lwh \\ 0 \\ 0 \end{Bmatrix} \qquad C_2 = \begin{Bmatrix} A_{21} \\ A_{22} \\ 0 \end{Bmatrix} = \begin{Bmatrix} 33\dfrac{mL}{°C\Delta T} \\ 2800\dfrac{cal}{°C\Delta T} \\ 0 \end{Bmatrix}$$

$$\cos q = \frac{C_1 \bullet C_2}{|C_1| \bullet |C_2|} = \frac{(A_{11} * A_{21}) + (0*0) + (0*A_{22})}{(A_{11}^2 + 0^2 + 0^2)^{1/2} * (A_{21}^2 + 0^2 + A_{22}^2)^{1/2}} = \frac{A_{11} * A_{21}}{(A_{21}^2 + A_{22}^2)^{1/2} * A_{11}} \neq 0$$

To solve you must use this relationship

$$\text{R} = \cos(a\text{-}a') = \sin q$$

$$a = \tan^{-1}\left(\frac{-A_{12}}{A_{22}}\right) = \tan^{-1}\left(\frac{-0}{A_{22}}\right) = 0°$$

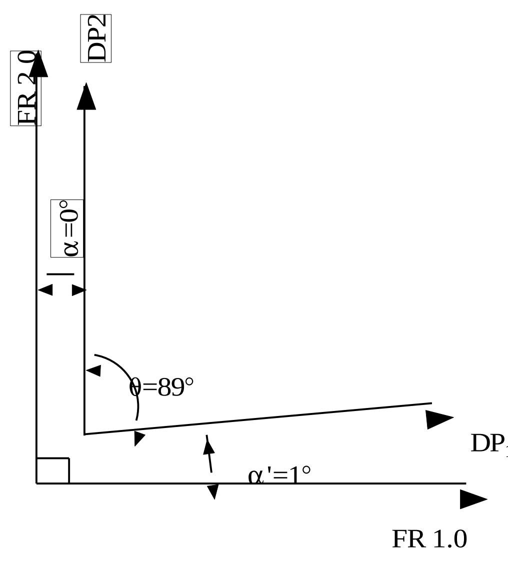
$$a' = \tan^{-1}\left(\frac{A_{21}}{A_{11}}\right) = \tan^{-1}\left(\frac{33\dfrac{mL}{°C\Delta T}}{2000mL}\right) = 1°$$

$$\text{R} = \cos(0\text{-}1) = \cos(-1) = 0.9998$$

$$\sin q = 0.9998$$

$$q = 89°$$

$$\cos(89°) = 0.017$$



$$FR_{3,2} = C_3 DP_3 + C_2 DP_2$$

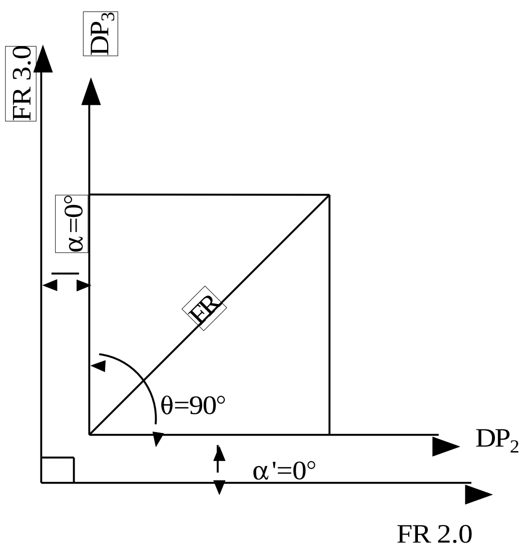
$$C_2 = \begin{Bmatrix} A_{21} \\ A_{22} \\ 0 \end{Bmatrix} = \begin{Bmatrix} 33\dfrac{mL}{°C\Delta T} \\ 2800\dfrac{cal}{°C\Delta T} \\ 0 \end{Bmatrix} \qquad C_3 = \begin{Bmatrix} 0 \\ 0 \\ A_{33} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ M_t > 5500g \end{Bmatrix}$$

$$\cos\theta = \frac{C_2 \bullet C_3}{|C_2| \bullet |C_3|} = \frac{(A_{21} * 0) + (A_{22} * 0) + (0 * A_{33})}{\left(A_{21}^2 + A_{22}^2 + 0^2\right)^{1/2} * \left(0^2 + 0^2 + A_{33}^2\right)^{1/2}} = \frac{0}{\left(A_{21}^2 + A_{22}^2\right)^{1/2} * A_{33}} = 0$$

$$\theta = 90°$$

$$\alpha = \tan^{-1}\left(\frac{-A_{23}}{A_{33}}\right) = \tan^{-1}\left(\frac{-0}{A_{33}}\right) = 0°$$

$$\alpha' = \tan^{-1}\left(\frac{A_{32}}{A_{22}}\right) = \tan^{-1}\left(\frac{0}{A_{22}}\right) = 0°$$



Semangularity, S

$$S = \prod_{j=1}^{n}\left(\frac{|A_{jj}|}{A_{j1}^2 + A_{j2}^2 + A_{j3}^2}\right) = \left(\frac{|A_{11}|}{A_{11}^2 + A_{12}^2 + A_{13}^2}\right)\left(\frac{|A_{22}|}{A_{21}^2 + A_{22}^2 + A_{23}^2}\right)\left(\frac{|A_{33}|}{A_{31}^2 + A_{32}^2 + A_{33}^2}\right)$$

$$= \left\{\frac{|\{2L = lwh\}|}{\left(\{2L = lwh\}^2 + 0^2 + 0^2\right)^{1/2}}\right\}\left\{\frac{\left|2800\frac{cal}{°C\Delta T}\right|}{\left[\left(33\frac{mL}{°C\Delta T}\right)^2 + \left(2800\frac{cal}{°C\Delta T}\right)^2 + 0^2\right]}\right\}\left\{\frac{|M_t > 5500g|}{\left(0^2 + 0^2 + (M_t > 5500g)^2\right)^{1/2}}\right\}$$

$$= 1*\left\{\frac{\left|2800\frac{cal}{°C\Delta T}\right|}{\left[\left(33\frac{mL}{°C\Delta T}\right)^2 + \left(2800\frac{cal}{°C\Delta T}\right)^2 + 0^2\right]}\right\}*1$$

Signal-to-Noise Ratio (S/N) and Bias, b

$$\begin{bmatrix} FR_{1a} \\ FR_{1b} \\ FR_{1c} \end{bmatrix} = [A_i]\begin{bmatrix} DP_{1a} \\ DP_{1b} \\ DP_{1c} \end{bmatrix} = \begin{bmatrix} 2.5L \\ 2.0L \\ 1.5L \end{bmatrix}$$

$$\overline{FR_1} = m = \frac{1}{n}\sum_{p=1}^{n} FR_1^p = \frac{1}{3}\sum(2.5 + 2.0 + 1.5) = \frac{6}{3} = 2$$

$$s^2 = \text{var}\,iance = \frac{1}{n-1}\sum_{p=1}^{n}\left(FR_1^p - \overline{FR_1}\right)^2 = \frac{1}{2}\sum(2.5-2)^2 + (2-2)^2 + (1.5-2)^2 = \frac{1}{4}$$

$$S/N = 10\log_{10}\left(\frac{m^2}{s^2}\right) = 10\log_{10}\left(\frac{2^2}{1/4}\right) = 12db$$

$$b = (FR_1)_0 - \overline{FR_1} = 2 - 2 = 0$$

$$\begin{bmatrix} FR_{2a} \\ FR_{2b} \\ FR_{2c} \end{bmatrix} = [A_i] \begin{bmatrix} DP_{2a} \\ DP_{2b} \\ DP_{2c} \end{bmatrix} = \begin{bmatrix} 90°C \\ 68°C \\ 65°C \end{bmatrix}$$

$$\overline{FR_2} = m = \frac{1}{n}\sum_{p=1}^{n} FR_2^p = \frac{1}{3}\sum(90+68+65) = 74$$

$$s^2 = \text{var}\,iance = \frac{1}{n-1}\sum_{p=1}^{n}\left(FR_2^p - \overline{FR_2}\right)^2 = \frac{1}{2}\sum(90-74)^2 + (68-74)^2 + (65-74)^2 = 186.5$$

$$S/N = 10\log_{10}\left(\frac{m^2}{s^2}\right) = 10\log_{10}\left(\frac{74^2}{186.5}\right) = 14.7db$$

$$b = (FR_2)_0 - \overline{FR} = 60 - 74 = -14$$

$$\begin{bmatrix} FR_{3a} \\ FR_{3b} \\ FR_{3c} \end{bmatrix} = [A_i] \begin{bmatrix} DP_{3a} \\ DP_{3b} \\ DP_{3c} \end{bmatrix} = \begin{bmatrix} 6408g \\ 5500g \\ 4365g \end{bmatrix}$$

$$\overline{FR_3} = m = \frac{1}{n}\sum_{p=1}^{n} FR_3^p = \frac{1}{3}\sum(6408+5500+4365) = 5424g$$

$$s^2 = \text{var}\,iance = \frac{1}{n-1}\sum_{p=1}^{n}\left(FR_3^p - \overline{FR_3}\right)^2 = \frac{1}{2}\sum(6408-5424)^2 + (5500-5424)^2 + (4365-5424)^2 = 1{,}048{,}$$

$$S/N = 10\log_{10}\left(\frac{m^2}{s^2}\right) = 10\log_{10}\left(\frac{5424^2}{1{,}048{,}000}\right) = 14.5db$$

$$b = (FR_3)_0 - \overline{FR_3} = 5500 - 5424 = 76$$

Summary Table

	FR_1	FR_2	FR_3
m	2	74	5424
b	0	-14	76
S/N	12 db	14.7 db	14.5 db
R	FR_1:FR_2=0.9998 FR_1:FR_3=1	FR_2:FR_3=1	
S	<1	<1	<1

$$\begin{Bmatrix} FR_1 \\ FR_2 \\ FR_3 \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{Bmatrix} DP_1 \\ DP_2 \\ DP_3 \end{Bmatrix}$$

$$Ideality = \frac{\sum F_u}{\sum \left(F_h + \text{cost}\right)} = \frac{\left(A_{11} + A_{22} + A_{33}\right)}{\left(A_{12} + A_{13} + A_{21} + A_{23} + A_{31} + A_{32}\right) + \text{cost}\left(\text{DP}_1 + DP_2 + DP_3\right)}$$

Capability

$$C_{p_{DP1}} = \frac{USL - LSL}{6s} = \frac{DP_{1a} - DP_{1c}}{6s};$$

$$C_{p_{DP2}} = \frac{USL - LSL}{6s} = \frac{DP_{2a} - DP_{2c}}{6s};$$

$$C_{p_{DP3}} = \frac{USL - LSL}{6s} = \frac{DP_{3a} - DP_{3c}}{6s};$$

$$C_{system} = \min\left(C_{p_{DP1}}, C_{p_{DP2}}, C_{p_{DP3}}\right)$$

$$s^2 = \text{var}\,iance = \frac{1}{n-1}\sum_{p=1}^{n}\left(FR_j^p - \bar{\bar{F}}\bar{\bar{R}}_i\right)^2$$

$$s_{FR1}^2 = \frac{1}{2}\left[\left(A_{11a}DP_1 - \bar{\bar{F}}\bar{\bar{R}}_1\right)^2 + \left(A_{11b}DP_1 - \bar{\bar{F}}\bar{\bar{R}}_1\right)^2 + \left(A_{11c}DP_1 - \bar{\bar{F}}\bar{\bar{R}}_1\right)^2\right]$$

where

$$\bar{\bar{F}}\bar{\bar{R}}_1 = \frac{1}{2}\left(FR_{1a} + FR_{1b} + FR_{1c}\right)$$

$$s_{FR2}^2 = \frac{1}{2}\left[\left(A_{22a}DP_2 - \bar{\bar{F}}\bar{\bar{R}}_2\right)^2 + \left(A_{22b}DP_2 - \bar{\bar{F}}\bar{\bar{R}}_2\right)^2 + \left(A_{22c}DP_2 - \bar{\bar{F}}\bar{\bar{R}}_2\right)^2\right]$$

where

$$\bar{\bar{F}}\bar{\bar{R}}_2 = \frac{1}{2}\left(FR_{2a} + FR_{2b} + FR_{2c}\right)$$

$$s_{FR3}^2 = \frac{1}{2}\left[\left(A_{33a}DP_3 - \bar{\bar{F}}\bar{\bar{R}}_3\right)^2 + \left(A_{33b}DP_3 - \bar{\bar{F}}\bar{\bar{R}}_3\right)^2 + \left(A_{33c}DP_3 - \bar{\bar{F}}\bar{\bar{R}}_3\right)^2\right]$$

where

$$\bar{\bar{F}}\bar{\bar{R}}_3 = \frac{1}{2}\left(FR_{3a} + FR_{3b} + FR_{3c}\right)$$

$$S = \prod_{j=1}^{n} \frac{|A_{jj}|}{\left(\sum_{k=1}^{n} A_{kj}^2\right)} = \left(\frac{|A_{11}|}{\left(A_{11}^2 + A_{21}^2 + A_{31}^2\right)^{1/2}} \right) \left(\frac{|A_{22}|}{\left(A_{21}^2 + A_{22}^2 + A_{23}^2\right)^{1/2}} \right) \left(\frac{|A_{33}|}{\left(A_{31}^2 + A_{32}^2 + A_{33}^2\right)^{1/2}} \right)$$

$$R_{12} = \cos(a - a'); \quad a = \tan^{-1}\left(\frac{-A_{12}}{A_{22}}\right); \quad a' = \tan^{-1}\left(\frac{A_{21}}{A_{11}}\right)$$

$$R_{13} = \cos(a - a'); \quad a = \tan^{-1}\left(\frac{-A_{13}}{A_{33}}\right); \quad a' = \tan^{-1}\left(\frac{A_{31}}{A_{11}}\right)$$

$$R_{23} = \cos(a - a'); \quad a = \tan^{-1}\left(\frac{-A_{23}}{A_{33}}\right); \quad a' = \tan^{-1}\left(\frac{A_{32}}{A_{22}}\right)$$

$$R_{21} = \cos(a - a'); \quad a = \tan^{-1}\left(\frac{-A_{21}}{A_{22}}\right); \quad a' = \tan^{-1}\left(\frac{A_{12}}{A_{11}}\right)$$

$$R_{31} = \cos(a - a'); \quad a = \tan^{-1}\left(\frac{-A_{31}}{A_{33}}\right); \quad a' = \tan^{-1}\left(\frac{A_{13}}{A_{11}}\right)$$

$$R_{32} = \cos(a - a'); \quad a = \tan^{-1}\left(\frac{-A_{32}}{A_{33}}\right); \quad a' = \tan^{-1}\left(\frac{A_{23}}{A_{22}}\right)$$

$$b_1 = (FR_1)_0 - \frac{1}{2}(FR_{1a} + FR_{1b} + FR_{1c})$$

$$b_2 = (FR_2)_0 - \frac{1}{2}(FR_{2a} + FR_{2b} + FR_{2c})$$

$$b_3 = (FR_3)_0 - \frac{1}{2}(FR_{3a} + FR_{3b} + FR_{3c})$$

$$SNR_1 = \frac{(FR_{1a} + FR_{1b} + FR_{1c})}{\left(A_{11a}DP_1 - \frac{1}{2}(FR_{1a} + FR_{1b} + FR_{1c})\right)^2 + \left(A_{11b}DP_1 - \frac{1}{2}(FR_{1a} + FR_{1b} + FR_{1c})\right)^2 + \left(A_{11c}DP_1 - \frac{1}{2}(FR_{1a} + FR_{1b} + FR_{1c})\right)^2}$$

$$SNR_2 = \frac{(FR_{2a} + FR_{2b} + FR_{2c})}{\left(A_{22a}DP_2 - \frac{1}{2}(FR_{2a} + FR_{2b} + FR_{2c})\right)^2 + \left(A_{22b}DP_2 - \frac{1}{2}(FR_{2a} + FR_{2b} + FR_{2c})\right)^2 + \left(A_{22c}DP_2 - \frac{1}{2}(FR_{2a} + FR_{2b} + FR_{2c})\right)^2}$$

$$SNR_3 = \frac{(FR_{3a} + FR_{3b} + FR_{3c})}{\left(A_{33a}DP_3 - \frac{1}{2}(FR_{3a} + FR_{3b} + FR_{3c})\right)^2 + \left(A_{33b}DP_3 - \frac{1}{2}(FR_{3a} + FR_{3b} + FR_{3c})\right)^2 + \left(A_{33c}DP_3 - \frac{1}{2}(FR_{3a} + FR_{3b} + FR_{3c})\right)^2}$$

Combining Ideality and Reangularity

$$Ideality = \frac{\sum F_u}{\sum \left(F_h + \text{cost} \right)} = \frac{\left(A_{11} + A_{22} + A_{33} \right)}{\left(\frac{1}{R_{12}} + \frac{1}{R_{13}} + \frac{1}{R_{21}} + \frac{1}{R_{23}} + \frac{1}{R_{31}} + \frac{1}{R_{32}} \right) + \text{cost}\left(\mathrm{DP}_1 + DP_2 + DP_3 \right)}$$