MODULE 1: VECTORS AND MATRICES

1. Vectors

Vectors of n-dimensions are an ordered collection of n numbers. They represent a direction and magnitude. They are very useful for conveniently representing many quantities. For instance, GPS co-ordinates [latitude, longitude] represent any point on earth. The number of components in a vector represent the dimensionality of the vector. You might be familiar with programming languages representing vectors using a 1-dimensional arrays of numbers. However, let's be clear that vectors are n-dimensional quantities and the length of the vector (in the programming language sense) is the dimensionality of the vector.

There are two fundamental operations with Vectors that we need to be intimately familiar with. First is a scalar multiplication of a vector and the second is adding two vectors. These two operations form the basis of Linear Algebra.

2. Multiply Vectors by Scalars

Given a vector of n-dimensions, multiplying it with a scalar uniformly scales every component of that vector by that scalar. For instance, if we have $v = [1, 4, -3]$. Then $3v = [3, 12, -9]$. You multiply every component of the vector by the scalar. It is as simple as that. In Octave, you can set up a vector thus:

```
> v = [1,4,-3];
```

Note that the $>$ symbol represents the Octave prompt. Semicolons suppress the interactive output that Octave produces and % represent comments.

```
> v2 = 3 * v
```

This will reproduce our example in Octave. In general, we'll represent vector v by a set of n variables thus: $v = [v_1, v_2, v_3, \ldots, v_n]$ where each v_k is a scalar number.

3. Adding two vectors

$v + w$ represents the addition of two n-dimensional vectors. Note that we can only add two vectors together if they have the same dimensions. To perform the vector addition, you add each component of the two vectors one by one. For instance if we have,

$v = [1, 4, -3]$ & $w = [-2, 5, 7]$, then $v + w = [-1, 9, 4]$. You can verify this in Octave typing

```
> v - [1, 4, -3];
> w = [-2, 5, 7];
> v + w
```

4. Linear combination of two vectors

Putting the above two operations together gets us to linear combinations. Simple as they may look, linear combinations are crucial for a great understanding of Linear Algebra.

Try to compute $3v - 2w$ in Octave.

In 2- or 3-dimensions, vectors can be visualized as rays that go from the origin to the co-ordinates represented by the vector. For instance $(2, 3)$ represents a ray that goes from origin to a point 2 units along the x-axis and 3 units along the y-axis. When you add two vectors together, you start at the origin and move along the first vector and then from that point, move along the second vector. For instance $(2, 3) + (-1, 2)$ is equivalent to starting at the origin of the co-ordinate system and going 2 units along x-axis & 3 units along the y-axis and then going 1 unit down the x-axis and 2 units up the y-axis reaching a final destination of $(1, 5)$. You can work out a similar geometric interpretation for linear combinations.

5. Dot-product of two vectors

We take the dot-product of two vectors, also known as the inner product of two vectors (of the same dimensions, of course!) thus:

$v.w = v_1w_1 + v_2w_2 + v_3w_3 + \ldots + v_nw_n$. You can do this is Octave also using the same notation:

```
> dot(v,w)
```

Alternatively, you can also type $v \star w'$ and get the dot-product, by taking the transpose of w. If you don't yet know what a transpose is, we'll cover it later. For now, we'll stick to taking the dot-product by using the dot function.

6. Length of a vector

We'll define the length of vector (or the Euclidean norm of the vector) as the square root of the inner-product of a vector with itself. The dot product of a vector gives the length of the vector squared.

6.1. **Unit vector:** A unit vector is a vector of length 1 unit.

6.2. **Angle between two vectors:** The dot-product between two vectors is proportional to their lengths and to the angle between them. In particular, the cosine of the angle between them. If you have two unit vectors, then the dot-product between them is simply the cosine of the angle between them.

6.3. **Orthogonal vectors:** Two vectors are considered orthogonal if they are 90 degrees apart. In other words, their dot-product will be zero. This can be seen from trigonometry – $cos(90) = 0$.

6.4. **Orthonormal vectors:** Two unit vectors that are orthogonal to each other are called orthonormal vectors. For instance, think the co-ordinate system. A unit vector along the x-axis is orthonormal to a unit vector along the y-axis.

Orthonormal vectors are special. If you have a complete set of orthonormal vectors, you can define any other vector in that space in terms of a linear combination of these orthonormal vectors. A complete set of orthonormal vectors is called a basis. Complete implies that there is one orthonormal vector representing each dimension of the space. E.g, in 2-D space, we have two orthonormal vectors and in 3-D space, we have 3 orthonormal vectors and so forth. Orthonormal basis are extremely important in linear algebra. You'll find them cropping up everywhere.

7. Matrices

An $m{\star}n$ matrix is a rectangular array of numbers arranged in m rows and n columns. We can think of matrices as either n column-vectors (each vector being 1 column of the matrix) concatenated together or as m row-vectors concatenated together to form the matrix.

8. Matrix multiplied with a vector

You can multiply an $m \star n$ matrix A with an m-dimensional vector x as follows. When the matrix A is applied to x, the matrix A is split into m column vectors and each column is multiplied by one component of the vector, in order. That is, the first column vector of A is multiplied by the first element of x, second column vector of A by the second element of x, and finally all the resulting vectors are added together to produce a new vector b.

Another way to think about a matrix and vector multiplication is to take the dot-product between the row-vectors of the matrix A and the vector x. Both of these are completely equivalent. In this course, we'll jump between both of these ways of thinking about matrix-vector multiplication.

9. Summary

I am sure this material is familiar to you from undergraduate linear algebra. We use matrices and vectors quite a bit in data analytics and machine learning. For instance, very important class of models such as linear support vector machines, neural networks, logistic regression all have matrices and vectors at their heart. Solving linear regression equations are at the very foundation of these machine learning techniques.