

# R Lab for Regression Analysis

*Young-geun Kim*

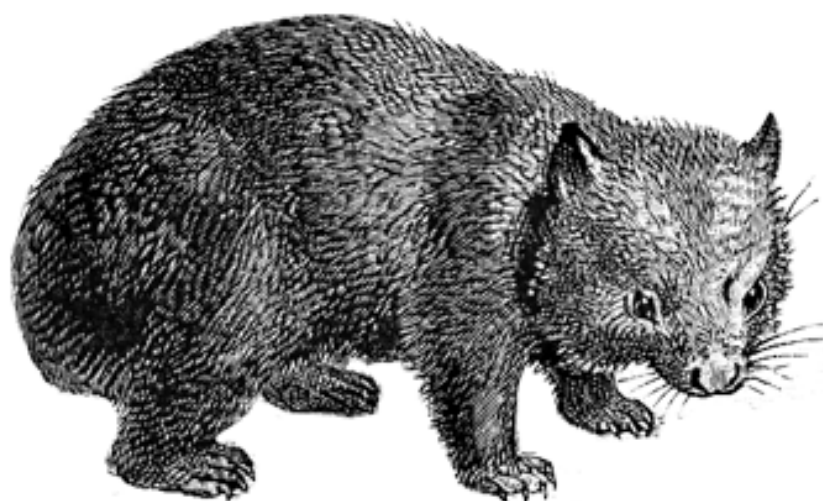
*Department of Statistics, SKKU*

*dudrms33@g.skku.edu*

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*YOUNG-GEUN statistics*



# Regression Analysis

*R Lab*

O RLY?

*Young-geun Kim*

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# Welcome

This book aims at covering materials of regression analysis. Also, there will be R programming for regression.



# Linear Regression Analysis

```
data(BioOxyDemand, package = "MPV")
(BioOxyDemand <-
  BioOxyDemand %>%
  tbl_df())
```

```
# A tibble: 14 x 2
```

	x	y
	<int>	<int>
1	3	4
2	8	7
3	10	8
4	11	8
5	13	10
6	16	11
7	27	16
8	30	26
9	35	21
10	37	9
11	38	31
12	44	30
13	103	75
14	142	90

## 0.1 Relation

We wonder how  $x$  affects  $y$ , especially linearly.

- Functional relation: mathematical equation,

$$y = \beta_0 + \beta_1 x$$

- Statistical relation: embedded with noise

So we try to estimate

$$y = \beta_0 + \beta_1 x + \epsilon$$

```
BioOxyDemand %>%
  ggplot(aes(x, y)) +
  geom_point()
```



Looking just with the eyes, we can see the linear relationship. Regression analysis estimates the relationship statistically.

```
BioOxyDemand %>%  
  ggplot(aes(x, y)) +  
  geom_smooth(method = "lm") +  
  geom_point()
```





# Simple Linear Regression

## 0.2 Model

```
delv <- MPV::p2.9 %>% tbl_df()
```

```
delv %>%  
  ggplot(aes(x = x, y = y)) +  
  geom_point() +  
  labs(  
    x = "Number of Cases",  
    y = "Delivery Time"  
  )
```

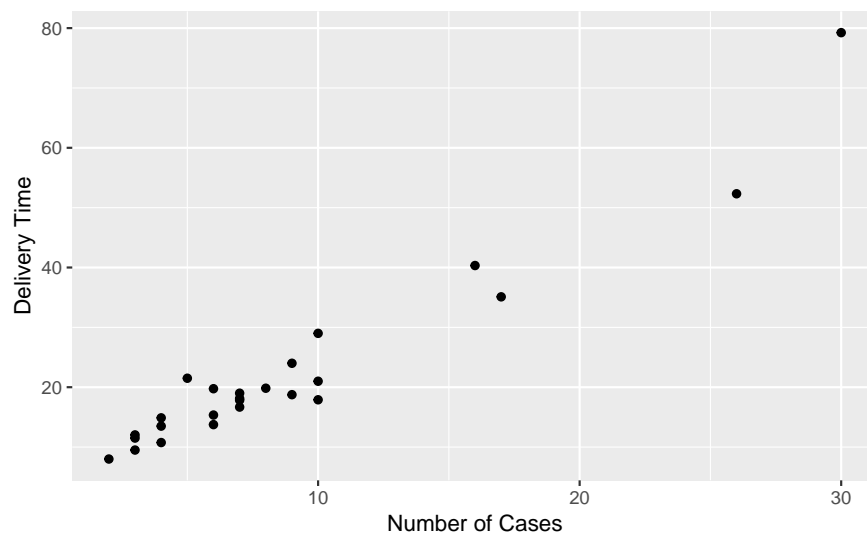


Figure 1: The Delivery Time Data

Given data  $(x_1, Y_1), \dots, (x_n, Y_n)$ , we try to fit linear model

$$Y_i = \beta_0 + \beta_1 x_i + \epsilon_i$$

Here  $\epsilon_i$  is a error term, which is a random variable.

$$\epsilon \stackrel{iid}{\sim} (0, \sigma^2)$$

It gives the problem of estimating three parameters  $(\beta_0, \beta_1, \sigma^2)$ . Before estimating these, we set some assumptions.

1. linear relationship
2.  $\epsilon_i$ s are independent
3.  $\epsilon_i$ s are identically distributed, i.e. *constant variance*
4. In some setting,  $\epsilon_i \sim N$

### 0.3 Least Squares Estimation

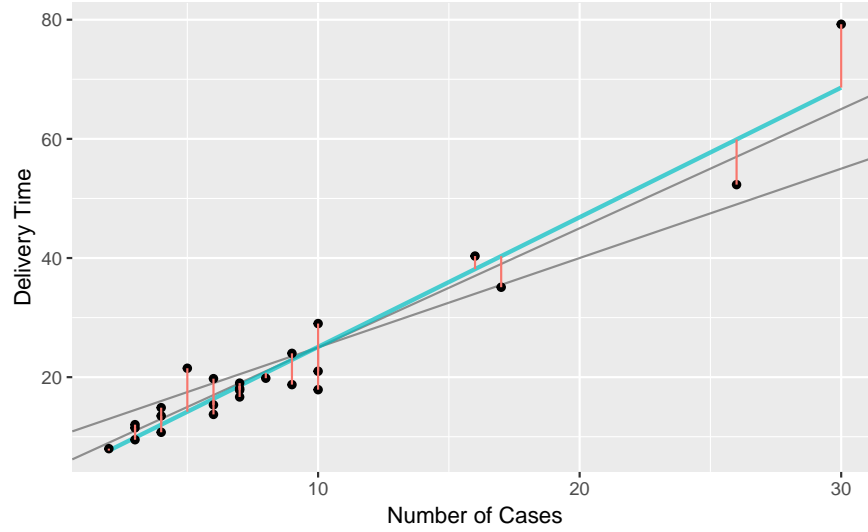


Figure 2: Idea of the least square estimation

We try to find  $\beta_0$  and  $\beta_1$  that minimize the sum of squares of the vertical distances, i.e.

$$(\beta_0, \beta_1) = \arg \min \sum_{i=1}^n (Y_i - \beta_0 - \beta_1 x_i)^2 \quad (1)$$

#### 0.3.1 Normal equations

Denote that Equation (1) is quadratic. Then we can find its minimum by find the zero point of the first derivative. Set

$$Q(\beta_0, \beta_1) := \sum_{i=1}^n (Y_i - \beta_0 - \beta_1 x_i)^2$$

Then we have

$$\frac{\partial Q}{\partial \beta_0} = -2 \sum_{i=1}^n (Y_i - \beta_0 - \beta_1 x_i) = 0 \quad (2)$$

and

$$\frac{\partial Q}{\partial \beta_1} = -2 \sum_{i=1}^n (Y_i - \beta_0 - \beta_1 x_i) x_i = 0 \quad (3)$$

From (2),

$$\sum_{i=1}^n Y_i - n\hat{\beta}_0 - \hat{\beta}_1 \sum_{i=1}^n x_i = 0$$

Thus,

$$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{x}$$

(3) gives

$$\sum_{i=1}^n x_i (Y_i - \bar{Y} + \hat{\beta}_1 \bar{x} - \hat{\beta}_1 x_i) = \sum_{i=1}^n x_i (Y_i - \bar{Y}) - \hat{\beta}_1 \sum_{i=1}^n x_i (x_i - \bar{x}) = 0$$

Thus,

$$\hat{\beta}_1 = \frac{\sum_{i=1}^n x_i (Y_i - \bar{Y})}{\sum_{i=1}^n x_i (x_i - \bar{x})}$$

*Remark.*

$$\hat{\beta}_1 = \frac{S_{XY}}{S_{XX}}$$

where  $S_{XX} := \sum_{i=1}^n (x_i - \bar{x})^2$  and  $S_{XY} := \sum_{i=1}^n (x_i - \bar{x})(Y_i - \bar{Y})$

*Proof.* Note that  $\bar{x}^2 = \frac{1}{n^2} \left( \sum_{i=1}^n x_i \right)^2$ . Then we have

$$\begin{aligned} S_{XX} &= \sum_{i=1}^n (x_i - \bar{x})^2 \\ &= \sum_{i=1}^n x_i^2 - 2 \sum_{i=1}^n x_i \bar{x} + \sum_{i=1}^n \bar{x}^2 \\ &= \sum_{i=1}^n x_i^2 - \frac{2}{n} \left( \sum_{i=1}^n x_i \right)^2 + \frac{1}{n} \left( \sum_{i=1}^n x_i \right)^2 \\ &= \sum_{i=1}^n x_i^2 - \frac{1}{n} \left( \sum_{i=1}^n x_i \right)^2 \end{aligned} \quad (4)$$

It follows that

$$\begin{aligned}
\hat{\beta}_1 &= \frac{\sum x_i(Y_i - \bar{Y})}{\sum x_i(x_i - \bar{x})} \\
&= \frac{\sum x_i(Y_i - \bar{Y}) - \bar{x} \sum (Y_i - \bar{Y})}{\sum x_i^2 - \frac{1}{n}(\sum x_i)^2} \quad \because \sum (Y_i - \bar{Y}) = 0 \\
&= \frac{\sum (x_i - \bar{x})(Y_i - \bar{Y})}{\sum x_i^2 - \frac{1}{n}(\sum x_i)^2} \\
&= \frac{S_{XY}}{S_{XX}}
\end{aligned}$$

□

```
lm(y ~ x, data = delv)
```

Call:

```
lm(formula = y ~ x, data = delv)
```

Coefficients:

(Intercept)	x
3.32	2.18

### 0.3.2 Prediction and Mean response

“Essentially, all models are wrong, but some are useful.”

—George Box

Recall that we have assumed the **linear assumption** between the predictor and the response variables, i.e. the true model. Estimating  $\beta_0$  and  $\beta_1$  is same as estimating the *assumed true model*.

**Definition 0.1** (Mean response).

$$E(Y \mid X = x) = \beta_0 + \beta_1 x$$

We can estimate this mean response by

$$\widehat{E(Y \mid x)} = \hat{\beta}_0 + \hat{\beta}_1 x \quad (5)$$

However, in practice, the model might not be true, which is included in  $\epsilon$  term.

$$Y_i = \beta_0 + \beta_1 x_i + \epsilon_i$$

Our real problem is predicting individual  $Y$ , not the mean. The *prediction* of response can be done by

$$\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i \quad (6)$$

Observe that the values of Equation (5) and (6) are same. However, due to the **error term in the prediction**, it has larger standard error.

### 0.3.3 Properties of LSE

Parameters  $\beta_0$  and  $\beta_1$  have some properties related to the expectation and variance. We can notice that these lse's are **unbiased linear estimator**. In fact, these are the *best unbiased linear estimator*. This will be covered in the Gauss-Markov theorem.

**Lemma 0.1.**

$$S_{XX} = \sum_{i=1}^n x_i^2 - \frac{1}{n} \left( \sum_{i=1}^n x_i \right)^2 = \sum_{i=1}^n x_i (x_i - \bar{x})$$

$$S_{XY} = \sum_{i=1}^n x_i Y_i - \frac{1}{n} \left( \sum_{i=1}^n x_i \right) \left( \sum_{i=1}^n Y_i \right) = \sum_{i=1}^n Y_i (x_i - \bar{x})$$

*Proof.* We already proven the first part of  $S_{XX}$ . See the Equation (4). The second part is tivial. Since  $\sum (x_i - \bar{x}) = 0$ ,

$$S_{XX} = \sum_{i=1}^n (x_i - \bar{x})^2 = \sum_{i=1}^n (x_i - \bar{x}) x_i$$

For the first part of  $S_{XY}$ ,

$$\begin{aligned} S_{XY} &= \sum_{i=1}^n (x_i - \bar{x})(Y_i - \bar{Y}) \\ &= \sum_{i=1}^n x_i Y_i - \bar{x} \sum_{i=1}^n Y_i - \bar{Y} \sum_{i=1}^n x_i + n \bar{x} \bar{Y} \\ &= \sum_{i=1}^n x_i Y_i - \frac{1}{n} \left( \sum_{i=1}^n x_i \right) \left( \sum_{i=1}^n Y_i \right) \end{aligned}$$

Second part of  $S_{XY}$  also can be proven from the definition.

$$\begin{aligned} S_{XY} &= \sum_{i=1}^n (x_i - \bar{x})(Y_i - \bar{Y}) \\ &= \sum_{i=1}^n Y_i (x_i - \bar{x}) - \bar{Y} \sum_{i=1}^n (x_i - \bar{x}) \\ &= \sum_{i=1}^n Y_i (x_i - \bar{x}) \quad \because \sum_{i=1}^n (x_i - \bar{x}) = 0 \end{aligned}$$

□

**Lemma 0.2** (Linearity). *Each coefficient is a linear estimator.*

$$\hat{\beta}_1 = \sum_{i=1}^n \frac{(x_i - \bar{x})}{S_{XX}} Y_i$$

$$\hat{\beta}_0 = \sum_{i=1}^n \left( \frac{1}{n} - \frac{(x_i - \bar{x})}{S_{XX}} \right) Y_i$$

*Proof.* From lemma 0.1,

$$\begin{aligned}\hat{\beta}_1 &= \frac{S_{XY}}{S_{XX}} \\ &= \frac{1}{S_{XX}} \sum_{i=1}^n (x_i - \bar{x}) Y_i\end{aligned}$$

It gives that

$$\begin{aligned}\hat{\beta}_0 &= \bar{Y} - \hat{\beta}_1 \bar{x} \\ &= \frac{1}{n} \sum_{i=1}^n Y_i - \bar{x} \sum_{i=1}^n \frac{(x_i - \bar{x})}{S_{XX}} Y_i \\ &= \sum_{i=1}^n \left( \frac{1}{n} - \frac{(x_i - \bar{x}) \bar{x}}{S_{XX}} \right) Y_i\end{aligned}$$

□

**Proposition 0.1** (Unbiasedness). *Both coefficients are unbiased.*

(a)  $E\hat{\beta}_1 = \beta_1$

(b)  $E\hat{\beta}_0 = \beta_0$

From the model,  $Y_1, \dots, Y_n \stackrel{indep}{\sim} (\beta_0 + \beta_1 x_i, \sigma^2)$ .

*Proof.* From lemma 0.1,

$$\begin{aligned}E\hat{\beta}_1 &= \sum_{i=1}^n \left[ \frac{(x_i - \bar{x})}{S_{XX}} E(Y_i) \right] \\ &= \sum_{i=1}^n \frac{(x_i - \bar{x})}{S_{XX}} (\beta_0 + \beta_1 x_i) \\ &= \frac{\beta_1 \sum (x_i - \bar{x}) x_i}{\sum (x_i - \bar{x}) x_i} \quad \because \sum (x_i - \bar{x}) = 0 \\ &= \beta_1\end{aligned}$$

It follows that

$$\begin{aligned}E\hat{\beta}_0 &= E(\bar{Y} - \hat{\beta}_1 \bar{x}) \\ &= E(\bar{Y}) - \bar{x} E(\hat{\beta}_1) \\ &= E(\beta_0 + \beta_1 \bar{x} + \bar{\epsilon}) - \beta_1 \bar{x} \\ &= \beta_0 + \beta_1 \bar{x} - \beta_1 \bar{x} \\ &= \beta_0\end{aligned}$$

□

**Proposition 0.2** (Variances). *Variances and covariance of coefficients*

$$(a) \text{Var} \hat{\beta}_1 = \frac{\sigma^2}{S_{XX}}$$

$$(b) \text{Var} \hat{\beta}_0 = \left( \frac{1}{n} + \frac{\bar{x}^2}{S_{XX}} \right) \sigma^2$$

$$(c) \text{Cov}(\hat{\beta}_0, \hat{\beta}_1) = -\frac{\bar{x}}{S_{XX}} \sigma^2$$

*Proof.* Proving is just arithmetic.

(a)

$$\begin{aligned} \text{Var} \hat{\beta}_1 &= \frac{1}{S_{XX}^2} \sum_{i=1}^n \left[ (x_i - \bar{x})^2 \text{Var}(Y_i) \right] + \frac{1}{S_{XX}^2} \sum_{j \neq k}^n \left[ (x_j - \bar{x})(x_k - \bar{x}) \text{Cov}(Y_j, Y_k) \right] \\ &= \frac{\sigma^2}{S_{XX}} \quad \because \text{Cov}(Y_j, Y_k) = 0 \text{ if } j \neq k \end{aligned}$$

(b)

$$\begin{aligned} \text{Var} \hat{\beta}_0 &= \sum_{i=1}^n \left( \frac{1}{n} - \frac{(x_i - \bar{x})\bar{x}}{S_{XX}} \right)^2 \text{Var}(Y_i) + \sum_{j \neq k} \left( \frac{1}{n} - \frac{(x_j - \bar{x})\bar{x}}{S_{XX}} \right) \left( \frac{1}{n} - \frac{(x_k - \bar{x})\bar{x}}{S_{XX}} \right) \text{Cov}(Y_j, Y_k) \\ &= \frac{\sigma^2}{n} - 2\sigma^2 \frac{\bar{x}}{S_{XX}} \sum_{i=1}^n (x_i - \bar{x}) + \frac{\sigma^2 \bar{x}^2 \sum_{i=1}^n (x_i - \bar{x})^2}{S_{XX}^2} \\ &= \left( \frac{1}{n} + \frac{\bar{x}^2}{S_{XX}} \right) \sigma^2 \quad \because \sum (x_i - \bar{x}) = 0 \end{aligned}$$

(c)

$$\begin{aligned} \text{Cov}(\hat{\beta}_0, \hat{\beta}_1) &= \text{Cov}(\bar{Y} - \hat{\beta}_1 \bar{x}, \hat{\beta}_1) \\ &= -\bar{x} \text{Var} \hat{\beta}_1 \\ &= -\frac{\bar{x}}{S_{XX}} \sigma^2 \end{aligned}$$

□

### 0.3.4 Gauss-Markov Theorem

Chapter 0.3.3 shows that the  $\beta_0^{LSE}$  and  $\beta_1^{LSE}$  are the **linear unbiased estimators**. Are these good? Good compared to *what estimators*? Here we consider *linear unbiased estimator*. If variances in the proposition 0.2 are lower than any parameters in this parameter family,  $\beta_0^{LSE}$  and  $\beta_1^{LSE}$  are the **best linear unbiased estimators**.

**Theorem 0.1** (Gauss Markov Theorem).  *$\hat{\beta}_0$  and  $\hat{\beta}_1$  are BLUE, i.e. the best linear unbiased estimator.*

$$\text{Var}(\hat{\beta}_0) \leq \text{Var}\left(\sum_{i=1}^n a_i Y_i\right) \forall a_i \in \mathbb{R} \text{ s.t. } E\left(\sum_{i=1}^n a_i Y_i\right) = \beta_0$$

$$\text{Var}(\hat{\beta}_1) \leq \text{Var}\left(\sum_{i=1}^n b_i Y_i\right) \forall b_i \in \mathbb{R} \text{ s.t. } E\left(\sum_{i=1}^n b_i Y_i\right) = \beta_1$$

*Bestness of beta1.* Consider  $\Theta := \left\{ \sum_{i=1}^n b_i Y_i \in \mathbb{R} : E\left(\sum_{i=1}^n b_i Y_i\right) = \beta_1 \right\}$ .

Claim:  $Var(\sum b_i Y_i) - Var(\hat{\beta}_1) \geq 0$

Let  $\sum b_i Y_i \in \Theta$ . Then  $E(\sum b_i Y_i) = \beta_1$ .

Since  $E(Y_i) = \beta_0 + \beta_1 x_i$ ,

$$\beta_0 \sum b_i + \beta_1 \sum b_i x_i = \beta_1$$

It gives

$$\begin{cases} \sum b_i = 0 \\ \sum b_i x_i = 1 \end{cases} \quad (7)$$

Then

$$\begin{aligned} 0 &\leq Var\left(\sum b_i Y_i - \hat{\beta}_1\right) = Var\left(\sum b_i Y_i - \sum \frac{(x_i - \bar{x})}{S_{XX}} Y_i\right) \\ &\stackrel{indep}{=} \sum \left(b_i - \frac{(x_i - \bar{x})}{S_{XX}}\right)^2 \sigma^2 \\ &= \sum \left(b_i^2 - \frac{2b_i(x_i - \bar{x})}{S_{XX}} + \frac{(x_i - \bar{x})^2}{S_{XX}^2}\right) \sigma^2 \\ &= \sum b_i^2 \sigma^2 - \frac{2\sigma^2}{S_{XX}} \sum b_i x_i + \frac{2\bar{x}\sigma^2}{S_{XX}} \sum b_i + \sigma^2 \frac{\sum (x_i - \bar{x})^2}{S_{XX}^2} \\ &= \sum b_i^2 \sigma^2 - \frac{\sigma^2}{S_{XX}} \quad \because (7) \text{ and } S_{XX} = \sum (x_i - \bar{x})^2 \\ &= Var(\sum b_i Y_i) - Var(\hat{\beta}_1) \end{aligned}$$

Hence,

$$Var(\sum b_i Y_i) \geq Var(\hat{\beta}_1)$$

□

*Bestness of beta0.* Consider  $\Theta := \left\{ \sum_{i=1}^n a_i Y_i \in \mathbb{R} : E\left(\sum_{i=1}^n a_i Y_i\right) = \beta_0 \right\}$ .

Claim:  $Var(\sum a_i Y_i) - Var(\hat{\beta}_0) \geq 0$

Let  $\sum a_i Y_i \in \Theta$ . Then  $E(\sum a_i Y_i) = \beta_0$ .

Since  $E(Y_i) = \beta_0 + \beta_1 x_i$ ,

$$\beta_0 \sum a_i + \beta_1 \sum a_i x_i = \beta_0$$

It gives

$$\begin{cases} \sum a_i = 1 \\ \sum a_i x_i = 0 \end{cases} \quad (8)$$



Then

$$\begin{aligned}
0 \leq \text{Var}\left(\sum a_i Y_i - \hat{\beta}_0\right) &= \text{Var}\left[\sum a_i Y_i - \sum \left(\frac{1}{n} - \frac{(x_i - \bar{x})\bar{x}}{S_{XX}}\right) Y_i\right] \\
&= \sum \left(a_i - \frac{1}{n} + \frac{(x_i - \bar{x})\bar{x}}{S_{XX}}\right)^2 \sigma^2 \\
&= \sum \left[a_i^2 - 2a_i\left(\frac{1}{n} - \frac{(x_i - \bar{x})\bar{x}}{S_{XX}}\right) + \left(\frac{1}{n} - \frac{(x_i - \bar{x})\bar{x}}{S_{XX}}\right)^2\right] \sigma^2 \\
&= \sum a_i^2 \sigma^2 - \frac{2\sigma^2}{n} \sum a_i + \frac{2\bar{x}\sigma^2 \sum a_i x_i}{S_{XX}} - \frac{2\bar{x}^2 \sigma^2 \sum a_i}{S_{XX}} \\
&\quad + \sigma^2 \left(\frac{1}{n} - \frac{2\bar{x}}{nS_{XX}} \sum (x_i - \bar{x}) + \frac{\bar{x}^2 \sum (x_i - \bar{x})^2}{S_{XX}^2}\right) \\
&= \sum a_i^2 \sigma^2 - \frac{2\sigma^2}{n} - \frac{2\bar{x}^2 \sigma^2}{S_{XX}} \quad \because (8) \\
&\quad + \left(\frac{1}{n} + \frac{\bar{x}^2}{S_{XX}}\right) \sigma^2 \quad \because \sum (x_i - \bar{x}) = 0 \text{ and } S_{XX} := \sum (x_i - \bar{x})^2 \\
&= \sum a_i^2 \sigma^2 - \left(\frac{1}{n} + \frac{\bar{x}^2}{S_{XX}}\right) \sigma^2 \\
&= \text{Var}\left(\sum a_i Y_i\right) - \text{Var}\hat{\beta}_0
\end{aligned}$$

Hence,

$$\text{Var}\left(\sum a_i Y_i\right) \geq \text{Var}(\hat{\beta}_0)$$

□

**Example 0.1.** Show that  $\sum (Y_i - \hat{Y}_i) = 0$ ,  $\sum x_i (Y_i - \hat{Y}_i) = 0$ , and  $\sum \hat{Y}_i (Y_i - \hat{Y}_i) = 0$ .

*Solution.* Consider the two normal equations (2) and (3). Note that  $\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i$ .

From the Equation (2), we have  $\sum (Y_i - \hat{Y}_i) = 0$ .

From the Equation (3), we have  $\sum x_i (Y_i - \hat{Y}_i) = 0$ .

It follows that

$$\begin{aligned}
\sum \hat{Y}_i (Y_i - \hat{Y}_i) &= \sum (\hat{\beta}_0 + \hat{\beta}_1 x_i) (Y_i - \hat{Y}_i) \\
&= \hat{\beta}_0 \sum (Y_i - \hat{Y}_i) + \hat{\beta}_1 \sum x_i (Y_i - \hat{Y}_i) \\
&= 0
\end{aligned}$$

### 0.3.5 Estimation of $\sigma^2$

There is the last parameter,  $\sigma^2 = \text{Var}(Y_i)$ . In the *least squares estimation literary*, we estimate  $\sigma^2$  by

$$\hat{\sigma}^2 = \frac{1}{n-2} \sum_{i=1}^n (Y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2 \quad (9)$$

Why  $n-2$ ? This makes the estimator unbiased.

**Proposition 0.3** (Unbiasedness).

$$E(\hat{\sigma}^2) = \sigma^2$$

*Proof.* Note that

$$(Y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) = (Y_i - \bar{Y}) - \hat{\beta}_1 (x_i - \bar{x})$$

Then

$$\begin{aligned} E(\hat{\sigma}^2) &= \frac{1}{n-2} E \left[ \sum (Y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2 \right] \\ &= \frac{1}{n-2} E \left[ \sum (Y_i - \bar{Y})^2 + \hat{\beta}_1^2 \sum (x_i - \bar{x})^2 - 2\hat{\beta}_1 \sum (Y_i - \bar{Y})(x_i - \bar{x}) \right] \\ &= \frac{1}{n-2} E(S_{YY} + \hat{\beta}_1^2 S_{XX} - 2\hat{\beta}_1 S_{XY}) \\ &= \frac{1}{n-2} E(S_{YY} - \hat{\beta}_1^2 S_{XX}) \quad \because S_{XY} = \hat{\beta}_1 S_{XX} \\ &= \frac{1}{n-2} \left( \underbrace{E S_{YY}}_{(a)} - S_{XX} \underbrace{E \hat{\beta}_1^2}_{(b)} \right) \end{aligned}$$

(a)

$$\begin{aligned} E S_{YY} &= E \left[ \sum (Y_i - \bar{Y})^2 \right] \\ &= E \left[ \sum \left( (\beta_0 + \beta_1 x_i + \epsilon_i) - (\beta_0 + \beta_1 \bar{x} + \bar{\epsilon}) \right)^2 \right] \\ &= E \left[ \sum \left( \beta_1 (x_i - \bar{x}) + (\epsilon_i - \bar{\epsilon}) \right)^2 \right] \\ &= \beta_1^2 S_{XX} + E \left( \sum (\epsilon_i - \bar{\epsilon})^2 \right) + 2\beta_1 \sum (x_i - \bar{x}) E(\epsilon_i - \bar{\epsilon}) \\ &= \beta_1^2 S_{XX} + E \left( \sum (\epsilon_i - \bar{\epsilon})^2 \right) \end{aligned}$$

Since  $E(\bar{\epsilon}) = 0$  and  $Var(\bar{\epsilon}) = \frac{\sigma^2}{n}$ ,

$$\begin{aligned} E \left( \sum (\epsilon_i - \bar{\epsilon})^2 \right) &= E \left( \sum (\epsilon_i^2 + \bar{\epsilon}^2 - 2\epsilon_i \bar{\epsilon}) \right) \\ &= \sum E(\epsilon_i^2) - nE(\bar{\epsilon}^2) \quad \because \sum \epsilon = n\bar{\epsilon} \\ &= \sum (Var(\epsilon_i) + E(\epsilon_i)^2) - n(Var(\bar{\epsilon}) + E(\bar{\epsilon})^2) \\ &= n\sigma^2 - \sigma^2 \\ &= (n-1)\sigma^2 \end{aligned}$$

Thus,

$$E S_{YY} = \beta_1^2 S_{XX} + (n-1)\sigma^2$$

(b)

$$\begin{aligned}
E\hat{\beta}_1^2 &= \text{Var}\hat{\beta}_1 + E(\hat{\beta}_1)^2 \\
&= \frac{\sigma^2}{S_{XX}} + \beta_1^2
\end{aligned}$$

It follows that

$$\begin{aligned}
E(\hat{\sigma}^2) &= \frac{1}{n-2} \left( \underbrace{ES_{YY}}_{(a)} - S_{YY} \underbrace{E\hat{\beta}_1^2}_{(b)} \right) \\
&= \frac{1}{n-2} \left( \left( \beta_1^2 S_{XX} + (n-1)\sigma^2 \right) - S_{XX} \left( \frac{\sigma^2}{S_{XX}} + \beta_1^2 \right) \right) \\
&= \frac{1}{n-2} ((n-2)\sigma^2) \\
&= \sigma^2
\end{aligned}$$

□

## 0.4 Maximum Likelihood Estimation

In this section, we add an assumption to an random errors  $\epsilon_i$ .

$$\epsilon_i \stackrel{iid}{\sim} N(0, \sigma^2)$$

**Example 0.2** (Gaussian Likelihood). Note that  $Y_i \stackrel{indep}{\sim} N(\beta_0 + \beta_1 x_i, \sigma^2)$ . Then the likelihood function is

$$L(\beta_0, \beta_1, \sigma^2) = \prod_{i=1}^n \left( \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left( -\frac{(Y_i - \beta_0 - \beta_1 x_i)^2}{2\sigma^2} \right) \right)$$

and so the log-likelihood function can be computed as

$$l(\beta_0, \beta_1, \sigma^2) = -\frac{n}{2} \ln(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^n (Y_i - \beta_0 - \beta_1 x_i)^2$$

### 0.4.1 Likelihood equations

**Definition 0.2** (Maximum Likelihood Estimator).

$$(\hat{\beta}_0^{MLE}, \hat{\beta}_1^{MLE}, \hat{\sigma}^{2MLE}) := \arg \sup L(\beta_0, \beta_1, \sigma^2)$$

Since  $l(\cdot) = \ln L(\cdot)$  is monotone,

*Remark.*

$$(\hat{\beta}_0^{MLE}, \hat{\beta}_1^{MLE}, \hat{\sigma}^{2MLE}) = \arg \sup l(\beta_0, \beta_1, \sigma^2)$$

We can find the maximum of this *quadratic* function by making first derivative.

$$\frac{\partial l}{\partial \beta_0} = \frac{1}{\sigma^2} \sum_{i=1}^n (Y_i - \beta_0 - \beta_1 x_i) = 0 \tag{10}$$

$$\frac{\partial l}{\partial \beta_1} = \frac{1}{\sigma^2} \sum_{i=1}^n x_i (Y_i - \beta_0 - \beta_1 x_i) = 0 \quad (11)$$

$$\frac{\partial l}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4} \sum_{i=1}^n (Y_i - \beta_0 - \beta_1 x_i)^2 = 0 \quad (12)$$

Denote that Equations (10) and (11) given  $\hat{\sigma}^2$  are equivalent to the normal equations. Thus,

$$\hat{\beta}_0^{MLE} = \hat{\beta}_0^{LSE}, \quad \hat{\beta}_1^{MLE} = \hat{\beta}_1^{LSE}$$

From (12),

$$\hat{\sigma}^{2MLE} = \frac{1}{n} \sum_{i=1}^n (Y_i - \beta_0 - \beta_1 x_i)^2 = \frac{n-2}{n} \hat{\sigma}^{2LSE}$$

Recall that  $\hat{\sigma}^{2LSE}$  is an unbiased, i.e. this *MLE is not an unbiased estimator*. Since  $\hat{\sigma}^{2MLE} \approx \hat{\sigma}^{2LSE}$  for large  $n$ , however, it is *asymptotically unbiased*.

**Theorem 0.2** (Rao-Cramer Lower Bound, univariate case). *Let  $X_1, \dots, X_n \stackrel{iid}{\sim} f(x; \theta)$ . If  $\hat{\theta}$  is an unbiased estimator of  $\theta$ ,*

$$Var(\hat{\theta}) \geq \frac{1}{I_n(\theta)}$$

$$\text{where } I_n(\theta) = -E\left(\frac{\partial^2 l(\theta)}{\partial \theta^2}\right)$$

To apply this theorem 0.2 in the simple linear regression setting, i.e.  $(\beta_0, \beta_1)$ , we need to look at the *bivariate case*.

**Theorem 0.3** (Rao-Cramer Lower Bound, bivariate case). *Let  $X_1, \dots, X_n \stackrel{iid}{\sim} f(x; \theta_1, \theta_2)$  and let  $\boldsymbol{\theta} = (\theta_1, \theta_2)^T$ . If each  $\hat{\theta}_1, \hat{\theta}_2$  is an unbiased estimator of  $\theta_1$  and  $\theta_2$ , then*

$$Var(\boldsymbol{\theta}) := \begin{bmatrix} Var(\hat{\theta}_1) & Cov(\hat{\theta}_1, \hat{\theta}_2) \\ Cov(\hat{\theta}_1, \hat{\theta}_2) & Var(\hat{\theta}_2) \end{bmatrix} \geq I_n^{-1}(\theta_1, \theta_2)$$

where

$$I_n(\theta_1, \theta_2) = - \begin{bmatrix} E\left(\frac{\partial^2 l(\theta_1, \theta_2)}{\partial \theta_1^2}\right) & E\left(\frac{\partial^2 l(\theta_1, \theta_2)}{\partial \theta_1 \partial \theta_2}\right) \\ E\left(\frac{\partial^2 l(\theta_1, \theta_2)}{\partial \theta_1 \partial \theta_2}\right) & E\left(\frac{\partial^2 l(\theta_1, \theta_2)}{\partial \theta_2^2}\right) \end{bmatrix}$$

Assume that  $\sigma^2$  is **known**. From the Equations (10) and (11),

$$\begin{cases} \frac{\partial^2 l}{\partial \beta_0^2} = -\frac{n}{\sigma^2} \\ \frac{\partial^2 l}{\partial \beta_1^2} = -\frac{\sum x_i^2}{\sigma^2} \\ \frac{\partial^2 l}{\partial \beta_0 \partial \beta_1} = -\frac{\sum x_i}{\sigma^2} \end{cases}$$

Thus,

$$I_n(\beta_0, \beta_1) = \begin{bmatrix} \frac{n}{\sigma^2} & \frac{\sum x_i}{\sigma^2} \\ \frac{\sum x_i}{\sigma^2} & \frac{\sum x_i^2}{\sigma^2} \end{bmatrix}$$

Applying gaussian elimination,

$$\begin{aligned} \left[ \begin{array}{cc|cc} \frac{n}{\sigma^2} & \frac{\sum x_i}{\sigma^2} & 1 & 0 \\ \frac{\sum x_i}{\sigma^2} & \frac{\sum x_i^2}{\sigma^2} & 0 & 1 \end{array} \right] &\leftrightarrow \left[ \begin{array}{cc|cc} \frac{n}{\sigma^2} & \frac{\sum x_i}{\sigma^2} & 1 & 0 \\ \frac{\sum x_i}{\sigma^2} & \frac{\sum x_i^2}{\sigma^2} & 0 & 1 \end{array} \right] \\ &\leftrightarrow \left[ \begin{array}{cc|cc} \frac{n}{\sigma^2} & \frac{\sum x_i}{\sigma^2} & 1 & 0 \\ 0 & \frac{\sum x_i^2 - \frac{(\sum x_i)^2}{n}}{\sigma^2} & -1 & \frac{1}{x} \end{array} \right] \\ &\leftrightarrow \left[ \begin{array}{cc|cc} 1 & \bar{x} & \frac{\sigma^2}{n} & 0 \\ 0 & 1 & -\frac{\bar{x}}{S_{XX}} \sigma^2 & \frac{\sigma^2}{S_{XX}} \end{array} \right] \\ &\leftrightarrow \left[ \begin{array}{cc|cc} 1 & 0 & \left( \frac{1}{n} + \frac{\bar{x}^2}{S_{XX}} \right) \sigma^2 & -\frac{\bar{x}}{S_{XX}} \sigma^2 \\ 0 & 1 & -\frac{\bar{x}}{S_{XX}} \sigma^2 & \frac{\sigma^2}{S_{XX}} \end{array} \right] \end{aligned}$$

Hence,

$$I_n^{-1}(\beta_0, \beta_1) = \begin{bmatrix} \left( \frac{1}{n} + \frac{\bar{x}^2}{S_{XX}} \right) \sigma^2 & -\frac{\bar{x}}{S_{XX}} \sigma^2 \\ -\frac{\bar{x}}{S_{XX}} \sigma^2 & \frac{\sigma^2}{S_{XX}} \end{bmatrix} = \begin{bmatrix} \text{Var}(\hat{\beta}_0) & \text{Cov}(\hat{\beta}_0, \hat{\beta}_1) \\ \text{Cov}(\hat{\beta}_0, \hat{\beta}_1) & \text{Var}(\hat{\beta}_1) \end{bmatrix}$$

Since  $\text{Var}(\hat{\beta}) - I^{-1} = 0$  is non-negative definite, each  $\text{Var}(\hat{\beta}_0) = \left( \frac{1}{n} + \frac{\bar{x}^2}{S_{XX}} \right) \sigma^2$  and  $\text{Var}(\hat{\beta}_1) = \frac{\sigma^2}{S_{XX}}$  is a theoretical bound.

*Remark.* This says that  $\hat{\beta}_0^{LSE} = \hat{\beta}_0^{MLE}$  and  $\hat{\beta}_1^{LSE} = \hat{\beta}_1^{MLE}$  have the smallest variance among all unbiased estimator.

This result is *stronger than Gauss-Markov theorem* 0.1, where the LSE has the smallest variance among all *linear unbiased* estimators. It can be simply obtained from the *Lehmann-Scheffe Theorem*: If some unbiased estimator is a function of complete sufficient statistic, then this estimator is the unique MVUE (Hogg et al., 2018).

*Remark* (Lehmann and Scheffe for regression coefficients).  $u\left(\sum Y_i, S_{XY}\right)$  is CSS in this regression problem, i.e. known  $\sigma^2$ .

*Proof.* From the example 0.2,

$$\begin{aligned} L(\beta_0, \beta_1) &= (2\pi\sigma^2)^{-\frac{n}{2}} \exp \left[ -\frac{1}{2\sigma^2} \sum (Y_i - \beta_0 - \beta_1 x_i)^2 \right] \\ &= (2\pi\sigma^2)^{-\frac{n}{2}} \exp \left[ -\frac{1}{2\sigma^2} \sum \left( Y_i^2 - (\beta_0 + \beta_1 x_i) Y_i + (\beta_0 + \beta_1 x_i)^2 \right) \right] \\ &= (2\pi\sigma^2)^{-\frac{n}{2}} \exp \left[ -\frac{1}{2\sigma^2} \left( -\beta_0 \sum Y_i - \beta_1 \sum x_i Y_i \right) \right] \exp \left[ -\frac{1}{2\sigma^2} \left( \sum Y_i^2 + (\beta_0 + \beta_1 x_i)^2 \right) \right] \end{aligned}$$

By the Factorization theorem, both  $\sum Y_i$  and  $\sum x_i Y_i$  are sufficient statistics. Since  $S_{XY}$  is one-to-one function of  $\sum x_i Y_i$ , it is also a sufficient statistic.

Denote that the normal distribution is in exponential family.

Hence,  $(\sum Y_i, S_{XY})$  are CSS. □

## 0.5 Residuals

**Definition 0.3** (Residuals).

$$e_i := Y_i - \hat{Y}_i$$

### 0.5.1 Prediction error

```
delv %>%
  mutate(yhat = predict(lm(y ~ x))) %>%
  ggplot(aes(x = x, y = y)) +
  geom_smooth(method = "lm", se = FALSE) +
  geom_point() +
  geom_linerange(aes(ymin = y, ymax = yhat), col = I("red"), alpha = .7) +
  labs(
    x = "Number of Cases",
    y = "Delivery Time"
  )
)
```

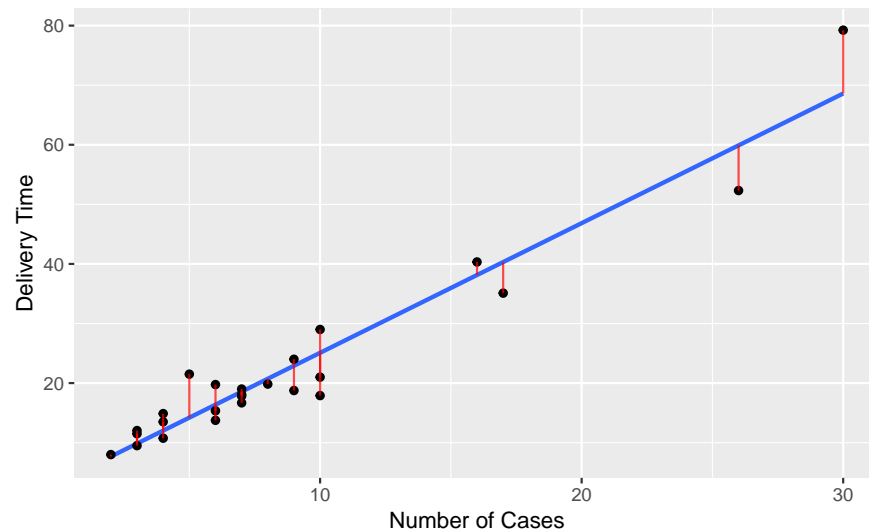


Figure 3: Fit and residuals

See Figure 3. Each red line is  $e_i$ . As we can see,  $e_i$  represents the difference between *observed* response and *predicted* response. A large  $|e_i|$  indicates a large prediction error. You can call this  $e_i$  for each  $Y_i$  by `lm()$residuals` or `residuals()`.

```
delv_fit <- lm(y ~ x, data = delv)
delv_fit$residuals
```

1	2	3	4	5	6	7	8	9	10
-1.874	1.651	2.181	2.855	-2.628	-0.444	0.327	-0.724	10.634	7.298
11	12	13	14	15	16	17	18	19	20

2.191	-4.082	1.475	3.372	1.094	3.918	-1.028	0.446	-0.349	-5.216
21	22	23	24	25					
-7.182	-7.581	-4.156	-0.900	-1.275					

$\sum e_i^2$ , which has been minimized in the procedure of LSE, can be used to see *overall size of prediction errors*.

**Definition 0.4** (Error Sums of Squares).

$$SSE := \sum_{i=1}^n e_i^2$$

### 0.5.2 Residuals and the variance

$e_i$  is a random quantity, which contains the information for  $\epsilon_i$ .  $\sum e_i^2$  can give information about  $\sigma^2 = \text{Var}(\epsilon_i)$ . For this, it is expected that  $e_i$  and  $\epsilon_i$  have similar feature.

**Lemma 0.3.** *Covariance between  $Y$  and each coefficient*

$$(a) \text{Cov}(\hat{\beta}_0, Y_i) = \left( \frac{1}{n} - \frac{(x_i - \bar{x})\bar{x}}{S_{XX}} \right) \sigma^2$$

$$(b) \text{Cov}(\hat{\beta}_1, Y_i) = \frac{(x_i - \bar{x})}{S_{XX}} \sigma^2$$

*Proof.* (a)

$$\begin{aligned} \text{Cov}(\hat{\beta}_0, Y_i) &= \text{Cov}\left(\sum a_i Y_i, Y_i\right) \\ &= \left( \frac{1}{n} - \frac{(x_i - \bar{x})\bar{x}}{S_{XX}} \right) \sigma^2 \end{aligned}$$

(b)

$$\begin{aligned} \text{Cov}(\hat{\beta}_1, Y_i) &= \text{Cov}\left(\sum b_i Y_i, Y_i\right) \\ &= \frac{(x_i - \bar{x})}{S_{XX}} \sigma^2 \end{aligned}$$

□

**Proposition 0.4** (Properties of residuals). *Mean and variance of the residual*

$$(a) E(e_i) = 0$$

$$(b) \text{Var}(e_i) \neq \sigma^2$$

$$(c) \forall i \neq j : \text{Cov}(e_i, e_j) \neq 0$$

*Proof.* (a) Recall that this is the assumption of the regression model.

(b) Lemma 0.3 implies that

$$\begin{aligned} \text{Cov}(\bar{Y}, \hat{\beta}_1) &= \text{Cov}\left(\frac{1}{n} \sum Y_i, \hat{\beta}_1\right) \\ &= \frac{1}{n} \sum_{i=1}^n \frac{(x_i - \bar{x})}{S_{XX}} \sigma^2 \\ &= 0 \quad \because \sum (x_i - \bar{x}) = 0 \end{aligned}$$

Then

$$\begin{aligned}
 Var(\hat{Y}_i) &= Var(\hat{\beta}_0 + \hat{\beta}_1 x_i) \\
 &= Var\left[\bar{Y} + (x_i - \bar{x})\hat{\beta}_1\right] \quad \because \hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{x} \\
 &= Var(\bar{Y}) + (x_i - \bar{x})^2 Var(\hat{\beta}_1) + 2(x_i - \bar{x})Cov(\bar{Y}, \hat{\beta}_1) \\
 &= \frac{\sigma^2}{n} + (x_i - \bar{x})^2 \frac{\sigma^2}{S_{XX}} + 0 \\
 &= \left(\frac{1}{n} + \frac{(x_i - \bar{x})^2}{S_{XX}}\right) \sigma^2
 \end{aligned} \tag{13}$$

From the same lemma 0.3,

$$\begin{aligned}
 Cov(Y_i, \hat{Y}_i) &= Cov(Y_i, \bar{Y} + (x_i - \bar{x})\hat{\beta}_1) \\
 &= Cov(Y_i, \bar{Y}) + (x_i - \bar{x})Cov(Y_i, \hat{\beta}_1) \\
 &= \frac{\sigma^2}{n} + \frac{(x_i - \bar{x})^2}{S_{XX}} \sigma^2 \quad \because Cov(Y_i, \hat{\beta}_1) = \frac{(x_i - \bar{x})}{S_{XX}} \sigma^2 \\
 &= \left(\frac{1}{n} + \frac{(x_i - \bar{x})^2}{S_{XX}}\right) \sigma^2
 \end{aligned} \tag{14}$$

These (13) and (14) give that

$$\begin{aligned}
 Var(e_i) &= Var(Y_i) + Var(\hat{Y}_i) - 2Cov(Y_i, \hat{Y}_i) \\
 &= \sigma^2 + \left(\frac{1}{n} + \frac{(x_i - \bar{x})^2}{S_{XX}}\right) \sigma^2 - 2\left(\frac{1}{n} + \frac{(x_i - \bar{x})^2}{S_{XX}}\right) \sigma^2 \\
 &= \left(1 - \frac{1}{n} - \frac{(x_i - \bar{x})^2}{S_{XX}}\right) \sigma^2 \\
 &\neq \sigma^2
 \end{aligned}$$

(c) Let  $i \neq j$ . Then

$$\begin{aligned}
 Cov(e_i, e_j) &= Cov\left(Y_i - (\hat{\beta}_0 + \hat{\beta}_1 x_i), Y_j - (\hat{\beta}_0 + \hat{\beta}_1 x_j)\right) \\
 &= Cov(Y_i, Y_j) - Cov\left(Y_i, (\hat{\beta}_0 + \hat{\beta}_1 x_j)\right) - Cov\left((\hat{\beta}_0 + \hat{\beta}_1 x_i), Y_j\right) + Cov\left((\hat{\beta}_0 + \hat{\beta}_1 x_i), (\hat{\beta}_0 + \hat{\beta}_1 x_j)\right) \\
 &= 0 - \left(\frac{1}{n} - \frac{(x_i - \bar{x})\bar{x}}{S_{XX}}\right) \sigma^2 - \frac{(x_i - \bar{x})x_j}{S_{XX}} \sigma^2 \\
 &\quad - \left(\frac{1}{n} - \frac{(x_j - \bar{x})\bar{x}}{S_{XX}}\right) \sigma^2 - \frac{(x_i - \bar{x})x_i}{S_{XX}} \sigma^2 \\
 &\quad + \left(\frac{1}{n} + \frac{\bar{x}^2 + x_i x_j - \bar{x}(x_i + x_j)}{S_{XX}}\right) \sigma^2 \\
 &= -\left(\frac{1}{n} + \frac{\bar{x}^2 + x_i x_j - \bar{x}(x_i + x_j)}{S_{XX}}\right) \sigma^2 \\
 &\neq 0
 \end{aligned}$$

□



## 0.6 Decomposition of Total Variability

**Definition 0.5** (Uncorrected Total Sum of Squares).

$$SST_{uncor} := \sum_{i=1}^n Y_i^2$$

**Definition 0.6** (Corrected Total Sum of Squares).

$$SST := \sum_{i=1}^n (Y_i - \bar{Y})^2$$

What does this total sum of squares mean? To know this, we should know  $\bar{Y}$  first.

```
delv %>%
  ggplot(aes(x = x, y = y)) +
  geom_smooth(method = "lm", formula = y ~ 1, se = FALSE) +
  geom_point() +
  labs(
    x = "Number of Cases",
    y = "Delivery Time"
  )
```

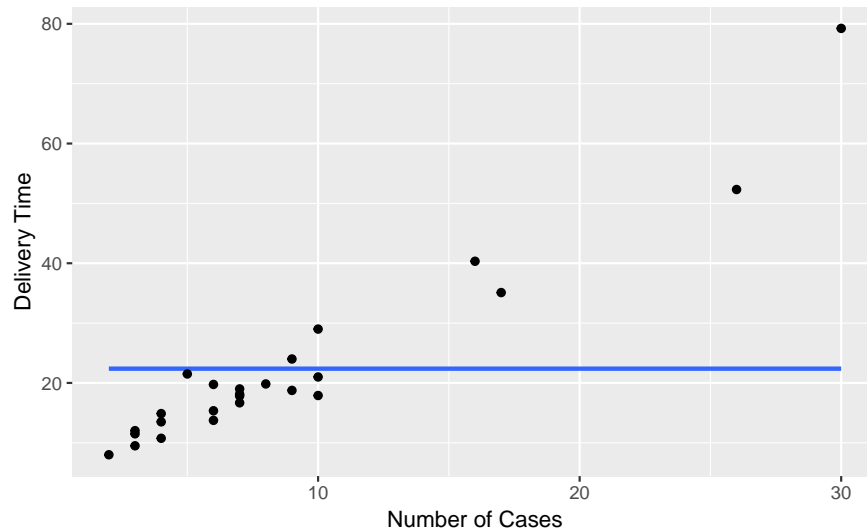


Figure 4: Regression without predictor

See Figure 4. The line represents the closest line when we use only intercept term for the regression model. In other words, *if we use no information for the response*, i.e. no predictor variables, we will get just average of the response variable. Consider

$$Y_i = \beta_0 + \epsilon_i$$

Then we can get only one normal equation

$$\sum (Y_i - \hat{\beta}_0) = 0$$

Hence,

$$\hat{\beta}_0 = \frac{1}{n} \sum_{i=1}^n Y_i \equiv \bar{Y}$$

From this fact,  $SST$  implies

**Proposition 0.5** (Decomposition of SST).

$$SST = SSR + SSE$$

where  $SSR = \sum(\hat{Y}_i - \bar{Y})^2$  and  $SSE = \sum(Y_i - \hat{Y}_i)^2$

## 0.7 Geometric Interpretations

# Bibliography

Hogg, R. V., McKean, J. W., and Craig, A. T. (2018). *Introduction to Mathematical Statistics*. Pearson College Division, 8 edition.