01-F2

绪论

动态规划：最长公共子序列

世上一切都无独有偶，为什么你与我却否？

Make it work, make it right, make it fast.

- Kent Beck

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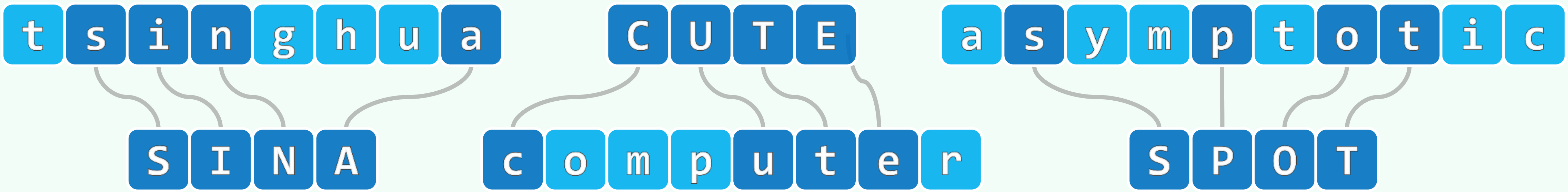
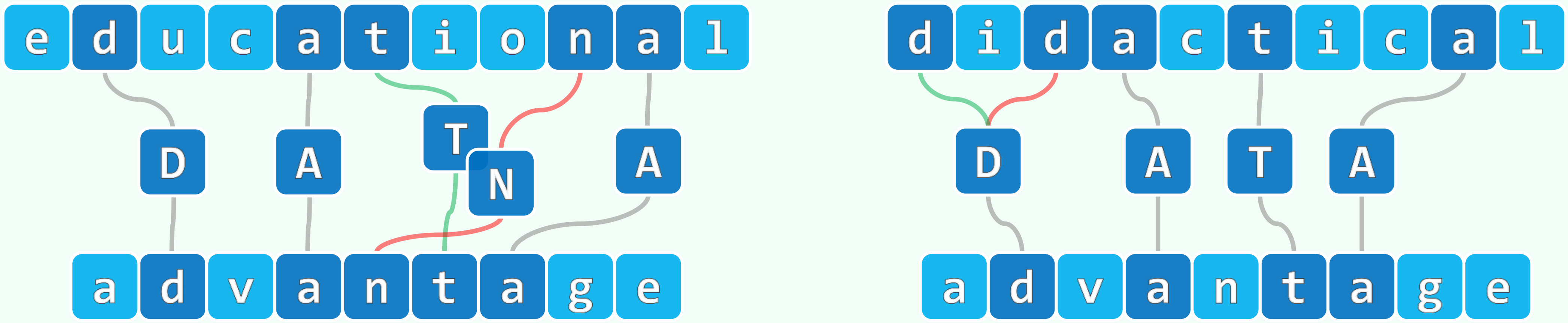
问题定义

❖ 子序列（Subsequence）：由序列中若干字符，按原相对次序构成



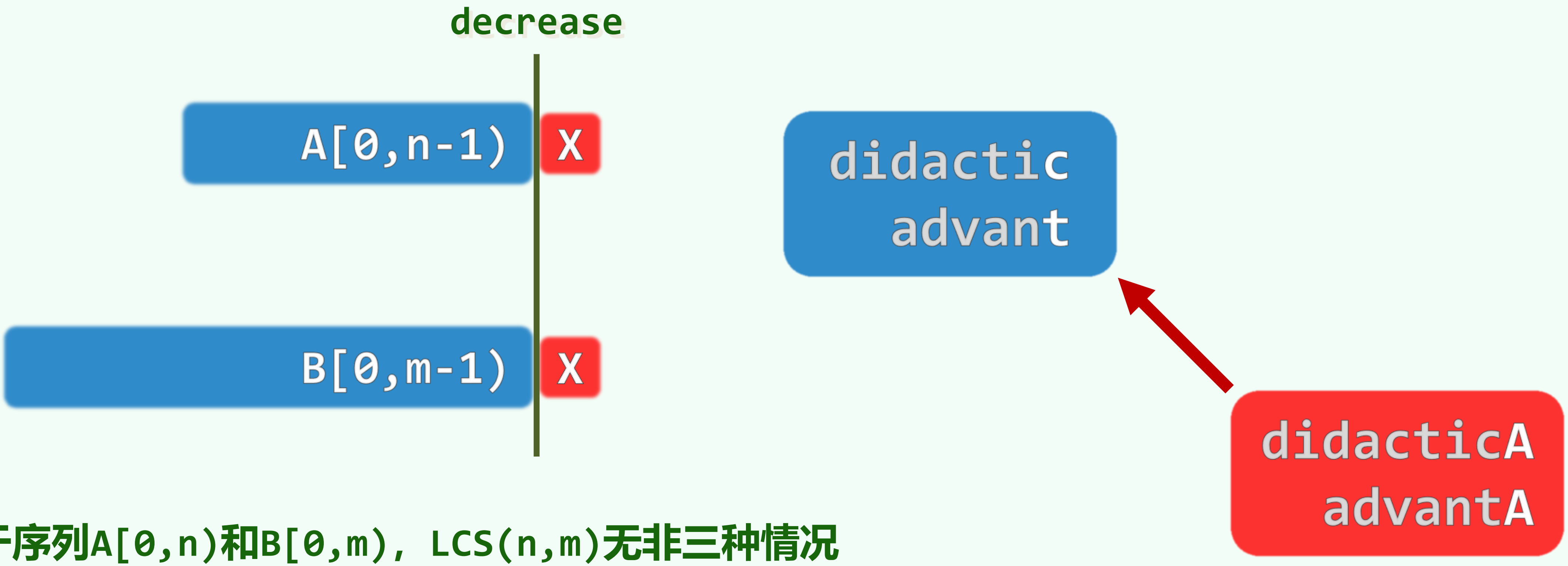
❖ 最长公共子序列（Longest Common Subsequence）：两个序列公共子序列中的最长者



可能有多个

可能有歧义

减治递归

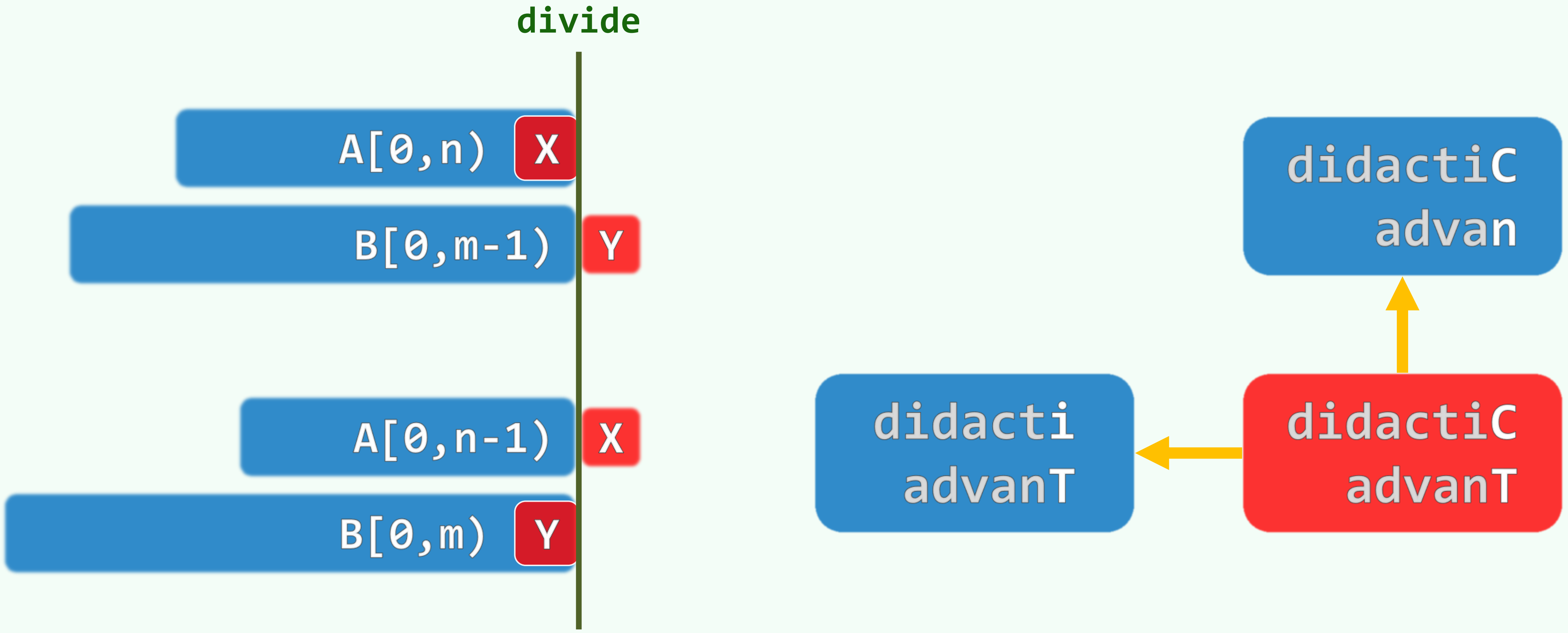


对于序列A[0,n)和B[0,m)，LCS(n,m)无非三种情况

0) 若 n = 0 或 m = 0，则取作空序列（长度为零）　//递归基：必然总能抵达

1) 若A[n-1] = 'X' = B[m-1]，则取作：LCS(n-1,m-1) + 'X'

分治递归



2) A[n-1] ≠ B[m-1]，则在 LCS(n,m-1) 与 LCS(n-1,m) 中取更长者

描述：伪代码

```
Input: two strings A and B of length n and m resp.,

Output: (the length of) the longest common subsequence of A and B

lcs( A[], n, B[], m )

   Compare the last character of A and B, i.e., A[n-1] and B[m-1]

   If A[n-1] = B[m-1]

      Compute x = lcs(A, n-1, B, m-1) recursively and return 1 + x

   Else

      Compute x = lcs(A, n-1, B, m) & y = lcs(A, n, B, m-1) and return max(x, y)

   As the recursion base, return 0 when either n or m is 0
```

实现：递归版

```
unsigned int lcs( char const * A, int n, char const * B, int m ) {
   if (n < 1 || m < 1) //trivial cases
      return 0;
   else if ( A[n-1] == B[m-1] ) //decrease & conquer
      return 1 + lcs(A, n-1, B, m-1);
   else //divide & conquer
      return max( lcs(A, n-1, B, m), lcs(A, n, B, m-1) );
}
```

理解

❖ LCS的每一个解，对应于(0,0)与(n,m)之间的一条**单调通路**；反之亦然

		E	D	U	C	A	T	I	O	N	A	L
	0	0	0	0	0	0	0	0	0	0	0	0
A	0	0	0	0	0	1	1	1	1	1	1	1
D	0	0	1	1	1	1	1	1	1	1	1	1
V	0	0	1	1	1	1	1	1	1	1	1	1
A	0	0	1	1	1	2	2	2	2	2	2	2
N	0	0	1	1	1	2	2	2	2	3	3	3
T	0	0	1	1	1	2	3	3	3	3	3	3
A	0	0	1	1	1	2	3	3	3	3	4	4
G	0	0	1	1	1	2	3	3	3	3	4	4
E	0	1	1	1	1	2	3	3	3	3	4	4

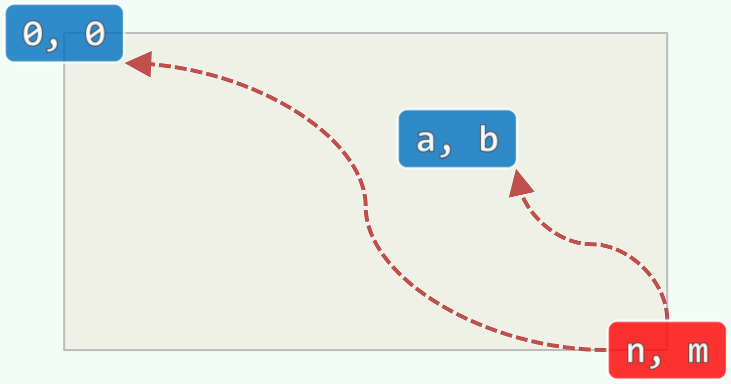
多解

		D	I	D	A	C	T	I	C	A	L
	0	0	0	0	0	0	0	0	0	0	0
A	0	0	0	0	1	1	1	1	1	1	1
D	0	1	1	1	1	1	1	1	1	1	1
V	0	1	1	1	1	1	1	1	1	1	1
A	0	1	1	1	2	2	2	2	2	2	2
N	0	1	1	1	2	2	2	2	2	2	2
T	0	1	1	1	2	2	3	3	3	3	3
A	0	1	1	1	2	2	3	3	3	4	4
G	0	1	1	1	2	2	3	3	3	4	4
E	0	1	1	1	2	2	3	3	3	4	4

歧义

复杂度

- 单调性：每经一次比对，至少一个序列的长度缩短一个单位
- 最好情况，只需 $\mathcal{O}(n+m)$ 时间 //比如...
- 然而最坏情况下，子问题数量不仅会增加，且可能大量雷同



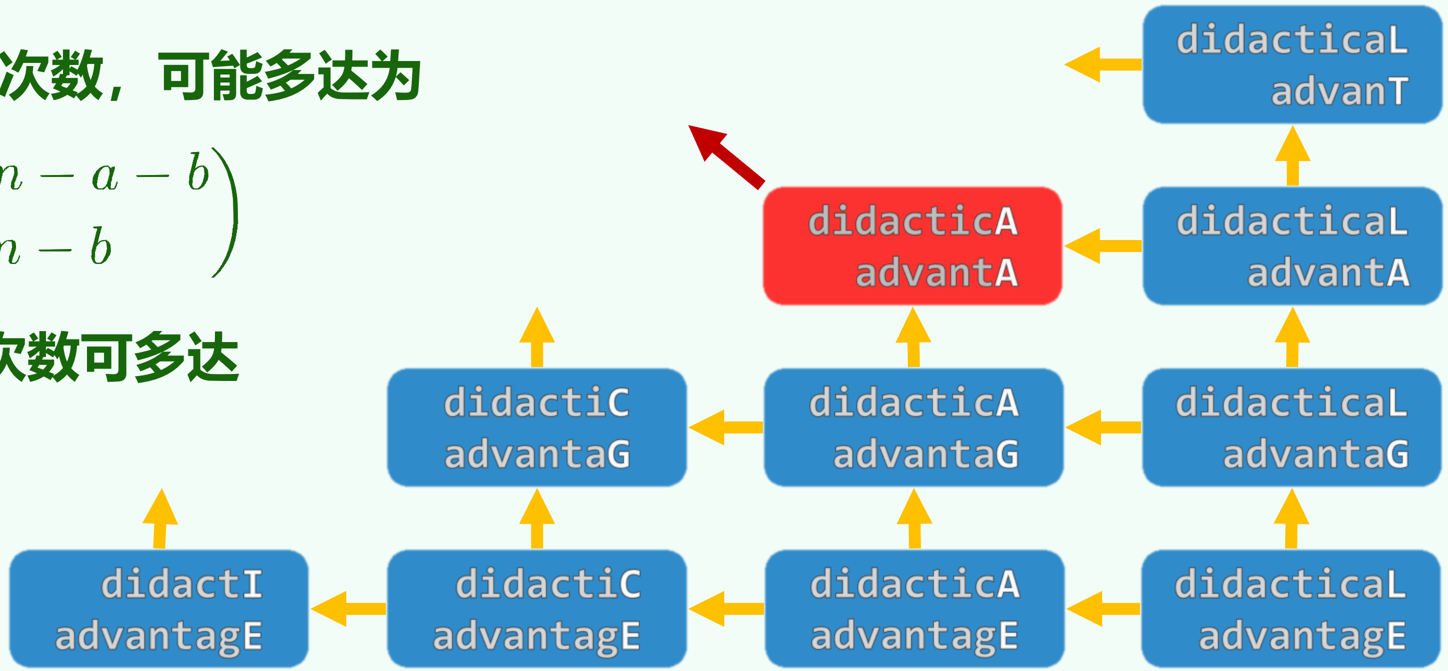
子任务LCS(A[a],B[b])重复的次数，可能多达为

$$\binom{n+m-a-b}{n-a} = \binom{n+m-a-b}{m-b}$$

特别地，LCS(A[0], B[0])的次数可多达

$$\binom{n+m}{n} = \binom{n+m}{m}$$

当n = m时，为 $\Omega(2^n)$



实现：记忆化版 (1/2)

```
unsigned int lcsMemo(char const* A, int n, char const* B, int m) {
   unsigned int * lcs = new unsigned int[n*m]; //lookup-table of sub-solutions
   memset(lcs, 0xFF, sizeof(unsigned int)*n*m); //initialized with n*m UINT_MAX's
   unsigned int solu = lcsM(A, n, B, m, lcs, m);
   delete[] lcs;
   return solu;
}
```

实现：记忆化版（2/2）

```
unsigned int lcsM( char const * A, int n, char const * B, int m,
                   unsigned int * const lcs, int const M ) {
   if (n < 1 || m < 1)
      return 0; //trivial cases
   if (UINT_MAX != lcs[(n-1)*M + m-1])
      return lcs[(n-1)*M + m-1]; //recursion stops
   else
      return lcs[(n-1)*M + m-1] = (A[n-1] == B[m-1]) ?
                  1 + lcsM(A, n-1, B, m-1, lcs, M) :
                  max( lcsM(A, n-1, B, m, lcs, M), lcsM(A, n, B, m-1, lcs, M) );
}
```

动态规划

- ❖ 与fib()类似，这里也有大量重复的递归实例（子问题）

  各子问题，分别对应于A和B的某个前缀组合

  因此实际上，总共不过 $\mathcal{O}(n \cdot m)$ 种

- ❖ 采用动态规划的策略

  只需 $\mathcal{O}(n \cdot m)$ 时间即可计算出所有子问题

- ❖ 为此，只需
  - 将所有子问题（假想地）列成一张表
  - 颠倒计算方向：从LCS(0,0)出发，依次计算出所有项——直至LCS(n,m)

		d	i	d	a	c	t	i	c	a	l
	0	0	0	0	0	0	0	0	0	0	0
a	0	0	0	0	1	1	1	1	1	1	1
d	0	1	1	1	1	1	1	1	1	1	1
v	0	1	1	1	1	1	1	1	1	1	1
a	0	1	1	1	2	2	2	2	2	2	2
n	0	1	1	1	2	2	2	2	2	2	2
t	0	1	1	1	2	2	3	3	3	3	3
a	0	1	1	1	2	2	3	3	3	4	4
g	0	1	1	1	2	2	3	3	3	4	4
e	0	1	1	1	2	2	3	3	3	4	4

实现：迭代（动态规划)版

```
unsigned int lcs(char const * A, int n, char const * B, int m) {
   if (n < m) { swap(A, B); swap(n, m); } //make sure m <= n
   unsigned int* lcs1 = new unsigned int[m+1]; //the current two rows are
   unsigned int* lcs2 = new unsigned int[m+1]; //buffered alternatively
   memset(lcs1, 0x00, sizeof(unsigned int) * (m+1));
   memset(lcs2, 0x00, sizeof(unsigned int) * (m+1));
   for (int i = 0; i < n; swap(lcs1, lcs2), i++)
      for (int j = 0; j < m; j++)
         lcs2[j+1] = (A[i] == B[j]) ? 1 + lcs1[j] : max(lcs2[j], lcs1[j+1]);
   unsigned int solu = lcs1[m]; delete[] lcs1; delete[] lcs2; return solu;
}
```