INTRODUCTION TO CALCULUS: LIMITS, DERIVATIVES, AND INTEGRALS

1. Limits

Let's start with a motivating example for calculus. You may be familiar with this from high school physics. We define the average velocity of an automobile as the change in position divided by the change in time. That is, if the automobile traveled 120 miles in a span of 2 hours, we say that it traveled at

$$v_{avg} = \frac{\Delta s}{\Delta t} = \frac{120\ miles}{2\ hours} = 60\ mph \tag{1}$$

where Δ represents the *change in* distance or time (the quantities of interest, in this case).

While it is true that the average velocity is 60 mph, it is clear that at any point of time, the automobile might have gone faster or slower or even reversed directions. We simply cannot tell how the velocity evolved at any time instant. In order for us to do that, we'll need the distance at each time instant. If we could represent the distance of the automobile as a function of time, e.g. $d(t)$, we could estimate the velocity at an instance τ by considering the change in distance around a small neighborhood of time τ. Let δ represent a small time interval (say 1 second). We can then approximate the velocity at τ, or $v(\tau)$ as follows:

$$v(\tau) \approx \frac{d(\tau+\delta) - d(\tau)}{\tau + \delta - \tau} = \frac{d(\tau+\delta) - d(\tau)}{\delta} \tag{2}$$

As δ becomes smaller and smaller, our approximation becomes better and better. However, we are interested in the velocity at τ and that means δ is zero, causing an algebraic issue.

We'll define *Limits* informally as follows: When we write

$$\lim_{t \to \tau} v(t) = V \tag{3}$$

we imply that $v(t)$ gets arbitrarily close to V as t gets arbitrarily close to τ. In order for this to be precise and work in mathematical proofs, we need to define what we mean by *arbitrarily close*. For now, we will accept the intuition that we are working on a small neighborhood around the point of interest and we examine how the function of interest converges (or diverges) around that point of interest.

Let's see how we can use limits to find the slope of a function at a point. To be precise, it is the slope of the tangent line to the function at that point. We know that the slope of a line is defined as follows:

$$s = \frac{y_2 - y_1}{x_2 - x_1} \tag{4}$$

We know that the tangent touches the function at the point of interest. Let's look at the function $y = x^2$ and compute its slope at $x = 3$. The slope of the tangent line at $x = 3$ can be defined using limits as:

$$\lim_{x\to 3} \frac{x^2 - 3^2}{x - 3} \tag{5}$$

which can be simplified as

$$\lim_{x\to 3} \frac{x^2 - 3^2}{x - 3} = \lim_{x\to 3} \frac{(x+3)(x-3)}{x-3} = \lim_{x\to 3}(x+3) = 6 \tag{6}$$

Here, we were able to exploit algebra and cancel out terms as the denominator was not zero. It got arbitrarily close to zero but was not zero. So, we could cancel out terms with impunity. This is what limits allow us to do. It lets us get arbitrarily close and use algebraic manipulations for an long as possible.

We will now define limits precisely in terms of two real numbers ϵ and δ. The statement $\lim_{x\to a} f(x) = L$ has the following precise definition. Given any real number $\epsilon > 0$, there exists another real number $\delta > 0$ so that if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$.

Let's use the above definition to prove the existence of limits so that we can see the definition in action. Let's prove $\lim_{x\to 10}(2x - 5) = 15$.

$$\begin{aligned} |f(x) - 15| < \epsilon &\iff |(2x-5) - 15| < \epsilon \\ \iff |2x - 20| < \epsilon &\iff |2(x-10)| < \epsilon \\ \iff |2||x - 10| < \epsilon &\iff |x - 10| < \frac{\epsilon}{2} \end{aligned} \tag{7}$$

Now, if we choose $\delta = \epsilon/2$, we can show that if $0 < |x - 10| < \delta$, $|f(x) - 15| < \epsilon$ and we are done.

Limits give us an algorithm for computing quantities of interest. We can approximate limits by considering small numeric intervals and calculating values at those points and then solving for the equations numerically. As long as we are within the precision limits of the particular computer we are on, we can get arbitrarily close answers.

2. Derivatives

Velocity is one example of a change in a function. In general, we are interested in how the function $f(x)$ changes as x changes. We can calculate this instantaneous rate of change using limits and it is called the *derivative*, if it exists. The derivative of a function $f(x)$ at a point a is defined as follows:

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \tag{8}$$

provided the limit exists. This defines the derivative at a point $x = a$. In general, the derivative can also be thought of as a function of x, $f'(x)$. When we consider the derivative as a function of x, we modify the definition a little bit as follows:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \tag{9}$$

for all values of x where this limit exists. If $f'(x)$ exists over the entire domain of $f(x)$, then $f(x)$ is called an analytical function. An example of an analytical function is $f(x) = x^2 + 2x + 1$. It is easy to show that the derivative exists for all values of x. As a contrast, consider $|x|$, which changes abruptly when x goes from being positive to being negative. At $x = 0$, the limit is undefined and $|x|$ is an example of a function that is not analytical.

Let's compute the derivative of $f(x) = x^2 + 2x + 1$ using limits.

$$f'(x) = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 + 2x + 2h + 1 - x^2 - 2x - 1}{h} \tag{10}$$

which we got by expanding out $(x + h)^2 + 2(x + h) + 1$. Simplifying the above equation, we get

$$f'(x) = \lim_{h \to 0} \frac{2xh + 2h + h^2}{h} = lim_{h \to 0} 2x + 2 + h = 2x + 2 \tag{11}$$

We'll revisit derivatives again when we deal with multivariate functions (e.g functions of two or more variables). For now, we have enough of a refresher on derivatives to be able to write programs to compute them numerically for arbitrary functions.

3. Integrals

Let's consider the converse problem to the one we started with. If we knew the velocity at every time instant, can we compute the distance traveled in a two-hour time frame? We intuitively know that we can do this. We can certainly approximate it by taking small intervals of time and calculating the distance traveled in this time interval and then summing them all up. For instance, to calculate the distance traveled between time t_1 and t_2, we can do the following:

$$D \approx \sum_{t=t_1}^{t_2} v(t) \Delta t \tag{12}$$

where Δt is a small time interval. This gives us an approximate estimate of the distance traveled from the values of velocity at every instant. This is true for not only velocity and distance but we can similarly add up any function that is changing and compute the area under the curve created by that function. We can use this intuitive understanding to define the integral as

$$\int_a^b f(x) dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x \tag{13}$$

We are essentially making smaller and smaller intervals and the number of such intervals tends to infinity and Δx tends to zero at the same time. This becomes a finer and finer approximation and if such a limit exists, it is known as the definite integral of the function between points $x_1 = a$ and $x_n = b$. If this limit exists, we say the function $f(x)$ is integrable between a and b.

We can see that integrals and derivatives are complementary to each other. In fact, an integral without the upper and lower limits is also known as the anti-derivative. An integral of the form:

$$\int f(x)dx \tag{14}$$

is called the indefinite integral of $f(x)$. The first fundamental theorem of calculus relates the definite and indefinite integrals thus: If $F(x)$ is the indefinite integral of $f(x)$ then

$$\int_a^b f(x)dx = F(b) - F(a) \tag{15}$$

Since the derivative of a constant term is 0, we write the anti-derivate as the non unique inverse of the derivative.That is:

$$\int f(x)dx = F(x) + C \tag{16}$$

where C is any constant.

We can approximate the definite integral numerically by computing a sum of increasingly small intervals and then returning back this sum. This will be a reasonable approximation to the integral of a function.