Vehicle Routing Problem with Pickup and Delivery

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1 Problem Definition

In the Capacitated Vehicle Routing Problem with Pickup and Delivery (CVRPPD), a fleet of vehicles with limited capacity and based at multiple depots must fulfill a set of orders. Each order involves transporting a load from a specific pickup node to a designated delivery node. The problem consists of optimizing the routes so that each vehicle starts and ends at its original depot while efficiently fulfilling all orders within its capacity constraints.

1.1 Mathematical Formulation

D is the set of depots. P and C denote the sets of pickup and delivery (clients) nodes, respectively. With this, we define a complete graph $G = (V, E)$ where $V = N \cup D$ is the set of vertices (with $N = P \cup C$), and $E = \{(i, j) : i, j \in V\}$ is the set of edges.

O is the set of orders. Each $o \in O$ has a pickup node $p(o) \in P$, a delivery node $c(o) \in C$ and a load to be transported that is represented as l_p and l_c in the pickup and delivery nodes, respectively, with $l_p = l_c$.

Finally, K is the set of vehicles. Each $k \in K$ has a limited capacity c_k, an origin node $d(k) \in D$, a usage cost u_k, and a time and cost matrices $t_{i,j,k}$ and $f_{i,j,k}$ that represent the time and cost to traverse edge $(i, j) \in E$, respectively.

2 Mathematical Optimization Model

This section shows the Mixed-Integer Optimization problem for the CVRPPD for the Medical Waste Collection Problem.

2.1 Variables

In this section, we show the set of decision variables.

- $x_{i,j,k}$: Binary variable indicating if vehicle k traverses from node i to node j.
- y_k: Binary variable indicating if vehicle k is used.
- $T_{i,k}$: Continuous variable denoting the time for vehicle k to arrive at node i.
- $L_{i,k}$: Continuous variable denoting the amount of load in vehicle k after leaving node i.

2.2 Objective Function

The objective function is determined by the sum of the traveling and usage costs of every vehicle:

$$\min : f = \sum_{k \in K} \left[u_k \cdot y_k + \sum_{(i,j) \in E} \left(c_{i,j,k} \cdot x_{i,j,k} \right) \right] \tag{1}$$

2.3 Constraints

$$\sum_{p \in P} x_{d(k),p,k} = y_k, \quad \forall k \in K \tag{2}$$

$$\sum_{d \in D \setminus \{d(k)\}} \sum_{j \in V} x_{d,j,k} = 0, \quad \forall k \in K \tag{3}$$

$$M y_k \geq \sum_{(i,j) \in E} x_{i,j,k}, \quad \forall k \in K \tag{4}$$

$$\sum_{j \in V} x_{i,j,k} \leq 1, \quad \forall i \in V, \forall k \in K \tag{5}$$

$$\sum_{j \in V \setminus \{i\}} x_{j,i,k} - \sum_{j \in V \setminus \{i\}} x_{i,j,k} = 0, \quad \forall i \in V, \forall k \in K \tag{6}$$

$$\sum_{i \in V \setminus \{j\}} \sum_{k \in K} x_{i,j,k} = 1, \quad \forall j \in N \tag{7}$$

$$\sum_{j \in V \setminus \{p(o)\}} x_{p(o),j,k} = \sum_{i \in V \setminus \{d(o)\}} x_{d(o),i,k}, \quad \forall o \in O, \forall k \in K \tag{8}$$

$$T_{i,k} + t_{i,j,k} \leq T_{j,k} + M(1 - x_{i,j,k}), \quad \forall i \in V, \forall j \in N \setminus \{i\}, \forall k \in K \tag{9}$$

$$T_{p(o),k} + t_{p(o),c(o),k} \leq T_{c(o),k} + M \left(1 - \sum_{i \in N} x_{i,c(o),k} \right), \quad \forall o \in O, \forall k \in K \tag{10}$$

$$L_{p,k} \geq L_{i,k} + l_p - M(1 - x_{i,p,k}), \quad \forall p \in P, \forall i \in V \setminus \{p\}, \forall k \in K \tag{11}$$

$$L_{c,k} \geq L_{i,k} - l_c - M(1 - x_{i,c,k}), \quad \forall c \in C, \forall i \in V \setminus \{c\}, \forall k \in K \tag{12}$$

$$L_{i,k} \leq c_k, \quad \forall i \in V, \forall k \in K \tag{13}$$

$$x_{i,j,k} \in \{0, 1\}, \quad \forall (i,j) \in E, \forall k \in K \tag{14}$$

$$y_k \in \{0, 1\}, \quad \forall k \in K \tag{15}$$

$$T_{i,k} \geq 0, \quad \forall i \in V, \forall k \in K \tag{16}$$

$$L_{i,k} \geq 0, \quad \forall i \in V, \forall k \in K \tag{17}$$

Constraint (2) ensures that, if used, all vehicles start their routes at the origin depot. Constraint (3) forbids the vehicle from departure from an origin depot different than the assigned one. Constraint (4) states that a vehicle must be labeled as used to make a route. Constraint (5) states that no vehicle can traverse more than one edge at a time. Constraint (6) ensures vehicles' flow balance. Constraint (7) states that all pickup and delivery nodes must be visited exactly once. Constraint (8) requires the same vehicle to pick up and deliver each order. Constraint (9) defines the arrival time in the routes. Constraint (10) demands that the vehicle in charge of an order must visit the pickup node before visiting the delivery node. Constraints (11) and (12) determine the load quantity through the route. Constraint (13) demands that no vehicle exceeds its load capacity. Constraints (14) to (17) are the variables domain constraints.