Global Shallow Water Spectral Model in Python

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Introduction

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Model Governing Equations

The shallow water equations (or barotropical primitive equations) over a spherical planet with topography written in the vorticity-divergence forms are:

$$\frac{\partial \zeta}{\partial t} = -\frac{1}{a(1-\mu^2)}\frac{\partial}{\partial \lambda}[U(\zeta+f)] - \frac{1}{a}\frac{\partial}{\partial \mu}[V(\zeta+f)] \tag{2.1}$$

$$\frac{\partial \delta}{\partial t} = +\frac{1}{a(1-\mu^2)}\frac{\partial}{\partial \lambda}[V(\zeta+f)] - \frac{1}{a}\frac{\partial}{\partial \mu}[U(\zeta+f)] - \nabla^2\left[\Phi' + \Phi_s + \frac{U^2+V^2}{2(1-\mu^2)}\right] \tag{2.2}$$

$$\frac{\partial \Phi'}{\partial t} = -\frac{1}{a(1-\mu^2)}\frac{\partial}{\partial \lambda}(U\Phi') - \frac{1}{a}\frac{\partial}{\partial \mu}(V\Phi') - \bar{\Phi}\delta \tag{2.3}$$

Where, a is the planet radius, λ is the longitude, and $\mu = \sin\varphi$ (where φ is the latitude); Coriolis parameter $f = 2\Omega\sin\varphi$ (where Ω is the angular velocity of the planet); latitude-scaled zonal and meridional wind components $(U, V) = (u\cos\varphi, v\cos\varphi)$; free surface geopotential $\Phi = \Phi_s + \bar{\Phi} + \Phi'$, where Φ_s is the topographic geopotential, $\bar{\Phi}$ is the mean geopotential of the atmosphere (constant), and Φ' is the geopotential perturbation; relative vorticity (ζ) and divergence (δ) are:

$$\zeta = \frac{1}{a(1-\mu^2)}\frac{\partial V}{\partial \lambda} - \frac{1}{a}\frac{\partial U}{\partial \mu}$$

$$\delta = \frac{1}{a(1-\mu^2)}\frac{\partial U}{\partial \lambda} + \frac{1}{a}\frac{\partial V}{\partial \mu}$$

Redefine the nonlinear products as: $A \equiv U(\zeta+f), B \equiv V(\zeta+f), C \equiv U\Phi', D \equiv V\Phi', E \equiv \frac{U^2+V^2}{2(1-\mu^2)}$, then the above shallow water equations (2.1), (2.2), and (2.3) become:

$$\frac{\partial \zeta}{\partial t} = -\frac{1}{a(1-\mu^2)}\frac{\partial A}{\partial \lambda} - \frac{1}{a}\frac{\partial B}{\partial \mu} \tag{2.4}$$

$$\frac{\partial \delta}{\partial t} = +\frac{1}{a(1-\mu^2)}\frac{\partial B}{\partial \lambda} - \frac{1}{a}\frac{\partial A}{\partial \mu} - \nabla^2(\Phi_s + E) - \nabla^2\Phi' \tag{2.5}$$

$$\frac{\partial \Phi'}{\partial t} = -\frac{1}{a(1-\mu^2)}\frac{\partial C}{\partial \lambda} - \frac{1}{a}\frac{\partial D}{\partial \mu} - \bar{\Phi}\delta \tag{2.6}$$

Chapter 3

Numerical Algorithms

3.1 Governing Equations in Spectral Form

Performing the spherical harmonic transforming (details can be found in appendix A) on (2.4), (2.5), and (2.6) yields to the spectral form of the governing equations:

$$\frac{\partial \zeta_n^m}{\partial t} = -\sum_{j=1}^{J}[imA^m(\mu_j)P_n^m(\mu_j) - B^m(\mu_j)H_n^m(\mu_j)]\frac{w_j}{a(1-\mu_j^2)} \tag{3.1}$$

$$\begin{aligned}\frac{\partial \delta_n^m}{\partial t} = &+\sum_{j=1}^{J}[imB^m(\mu_j)P_n^m(\mu_j) + A^m(\mu_j)H_n^m(\mu_j)]\frac{w_j}{a(1-\mu_j^2)} \\ &+ \frac{n(n+1)}{a^2}(E_n^m + \Phi_{sn}^{\ m}) + \frac{n(n+1)}{a^2}\Phi'^m_n\end{aligned} \tag{3.2}$$

$$\begin{aligned}\frac{\partial \Phi'^m_n}{\partial t} = &-\sum_{j=1}^{J}[imC^m(\mu_j)P_n^m(\mu_j) - D^m(\mu_j)H_n^m(\mu_j)]\frac{w_j}{a(1-\mu_j^2)} \\ &- \bar{\Phi}\delta_n^m\end{aligned} \tag{3.3}$$

Where, m is the zonal number, n is the total number; j is the latitude index, J is the number of Gaussian latitudes, w_j is the Gaussian weights; $P_n^m(\mu)$ is the normalized associated Legendre polynomials, its first-order derivative $H_n^m(\mu) = (1-\mu^2)\dfrac{\partial P_n^m(\mu)}{\partial \mu}$.

3.2 Semi-Implicity Time Integration

Rewrite the nonlinear terms (and topographic forcing term): the right-hand side of (3.1), the first and second term on the right-hand side of (3.2), and the first term on the right-hand side of (3.3), as $\mathcal{O}_n^m$, $\mathcal{P}_n^m$, and $\mathcal{Q}_n^m$, respectively; And then, discretize (3.1), (3.2), and (3.3) in time using centered difference for thesee nonlinear terms (slow geostrophic evolution) but time-averaged scheme for linear terms (fast geostrophic adjustment, like gravity waves):

$$\frac{(\zeta_n^m)^{\tau+1} - (\zeta_n^m)^{\tau-1}}{2\Delta t} = (\mathcal{O}_n^m)^\tau \tag{3.4}$$

$$\frac{(\delta_n^m)^{\tau+1} - (\delta_n^m)^{\tau-1}}{2\Delta t} = (\mathcal{P}_n^m)^\tau + \frac{n(n+1)}{a^2}\frac{(\Phi'^m_n)^{\tau+1} + (\Phi'^m_n)^{\tau-1}}{2} \tag{3.5}$$

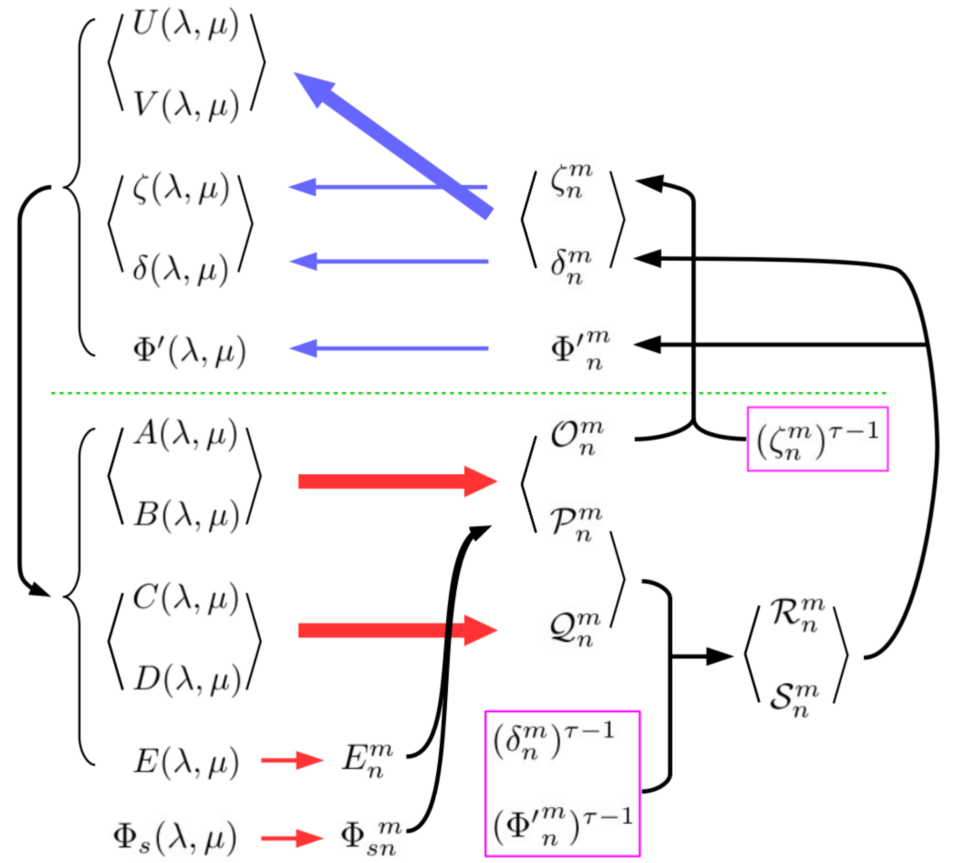

Figure 3.1: The procedure of the global shallow water spectral model. Red and blue arrows represent the spectral transforms from physical space to spectral space and from spectral space to physical space, respectively. Black flows do not contain spectral transform.

$$\frac{(\Phi'^m_n)^{\tau+1}-(\Phi'^m_n)^{\tau-1}}{2\Delta t}=(\mathcal{Q}_n^m)^{\tau}-\bar{\Phi}\frac{(\delta_n^m)^{\tau+1}+(\delta_n^m)^{\tau-1}}{2} \tag{3.6}$$

Where, the superscript $\tau-1,\tau,\tau+1$ represent the previous, current, next time step, respectively.

The solutions of the above linear algebra equations (3.4), (3.5), and (3.6) are:

$$(\zeta_n^m)^{\tau+1}=(\zeta_n^m)^{\tau-1}+2\Delta t(\mathcal{O}_n^m)^{\tau} \tag{3.7}$$

$$(\delta_n^m)^{\tau+1}=\frac{\mathcal{R}_n^m+\Delta t\frac{n(n+1)}{a^2}\mathcal{S}_n^m}{\mathcal{G}_n^m} \tag{3.8}$$

$$(\Phi'^m_n)^{\tau+1}=\frac{\mathcal{S}_n^m-\Delta t\bar{\Phi}\mathcal{R}_n^m}{\mathcal{G}_n^m} \tag{3.9}$$

Where,

$$\mathcal{R}_n^m=(\zeta_n^m)^{\tau-1}+2\Delta t(\mathcal{P}_n^m)^{\tau}+\Delta t\frac{n(n+1)}{a^2}(\Phi'^m_n)^{\tau-1}$$
$$\mathcal{S}_n^m=(\Phi'^m_n)^{\tau-1}+2\Delta t(\mathcal{Q}_n^m)^{\tau}-\Delta t\bar{\Phi}(\delta_n^m)^{\tau-1}$$
$$\mathcal{G}_n^m=1+(\Delta t)^2\frac{n(n+1)}{a^2}\bar{\Phi}$$

As a summary, the procedure diagram of this spectral model is shown in Figure 3.1.

Chapter 4

Technical Notes

Appendix A

Spectral Transform Method

A.1 Spherical harmonic transform (from physical space to spectral space)

The spherical harmonic transform contains two steps:

1) Fourier transform:

$$A^m(\mu) = \frac{1}{2\pi}\int_0^{2\pi} A(\lambda,\mu)e^{-im\lambda}d\lambda \tag{A.1}$$

2) Legendre transform:

$$A_n^m = \int_{-1}^{1} A^m(\mu)P_n^m(\mu)d\mu \tag{A.2}$$

where, $P_n^m(\mu)$ is the normalized associated Legendre polynomial.
Combining step 1) and 2), the spherical harmonic transform can be written as:

$$A_n^m = \frac{1}{2\pi}\int_{-1}^{1}\int_0^{2\pi} A(\lambda,\mu)Y_n^{m*}(\lambda,\mu)d\lambda d\mu \tag{A.3}$$

Where, $Y_n^{m*}(\lambda,\mu) = P_n^m(\mu)e^{-im\lambda}$ is the conjugate spherical harmonic function.
Practically, step 1) and 2) are performed discretely as:

$$A^m(\mu_j) = \frac{1}{2\pi}\sum_{i=1}^{I} A(\lambda_i,\mu_j)e^{-\mathbf{i}m\lambda_i} \tag{A.4}$$

$$A_n^m = \sum_{j=1}^{J} A^m(\mu_j)P_n^m(\mu_j)w(\mu_j) \tag{A.5}$$

Where, $\mathbf{i} = \sqrt{-1}$; i and j are the indexes of longitude and latitude; $w(\mu_j)$ are the Gaussian weights.

A.2 Inverse spherical harmonic transform (from spectral space to physical space)

The spherical harmonic transform also contains two steps, but in discrete form only:

1) Inverse Legendre transform:

$$A^m(\mu) = \begin{cases} \sum\limits_{n=|m|}^{N} A_n^m P_n^m(\mu) & \text{if } m \geq 0 \\ \sum\limits_{n=|m|}^{N} A_n^{m*} P_n^m(\mu) & \text{if } m < 0 \end{cases} \tag{A.6}$$

2) Inverse Fourier transform:

$$A(\lambda, \mu) = Re\left[\sum_{m=-M}^{M} A^m(\mu) e^{\mathbf{i}m\lambda}\right] \tag{A.7}$$

Where, A_n^{m*} is the conjugate of A_n^m; Re means using the real part only.

Appendix B

Spectral Transform on Global Wind Field

B.1 Finite difference form in physical space

The transform in physical space among latitude-scaled wind field pair (U,V), stream-potential function pair (ψ, χ), and vorticity-divergence pair (ζ, δ) are:

$$\begin{aligned} U &= \frac{1}{a}\frac{\partial \chi}{\partial \lambda} - \frac{1-\mu^2}{a}\frac{\partial \psi}{\partial \mu} \\ V &= \frac{1}{a}\frac{\partial \psi}{\partial \lambda} + \frac{1-\mu^2}{a}\frac{\partial \chi}{\partial \mu} \end{aligned} \tag{B.1}$$

$$\begin{aligned} \zeta &= \frac{1}{a(1-\mu^2)}\frac{\partial V}{\partial \lambda} - \frac{1}{a}\frac{\partial U}{\partial \mu} = \nabla^2 \psi \\ \delta &= \frac{1}{a(1-\mu^2)}\frac{\partial U}{\partial \lambda} + \frac{1}{a}\frac{\partial V}{\partial \mu} = \nabla^2 \chi \end{aligned} \tag{B.2}$$

Where, the Laplace operator $\nabla^2 = \dfrac{1}{a^2(1-\mu^2)}\dfrac{\partial^2}{\partial \lambda^2} + \dfrac{1}{a^2}\dfrac{\partial}{\partial \mu}\left[(1-\mu^2)\dfrac{\partial}{\partial \mu}\right]$.

B.2 Spectral transform via the spherical harmonic function

$$\begin{aligned} U^m(\mu_j) &= \frac{1}{a}\sum_{n=|m|}^{N}\left[im\chi_n^m P_n^m(\mu_j) - \psi_n^m H_n^m(\mu_j)\right] \\ V^m(\mu_j) &= \frac{1}{a}\sum_{n=|m|}^{N}\left[im\psi_n^m P_n^m(\mu_j) + \chi_n^m H_n^m(\mu_j)\right] \end{aligned} \tag{B.3}$$

$$\begin{aligned} \zeta^m(\mu_j) &= \frac{1}{a(1-\mu_j^2)}\sum_{n=|m|}^{N}\left[imV_n^m P_n^m(\mu_j) - U_n^m H_n^m(\mu_j)\right] \\ \delta^m(\mu_j) &= \frac{1}{a(1-\mu_j^2)}\sum_{n=|m|}^{N}\left[imU_n^m P_n^m(\mu_j) + V_n^m H_n^m(\mu_j)\right] \end{aligned} \tag{B.4}$$

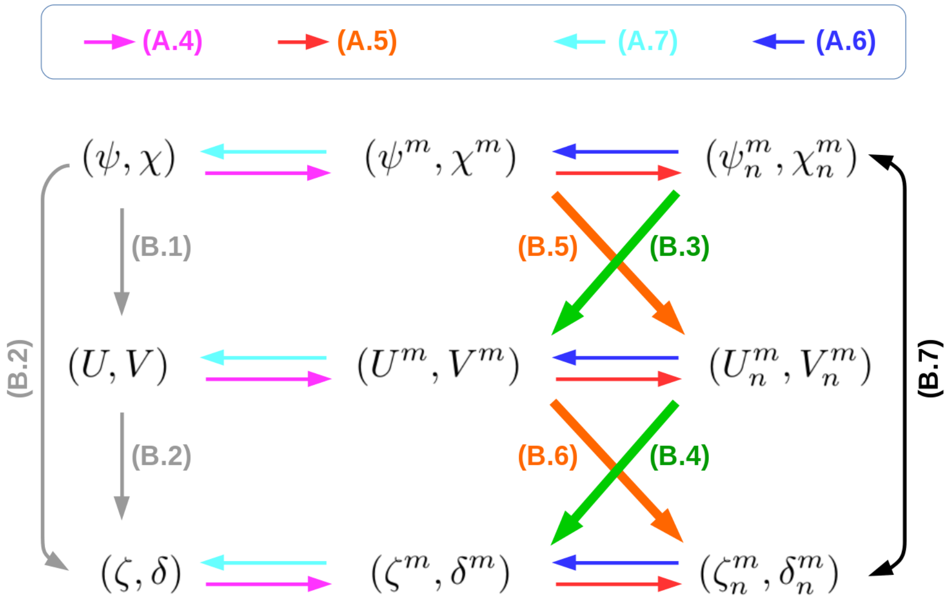

Figure B.1: The transform of global wind field. Each arrow is corresponding with an equation in the appendix.

$$\begin{aligned} U_n^m &= \frac{1}{a}\sum_{j=1}^{J}\left\{im\chi^m(\mu_j)P_n^m(\mu_j) + \psi^m(\mu_j)[H_n^m(\mu_j) - 2\mu_j P_n^m(\mu_j)]\right\} w(\mu_j) \\ V_n^m &= \frac{1}{a}\sum_{j=1}^{J}\left\{im\psi^m(\mu_j)P_n^m(\mu_j) - \chi^m(\mu_j)[H_n^m(\mu_j) - 2\mu_j P_n^m(\mu_j)]\right\} w(\mu_j) \end{aligned} \tag{B.5}$$

$$\begin{aligned} \zeta_n^m &= \frac{1}{a}\sum_{j=1}^{J}\left[imV^m(\mu_j)P_n^m(\mu_j) + U^m(\mu_j)H_n^m(\mu_j)\right]\frac{w(\mu_j)}{1-\mu_j^2} \\ \delta_n^m &= \frac{1}{a}\sum_{j=1}^{J}\left[imU^m(\mu_j)P_n^m(\mu_j) - V^m(\mu_j)H_n^m(\mu_j)\right]\frac{w(\mu_j)}{1-\mu_j^2} \end{aligned} \tag{B.6}$$

$$\begin{aligned} \zeta_n^m &= -\frac{n(n+1)}{a^2}\psi_n^m \\ \delta_n^m &= -\frac{n(n+1)}{a^2}\chi_n^m \end{aligned} \tag{B.7}$$

These spectral transforms on wind field are shown in Figure B.1, where the key wind transforms used in the shallow water model are B.3, B.6, and B.7.