08-A4

高级搜索树

伸展树：分摊分析

所谓物价，其实就是我称之为生命的那部分，必须在交换时支付：要么立即支付，要么以后支付

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S的势能

- （任何时刻的）任何一棵伸展树 S，都可以**假想地**被认为具有势能：

$$\Phi(S) = \log\left(\prod_{v\in S} size(v)\right) = \sum_{v\in S} \log\left(size(v)\right) = \sum_{v\in S} rank(v) = \sum_{v\in S} \log V$$

- 直觉：越**平衡**/**倾侧**的树，势能越**小**/**大**

  - **单链**： $\Phi(S) = \log n! = \mathcal{O}(n\log n)$

  - **满树**： $\Phi(S) = \log\prod_{d=0}^{h}(2^{h-d+1}-1)^{2^d} \le \log\prod_{d=0}^{h}(2^{h-d+1})^{2^d}$

$$= \log\prod_{d=0}^{h} 2^{(h-d+1)\cdot 2^d} = \sum_{d=0}^{h}(h-d+1)\cdot 2^d = (h+1)\cdot\sum_{d=0}^{h}2^d - \sum_{d=0}^{h} d\cdot 2^d$$

$$= (h+1)\cdot(2^{h+1}-1) - [(h-1)\cdot 2^{h+1} + 2] = 2^{h+2} - h - 3 = \mathcal{O}(n)$$

T的上界

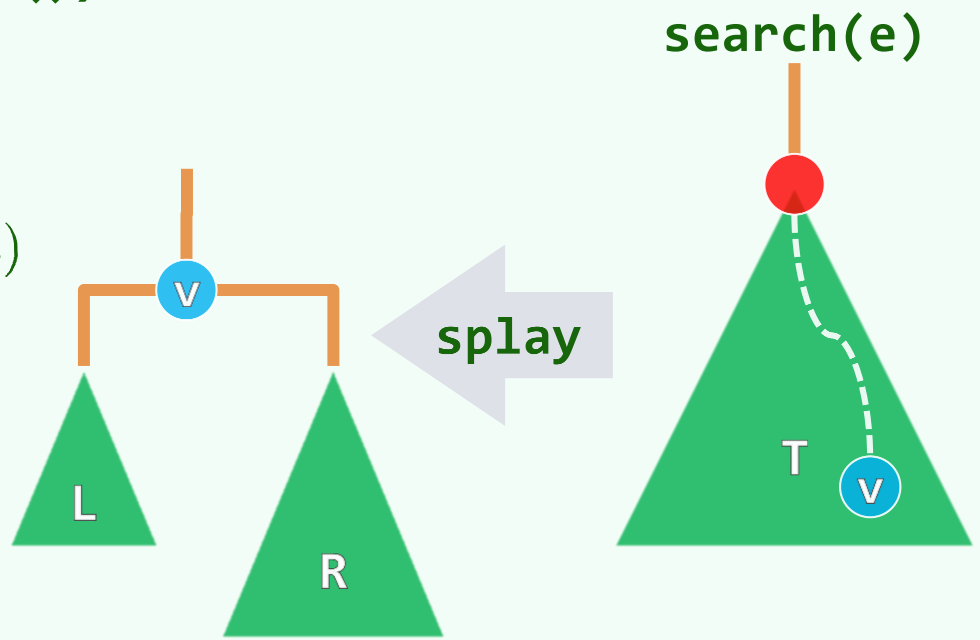
- 考查对伸展树 S 的 $m \gg n$ 次连续访问（不妨仅考查search()）
- 若记：$A^{(k)} = T^{(k)} + \Delta\Phi^{(k)},\ k = 0, 1, 2, \dots, m$

  则有：$A - \mathcal{O}(n\log n) \le \underline{T = A - \Delta\Phi} \le A + \mathcal{O}(n\log n)$
- 故若能证明：$A = \mathcal{O}(m\log n)$

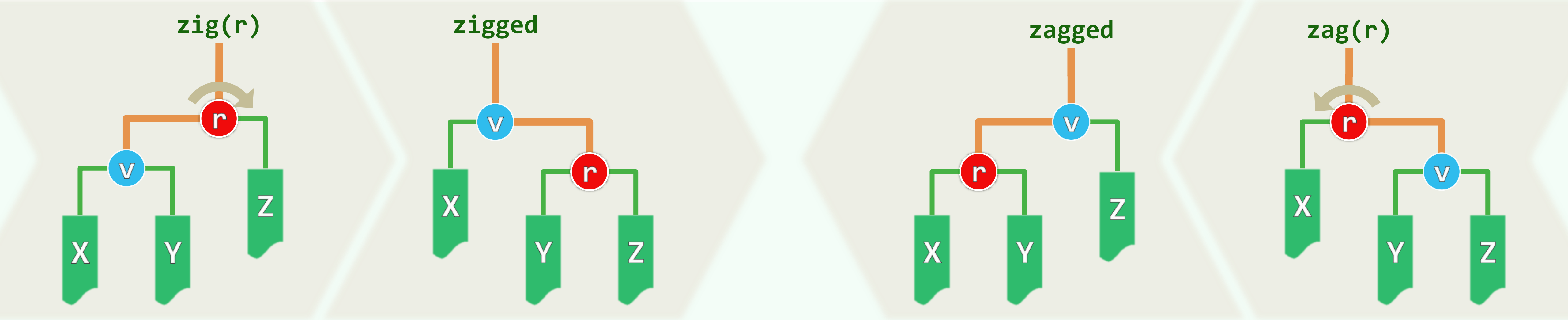
  则必有：$T = \mathcal{O}(m\log n)$
- 好消息是，尽管 $T^{(k)}$ 的变化幅度可能很大，我们却能证明：

$A^{(k)}$ 都不致超过节点v的势能变化量，即：$\mathcal{O}(\ rank^{(k)}(v) - rank^{(k-1)}(v)\ ) = \mathcal{O}(\log n)$

- 事实上，$A^{(k)}$ 不过是v的若干次连续伸展操作（时间成本）的累积，这些操作无非三种情况...



Zig / Zag

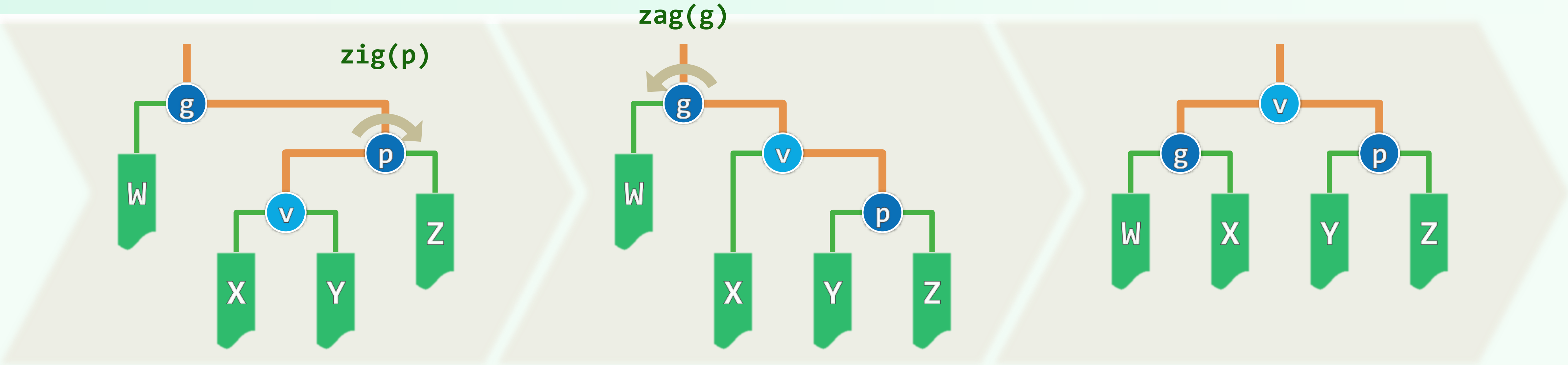


$$A_i^{(k)} = T_i^{(k)} + \Delta\Phi(S_i^{(k)}) = 1 + \Delta rank_i(v) + \Delta rank_i(r)$$

$$= 1 + [rank_i(v) - rank_{i-1}(v)] + [\underline{rank_i(r) - rank_{i-1}(r)}]$$

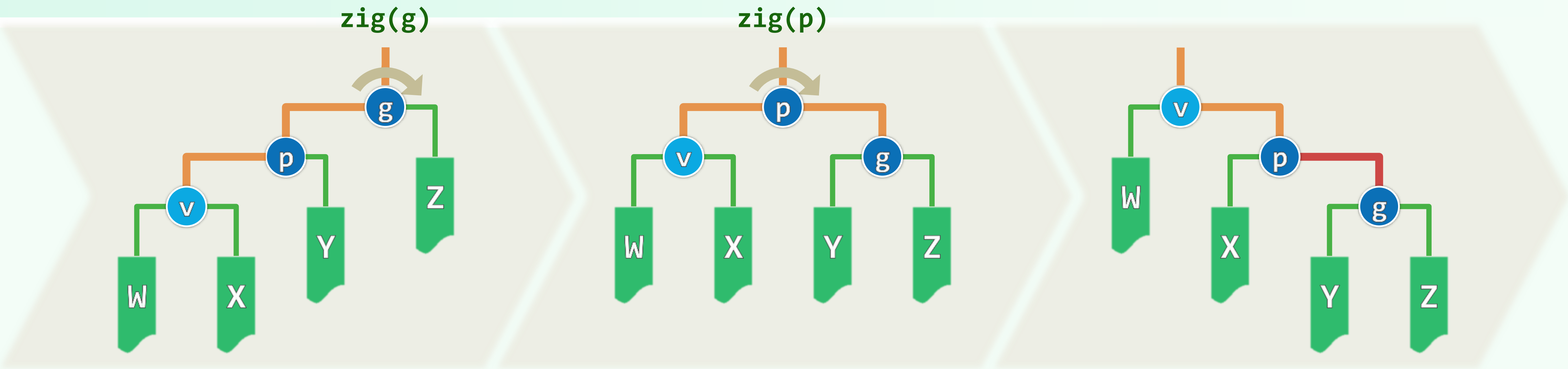
$$< 1 + [rank_i(v) - rank_{i-1}(v)]$$

zig-zag / zag-zig



$$
\begin{aligned}
A_i^{(k)} &= T_i^{(k)} + \Delta\Phi(S_i^{(k)}) = 2 + \Delta rank_i(g) + \Delta rank_i(p) + \Delta rank_i(v) \\
&= 2 + [rank_i(g) - \cancel{rank_{i-1}(g)}] + [rank_i(p) - \underline{rank_{i-1}(p)}] + [\cancel{rank_i(v)} - \underline{rank_{i-1}(v)}] \\
&< 2 + \underline{rank_i(g) + rank_i(p)} - 2 \cdot rank_{i-1}(v) \qquad (\because\ rank_{i-1}(p) > rank_{i-1}(v)\ ) \\
&< \cancel{2} + \underline{2 \cdot rank_i(v) - \cancel{2}} - 2 \cdot rank_{i-1}(v) \qquad (\because\ \frac{\log G_i + \log P_i}{2} \le \log \frac{G_i + P_i}{2} < \log \frac{V_i}{2}\ ) \\
&= 2 \cdot (rank_i(v) - rank_{i-1}(v))
\end{aligned}
$$

zig-zig / zag-zag



$$A_i^{(k)} = T_i^{(k)} + \Delta\Phi(S_i^{(k)}) = 2 + \Delta rank_i(g) + \Delta rank_i(p) + \Delta rank_i(v)$$

$$= 2 + [rank_i(g) - \cancel{rank_{i-1}(g)}] + [rank_i(p) - \underline{rank_{i-1}(p)}] + [\cancel{rank_i(v)} - \underline{rank_{i-1}(v)}]$$

$$< 2 + rank_i(g) + \underline{rank_i(p)} - 2 \cdot rank_{i-1}(v) \qquad (\because\ rank_{i-1}(p) > rank_{i-1}(v)\ )$$

$$< 2 + rank_i(g) + \underline{rank_i(v)} - 2 \cdot rank_{i-1}(v) \qquad (\because\ rank_i(p) < rank_i(v)\ )$$

$$< 3 \cdot (rank_i(v) - rank_{i-1}(v)) \qquad (\because\ \frac{\log G_i + \log V_{i-1}}{2} \leq \log \frac{G_i + V_{i-1}}{2} < \log \frac{V_i}{2}\ )$$