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Technical Note—A Robust Perspective on Transaction Costs in Portfolio Optimization

Olivares-Nadal, Alba V., and Victor DeMiguel
Operations Research, 2018

解读者：屠雪永

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1. Introduction-- Motivation

- Considerable advances starting with the pathbreaking work of Markowitz (1952) to handle estimation error.

- Understand the conditions under which mean-variance optimal portfolio models can be expected to perform well even in the presence of estimation risk.

1. Introduction-- Literature

- Bayesian approach
  - relying on diffuse-priors (Barry, 1974; Bawa, Brown, and Klein, 1979)
  - “shrinkage estimators” (Jobson, Korkie, and Ratti, 1979; Jobson and Korkie, 1980; Jorion, 1985, 1986)
  - asset-pricing model for establishing a prior (Pastor, 2000; Pastor and Stambaugh, 2000)

1. Introduction-- Literature

- Non-Bayesian approaches
  - **"robust"** portfolio allocation rules (Goldfarb and Iyengar, 2003; Garlappi, Uppal, and Wang, 2007)
  - portfolio rules designed to optimally **diversify** across market and estimation risk (Kan and Zhou, 2007)
  - portfolios that exploit the **moment restrictions** imposed by the factor structure of returns (MacKinlay and Pastor, 2000)
  - methods that focus on reducing the error in estimating the **covariance matrix** (Best and Grauer, 1992; Chan, Karceski, and Lakonishok, 1999; Ledoit and Wolf, 2004a, 2004b)
  - portfolio rules that **impose short selling constraints** (Frost and Savarino, 1988; Chopra, 1993; Jagannathan and Ma, 2003)

1. Introduction-- contributions

- Our first contribution is to show that **none is consistently better than** the naive 1/N benchmark in terms of Sharpe ratio, certainty-equivalent return, or turnover.

- Our second contribution is to derive the **critical length of the estimation window** that is needed for the sample-based mean-variance strategy outperform the 1/N strategy.

- Our third contribution is to show that these models too need **very long estimation windows** before they can be expected to outperform the 1/N policy.

1. Introduction-- conclusions

- Portfolio strategies from the optimizing models are expected to outperform the 1/N benchmark if:
  - (i) the estimation window is **long**;
  - (ii) the ex ante (true) **Sharpe ratio** of the mean-variance efficient portfolio is substantially **higher than** that of the 1/N portfolio;
  - (iii) the number of assets is **small**.

List of various asset-allocation models considered

#	Model	Abbreviation
Naive		
0.	$1/N$ with rebalancing *(benchmark strategy)*	ew or $1/N$
Classical approach that ignores estimation error		
1.	Sample-based mean-variance	mv
Bayesian approach to estimation error		
2.	Bayesian diffuse-prior	Not reported
3.	Bayes-Stein	bs
4.	Bayesian Data-and-Model	dm
Moment restrictions		
5.	Minimum-variance	min
6.	Value-weighted market portfolio	vw
7.	MacKinlay and Pastor's (2000) missing-factor model	mp
Portfolio constraints		
8.	Sample-based mean-variance with shortsale constraints	mv-c
9.	Bayes-Stein with shortsale constraints	bs-c
10.	Minimum-variance with shortsale constraints	min-c
11.	Minimum-variance with generalized constraints	g-min-c
Optimal combinations of portfolios		
12.	Kan and Zhou's (2007) "three-fund" model	mv-min
13.	Mixture of minimum-variance and $1/N$	ew-min
14.	Garlappi, Uppal, and Wang's (2007) multi-prior model	Not reported

2. Asset-Allocation Models

- Mean-variance(mv)

$$\max_{\mathbf{x}_t} \mathrm{x}_t^\top \mu_t - \frac{\gamma}{2} \mathrm{x}_t^\top \Sigma_t \mathrm{x}_t,$$

$$\mathrm{w}_t = \frac{\Sigma_t^{-1} \mu_t}{\mathbf{1}_N \Sigma_t^{-1} \mu_t}.$$

- Bayesian diffuse-prior portfolio

$$p(\mu, \Sigma) \propto |\Sigma|^{-(N+1)/2}$$

$$\hat{\mu} \qquad \hat{\Sigma}\,(1 + 1/M)$$

- Bayes-Stein shrinkage portfolio(bs)

$$\hat{\mu}_t^{\mathrm{bs}} = (1 - \hat{\phi}_t)\, \hat{\mu}_t + \hat{\phi}_t\, \hat{\mu}_t^{\min},$$

$$\hat{\phi}_t = \frac{N+2}{(N+2) + M(\hat{\mu}_t - \mu_t^{\min})^\top \hat{\Sigma}_t^{-1} (\hat{\mu}_t - \mu_t^{\min})},$$

$$\hat{\mu}_t^{\min} \equiv \hat{\mu}_t^\top \hat{\mathrm{w}}_t^{\min}$$

2. Asset-Allocation Models

- Minimum-variance portfolio(min)

$$\min_{\mathrm{w}_t} \ \mathrm{w}_t^\top \Sigma_t \mathrm{w}_t, \quad \text{s.t.} \quad \mathbf{1}_N^\top \mathrm{w}_t = 1$$

- Portfolio implied by asset-pricing models with unobservable factors(mp)

$$\Sigma = \nu \mu \mu^\top + \sigma^2 I_N,$$

- The Kan and Zhou (2007) three-fund portfolio(mv-min)

$$\hat{\mathrm{x}}_t^{\text{mv-min}} = \frac{1}{\gamma}\left(c\, \hat{\Sigma}_t^{-1} \hat{\mu}_t + d\, \hat{\Sigma}_t^{-1} \mathbf{1}_N\right)$$

- Mixture of equally weighted and minimum-variance portfolios. (ew-min)

$$\hat{\mathrm{w}}^{\text{ew-min}} = c\, \frac{1}{N} \mathbf{1}_N + d\, \hat{\Sigma}^{-1} \mathbf{1}_N, \quad \text{s.t.} \quad \mathbf{1}_N^\top \hat{\mathrm{w}}^{\text{ew-min}} = 1$$

2. Methodology for Evaluating Performance

- Sharpe ratios $$\widehat{\mathrm{SR}}_k = \frac{\hat{\mu}_k}{\hat{\sigma}_k}$$ ↑
- Certainty-equivalent (CEQ) return $$\widehat{\mathrm{CEQ}}_k = \hat{\mu}_k - \frac{\gamma}{2}\hat{\sigma}_k^2$$ ↑
- Turnover $$\mathrm{Turnover} = \frac{1}{T-M}\sum_{t=1}^{T-M}\sum_{j=1}^{N}\left(\left|\hat{w}_{k,j,t+1} - \hat{w}_{k,j,t^+}\right|\right)$$ ↓
- Transaction cost $$W_{k,t+1} = W_{k,t}(1 + R_{k,p})\left(1 - c \times \sum_{j=1}^{N}|\hat{w}_{k,j,t+1} - \hat{w}_{k,j,t^+}|\right)$$ ↑
- Return-loss $$\text{return-loss}_k = \frac{\mu_{ew}}{\sigma_{ew}} \times \sigma_k - \mu_k$$ ↓

3. Results from the Empirical Datasets

Sharpe ratios for empirical data

Strategy	S&P sectors $N=11$	Industry portfolios $N=11$	Inter'l portfolios $N=9$	Mkt/ SMB/HML $N=3$	FF 1-factor $N=21$	FF 4-factor $N=24$
$1/N$	0.1876	0.1353	0.1277	0.2240	0.1623	0.1753
mv (in sample)	0.3848	0.2124	0.2090	0.2851	0.5098	0.5364
mv	0.0794	0.0679	−0.0332	0.2186	0.0128	0.1841
	(0.12)	(0.17)	(0.03)	(0.46)	(0.02)	(0.45)
bs	0.0811	0.0719	−0.0297	0.2536	0.0138	0.1791
	(0.09)	(0.19)	(0.03)	(0.25)	(0.02)	(0.48)
dm ($\sigma_\alpha = 1.0\%$)	0.1410	0.0581	0.0707	0.0016	0.0004	0.2355
	(0.08)	(0.14)	(0.08)	(0.00)	(0.01)	(0.17)
min	0.0820	0.1554	0.1490	0.2493	0.2778	−0.0183
	(0.05)	(0.30)	(0.21)	(0.23)	(0.01)	(0.01)
vw	0.1444	0.1138	0.1239	0.1138	0.1138	0.1138
	(0.09)	(0.01)	(0.43)	(0.00)	(0.01)	(0.00)
mp	0.1863	0.0533	0.0984	−0.0002	0.1238	0.1230
	(0.44)	(0.04)	(0.15)	(0.00)	(0.08)	(0.03)

4. Results from Studying Analytically the Estimation Error

- Mean-variance utility $U(\mathrm{x}) = \mathrm{x}^\top \mu - \frac{\gamma}{2} \mathrm{x}^\top \Sigma \mathrm{x}$
- The optimal weight is $\mathrm{x}^* = \frac{1}{\gamma} \Sigma^{-1} \mu$
- The corresponding optimized utility is $U(\mathrm{x}^*) = \frac{1}{2\gamma} \mu \Sigma^{-1} \mu \equiv \frac{1}{2\gamma} S_*^2$
- The expected loss from using a particular estimator of the weight $\hat{x}$ as

$$L(\mathrm{x}^*, \hat{\mathrm{x}}) = U(\mathrm{x}^*) - E[U(\hat{\mathrm{x}})]$$

- Define **the critical M_{mv}^*** of the sample-based mean-variance strategy as the **smallest number of estimation periods** necessary for the mean-variance portfolio to outperform, the 1/N rule. Formally

$$M_{\mathrm{mv}}^* \equiv \inf\{M : L_{\mathrm{mv}}(\mathrm{x}^*, \hat{\mathrm{x}}) < L_{\mathrm{ew}}(\mathrm{x}^*, \mathrm{w}^{\mathrm{ew}})\}$$

Proposition 1. *Let* $S_*^2 = \mu\Sigma^{-1}\mu$ *be the squared Sharpe ratio of the tangency (mean-variance) portfolio of risky assets and* $S_{ew}^2 = (\mathbf{1}_N^\top\mu)^2/\mathbf{1}_N^\top\Sigma\mathbf{1}_N$ *the squared Sharpe ratio of the* $1/N$ *portfolio. Then:*

1. *If μ is unknown and Σ is known, the sample-based mean-variance strategy has a lower expected loss than the 1/N strategy if:*

$$S_*^2 - S_{ew}^2 - \frac{N}{M} > 0. \tag{23}$$

2. *If μ is known and Σ is unknown, the sample-based mean-variance strategy has a lower expected loss than the* $1/N$ *strategy if:*

$$kS_*^2 - S_{ew}^2 > 0, \tag{24}$$

where

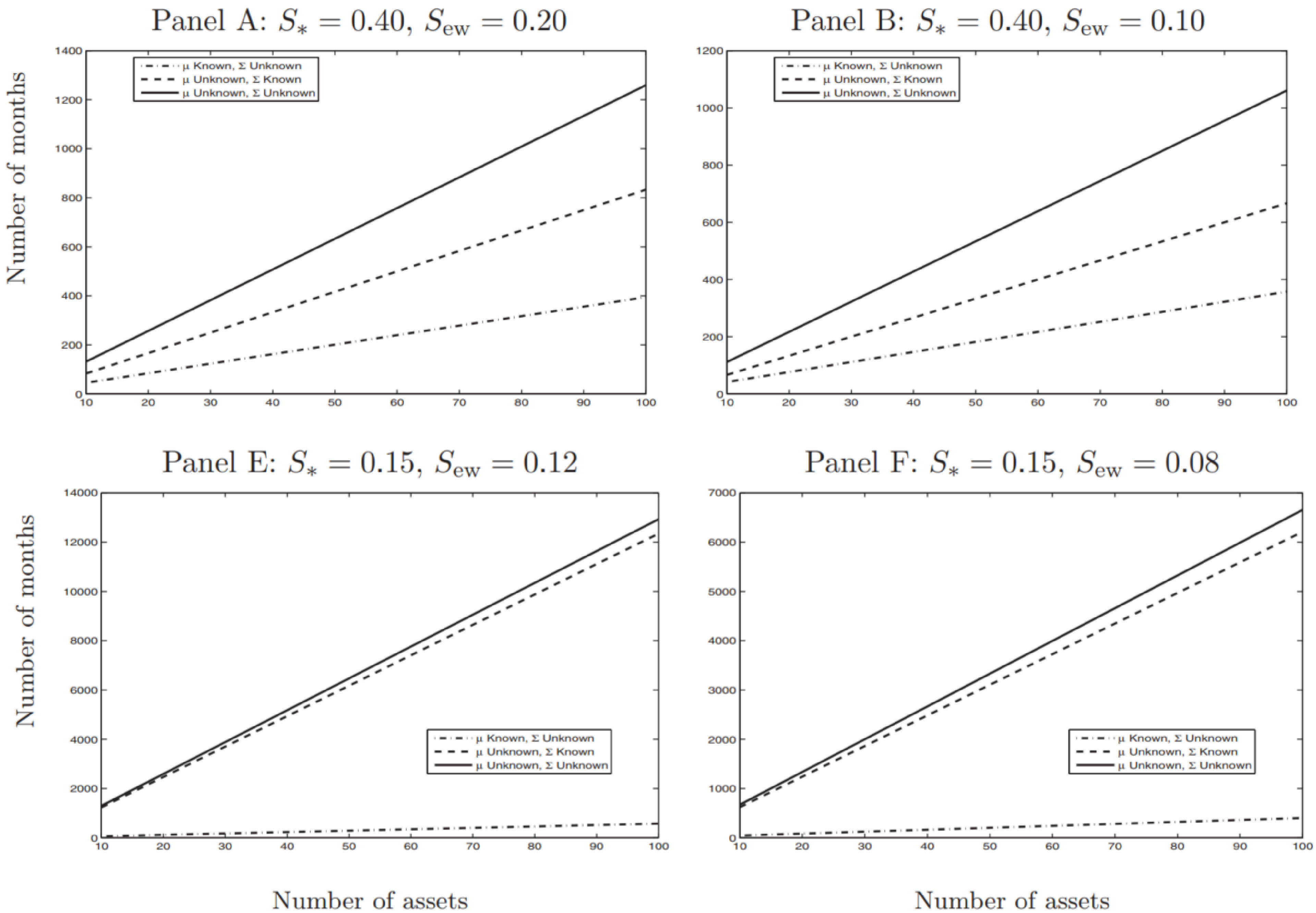
$$k = \left(\frac{M}{M-N-2}\right)\left(2 - \frac{M(M-2)}{(M-N-1)(M-N-4)}\right) < 1. \tag{25}$$

3. *If both μ and Σ are unknown, the sample-based mean-variance strategy has a lower expected loss than the* $1/N$ *strategy if:*

$$kS_*^2 - S_{ew}^2 - h > 0, \tag{26}$$

where

$$h = \frac{NM(M-2)}{(M-N-1)(M-N-2)(M-N-4)} > 0. \tag{27}$$



Panel A: $S_* = 0.40$, $S_{ew} = 0.20$
Panel B: $S_* = 0.40$, $S_{ew} = 0.10$
Panel E: $S_* = 0.15$, $S_{ew} = 0.12$
Panel F: $S_* = 0.15$, $S_{ew} = 0.08$
μ Known, Σ Unknown
μ Unknown, Σ Known
μ Unknown, Σ Unknown
Number of months
Number of assets

Table 6
Sharpe ratios for simulated data

Strategy	$N = 10$			$N = 25$			$N = 50$		
	$M = 120$	$M = 360$	$M = 6000$	$M = 120$	$M = 360$	$M = 6000$	$M = 120$	$M = 360$	$M = 6000$
$1/N$	0.1356	0.1356	0.1356	0.1447	0.1447	0.1447	0.1466	0.1466	0.1466
mv (true)	0.1477	0.1477	0.1477	0.1477	0.1477	0.1477	0.1477	0.1477	0.1477
	(0.00)	(0.00)	(0.00)	(0.03)	(0.03)	(0.03)	(0.15)	(0.15)	(0.15)
mv	−0.0019	0.0077	0.1416	0.0027	0.0059	0.1353	0.0078	−0.0030	0.1212
	(0.00)	(0.00)	(0.03)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
bs	−0.0021	0.0087	0.1416	0.0031	0.0074	0.1363	0.0076	−0.0035	0.1229
	(0.00)	(0.00)	(0.03)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)

- 尽管模拟数据表明预计所用的窗口期越长，样本外表现越好，但是一般在实证中选择60个月或者120个月。这是由于模拟数据没有考虑到经济环境与企业经营状况的时变性，而且百年企业数量较少，现实中无法做到大量存在上千个月的历史数据。

A Generalized Approach to Portfolio Optimization: Improving Performance by Constraining Portfolio Norms

DeMiguel, Victor, Lorenzo Garlappi, Francisco J. Nogales and Raman Uppal.

Management Science, 2009.

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1. Introduction-- Motivation

- The **minimum-variance portfolio** usually performs better out of sample than any other mean-variance portfolio.

- Although the estimation error associated with the **sample covariances** is smaller than that for sample mean returns, it can still be substantial.

- We develop a new approach for determining the optimal **portfolio weights** in the presence of estimation error.

1. Introduction-- Literature

$$\min_{w} \; w^\top \widehat{\Sigma} w, \quad \text{s.t.} \; w^\top e = 1,$$

- Deal with the problem of estimating the large number of elements in the covariance matrix:
- Use **higher frequency data**, say, daily instead of monthly returns (see Jagannathan and Ma 2003).
- Impose some **factor structure** on the estimator of the covariance matrix (Chan et al. 1999, Green and Hollifield 1992).
- Use as an estimator a **weighted average** of the sample covariance matrix and another estimator, such as the 1-factor covariance matrix or the identity matrix. (Ledoit and Wolf ,2003, 2004)
- Impose **shortsale constraints** on the portfolio weights (see Frost and Savarino1988, Chopra 1993).

1. Introduction-- contributions

- 1. We provide a **general framework** for finding portfolios that perform well out-of-sample in the presence of estimation error.
  - Our framework nests as special cases the shrinkage approaches of Jagannathan and Ma(2003)(Risk reduction in large portfolios: Why imposing the wrong constraints helps)
  - Ledoit and Wolf (2003)(Improved estimation of the covariance matrix of stock returns with an application to portfolio selection.)
  - The 1/N portfolio studied in DeMiguel et al.2009(Optimal versus naive diversification: How inefficient is the 1/N portfolio strategy?)

- 2. We also use our framework to propose **several new portfolio strategies**.

1. Introduction-- contributions

- 3. We give a **Bayesian interpretation** for the norm-constrained portfolios.

- 4. Optimal portfolio weights in an unconstrained mean- or minimum variance problem can be thought of as coefficients of **an ordinary least squares regression.**

- 5. the generalized framework allows one to **calibrate** the model using historical data to improve its out-of-sample performance.

2. Existing Approaches: Shrinking the Sample Covariance Matrix

$$\min_{\mathrm{w}} \ \mathrm{w}^\top \widehat{\Sigma} \mathrm{w},$$
$$\text{s.t.} \ \mathrm{w}^\top e = 1,$$

- Jagannathan and Ma (2003) study the shortsale-constrained minimum-variance portfolio ($w \geq 0$)

$$\widehat{\Sigma}_{\mathrm{JM}} = \widehat{\Sigma} - \lambda e^\top - e \lambda^\top,$$

  - where $\lambda \geq 0$ is the vector of Lagrange multipliers for the shortsale constraint.

- Ledoit and Wolf (2003, 2004) propose replacing the sample covariance matrix with a **weighted average of the sample covariance matrix and a low-variance target estimator**

$$\widehat{\Sigma}_{\mathrm{LW}} = \frac{1}{1+\nu} \widehat{\Sigma} + \frac{\nu}{1+\nu} \widehat{\Sigma}_{\mathrm{target}}$$

3. Norm-constrained minimum-variance portfolio

$$\min_{\mathrm{w}} \quad \mathrm{w}^\top \widehat{\Sigma} \mathrm{w},$$
$$\text{s.t.} \quad \mathrm{w}^\top e = 1,$$

$$\|\mathrm{w}\| \le \delta$$

- 1-norm $\quad \|\mathrm{w}\|_1 = \sum_{i=1}^{N} |\mathrm{w}_i|$
- A-norm $\quad \|\mathrm{w}\|_A = (\mathrm{w}^\top A \mathrm{w})^{1/2}$

3. The 1-Norm-Constrained Portfolios

- 1-norm $\|\mathrm{w}\|_1 = \sum_{i=1}^{N} |\mathrm{w}_i| \leq \delta$

PROPOSITION 1. *The solution to the 1-norm-constrained minimum-variance portfolio problem with* $\delta = 1$ *coincides with the solution to the shortsale-constrained problem*.

$$-\sum_{i \in \mathcal{N}(\mathbf{W})} \mathrm{w}_i \leq \frac{\delta - 1}{2}$$

3. The A-Norm-Constrained Portfolios

$$\min_{w} \ w^\top \hat{\Sigma} w,$$
$$\text{s.t.} \ w^\top e = 1,$$

PROPOSITION 2. *Provided $\hat{\Sigma}$ is nonsingular, for each $\nu \geq 0$ there exists a δ such that the solution to the minimum-variance problem in (1)–(2), with the sample covariance matrix, $\hat{\Sigma}$, replaced by $\hat{\Sigma}_{LW} = (1/(1+\nu))\hat{\Sigma} + \nu/(1+\nu)A$, coincides with the solution to the A-norm-constrained minimum-variance portfolio problem, which is the traditional minimum-variance problem subject to the additional constraint*

$$w^\top A w \leq \delta. \tag{6}$$

- A equals the 1-factor covariance matrix, $\widehat{\Sigma_F}$ →Ledoit and Wolf (2003).
- A equal to the identity matrix I. →Ledoit and Wolf (2004). 2-norm, $\|w\|_2$

$$\sum_{i=1}^{N}\left(w_i - \frac{1}{N}\right)^2 \leq \left(\delta - \frac{1}{N}\right)$$

- The 1/N portfolio is a special case of the 2-normconstrained portfolio with

$\delta = 1/N$

3. The Partial Minimum-Variance Portfolios

- we propose are obtained by applying the classical **conjugate-gradient method(共轭梯度法)** (Nocedal and Wright 1999, Chap. 5) to solve the minimum-variance problem.

- The conjugate-gradient method takes as a starting portfolio **some initial guess** (in our implementation we use the 1/N portfolio) and then **generates a sequence of N −1 portfolios** in which the terminal portfolio is the shortsale-unconstrained minimum-variance portfolio. We term each of these N − 1 intermediate portfolios a **partial minimum variance portfolio.**
- Proposition 3. The 2-norm of the kth partial minimum-variance portfolio is smaller than or equal to the **2-norm of the shortsale-unconstrained minimum-variance portfolio** for $k \le N - 1$.

3. A Bayesian Interpretation of the Norm-Constrained Portfolios

PROPOSITION 5. *Assume that asset returns are normally distributed. Moreover, assume that the investor believes a priori that the shortsale-unconstrained minimum-variance portfolio weights are distributed as a multivariate Normal distribution with probability density function*

$$\pi(\mathrm{w}) = (2\pi)^{-n/2}|A|^{1/2}e^{-\frac{1}{2}\mathrm{w}^\top A\mathrm{w}}. \quad (9)$$

Furthermore, assume that the investor believes a priori that the variance of the minimum-variance portfolio return, denoted by σ^2, has an independent prior distribution $\pi(\sigma^2)$. Then there exists a threshold parameter δ such that the weights of the A-norm-constrained minimum-variance portfolio are the mode of the posterior distribution of the minimum-variance portfolio weights.

3. A Moment-Shrinkage Interpretation of the Norm-Constrained Portfolios

PROPOSITION 6. *Let the solution to the 1-norm-constrained minimum-variance problem be such that* $(w_{NC1})_i \neq 0$ *for* $i = 1, \ldots, N$. *Then* w_{NC1} *is also the solution to the shortsale-unconstrained minimum-variance problem* (1)–(2) *if the sample covariance matrix,* $\widehat{\Sigma}$, *is replaced by the matrix*

$$\widehat{\Sigma}_{NC1} = \widehat{\Sigma} - \nu n e^{\top} - \nu e n^{\top}, \tag{10}$$

where $\nu \in \mathbb{R}$ *is the Lagrange multiplier for the* 1*-norm constraint at the solution to the* 1*-norm-constrained minimum-variance problem, and* $n \in \mathbb{R}^N$ *is a vector whose ith component is one if the weight assigned by the* 1*-norm-constrained portfolio to the ith asset is negative and zero otherwise.*

4. Out-of-Sample Evaluation

Table 2 List of Portfolios Considered

No.	Model	Abbreviation
	Panel A: Portfolio strategies developed in this paper	
1-norm-constrained minimum-variance portfolio		
	With δ calibrated using cross-validation over portfolio variance	NC1V
	With δ calibrated by maximizing portfolio return in previous period	NC1R
2-norm-constrained minimum-variance portfolio		
	With δ calibrated using cross-validation over portfolio variance	NC2V
	With δ calibrated by maximizing portfolio return in previous period	NC2R
$\widehat{\Sigma}_F$-norm-constrained minimum-variance portfolio		
	With δ calibrated using cross-validation over portfolio variance	NCFV
	With δ calibrated by maximizing portfolio return in previous period	NCFR
Partial minimum-variance portfolios		
	With k calibrated using cross-validation over portfolio variance	PARV
	With k calibrated by maximizing portfolio return in previous period	PARR
	Panel B: Portfolio strategies from the existing literature used for comparison	

4. Out-of-Sample Evaluation

Table 3 Portfolio Variances

Strategy	10Ind	48Ind	6FF	25FF	CRSP
	Panel A: Portfolio strategies developed in this paper				
NC1V	0.00134	0.00126	0.00156	0.00135	0.00074
	(0.07)	(0.01)	(0.46)	(0.65)	(0.06)
NC1R	0.00138	0.00135	0.00159	0.00143	0.00080
	(0.98)	(0.26)	(0.97)	(0.10)	(0.01)
NC2V	0.00134	0.00137	0.00156	0.00130	0.00066
	(0.08)	(0.21)	(0.13)	(0.43)	(0.51)
NC2R	0.00149	0.00176	0.00163	0.00152	0.00087
	(0.10)	(0.00)	(0.63)	(0.02)	(0.00)
NCFV	0.00135	0.00131	0.00162	0.00134	0.00052
	(0.39)	(0.03)	(0.47)	(0.82)	(0.00)
NCFR	0.00144	0.00166	0.00171	0.00170	0.00068
	(0.30)	(0.01)	(0.07)	(0.00)	(0.53)
PARV	0.00138	0.00141	0.00159	0.00133	0.00065
	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)
PARR	0.00153	0.00163	0.00161	0.00146	0.00085
	(0.02)	(0.01)	(0.77)	(0.12)	(0.00)
	Panel B: Portfolio strategies from existing literature				
1/N	0.00179	0.00221	0.00230	0.00249	0.00169
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)

4. Out-of-Sample Evaluation

Table 4 Portfolio Sharpe Ratios

Strategy	10Ind	48Ind	6FF	25FF	CRSP
	Panel A: Portfolio strategies developed in this paper				
NC1V	0.2854	0.2886	0.3385	0.3649	0.4013
	(0.06)	(0.32)	(0.01)	(0.00)	(0.11)
NC1R	0.2890	0.2831	0.3374	0.3553	0.3706
	(0.05)	(0.19)	(0.00)	(0.00)	(0.04)
NC2V	0.2919	0.2855	0.3527	0.4089	0.3994
	(0.08)	(0.22)	(0.10)	(0.22)	(0.07)
NC2R	0.3193	0.2891	0.3922	0.4278	0.4672
	(0.40)	(0.05)	(0.93)	(0.36)	(0.55)
NCFV	0.2927	0.2808	0.3479	0.3728	0.4463
	(0.21)	(0.27)	(0.08)	(0.03)	(0.48)
NCFR	0.3114	0.2723	0.3186	0.3815	0.4243
	(0.56)	(0.22)	(0.01)	(0.11)	(0.32)
PARV	0.2841	0.2823	0.3478	0.4077	0.3937
	(0.07)	(0.29)	(0.10)	(0.25)	(0.05)
PARR	0.3293	0.3166	0.3912	0.4403	0.4768
	(1.00)	(1.00)	(1.00)	(1.00)	(1.00)
	Panel B: Portfolio strategies from existing literature				
1/N	0.2541	0.2508	0.2563	0.2565	0.3326
	(0.02)	(0.10)	(0.00)	(0.00)	(0.00)

Size Matters: Optimal Calibration of Shrinkage Estimators for Portfolio Selection

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1. Introduction-- Motivation

- One of the most popular approaches to combat the impact of estimation error in portfolio selection is to use **shrinkage estimators.**

- Shrinkage estimators can help reduce estimation error, but the **shrinkage intensity** (size) matters.

2. Shrinkage estimators for portfolio selection

- The optimization problem

$$\max_{w} \quad w'\mu - \frac{\gamma}{2} w'\Sigma w,$$

$$w = \frac{1}{\gamma}\Sigma^{-1}\mu.$$

- 1. Shrinkage estimators of moments

$$\mu_{sh} = (1-\alpha)\mu_{sp} + \alpha v \mu_{tg},$$
$$\Sigma_{sh} = (1-\alpha)\Sigma_{sp} + \alpha v \Sigma_{tg},$$
$$\Sigma_{sh}^{-1} = (1-\alpha)\Sigma_{sp}^{-1} + \alpha v \Sigma_{tg}^{-1},$$

- 2. Shrinkage estimators of portfolio weights

$$w_{sh} = (1-\alpha)w_{sp} + \alpha v w_{tg},$$

2. Shrinkage estimators for portfolio selection

- The optimization problem

$$\max_{w} \quad w'\mu - \frac{\gamma}{2} w'\Sigma w,$$

$$w = \frac{1}{\gamma}\Sigma^{-1}\mu.$$

> 1. Shrinkage estimators of moments

$$\mu_{sh} = (1-\alpha)\mu_{sp} + \alpha v \mu_{tg},$$
$$\Sigma_{sh} = (1-\alpha)\Sigma_{sp} + \alpha v \Sigma_{tg},$$
$$\Sigma_{sh}^{-1} = (1-\alpha)\Sigma_{sp}^{-1} + \alpha v \Sigma_{tg}^{-1},$$

where α is the shrinkage intensity and v is a scaling parameter that we adjust to minimize the bias of the shrinkage target.

2. Shrinkage estimators for portfolio selection

- 1.1. Shrinkage estimator of mean returns $\hat{\mu}_t^{\min} \equiv \hat{\mu}_t^\top \hat{w}_t^{\min}$

➢ We choose the shrinkage intensity α to minimize the expected quadratic loss of the shrinkage estimator

$$\mu_{sh} = (1-\alpha)\mu_{sp} + \alpha \nu \mu_{tg},$$
$$\nu \mu_{tg} = \nu \iota$$

$$\min_{\alpha} \quad E\left[\|\mu_{sh} - \mu\|_2^2\right],$$

where $\|x\|_2^2 = \sum_{i=1}^N x_i^2$.

where ι is the vector of ones and ν is a scaling factor.

$$\nu_\mu = \text{argmin}_\nu \|\nu\iota - \mu\|_2^2 = (1/N)\sum_{i=1}^N \mu_i = \bar{\mu}$$

Proposition 1. *Assuming asset returns are iid, the shrinkage intensity α that minimizes the expected quadratic loss is:*

$$\alpha_\mu = \frac{E\left(\|\mu_{sp} - \mu\|_2^2\right)}{E\left(\|\mu_{sp} - \mu\|_2^2\right) + \|\nu_\mu \iota - \mu\|_2^2} = \frac{(N/T)\overline{\sigma^2}}{(N/T)\overline{\sigma^2} + \|\nu_\mu \iota - \mu\|_2^2}, \qquad (7)$$

where $\overline{\sigma^2} = \text{trace}(\Sigma)/N$.

2. Shrinkage estimators for portfolio selection

In this section, we prove the closed-form expression given in Proposition 1. In general, we consider that asset returns are independent and identically distributed. Then, from problem (6), we have:

$$\min_{\alpha} \quad E\left[\|\mu_{sh} - \mu\|_2^2\right] = (1-\alpha)^2 E\left[\|\mu_{sp} - \mu\|_2^2\right] + \alpha^2 \|\nu_\mu \imath - \mu\|_2^2. \quad \text{(A.5)}$$

Now, developing the optimality conditions of problem (A.5), we can obtain the optimal α that minimizes the expected quadratic loss:

$$\alpha_\mu = \frac{E\left(\|\mu_{sp} - \mu\|_2^2\right)}{E\left(\|\mu_{sp} - \mu\|_2^2\right) + \|\nu_\mu \imath - \mu\|_2^2}, \quad \text{(A.6)}$$

where $\nu_\mu = \text{argmin}_\nu \|\nu \imath - \mu\|_2^2 = \bar{\mu}$. We develop the expected value given in (A.6) to derive the closed-form expression:

$$E(\|\mu_{sp} - \mu\|^2) = E\left(\mu_{sp}' \mu_{sp}\right) - \mu' \mu. \quad \text{(A.7)}$$

Since μ_{sp} is a random variable with mean μ and covariance matrix $\frac{\Sigma}{T}$, we can use Lemma 1 to obtain the closed-form expression of $E(\|\mu_{sp} - \mu\|^2)$. Thus:

$$E\left(\|\mu_{sp} - \mu\|^2\right) = (N/T)\overline{\sigma^2}, \quad \text{(A.8)}$$

where $\overline{\sigma^2} = \text{trace}(\Sigma)/N$, and it completes the proof.

2. Shrinkage estimators for portfolio selection

- 1.2. Shrinkage estimator of the **covariance matrix**

➢ We choose the shrinkage intensity α to minimize the expected quadratic loss of the shrinkage estimator

$$E\left[\|\Sigma_{sh}-\Sigma\|_F^2\right]$$

$$\Sigma_{sh}=(1-\alpha)\Sigma_{sp}+\alpha v\Sigma_{tg},$$

$$\|X\|_F^2=\mathrm{trace}(X'X)$$

$$\alpha_\Sigma=\frac{E\left(\|\Sigma_{sp}-\Sigma\|_F^2\right)}{E\left(\|\Sigma_{sp}-\Sigma\|_F^2\right)+\|v_\Sigma I-\Sigma\|_F^2}.$$

$$v_{\Sigma^{-1}}=\mathrm{argmin}_v\|vI-\Sigma^{-1}\|_F^2=(1/N)\sum_{i=1}^N\sigma_i^{-2}=\overline{\sigma^{-2}}$$

2. Shrinkage estimators for portfolio selection

- 1.3. Shrinkage estimator of the **inverse covariance matrix**

➢ We choose the shrinkage intensity α to minimize the expected quadratic loss of the shrinkage estimator

$$E\left(\left\|\Sigma_{sh}^{-1}-\Sigma^{-1}\right\|_F^2\right)$$

$$\alpha_{\Sigma^{-1}}=\frac{E\left(\left\|\Sigma_{sp}^{-1}-\Sigma^{-1}\right\|_F^2\right)-E\left(<\Sigma_{sp}^{-1}-\Sigma^{-1},v_{\Sigma^{-1}}I-\Sigma^{-1}>\right)}{E\left(\left\|\Sigma_{sp}^{-1}-\Sigma^{-1}\right\|_F^2\right)+\left\|v_{\Sigma^{-1}}I-\Sigma^{-1}\right\|_F^2-2E\left(<\Sigma_{sp}^{-1}-\Sigma^{-1},v_{\Sigma^{-1}}I-\Sigma^{-1}>\right)}$$

$$\langle A, B\rangle = \mathrm{trace}(A'B)$$

2. Shrinkage estimators for portfolio selection

- 1.4. Shrinkage estimator of the **covariance matrix considering the condition number**

➢ **Relative improvement in average loss** (RIAL):measures the impact of estimation error on the portfolio weights.

$$\alpha_{\Sigma} = \frac{E\left(\|\Sigma_{sp} - \Sigma\|_F^2\right)}{E\left(\|\Sigma_{sp} - \Sigma\|_F^2\right) + \|\nu_{\Sigma} I - \Sigma\|_F^2}.$$

$$RIAL(\Sigma_{sh}) = \frac{E\left(\|\Sigma_{sp} - \Sigma\|_F^2\right) - E\left(\|\Sigma_{sh} - \Sigma\|_F^2\right)}{E\left(\|\Sigma_{sp} - \Sigma\|_F^2\right)}$$

➢ The condition number of the shrinkage covariance matrix is

$$\delta_{\Sigma_{sh}} = \frac{(1-\alpha)\lambda_{\max} + \alpha\nu_{\Sigma}}{(1-\alpha)\lambda_{\min} + \alpha\nu_{\Sigma}}$$

➢ To find an optimal shrinkage intensity

$$\alpha = \operatorname{argmin}\{\delta_{\Sigma_{sh}} - \phi RIAL(\Sigma_{sh})\}$$

2. Shrinkage estimators for portfolio selection

- **2. Shrinkage estimators of portfolio weights**

$$w_{sh} = (1-\alpha)w_{sp} + \alpha v w_{tg},$$

- Two new calibration criteria: the expected quadratic loss minimization criterion, and the Sharpe ratio maximization criterion.

$$\text{Expected quadratic loss (eql)}: \quad \min_{\alpha} \; E(f_{\mathrm{ql}}(w_{sh})) = \min_{\alpha} E\left(\|w_{sh} - w_{op}\|_2^2\right)$$

$$\text{Utility (ut)}: \quad \max_{\alpha} \; E(f_{\mathrm{ut}}(w_{sh})) = \max_{\alpha} \; E\left(w'_{sh}\mu - \frac{\gamma}{2} w'_{sh}\Sigma w_{sh}\right), \quad (15)$$

$$\text{Variance (var)}: \quad \min_{\alpha} \; E(f_{\mathrm{var}}(w_{sh})) = \min_{\alpha} \; E(w'_{sh}\Sigma w_{sh}), \quad (16)$$

$$\text{Sharpe ratio (SR)}: \quad \max_{\alpha} \; E(f_{\mathrm{SR}}(w_{sh})) = \max_{\alpha} \; \frac{E(w'_{sh}\mu)}{\sqrt{E(w'_{sh}\Sigma w_{sh})}}, \quad (17)$$

2. Shrinkage estimators for portfolio selection

- **3.Parametric calibration**
  - We characterize in **closed-form** the expectations required to compute the optimal shrinkage intensities under the assumption that returns are **iid normal**.

- **4. Nonparametric calibration of shrinkage estimators**
  - We generate B bootstrap samples by drawing observations with replacement from the original sample.
  - Then, for each bootstrap sample, we compute the statistic of interest.
  - Finally, we take the sample average among the B bootstrap statistics as an approximation to the expected value.

3. Out-of-sample performance

#	Policy	Abbreviation
Panel A: Benchmark portfolios		
1	Classical mean–variance portfolio	mv
2	Bayes-Stein mean–variance portfolio	bs
3	Kan-Zhou's (2007) three-fund portfolio	kz
4	Mixture of mean–variance and equally-weighted (Tu and Zhou, 2011)	tz
5	Mixture of minimum-variance and equally-weighted (DeMiguel et al. (2009))	dm
6	Minimum-Variance portfolio	min
7	Minimum-variance portfolio with Ledoit and Wolf (2004) shrinkage covariance matrix, which shrinks the sample covariance matrix to the identity matrix	lw
8	Minimum-variance portfolio with Ledoit and Wolf (2003) shrinkage covariance matrix, which shrinks the sample covariance matrix to the sample covariance matrix of a single-index factor model	lw-m
9	Equally-weighted portfolio	1/N or ew
Panel B: Portfolios estimated with new calibration procedures to shrink moments		
Shrinkage mean–variance portfolio		
10	Mean–variance portfolio formed with the shrinkage vector of means defined in Section 2.1.1	f-mv
Shrinkage minimum-variance portfolio		
11	Formed with Ledoit and Wolf (2004) shrinkage covariance matrix: calibrated under a parametric calibration assuming normality and calibrated under a bootstrap nonparametric approach	par-lw and npar-lw
12	Formed with the shrinkage inverse covariance matrix studied in Section 2.1.3: calibrated under a parametric calibration assuming normality and calibrated under a bootstrap nonparametric approach	par-ilw and npar-ilw
13	Formed with a shrinkage covariance matrix that accounts for the expected quadratic loss and the condition number: calibrated under a parametric calibration assuming normality and calibrated under a bootstrap nonparametric approach	par-clw and npar-clw
Panel C: Shrinkage portfolios		
14	Mixture of mean–variance and scaled minimum-variance portfolios	mv-min
15	Mixture of mean–variance and scaled equally-weighted portfolios	mv-ew
16	Mixture of minimum-variance and scaled equally-weighted portfolios	min-ew

3. Out-of-sample performance

Policy	5IndP	10IndP	38IndP
Panel A: Benchmark Portfolios			
Portfolios that consider the vector of means			
mv	0.591**	0.519***	−0.023***
bs	0.807	0.765***	0.203***
Portfolios that do not consider the vector of means			
min	0.895	0.934	0.478***
lw	0.893	0.961	0.752
lw-m	0.890	0.953	0.670***
Naïve Portfolios			
1/N	0.786	0.817	0.717
Panel B: Portfolios calibrated parametrically			
Portfolios that do not consider the vector of means			
par-lw	0.891	0.944	0.645***
par-ilw	0.902	0.918	0.730
par-clw	0.890	0.956	0.823**
Panel C: Portfolios calibrated nonparametrically			
Portfolios that consider the vector of means			
f-mv	0.854	0.797**	0.374***
Portfolios that do not consider the vector of means			
npar-lw	0.888**	0.957*	0.729***
npar-ilw	0.864	0.867	0.721
npar-clw	0.888	0.965	0.859***

3. Conclusions

- We provide a comprehensive investigation of shrinkage estimators for portfolio selection.
- The shrinkage estimator based on proposed criterion perform well in medium and large datasets.
- This paper is among the first to consider and compare different shrinkage estimators within the context of portfolio optimization.

Technical Note—A Robust Perspective on Transaction Costs in Portfolio Optimization

Olivares-Nadal, Alba V., and Victor DeMiguel

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1. Introduction-- Motivation

- Two crucial aspects in portfolio selection are
  - Estimation error
  - Transaction costs

- P-norm transaction costs can be equivalently reformulated as three different problems designed to **alleviate the impact of estimation error**
  - (i) a robust portfolio optimization problem
  - (ii) a regularized regression problem
  - (iii) a Bayesian portfolio problem

1. Introduction-- Motivation

- We propose a **data-driven approach** to portfolio selection that consists of using cross-validation to calibrate the transaction cost parameter from historical data

2. Transaction Costs and Robustness

- Mean-variance problem with p-norm transaction costs

$$\min_{\mathbf{w}} \ \left\{\tfrac{1}{2}\gamma \mathbf{w}^T \Sigma \mathbf{w} - \boldsymbol{\mu}^T \mathbf{w} + \kappa \|\Lambda(\mathbf{w} - \mathbf{w}_0)\|_p^p\right\}$$
$$\text{s.t.} \ \ \mathbf{w}^T \mathbf{1}_N = 1,$$

(i) *a robust portfolio problem*:

$$\min_{\mathbf{w}} \ \left\{\frac{\gamma}{2}\mathbf{w}^T \Sigma \mathbf{w} - \boldsymbol{\mu}^T \mathbf{w} + \max_{\hat{\boldsymbol{\mu}} \in U(\delta)} (\boldsymbol{\mu} - \hat{\boldsymbol{\mu}})^T (\mathbf{w} - \mathbf{w}_0)\right\}$$
$$\text{s.t.} \ \ \mathbf{w}^T \mathbf{1}_N = 1, \qquad (2)$$

where the uncertainty set for mean asset returns is $U(\delta) = \{\hat{\boldsymbol{\mu}} \colon \|\boldsymbol{\mu} - \hat{\boldsymbol{\mu}}\|_{q,\Lambda^{-T}} \leqslant \delta\}$;

2. Transaction Costs and Robustness

- Mean-variance problem with p-norm transaction costs

$$\min_{\mathbf{w}} \left\{ \tfrac{1}{2}\gamma \mathbf{w}^T \Sigma \mathbf{w} - \boldsymbol{\mu}^T \mathbf{w} + \kappa \|\Lambda(\mathbf{w} - \mathbf{w}_0)\|_p^p \right\}$$
$$\text{s.t.} \quad \mathbf{w}^T \mathbf{1}_N = 1,$$

(ii) *a regularized linear regression problem*:

$$\begin{aligned} \min_{\mathbf{w}} \quad & \left\{ \|\mathbf{1}_T - R\mathbf{w}\|_2^2 + \kappa' \|\Lambda(\mathbf{w} - \mathbf{w}_0)\|_p^p \right\} \\ \text{s.t.} \quad & \mathbf{w}^T \boldsymbol{\mu} = \mu_0, \\ & \mathbf{w}^T \mathbf{1}_N = 1, \end{aligned} \qquad (3)$$

where $R \in \mathbb{R}^{T \times N}$ is the matrix whose columns contain the historical returns for each of the N assets; or

2. Transaction Costs and Robustness

- Mean-variance problem with p-norm transaction costs

$$\min_{\mathbf{w}} \ \left\{\tfrac{1}{2}\gamma \mathbf{w}^T \Sigma \mathbf{w} - \boldsymbol{\mu}^T \mathbf{w} + \kappa \|\Lambda(\mathbf{w}-\mathbf{w}_0)\|_p^p\right\}$$
$$\text{s.t.} \ \ \mathbf{w}^T \mathbf{1}_N = 1,$$

(iii) *a Bayesian portfolio problem, where the investor believes a priori that the variance of the mean-variance portfolio return has an independent distribution* $\pi(\sigma^2)$*, that asset returns are normally distributed, and that the mean-variance portfolio weights are jointly distributed as a multivariate exponential power distribution, with probability density function*

$$\pi(\mathbf{w}) = \frac{p^N |\Lambda|}{2^N \alpha^N \Gamma(1/p)^N} e^{-\|\Lambda(\mathbf{w}-\mathbf{w}_0)\|_p^p/\alpha^p}, \qquad (4)$$

where α *is the scale parameter and* $\Gamma(\cdot)$ *is the gamma function.*

2. Transaction Costs and Robustness

$$\min_{\mathbf{w}} \left\{ \tfrac{1}{2}\gamma \mathbf{w}^T \Sigma \mathbf{w} - \boldsymbol{\mu}^T \mathbf{w} + \kappa \|\Lambda(\mathbf{w} - \mathbf{w}_0)\|_p^p \right\}$$
$$\text{s.t.} \ \ \mathbf{w}^T \mathbf{1}_N = 1,$$

Table 1. Sharpe Ratios

Strategy	10Ind	48Ind	6FF	25FF
Panel A: Portfolios that ignore transaction costs, $\kappa = 0$				
Minimum-variance, shortsale unconstrained	0.3007	0.1167	0.3480	0.3124
Minimum-variance, shortsale constrained	0.2953	0.2452*	0.2493*	0.2390**
Mean-variance, shortsale unconstrained	0.0686*	−0.0890*	0.2142*	−0.0076*
Mean-variance, shortsale constrained	0.2128**	0.1782	0.2502*	0.2382***
Panel B: Portfolios with nominal transaction costs, $0.005\|\Delta\mathbf{w}\|_1$				
Minimum-variance, shortsale unconstrained	0.2959	0.0955	0.3026**	0.3063
Minimum-variance, shortsale constrained	0.2420***	0.2601*	0.2374*	0.2318**
Mean-variance, shortsale unconstrained	0.2074**	−0.0523*	0.2631**	−0.0467*
Mean-variance, shortsale constrained	0.2214**	0.2018**	0.2505*	0.2588
Panel C: Data-driven portfolios with calibrated proportional transaction costs, $\kappa_{cv}\|\Delta\mathbf{w}\|_1$				
Minimum-variance, shortsale unconstrained	**0.3281*****	0.1505	0.3284	0.3745**
Minimum-variance, shortsale constrained	0.3006	**0.2925***	0.2479*	0.2563***
Mean-variance, shortsale unconstrained	0.2443	0.0039**	0.2436**	−0.0442*
Mean-variance, shortsale constrained	0.2693	0.2248**	0.2477*	0.2442***
Panel D: Data-driven portfolios with calibrated quadratic transaction costs, $\kappa_{cv}\|\Sigma^{1/2}\Delta\mathbf{w}\|_2^2$				
Minimum-variance, shortsale unconstrained	0.3234***	0.2349*	**0.3481**	**0.3761***
Minimum-variance, shortsale constrained	0.2983	0.2762*	0.2446*	0.2460***
Mean-variance, shortsale unconstrained	0.2565	0.0105***	0.2424**	0.0464*
Mean-variance, shortsale constrained	0.2748	0.2561*	0.2497*	0.2514***

V. Conclusion

- The data-driven portfolios perform favorably because they strike an optimal trade-off between rebalancing the portfolio to capture the information in recent historical return data

- and avoiding the large transaction costs and impact of estimation error associated with excessive trading.

Further research

- **Stock Return Serial Dependence and Out-of-Sample Portfolio Performance**, with F.J. Nogales and R. Uppal, The Review of Financial Studies, 27(4), 1031--1073 (2014).

- **A Transaction-Cost Perspective on the Multitude of Firm Characteristics**, with A. Martin-Utrera, F.J. Nogales, and R. Uppal, The Review of Financial Studies, 33(5), 2180--2222 (2020)

- **What Alleviates Crowding in Factor Investing?,** with A. Martin-Utrera and R. Uppal, LBS working paper (2022)

- **Comparing Factor Models with Price-Impact Costs**, with S.A. Li and A. Martin-Utrera, LBS working paper (2022)