3.5 Queue with Truncation (M/M/c/K)

$\not\exists$ | — limited System Size with K

$$P_n = \begin{cases} \dfrac{\lambda^n}{n!\,\mu^n} P_0 , & (0 \le n < c) \\[2ex] \dfrac{\lambda^n}{c^{n-c} c!\, \mu^n} P_0 , & (c \le n \le K) \end{cases}$$

$*\ \dfrac{\lambda}{\mu} = r ,\quad \rho = \dfrac{r}{c} = \dfrac{\lambda}{c\mu}$

$S_n = \sum_{n=1}^{n} a_1 r^n = \dfrac{a_1(1-r^n)}{1-r}$

<pf>

Known $\sum P_n = 1$

$$\Rightarrow P_0 = \left(\sum_{n=0}^{c-1} \frac{r^n}{n!} + \sum_{n=c}^{K} \frac{r^n}{c^{n-c} c!} \right)^{-1}$$

$$\Rightarrow \sum_{n=c}^{K} \frac{r^n}{c^{n-c} c!} = \sum_{n=c}^{K} \Big(\underbrace{\frac{r}{c}}_{\rho}\Big)^{n-c} \frac{r^c}{c!} \qquad S_n = \sum_{n=1}^{n} a_1 r^n = \frac{a_1(1-r^n)}{1-r}$$

$$= \frac{r^c}{c!} \sum_{n=1}^{K-c+1} \rho^{n-c+1}$$

$$= \frac{r^c}{c!} \frac{1-\rho^{K-c}}{1-\rho}$$

$$= \begin{cases} \dfrac{r^c}{c!}\left(\dfrac{1-\rho^{K-c+1}}{1-\rho}\right), & \rho \ne 1 \\[2ex] \dfrac{r^c}{c!}(K-c+1), & \rho = 1 \end{cases}$$

$$\therefore P_0 = \begin{cases} \left(\dfrac{r^c}{c!} \dfrac{1-\rho^{K-c+1}}{1-\rho} + \displaystyle\sum_{n=0}^{c-1} \frac{r^n}{n!} \right)^{-1} \\[2ex] \left(\dfrac{r^c}{c!}(K-c+1) + \displaystyle\sum_{n=0}^{c-1} \frac{r^n}{n!} \right)^{-1} \end{cases}$$

$*$ Erlang Type k :

$$E_k = \frac{k\mu(k\mu x)^{k-1}}{(k-1)!} e^{-k\mu x}$$

$$W_q(t) = \begin{cases} \displaystyle\sum_{n=0}^{c-1} q_n , & t = 0 \\[2ex] P_r\{T_q \le t\}, & t > 0 \end{cases}$$

waiting time: from n to $c-1$ $\Rightarrow E_{n-c+1}$

$\sum_{n=c}^{K-1} P_r\{ T_q \le t \mid \text{arr found } n \text{ customers in System} \} \times q_n + W_q(0)$

How ???

$$\Rightarrow \sum_{n=c}^{K-1} q_n \int_0^t \text{"}\frac{c\mu(c\mu x)^{n-c}}{(n-c)!} e^{-c\mu x}\text{"}\, dx + W_q(0)$$

$$= \sum_{n=c}^{K-1} q_n \left[1 - \int_t^\infty \frac{c\mu(c\mu x)^{n-c}}{(n-c)!} e^{-c\mu x} dx \right] + W_q(0)$$

$$\Rightarrow 1 - \sum_{n=c}^{K-1} q_n \sum_{i=0}^{n-c} \frac{(c\mu t)^i e^{-c\mu t}}{i!}$$

$$\int_t^\infty \frac{c\mu (c\mu x)^{n-c}}{(n-c)!} e^{-c\mu x}\, dx$$

$= Pr\{$ time to complete $(n-c+1)$ services $> t\}$

$= Pr\{(n-c)$ or less cust. completed in time $t\}$

$$= \sum_{i=0}^{n-c} \frac{(c\mu t)^i e^{-c\mu t}}{i!}$$

$q_n = Pr\{n$ cust. in Sys. $|$ arr. about to occur $\}$

$$= \frac{Pr\{\text{arr. about to occur} \mid n \text{ cust. in Sys.}\} \cdot P_n}{\sum_{i=0}^{K} Pr\{\text{arr about to occur} \mid i \text{ cust. in Sys.}\} \cdot P_i}$$

$$= \lim_{\Delta t \to 0} \frac{(\lambda \Delta t + o(\Delta t)) \cdot P_n}{\sum_{i=0}^{K} (\lambda \Delta t + o(\Delta t)) \cdot P_i + 0 \cdot P_K}$$

$$\frac{\Delta t}{\Delta t}) = \frac{\cancel{\lambda} P_n}{\cancel{\lambda} \sum_{i=0}^{K-1} P_i} = \frac{P_n}{1 - P_K} \qquad (n \leq K-1)$$

3.6 Erlang's Loss Formula $(M/M/c/\underset{\smile}{c})$

Special case of $M/M/c/K$ with $K=c$

limit Sys. Size

$$P_n = \frac{\frac{(\frac{\lambda}{\mu})^n}{n!}}{\sum_{i=0}^{c} \frac{(\frac{\lambda}{\mu})^i}{i!}} = \frac{\frac{r^n}{n!}}{\sum_{i=0}^{c} \frac{r^i}{i}}, \quad (0 \leq n \leq c)$$

3.7 $M/M/c/\infty$ $\Rightarrow$ impossible $\Rightarrow$ skip!

3.8 Finite-Source Queues

M machine , C repairman , Y spares

$$\lambda_n = \begin{cases} (M-n)\lambda , & 0 \le n \le M \\ 0 , & n \ge M \end{cases}$$

$$\mu_n = \begin{cases} n\mu , & 0 \le n \le c \\ c\mu , & n \ge c \end{cases}$$

$$p_n = \begin{cases} \binom{M}{n} r^n p_0 , & 1 \le n \le c \\ \binom{M}{n} \frac{n!}{c^{n-c} c!} r^n p_0 , & c \le n \le M \end{cases}$$

$$L = \sum_{n=0}^{\infty} n p_n \xrightarrow[L = \lambda' W]{\text{Little's law}} W$$

Effective arrival rate (λ' or λ_{eff})

$$W = \frac{L}{\lambda'} , \quad L_q = L - \frac{\lambda'}{\mu} , \quad W_q = \frac{L_q}{\lambda'}$$

$$\lambda' = \sum_{n=0}^{M} (M-n)\lambda p_n$$

$$= \lambda \left(M \sum_{n=0}^{M} p_n - n \sum_{n=0}^{M} p_n \right)$$

$$= \lambda (M - L)$$

$W_q(t)$

<pf>

$$W_q(t=0) = \sum_{n=0}^{c-1} q_n$$

$$W_q(t>0) = P_r\{ T_q \le t \}$$

$$= \sum_{n=c}^{Y+M-1} P_r \{ \underbrace{(n-c+1) \text{ Serv. completed in } t}_{E_{n-c+1}} \mid \text{arr found } n \text{ cust. in sys.} \}$$

$$= \sum_{n=c}^{Y+M-1} q_n \underbrace{\int_0^t \frac{c\mu (c\mu x)^{n-c}}{(n-c)!} e^{-c\mu x} dx}_{\text{to } 1 - \int_t^{\infty} \cdots} + W_q(0)$$

$$= \sum_{n=c}^{Y+M-1} q_n \left[1 - \int_t^{\infty} \frac{c\mu (c\mu x)^{n-c}}{(n-c)!} e^{-c\mu x}\, dx \right] + W_q(0)$$

$$\int_t^{\infty} \frac{c\mu (c\mu x)^{n-c}}{(n-c)!} e^{-c\mu x}\, dx$$

$= Pr\{$ time to complete $(n-c+1)$ services $> t\}$

$= Pr\{(n-c)$ or less cust. completed in time $t\}$

$$= \sum_{i=0}^{n-c} \frac{(c\mu t)^i e^{-c\mu t}}{i!}$$

$$= 1 - \sum_{n=c}^{Y+M-1} q_n \sum_{i=0}^{n-c} \frac{(c\mu t)^i}{i!} e^{-c\mu t}$$

q_n

When $Y=0$:

$q_n = Pr\{n$ cust. in Sys. $|$ arr is about to occur$\}$

$$= \frac{Pr\{\text{arr is about to occur} \mid n \text{ cust. in sys.}\} \cdot p_n}{\sum_{i=0}^{M} Pr\{\text{arr is about to occur} \mid i \text{ cust. in Sys.}\} \cdot p_i}$$

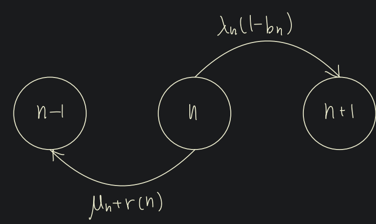
$$= \lim_{\Delta t \to 0} \frac{[(M-n)\lambda \Delta t + O(\Delta t)] \cdot p_n}{\sum_{i=0}^{M} [(M-i)\lambda \Delta t + o(\Delta t)] \cdot p_i}$$

$$\frac{\Delta t}{\Delta t} = \frac{(M-n)\lambda p_n}{\sum_{i=0}^{M} (M-i)\lambda p_i} = \frac{M-n}{M-L} p_n$$

When $Y>0$

$$q_n = \begin{cases} \dfrac{M p_n}{M - \sum_{i=Y}^{Y+M} (i-Y) p_i}, & 0 \le n \le Y-1 \\[2ex] \dfrac{(M-n+Y) p_n}{M - \sum_{i=Y}^{Y+M} (i-Y) p_i}, & Y \le n \le Y+M-1 \end{cases}$$

3.10 Queues with Impatience



Balking

- M/M/c/K
- b_n : $P_r\{$ reluctance of a cust. to join a queue upon arr $\}$

Reneging

- $r(n)$: the reneging func.

$$r(n) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} P_r\{\text{unit reneges during } \Delta t \text{ when there are } n \text{ cust. present}\}$$

$$\therefore\ r(0) = r(1) = 0$$

3.11 Transient Behavior of M/M/1/1

$p_n(t) =$

$$\begin{cases} P_1(t+\Delta t) = P_1(t)(1-\mu\Delta t) + P_0(t)\cdot\lambda\Delta t + o(\Delta t) \\ P_0(t+\Delta t) = P_0(t)(1-\mu\Delta t) + P_1(t)\cdot\mu\Delta t + o(\Delta t) \end{cases}$$

$-p_n(t), \div\Delta t, \lim$
$$\begin{cases} P_1'(t) = -\mu P_1(t) + \lambda P_0(t) \\ P_0'(t) = -\mu P_0(t) + \mu P_1(t) \end{cases}, \quad \forall t,\ P_0(t) + P_1(t) = 1$$

$$\Rightarrow \; p_1'(t) = -\mu p_1(t) + \lambda(1 - p_1(t))$$

$$\Rightarrow \; p_1'(t) + \underbrace{(\lambda+\mu)}_{}^{\alpha(x)} p_1(t) = \overset{r(x)}{\underline{\lambda}}$$

$$h = \int (\lambda+\mu)\, dt = (\lambda+\mu) t$$

$\ast$ 1st - ODE

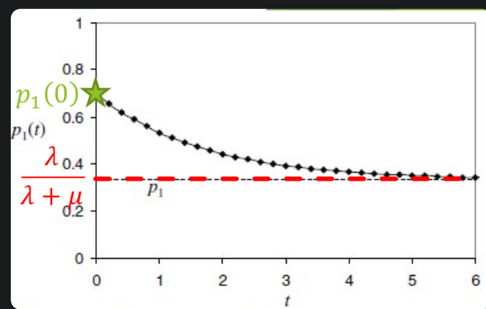
$$f'(x) = -\alpha(t) f(x) + r(x)$$

$$f(x) = c e^{-h} + e^{-h} \int e^{h} r(x)\, dx ,$$

$$h = \int \alpha(x)\, dx$$

1st - ODE $\Rightarrow \; c e^{-(\lambda+\mu)t} + e^{-(\lambda+\mu)t} \int e^{(\lambda+\mu)t} \lambda\, dt$

$$= c e^{-(\lambda+\mu)t} + e^{-(\lambda+\mu)t} \frac{\lambda}{\lambda+\mu} e^{(\lambda+\mu)t}$$



& $t = 0$ $\Rightarrow \; p_1(0) = c + \frac{\lambda}{\lambda+\mu}$ $\quad \therefore \; c = p_1(0) - \frac{\lambda}{\lambda+\mu}$

$$\therefore \; p_1(t) = \left(p_1(0) - \frac{\lambda}{\lambda+\mu}\right) e^{-(\lambda+\mu)t} + \frac{\lambda}{\lambda+\mu}$$

when $t \to \infty$, $p_1(t) = \frac{\lambda}{\lambda+\mu} = p_1$

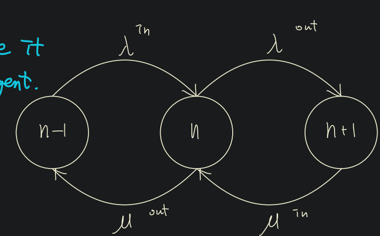
3.11.3 M/M/∞

Assume : $p_i(0) = 1$, $p_n(0) = 0 \quad \forall n \neq i$

By Generating Func.

$$P(z,t) \triangleq \sum_{n=0}^{\infty} z^n p_n(t) \; , \; |z| \leq 1$$

Make sure it would be divergent.



$$\frac{\partial P(z,t)}{\partial t} = \sum_{n=0}^{\infty} z^n p_n'(t)$$

Differetial - Diff. Eq.

$$z^n \cdot \left\{ \; p_n'(t) = -(\lambda + \mu) p_n(t) + \lambda p_{n-1}(t) + \mu p_{n+1}(t) \; , \quad n \geq 1 \right.$$

$$+) \quad z^0 \cdot \quad p_0'(t) = -\lambda p_0(t) + \mu p_1(t)$$

$$\sum_{n=0}^{\infty} z^n p_n'(t) = -\lambda \sum_{n=0}^{\infty} z^n p_n(t) - \mu \underbrace{\sum_{n=1}^{\infty} z^n p_n(t)}_{\sum_{n=0}^{\infty} z^n p_n(t) - z^0 p_0(t)} + \lambda \underbrace{\sum_{n=1}^{\infty} z^n p_{n-1}(t)}_{\sum_{n=0}^{\infty} z^{n+1} p_{n}(t)} + \mu \underbrace{\sum_{n=1}^{\infty} z^n p_{n+1}(t)}_{\sum_{n=0}^{\infty} z^{n-1} p_n(t) - z^{-1} p_0(t)}$$

$$= -\lambda P(z,t) - \mu\left(P(z,t) - p_0(t)\right) + \lambda z P(z,t) + \frac{\mu}{z}\left[P(z,t) - p_0(t)\right]$$

$$= \frac{1}{z}\left[(-\lambda z - \mu z + \lambda z^2 + \mu) P(z,t) + (-\mu z - \mu) p_0(t)\right]$$

$$\Rightarrow \frac{\partial}{\partial t} P(z,t) = \frac{1}{z}\left[\lambda z^2 - (\lambda+\mu) z + \mu\right] P(z,t) - \frac{\mu(1-z)}{z} p_0(t)$$

By Laplace Trans.

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} f(t) e^{-st} dt$$

$$\therefore \begin{cases} \bar{P}(z,s) \triangleq \mathcal{L}\{P(z,t)\} = \int_0^{\infty} e^{-st} P(z,t)\, dt, & Re(s) > 0 \\ \bar{p}_i(s) \triangleq \mathcal{L}\{p_i(t)\} = \int_0^{\infty} e^{-st} p_i(t)\, dt, & Re(s) > 0 \end{cases}$$

Make sure it wouldn't be divergent!

$\times e^{-st}$

$(f \circ g)' = f'g + fg'$

$$\Rightarrow \frac{\partial[e^{-st} P(z,t)]}{\partial t} = -s e^{-st} P(z,t) + e^{-st} \frac{\partial P(z,t)}{\partial t}$$

$\int_0^{\infty} dt$

$$\Rightarrow \underbrace{\int_0^{\infty} e^{-st} \frac{\partial P(z,t)}{\partial t} dt}_{\mathcal{L}\left\{\frac{\partial P(z,t)}{\partial t}\right\}} = \cancel{\frac{-s}{-s}} e^{-st} P(z,t) \Big|_0^{\infty} + s\int_0^{\infty} e^{-st} P(z,t)\, dt$$

$$= 0 - P(z,0) + s\bar{P}(z,s)$$

$$= -z^i + s\bar{P}(z,s)$$

$$\begin{cases} P(z,t) = \sum_{n=0}^{\infty} z^n p_n(t) \\ p_n(0) = 0,\ p_i(0) = 1\ \forall n \neq i \end{cases}$$

$$\therefore -P(z,0) = -\sum_{n=0}^{\infty} z^n p_n(0)$$

$$= -z^i p_i(0) + 0$$

$$= -z^i \#$$

$$\times z \Rightarrow -z^{i+1} + zs\bar{P}(z,s) = \left[\lambda z^2 - (\lambda+\mu) z + \mu\right] \bar{P}(z,s) - \mu(1-z)\bar{p}_0(s)$$

$$\therefore \bar{P}(z,s) = \frac{z^{i+1} - \mu(1-z)\bar{P}_0(s)}{-\lambda z^2 + (\lambda+\mu+s)z - \mu} \quad \text{By } \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$z_1, z_2 = \frac{(\lambda+\mu+s)\pm\sqrt{(\lambda+\mu+s)^2-4\lambda\mu}}{2\lambda}$$

$$\therefore z_1+z_2 = \frac{\lambda+\mu+s}{\lambda}, \quad z_1 z_2 = \frac{\mu}{\lambda}$$

Thm 3.2 Rouché's Thm

$f(z)$, $g(z)$ func. analytic inside & on a closed contour C, and

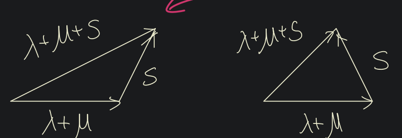
if $|g(z)| < |f(z)|$ on C,

then $f(z)+g(z)$ have the same number of zeros inside C

$$\$ \begin{cases} f(z) = (\lambda+\mu+s)z \\ g(z) = -\mu - \lambda z^2 \end{cases}$$

$|z|=1$ on C $\quad\because Re(s)>0$

$$\Rightarrow |f(z)| = |(\lambda+\mu+s)z| = |\lambda+\mu+s| > |\lambda+\mu|$$



$Re(s)>0$ $\quad$ $Re(s)<0$

$$\Rightarrow |\lambda z^2| + |\mu| \geq \underbrace{|\lambda z^2+\mu| = |g(z)|}$$

$$\therefore f(z) = (\lambda+\mu+s)z = 0 \Rightarrow z=0$$

$\Rightarrow$ only one zero inside the unit circle

for $f(z)+g(z) = (\lambda+\mu+s)\mu - \mu - \lambda z^2$

Known $\begin{cases} P(z,t) = \sum_{n=0}^{\infty} z^n p_n(t) \\ \bar{P}(z,s) = \sum_{n=0}^{\infty} z^n \bar{p}_n(t) \Rightarrow p_n(t) = \mathcal{L}^{-1}\{\bar{p}_n(s)\} \end{cases}$

$$= \frac{z^{i+1} - \mu(1-z)\bar{p}_0(s)}{(\lambda+\mu+s)z - \mu - \lambda z^2} = \frac{(z-z_1)\cdots}{-\lambda(z-z_1)(z-z_2)}$$

$\because |z_1| < |z_2| \Rightarrow z_1$ is within the unit circle

$$\Rightarrow z_1^{i+1} - \mu(1-z_1)\bar{p}_0(s) = 0$$

$$\Rightarrow \bar{p}_0(s) = \frac{z_1^{i+1}}{\mu(1-z_1)}$$

$$\Rightarrow \bar{P}(z,s) = \frac{z^{i+1} - \frac{1-z}{1-z_1} z_1^{i+1}}{\lambda(z-z_1)(z_2-z)}$$

want to split into 2 frac.

$$\frac{z^{i+1}(1-z_1) - z_1^{i+1}(1-z_1) + (z-z_1)z_1^{i+1}}{(z-z_1)(1-z_1)}$$

$$= \frac{(z^{i+1} - z_1^{i+1})(1-z_1) + (z-z_1)z_1^{i+1}}{(z-z_1)(1-z_1)}$$

$$= \frac{z^{i+1} - z_1^{i+1}}{z-z_1} + \frac{z_1^{i+1}}{1-z_1}$$

$\frac{\div z}{\div z}$

$$= \frac{z^i\left[1-\left(\frac{z_1}{z}\right)^{i+1}\right]}{1-\frac{z_1}{z}} + \frac{z_1^{i+1}}{1-z_1}$$

$S_n = \frac{a_1(1-r^n)}{1-r}$

unlimited $S_n = \frac{a_1(1-r^n)}{1-r}$

Since z_2 outside the unit circle.

$$= \frac{1}{\lambda z_2 \left(1-\frac{z}{z_2}\right)} \cdot \frac{z^{i+1}(1-z_1) - (1-z)z_1^{i+1}}{(z-z_1)(1-z_1)}$$

$$= \frac{1}{\lambda z_2}\sum_{k=0}^{\infty}\left(\frac{z}{z_2}\right)^k \cdot \left[z^i\left(1+\frac{z_1}{z}+\cdots+\left(\frac{z_1}{z}\right)^i\right) + \frac{z_1^{i+1}}{1-z_1}\right]$$

A B

$$= \frac{1}{\lambda z_2}\sum_{k=0}^{\infty}\left(\frac{z}{z_2}\right)^k \cdot \left[z^i + z_1 z^{i-1} + \cdots + z_1^i\right] + \frac{1}{\lambda z_2}\frac{z_1^{i+1}}{1-z_1}\sum_{k=0}^{\infty}\left(\frac{z}{z_2}\right)^k$$

$= \left(\frac{z}{z_2}\right)^n$

B: $$\frac{z_1^{i+1}}{\lambda z_2^{n+1}(1-z_1)} = \frac{z_1^{i+1}}{\lambda z_2^{n+1}}\left(1+z_1+z_1^2+\cdots\right)$$

Known $z_1 z_2 = \frac{\mu}{\lambda}$

$$= \frac{1}{\lambda}\left(\frac{1}{z_1 z_2}\right)^{n+1} z_1^{n+i+2} \sum_{k=0}^{\infty} z_1^k$$

$$= \frac{1}{\lambda}\left(\frac{1}{z_1 z_2}\right)^{n+1} \sum_{k=n+i+2}^{\infty} z_1^k z_2^k \times \frac{1}{z_2^k}$$

$z_1 z_2 = \frac{\mu}{\lambda}$ $$\Rightarrow \frac{1}{\lambda}\left(\frac{\lambda}{\mu}\right)^{n+1} \sum_{k=n+i+2}^{\infty} \left(\frac{\mu}{\lambda}\right)^k \frac{1}{z_2^k}$$

A: $$\frac{1}{\lambda z_2}\left[\frac{1}{z_2^{n-i}} + \frac{z_1}{z_2^{n-i+1}} + \cdots + \frac{z_1^{i}}{z_2^{n}}\right]$$