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COMP251 Assignment 2 Long Answer

4.1 Prove non-canonical by providing a counterexample.

Let a coin system C be $C = (10, 6, 1)$

Take x, the amount of money to be decomposed, to be 24.

Using the Greedy Algorithm,

since 10 is the largest in the 3-tuple coin system

and $10 < 24$,

We select 10 first, and the amount left to be decomposed is

$(24-10) = 14$.

Since 10 is the largest in the system and $10 < 14$

We select 10, and the amount left to be decomposed is

$(14-10) = 4$

Now, since $10 > 4$ and $6 > 4$ and $1 < 4$,

We can only decompose the remaing 4 cents into 1 cents.

Thus, using the greedy algorithm,

$24 = 10 + 10 + 1 + 1 + 1 + 1$

The number of coins needed is 6.

However, it is possible to decompse x, 24, in the following way:

$24 = 6 + 6 + 6 + 6$

This doesn't use the Greedy Algorithm as 6 is not the largest tuple in C that is less than or equal to 24.

In this case, the number of coins needed is 4.

Since $4 < 6$,

the solution given by the greedy algorithm is not the optimal solution for this system when $x = 24$, and 24 is a positive integer.

By definition, a coin system is canonical iff the solution given by the greedy algorithm is optimal for any positive integer x.

Thus, the coin system $C = (10, 6, 1)$ is non-canonical. $\square$

4.2

Claim: Let a_i denote the number of coins of denomination q^i (where $i \in \{0, 1, 2, \cdots, n\}$, and q, n are integers that are greater or equal to 2) used by the greedy solution. We claim that there is a unique optimal solution that has to agree with the greedy solution.

proof: Let $(a_0', a_1', \cdots, a_n')$ denote an optimal solution. of the corresponding system $(q^0, q^1, \ldots, q^n)$

Note that $a_i' < q$ for all $i < n$ since if we had $a_i' \geq q$ for some $i < n$ then we could replace q coins of denomination q^i by a single coin of denomination q^{i+1}, which leads to a smaller number of total coins, which improves upon the optimal solution and thus leads to a contradiction.

Then, we prove that $a_i' = a_i$ for all i (this would show that the greedy algorithm solution is the optimal solution).

Assume for contradiction that this is not the case.

Let j be the largest possible index such that $a_j' \neq a_j$, and by the way that our algorithm picks values a_i, $a_j' < a_j$.

Let $N = \Sigma_{i=0}^{j} a_i q^i = \Sigma_{i=0}^{j} a_i' q^i$,

this equality holds because the two solutions agree on $\{a_{j+1}, a_{j+2}, \cdots a_k\}$ by the assumption.

Thus,

$$N = \sum_{i=0}^{j-1} a_i' q^i + a_j' q^j \leq \sum_{i=0}^{j-1} (q-1) q^i + a_j' q^j$$

$$= (q-1) \frac{q^j - 1}{q-1} + a_j' q^j$$

$$= q^j - 1 + a_j' q^j$$

$$
\begin{aligned}
&= q^j (a'_j + 1) - 1 \\
&\leq a_j q^j - 1 \\
&\leq \sum_{i=0}^{j} a_i q^i - 1 \\
&= N-1
\end{aligned}
$$

N can not be less than or equal to $N-1$, so this leads to a contradiction.

Thus, the solution given by the greedy algorithm is the optimal solution of the system $C = (1, q, q^2, \ldots, q^{n-1}, q^n)$. (where q, n are integers $\geqslant 2$)

Thus, by the definition of a canonical system, the system C is canonical.

$\square$

4.3

Let X be the amount of money that we need to decompose.

$X = A \times 200 + B \times 100 + C \times 50 + D \times 20 + E \times 10 + F \times 5 + G \times 2 + H \times 1$

Where A, B, C, D, E, F, G are the numbers of bills / coins (200, 100, 50, 20, 10, 5, 2, 1, correspondingly) that we take in the optimal solution of decomposing X.

It is clear that $B \leq 1$ since we can get one 200 bill, increasing A by 1 and decreasing B by 2, and thus improving the solution, if otherwise.

Similarly, $C \leq 1$ since $2 \times 50 = 100$, we can get one 100 bill, increasing B by 1 and decreasing C by 2, and thus improving the solution, if otherwise.

Similarly, $D \leq 2$ since $3 \times 20 = 60 = 1 \times 5$, We can get one and one , increasing by 1 and increasing by 1 and decreasing by 3, thus needing one bill less, which improves the solution, if otherwise.

Similarly, $E \leq 1$ since $2 \times 10 = 20$, we can get one 20 bill, increasing D by 1 and decreasing E by 2, and thus improving the solution, if otherwise.

Similarly, $F \leq 1$ since $2 \times 5 = 10$, we can get one 10 bill, increasing E by 1 and decreasing F by 2, and thus improving the solution, if otherwise.

Similarly, $G \leq 2$ since $3 \times 2 = 6 = 5 + 1$, We can get one 5 and one 1, increasing F by 1 and increasing H by 1 and decreasing G by 3, thus needing one bill less, which improves the solution, if otherwise.

Similarly, $H \leq 1$ since $2 \times 1 = 2$, we can get one 2 bill, increasing G by 1 and decreasing H by 2, and thus improving the solution, if otherwise.

Let the solution given by the Greedy Algorithm be

$$X = a \times 200 + b \times 100 + c \times 50 + d \times 20 + e \times 10 + f \times 5 + g \times 2 + H \times 1$$

Where a, b, c, d, e, f, g, h are the numbers of bills / coins (200, 100, 50, 20, 5, 2, 1, correspondingly) that we take in the greedy algorithm solution for decomposing X.

Clearly, $b \leq 1$ since the algorithm would pick 200, which is larger than 100, if there is more than one 100.

Similarly, $C \leq 1$,

$d \leq 2$ since if d is 3 or greater, we are decomposing a number that is 60 or greater, and the greedy algorithm would pick 50, the largest bill that is smaller than 60, and 10 instead of 3 or more 20s. Similarly, $e \leq 1$, $f \leq 1$, $g \leq 2$, $H \leq 1$.

Thus, $0 \leq B \leq 1$, $0 \leq C \leq 1$, $0 \leq D \leq 2$, $0 \leq E \leq 1$

$0 \leq F \leq 1$, $0 \leq G \leq 2$, $0 \leq H \leq 1$.

$0 \leq b \leq 1$, $0 \leq c \leq 1$, $0 \leq d \leq 2$, $0 \leq e \leq 1$

$0 \leq f \leq 1$, $0 \leq g \leq 2$, $0 \leq h \leq 1$.

$$X = A \times 200 + B \times 100 + C \times 50 + D \times 20 + E \times 10$$
$$F \times 5 + G \times 2 + H \times 1$$
$$= (40A + 20B + 10C + 4D + 2E + F) \times 5 + 2 \times G + H$$

Let $a = 2G + H$

$a = X \bmod 5$

$$X = a \times 200 + b \times 100 + c \times 50 + d \times 20 + e \times 10 + f \times 5 + g \times 2 + h \times 1$$
$$= (40a + 20b + 10c + 4d + 2e + f) \times 5 + 2g + h$$

Let $b = 2g + h$

$b = X \bmod 5$

Thus, $a = b$.

Thus, $2G + H = 2g + h$

G, H and g, h correspond to the numbers of 1s and 2s needed.

From the given information, we know that all systems $(a, 1)$ with $a > 1$ are canonical. Thus, $(2, 1)$ is a canonical system.

Thus, the greedy algorithm solution is the optimal solution

$G = g$ and $H = h$.

Let $Y = \frac{(X - (2G+H))}{5} = \frac{(X - (2g+h))}{5}$

$Y = 40A + 20B + 10C + 4D + 2E + F$

$= (20A + 10B + 5C + 2D + E) \times 2 + F$

$F = Y \bmod 2$

$Y = 40a + 20b + 10c + 4d + 2e + f$

$= (20a + 10b + 5c + 2d + e) \times 2 + f$

$f = Y \bmod 2$

$F = f$

Let $P = \frac{Y-F}{2} = \frac{Y-f}{2}$

$= 20A + 10B + 5C + 2D + E$

$= (4A + 2B + C) \cdot 5 + 2D + E$

$= (4a + 2b + c) \cdot 5 + 2d + e$

$2D + E = 2d + e = P \bmod 5$

D, E and d, e are the numbers of 20 and 10 bills needed

Claim: (20, 10) is a canonical system.

proof: Let N be the amount of money needed to decompose

Let the optimal solution be: $N = 20 \times A + 10 \times B$

Let the greedy algorithm solution be: $N = 20 \times a + 10 \times b$

clearly $B \leq 1$ and $b \leq 1$, as explained above

$N = (2A + B) \times 10$

$B = \frac{N}{10} \bmod 2$

$N = (2a + b) \times 10$

$b = \frac{N}{10} \bmod 2$

$B = b$

$A = \frac{N - 10B}{20}$ and $a = \frac{N - 10b}{20}$

$A = a$

Thus, the solution given by the greedy algorithm is the optimal solution.

Thus, $D = d$ and $E = e$.

Let $Q = \dfrac{P-(2D+E)}{5} = \dfrac{P-(2d+e)}{5}$

$$Q = 4A + 2B + C$$
$$= (2A+B)\cdot 2 + C$$
$$C = Q \bmod 2$$
$$Q = 4a + 2b + c$$
$$= (2a+b)\cdot 2 + C$$
$$c = Q \bmod 2$$
$$c = C$$

Let $R = \dfrac{Q-c}{2}$

$$R = 2A + B$$
$$B = R \bmod 2$$
$$R = 2a + b$$
$$b = R \bmod 2$$
$$B = b$$

Let $S = \dfrac{R-B}{2}$

$$S = A$$
$$S = \frac{R-b}{2} = a$$
$$a = A$$

Therefore, $A = a$, $B = b$, $C = c$, $D = d$
$E = e$, $F = f$, $G = g$, $H = h$

The Greedy Algorithm solution is the Optimal solution

Thus, the system is canonical. $\square$

References: Yuexin (Virna) Xi 260716384
Yong gue Kim 260531176
Harsh Banthia
Sean Smith

I have referred to cs.toronto.edu/~denisp/csc373/docs/greedy-coin-change.pdf and referred to all the Reddit posts that are related to 4.2 and 4.3.

THANK YOU FOR YOUR TIME!