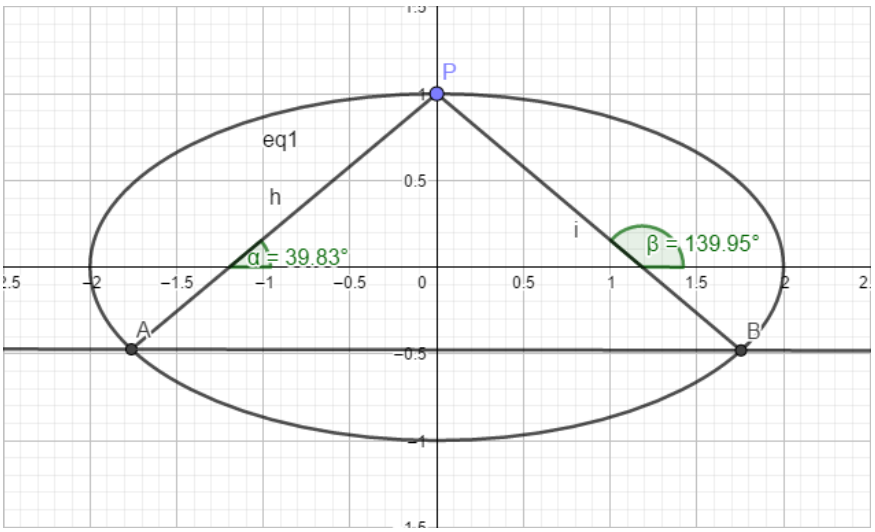
椭圆共点定角弦性质探究

> $P(x_0, y_0)$为椭圆$C: \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$上一定点，$PA$、$PB$为椭圆的两条动弦，其倾斜角分别为$\alpha$、$\beta$，且$\alpha+\beta=\theta$，$\theta \in [0, 2\pi)$.
>
> 
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结论

当$tan\theta=0$时，直线AB斜率$k=\frac{b^2}{a^2}\cdot\frac{x_0}{y_0}$为定值

当$tan\theta$不存在时，直线AB恒过定点$(\frac{x_0(a^2+b^2)}{a^2-b^2}, -\frac{y_0(a^2+b^2)}{a^2-b^2})$

当$tan\theta$存在且不为0时，直线AB恒过定点$(\frac{x_0(a^2+b^2)}{a^2-b^2}-\frac{2a^2y_0}{(a^2-b^2)tan\theta}, -\frac{y_0(a^2+b^2)}{a^2-b^2}-\frac{2b^2x_0}{(a^2-b^2)tan\theta})$

证明

设直线AB：$y=kx+m$，$A(x_1, y_1)$，$B(x_2, y_2)$，PA、PB斜率为$k_1=tan\alpha$、$k_2=tan\beta$，直线AB恒过定点Q

联立$\begin{cases} \frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \\ y=kx+m \end{cases}$ 得：$(a^2k^2+b^2)x^2+2a^2kmx+a^2(m^2-b^2)=0$

$x_1+x_2=\frac{-2a^2km}{a^2k^2+b^2}(1)$

$x_1x_2=\frac{a^2(m^2-b^2)}{a^2k^2+b^2}(2)$

$y_1+y_2=k(x_1+x_2)+2m=\frac{2b^2m}{a^2k^2+b^2}(3)$

$y_1y_2=(kx_1+m)(kx_2+m)=\frac{b^2(m^2-a^2k^2)}{a^2k^2+b^2}(4)$

$y_1x_2+y_2x_1=\frac{-2a^2b^2k}{a^2k^2+b^2}(5)$

因为$k_1=\frac{y_1-y_0}{x_1-x_0}, k_2=\frac{y_2-y_0}{x_2-x_0}$

所以

$k_1\cdot k_2=\frac{(y_1-y_0)(y_2-y_0)}{(x_1-x_0)(x_2-x_0)}=\frac{y_1y_2-y_0(y_1+y_2)+y_0^2}{x_1x_2-x_0(x_1+x_2)+x_0^2}(*)$

$k_1+k_2=\frac{y_1-y_0}{x_1-x_0}+\frac{y_2-y_0}{x_2-x_0}=\frac{(y_1-y_0)(x_2-x_0)+(y_2-y_0)(x_1-x_0)}{(x_1-x_0)(x_2-x_0)}=\frac{y_1x_2+y_2x_1-x_0(y_1+y_2)-y_0(x_1+x_2)+2x_0y_0}{x_1x_2-x_0(x_1+x_2)+x_0^2}(**)$

当tanθ = 0时

因为 $\alpha = \theta - \beta$

所以 $tan\alpha = tan(\theta - \beta) = \dfrac{tan\theta - tan\beta}{1 + tan\theta \cdot tan\beta} = -tan\beta$

即：$tan\alpha + tan\beta = 0$

即：$k_1 + k_2 = 0$

把(*)代入得

> $y_1x_2 + y_2x_1 - x_0(y_1 + y_2) - y_0(x_1 + x_2) + 2x_0y_0 = 0$

把(1)(2)(3)(4)(5)代入得

> $-2a^2b^2k - x_0(2b^2m) - y_0(-2a^2km) + 2x_0y_0(a^2k^2 + b^2) = 0$

整理得

> $(a^2y_0k - b^2x_0)m + a^2x_0y_0k^2 - a^2b^2k + b^2x_0y_0 = 0$

> $\begin{cases} a^2y_0k - b^2x_0 = 0 \\ a^2x_0y_0k^2 - a^2b^2k + b^2x_0y_0 = 0 \end{cases}$

由前式得 $k = \dfrac{b^2}{a^2} \cdot \dfrac{x_0}{y_0}$

代入后式检验得 $\dfrac{b^2x_0^2}{a^2y_0} - \dfrac{b^2}{y_0} + y_0 = 0$

整理得 $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ 成立

所以该方程的解为 $k = \dfrac{b^2}{a^2} \cdot \dfrac{x_0}{y_0}$

当tanθ不存在时

$\theta = \dfrac{\pi}{2}$或$\dfrac{3\pi}{2}$

因为 $\alpha = \theta - \beta$

所以 $tan\alpha = tan(\theta - \beta) = \dfrac{sin(\theta - \beta)}{cos(\theta - \beta)} = \dfrac{cos\beta}{sin\beta} = \dfrac{1}{tan\beta}$

即：$tan\alpha \cdot tan\beta = 1$

即：$k_1 \cdot k_2 = 1$

把(**)代入式得

> $y_1y_2 - y_0(y_1 + y_2) + y_0^2 - x_1x_2 + x_0(x_1 + x_2) - x_0^2 = 0$

把(1)(2)(3)(4)代入得

> $b^2(m^2 - a^2k^2) - y_0(2b^2m) - a^2(m^2 - b^2) + x_0(-2a^2km) + (y_0^2 - x_0^2)(a^2k^2 + b^2) = 0$

整理得

> $(a^2 - b^2)m^2 + 2(a^2x_0k + b^2y_0)m + a^2(b^2 + x_0^2 - y_0^2)k^2 + b^2(-a^2 + x_0^2 - y_0^2) = 0$

代入$a^2y^2 + b^2x^2 = a^2b^2$得

> $(a^2 - b^2)m^2 + 2(a^2x_0k + b^2y_0)m + x^2(a^2 + b^2)k^2 - y^2(a^2 + b^2) = 0$

设$m = \lambda k + \mu$并代入得

> $(a^2 - b^2)(\lambda^2k^2 + 2\lambda\mu k + \mu^2) + 2(a^2x_0k + b^2y_0)(\lambda k + \mu) + x^2(a^2 + b^2)k^2 - y^2(a^2 + b^2) = 0$

整理得

> $[(a^2 - b^2)\lambda^2 + 2a^2x_0\lambda + x_0^2(a^2 + b^2)]k^2 + [2(a^2 - b^2)\lambda\mu + 2a^2x_0\mu + 2b^2y_0\lambda]k + (a^2 - b^2)\mu^2 + 2b^2y_0\mu - y_0(a^2 + b^2) = 0$

> $$\left\{\begin{array}{l}(a^2-b^2)\lambda^2+2a^2x_0\lambda+x_0^2(a^2+b^2)=0\\2(a^2-b^2)\lambda\mu+2a^2x_0\mu+2b^2y_0\lambda=0\\(a^2-b^2)\mu^2+2b^2y_0\mu-y_0(a^2+b^2)=0\end{array}\right.$$

因式分解得

> $$\left\{\begin{array}{l}(\lambda+x_0)[(a^2-b^2)\lambda+x_0(a^2+b^2)]=0\\2(a^2-b^2)\lambda\mu+2a^2x_0\mu+2b^2y_0\lambda=0\\(\mu-y_0)[(a^2-b^2)\mu+y_0(a^2+b^2)]=0\end{array}\right.$$

解1式得

> $\lambda=-x_0$或$\lambda=-\dfrac{x_0(a^2+b^2)}{a^2-b^2}$

解3式得

> $\mu=y_0$或$\mu=-\dfrac{y_0(a^2+b^2)}{a^2-b^2}$

把$m=\lambda k+\mu$代入$y=kx+b$得

> $y=kx+\lambda k+\mu=k(x+\lambda)+\mu$恒过点$(-\lambda,\mu)$

经检验得

> 当$\{\lambda=-x_0,\mu=y_0\}$时，点为(x_0,y_0)与点P重合，舍去

故直线AB恒过定点为$(\dfrac{x_0(a^2+b^2)}{a^2-b^2},-\dfrac{y_0(a^2+b^2)}{a^2-b^2})$

当tanθ存在且不为0时

因为 $\alpha=\theta-\beta$
所以 $tan\alpha=tan(\theta-\beta)=\dfrac{tan\theta-tan\beta}{1+tan\theta\cdot tan\beta}$
即：$tan\theta\cdot tan\alpha\cdot tan\beta+tan\alpha+tan\beta-tan\theta=0$
即：$\tan\theta k_1k_2+k_1+k_2-\tan\theta=0$
把(*)(**)代入得

> > $tan\theta[y_1y_2-y_0(y_1+y_2)+y_0^2]+y_1x_2+y_2x_1-x_0(y_1+y_2)-y_0(x_1+x_2)+2x_0y_0-tan\theta[x_1x_2-x_0(x_1+x_2)+x_0^2]=0$

整理得

> > $tan\theta y_1y_2-tan\theta x_1x_2+y_1x_2+y_2x_1+(y_1+y_2)(-y_0tan\theta-x_0)+(x_1+x_2)(xtan\theta-y_0)+tan\theta(y_0^2-x_0^2)+2x_0y_0=0$

把$(1)(2)(3)(4)(5)$代入得

> > $tan\theta b^2(m^2-a^2k^2)-tan\theta a^2(m^2-b^2)+(-2a^2b^2k)+2b^2m(-y_0tan\theta-x_0)+(-2a^2km)(xtan\theta-y_0)+(a^2k^2+b^2)(tan\theta(y_0^2-x_0^2)+2x_0y_0)=0$

整理得

> > $(a^2-b^2)tan\theta\cdot m^2+[2a^2(x_0tan\theta-y_0)k+2b^2(y_0tan\theta+x_0)]m+a^2[tan\theta(b^2+x_0^2-y_0^2)-2x_0y_0]k^2+2a^2b^2k+b^2[tan\theta(-a^2+x_0^2-y_0^2)-2x_0y_0]=0$

代入$a^2y^2+b^2x^2=a^2b^2$得

> > $(a^2-b^2)tan\theta\cdot m^2+[2a^2(x_0tan\theta-y_0)k+2b^2(y_0tan\theta+x_0)]m+[x_0^2(a^2+b^2)tan\theta-2a^2x_0y_0]k^2+2a^2b^2k-[y_0^2(a^2+b^2)tan\theta+2b^2x_0y_0]=0$

设$m=\lambda k+\mu$并代入得

> $(a^2-b^2)\tan\theta(\lambda^2k^2+2\lambda\mu k+\mu^2)+[2a^2(x_0tan\theta-y_0)k+2b^2(y_0tan\theta+x_0)](\lambda k+\mu)+[x_0^2(a^2+b^2)tan\theta-2a^2x_0y_0]k^2+2a^2b^2k-[y_0^2(a^2+b^2)tan\theta+2b^2x_0y_0]=0$

整理得

> $[(a^2-b^2)\tan\theta\lambda^2+2a^2(x_0tan\theta-y_0)\lambda+x_0^2(a^2+b^2)tan\theta-2a^2x_0y_0]k^2+[2(a^2-b^2)\tan\theta\lambda\mu+2a^2(x_0\tan\theta-y_0)\mu+2b^2(y_0\tan\theta+x_0)\lambda+2a^2b^2]k+(a^2-b^2)\tan\theta\mu^2+2b^2(y_0tan\theta+x_0)\mu-y_0^2(a^2+b^2)tan\theta-2b^2x_0y_0=0$

> $$\left\{\begin{array}{l}(a^2-b^2)\tan\theta\lambda^2+2a^2(x_0tan\theta-y_0)\lambda+x_0^2(a^2+b^2)tan\theta-2a^2x_0y_0=0\\2(a^2-b^2)\tan\theta\lambda\mu+2a^2(x_0\tan\theta-y_0)\mu+2b^2(y_0\tan\theta+x_0)\lambda+2a^2b^2=0\\(a^2-b^2)\tan\theta\mu^2+2b^2(y_0tan\theta+x_0)\mu-y_0^2(a^2+b^2)tan\theta-2b^2x_0y_0=0\end{array}\right.$$

因式分解得

> $$\left\{\begin{array}{l}(\lambda+x_0)[(a^2-b^2)\tan\theta\lambda+x_0(a^2+b^2)-2a^2y_0]=0\\2(a^2-b^2)\lambda\mu+2a^2x_0\mu+2b^2y_0\lambda=0\\(\mu-y_0)[(a^2-b^2)\tan\theta\mu+y_0(a^2+b^2)+2b^2x_0]=0\end{array}\right.$$

解1式得

> $\lambda=-x_0$或$\lambda=-\dfrac{x_0(a^2+b^2)}{a^2-b^2}+\dfrac{2a^2y_0}{(a^2-b^2)\tan\theta}$

解3式得

> $\mu=y_0$或$\mu=-\dfrac{y_0(a^2+b^2)}{a^2-b^2}-\dfrac{2b^2x_0}{(a^2-b^2)\tan\theta}$

把$m=\lambda k+\mu$代入$y=kx+b$得

> $y=kx+\lambda k+\mu=k(x+\lambda)+\mu$恒过点$(-\lambda,\mu)$

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> 当$\{\lambda=-x_0,\mu=y_0\}$时，点为(x_0,y_0)与点P重合，舍去

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