MATH 458 Project #2

Due: November 14, 2019

Objective

As we have discussed during the lecture, the Newton's method for solving system of nonlinear equations converges quadratically when the initial guess x_0 is close to a solution x^*. In practice, when the initial solution is very far away from a solution, the behavior of the algorithm is more difficult to predict. In this project, we examine the behavior of the Newton's method is finding the roots of the following polynomial equation in complex plane: $z^4 - 1 = 0$. This equation can be written as a system of two nonlinear equations of two real-valued variables x, y where $z = x + iy$. The solutions of the equation are $1, -1, i, -i$ or $(x, y) = (1,0), (-1,0), (0,1)$ and $(0, -1)$.

Numerical Experiment

You need to create an implementation of the Newton's method and apply the algorithm for solving the equation $z^4 - 1 = 0$. Once you are convinced that the algorithm is working as intended, you should execute the code with a large number of starting point $(x_{0,k}, y_{0,j}) = (k/N, j/N)$ for $k, j = -2N, \cdots, 2N$. If the approximated solution given by the Newton's method converges to i, we can color the point $(x_{0,k}, y_{0,j})$ red. If the approximated solution given by the Newton's method converges to $-i$, we can color the point $(x_{0,k}, y_{0,j})$ blue. You can choose two more colors when the solutions converge to 1 or -1. This should give some quite interesting patterns.

One could use similar coloring method to show different aspects of the algorithm. For an example, if you record the number of iterations required for an approximated solution to be within ε of one of the 4 possible solutions starting from $(x_{0,k}, y_{0,j})$, we could generate an image for the number of iterations required to converge from different points in the domain $[-2,2] \times [-2,1]$.

Questions you may want to investigate

- Around each solution, there should be a region with the property that once an approximate solution is inside this region, the convergence rate is becoming quadratic. How should you characterize quadratic convergence? How should you determine the size of the quadratic convergence region?
- If starting from two nearby points, the Newton's method leads to two very different solutions, it may mean that the first several steps of the Newton's method are too large. If you limit the step-size of the Newton's method, that is

$$z_k = z_{k-1} - t_k \left(\nabla f(z_{k-1}) \right)^{-1} f(z_{k-1})$$

where t_k is selected so that $|z_k - z_{k-1}| \leq \delta$ for some selected value of δ, would the fractal image we created with the Newton's method change?

- Does the Newton's method always converge with the exception of starting from $z = 0$?