2.1 The Exponential Distribution

<Def> Exp. dist

$$f(t) = \lambda e^{-\lambda t}, \quad t \geq 0$$

$$\begin{cases} F(t) = \Pr\{T \leq t\} = 1 - e^{-\lambda t} \\ \bar{F}(t) = \Pr\{T > t\} = e^{-\lambda t} \\ E[T] = \dfrac{1}{\lambda} \\ Var[T] = \dfrac{1}{\lambda^2} \end{cases}$$

<Memoryless Property>

$$\Pr\{T > t+s \mid T > s\} = \Pr\{T > t\}$$

[Thm 2.1] Exp has Memoryless Property

<pf>

Known: Memoryless Property: $[\Pr\{T > t+s \mid T > s\} = \Pr\{T | t\}]$

Bayes' $\Rightarrow$ $\dfrac{\Pr\{T > t+s, \overset{\text{redun.}}{\cancel{T > s}}\}}{\Pr\{T > s\}}$

Known: Exp CCDF $[\bar{F}(t) = \Pr\{T > t\} = e^{-\lambda t}]$

$$\frac{e^{-\lambda(t+\cancel{s})}}{\cancel{e^{-\lambda s}}} = e^{-\lambda t} = \Pr\{T > t\} \quad \#$$

[Thm 2.2] Exp is the Only continous dist. has Memoryless

<pf>

Known: Memoryless $[\Pr\{T > t+s \mid T > s\} = \Pr\{T > t\}]$

$$\text{By } \bar{F}(t) \Rightarrow \frac{\bar{F}(t+s)}{\bar{F}(s)} = \bar{F}(t)$$

$$\Rightarrow \bar{F}(t+s) = \bar{F}(t)\,\bar{F}(s)$$

$$\ln \Rightarrow \ln \bar{F}(t+s) = \ln \bar{F}(t) + \ln \bar{F}(s)$$

let $g(x) \triangleq \ln \bar{F}(x)$

$$\Rightarrow \underbrace{g(t+s) = g(t) + g(s)}_{g(x) = \alpha x}$$

only when $g(x)$ is 1^{st} poly.

$$\Rightarrow \ln \bar{F}(x) = \alpha x$$

$$e \Rightarrow \bar{F}(x) = e^{\alpha x} \Rightarrow \underbrace{F(x) = 1 - e^{-\alpha x}}_{\text{Exp Dist}}$$

$\#$

<Def> Geometric Dist. $\Rightarrow$ Memoryless Property

$$\begin{cases} f(k) = \Pr\{X = k\} = q^{k-1}p, \quad q = 1-p \\ F(k) = \Pr\{X \le k\} = 1 - q^k \\ \bar{F}(k) = \Pr\{X > k\} = q^k \\ E[X] = \frac{1}{p} \\ Var[X] = \frac{q}{p^2} \end{cases}$$

<pf>

$$\Pr\{T > t+s \mid T > s\} = \Pr\{T > t\}$$

$$\frac{\Pr\{T > t+s\}}{\Pr\{T > s\}} = \Pr\{T > t\}$$

$$\frac{\bar{F}(t+s)}{\bar{F}(s)} = \bar{F}(t)$$

$$\frac{q^t \cdot q^s}{q^s} = q^t$$

$\#$

[Thm 2.3]

$\{T_1, \cdots T_n\}$, $\forall T_i \in$ Exp. with rates $\{\lambda_1, \cdots \lambda_n\}$

$$\Rightarrow \min\{T_1, \cdots T_n\} = \lambda_1 + \lambda_2 + \cdots + \lambda_n$$

<pf>

let T is $\min\{T_1, \cdots T_n\}$

$$\Rightarrow \Pr\{T > t\} = \Pr\{T_1 > t, T_2 > t, \cdots T_n > t\}$$

$$= e^{-\lambda_1 t} \cdot e^{-\lambda_2 t} \cdot \cdots \cdot e^{-\lambda_n t}$$

$$e^{-\lambda t} = e^{-(\lambda_1 + \lambda_2 + \cdots + \lambda_n)t}$$

$$\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_n \quad \#$$

[Thm 2.4] follow Thm 2.3

Prob. of picking min T

$$\Pr\{T_i = \min\{T_1, \cdots T_n\}\} = \frac{\lambda_i}{\lambda_1 + \cdots + \lambda_n}$$

<pf>

exclude T_i

$$\Pr\{T_i = \min\{T_1, \cdots T_n\}\} \Rightarrow \Pr\{T_i < T_j, \forall j \neq i\}$$

$\Rightarrow$ Find: when time from 0 to ∞

pick the T_i's prob.

$$= \int_0^\infty \Pr\{T_i < T_j, \forall i \neq j \mid T_i = t\} \cdot \lambda_i e^{-\lambda_i t}\, dt$$

$$\Downarrow$$

$$= \int_0^\infty \Pr\{T_j > t, \forall i \neq j\} \cdot \lambda_i e^{-\lambda_i t}\, dt$$

$\Pr\{T_1 > t, T_2 > t, \cdots T_n > t\}$

$$= \int_0^\infty \prod_{\forall j \neq i} e^{-\lambda_j t} \cdot \lambda_i e^{-\lambda_i t}\, dt$$

out

?

$$= \lambda_i \int_0^\infty e^{-\underbrace{(\lambda_1+\lambda_2+\cdots+\lambda_n)}_{\text{without } \lambda_i}t} \cdot e^{-\lambda_i t}\, dt$$

$$= \lambda_i \int_0^\infty e^{-(\lambda_1+\cdots+\lambda_n)t}\, dt$$

$$= \lambda_i \left. \frac{e^{-(\lambda_1+\cdots+\lambda_n)t}}{-(\lambda_1+\cdots+\lambda_n)} \right|_0^\infty$$

$$= \frac{\lambda_i(0-1)}{-(\lambda_1+\cdots+\lambda_n)} = \frac{\lambda_i}{\lambda_1+\cdots+\lambda_n} \quad /\!\!/$$

2.2 The Poisson Process

<Def> Stochastic Process

$\{N(t), t \geq 0\}$ is a collection of R.V. indexed by "Time"

<Def> Counting Process

a Stoch. Proc. in which

1. $N(t)$ with "value $\in \mathbb{N}_0$"
2. Nondecreasing in "Time"

<Def> Poi Disl

with rate $\lambda > 0$ is a Count. Proc. with

1. $N(0) = 0$ ⇒ start with 0

2. $Pr\{1 \text{ Event in } (t, t+\Delta t]\} = \lambda \Delta t + o(\Delta t)$ ⇒ only 1 Event would arr in Δt

3. $Pr\{\geq 2 \text{ Event in } (t, t+\Delta t]\} = o(\Delta t)$ ⇒ No ≥ 2 Events would arr in Δt

4. Indep. Incr. ⇒ Event with no overlapping Interval are Indep.

$$\begin{cases} f(A) = p_n = \dfrac{A^n}{n!} e^{-A} \quad (n \in \mathbb{N}_0) \\ E[X] = A \\ Var[X] = A \end{cases}$$

[Thm 2.6] Poi Dist

$N(t) \sim$ Poi, with rate λ.

Events happen by time $t \sim$ Poi, with mean λt

$$\Rightarrow \quad p_n(t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t} \quad (n \in \mathbb{N}_0)$$

$$\Rightarrow \quad p_n(t) = P_r\{N(t) = n\}$$

<pf>

By Indep. Incr.

$$\begin{aligned} P_n(t+\Delta t) = \; & Pr\{n \text{ happen in } t,\ 0 \text{ happen in } \Delta t\} \\ & + Pr\{n-1 \text{ happen in } t,\ 1 \text{ happen in } \Delta t\} \\ & + \cdots \\ & + Pr\{0 \text{ happen in } t,\ n \text{ happen in } \Delta t\} \end{aligned}$$

By Poi 2. & 3.

$$\begin{aligned} P_n(t+\Delta t) = \; & P_n(t) \cdot \Big(1 - \Big[\underbrace{(\lambda\Delta t + o(\Delta t))}_{1\text{ times}} + \underbrace{(o(\Delta t))}_{2\text{ times}} + (o(\Delta t)) + \cdots + \underbrace{(o(\Delta t))}_{n\text{ times}}\Big]\Big) \quad -① \\ & + P_{n-1}(t) \cdot \underbrace{(\lambda\Delta t + o(\Delta t))}_{1\text{-times}} \quad -② \\ & + P_{n-2}(t) \cdot \underbrace{o(\Delta t)}_{2\text{ times}} \quad -③ \\ & + \cdots \\ & + P_0(t) \cdot \underbrace{o(\Delta t)}_{n\text{ times}} \quad -(n+1) \end{aligned}$$

All posibles

$\Sigma o(\Delta t) = o(\Delta t)$

Split to 2 situation

$$P_n(t+\Delta t) = \begin{cases} \overset{①}{P_n(t)\cdot(1-\lambda\Delta t - o(\Delta t))} + \overset{②}{P_{n-1}(t)\cdot(\lambda\Delta t + o(\Delta t))} + \overset{③}{P_{n-2}(t)\cdot o(\Delta t)} + \cdots + \overset{ⓝ}{P_1(t)\cdot o(\Delta t)}, & n \geq 1 \quad -Ⓐ \\ P_0(t)\cdot(1-\lambda\Delta t - o(\Delta t)), & n = 0 \end{cases}$$

$\{\cdots\}\, o(\Delta t)$

Ⓐ move $o(\Delta t)$ out : $P_n(t)(1-\lambda\Delta t) + P_{n-1}(t)\cdot(\lambda\Delta t) + P_n(t)\cdot o(\Delta t) + P_{n-1}(t)\cdot o(\Delta t) + \cdots + P_1(t)\cdot o(\Delta t)$

$-P_n(t)$

$$\Rightarrow \begin{cases} P_n(t+\Delta t) - P_n(t) = P_n(t)\cdot(-\lambda\Delta t) + P_{n-1}(t)(\lambda\Delta t) + \{\cdots\}\, o(\Delta t), & n \geq 1 \\ P_0(t+\Delta t) - P_0(t) = P_0(t)\cdot(-\lambda\Delta t) + P_0(t)\cdot(-o(\Delta t)) \end{cases}$$

$\div \Delta t)$ $\lim_{\Delta t \to 0})$

$$\Rightarrow \begin{cases} \frac{d}{dt} P_n(t) = -\lambda P_n(t) + \lambda P_{n-1}(t) + \underbrace{\{\cdots\}}_{\leq 1} \underbrace{\lim_{\Delta t\to 0} \frac{o(\Delta t)}{\Delta t}}_{0}, & n \geq 1 \\ \frac{d}{dt} P_0(t) = -\lambda P_0(t) + \underbrace{P_0(t)}_{\leq 1} \underbrace{\lim_{\Delta t\to 0} \frac{-o(\Delta t)}{\Delta t}}_{0} \end{cases}$$

$$\Rightarrow \begin{cases} \frac{d}{dt} P_n(t) = -\lambda P_n(t) + \lambda P_{n-1}(t) \\ \frac{d}{dt} P_0(t) = -\lambda P_0(t) \end{cases}$$

By Boundary Cond. $\begin{cases} P_n(0) = 0, & n \in \mathbb{N} \\ P_0(0) = 1 \end{cases}$

Known 1^{st}-ODE : $f'(x) + \overset{\text{scaling}}{\alpha(x)} f(x) = g(x)$

$$\Rightarrow f(x) = c e^{-h} + e^{-h} \int e^{h} g(x)\, dx \quad , \quad h = \int \alpha(x)\, dx$$

First solve $p_0(t)$:

$$p_0'(t) + \overset{\alpha(x)}{\lambda} \cdot p_0(t) = \overset{g(x)}{0}$$

Therefore $h = \int \alpha(x)\, dx \Rightarrow \int \lambda\, dt = \lambda t$

$$\Rightarrow p_0(t) = c\, e^{-\lambda t} + e^{-\lambda t} \underbrace{\int e^{\lambda t} \cdot 0\, dt}_{C_2} \Rightarrow C_3 e^{-\lambda t}$$

So $p_0(t) = c\, e^{-\lambda t}$

By $P_0(0) = 1 \Rightarrow p_0(0) = C_3 \cdot e^{0} = 1$

$$\therefore C_3 = 1 \Rightarrow P_0(t) = e^{-\lambda t}$$

Then solve $p_n(t)$:

$$p_n'(t) + \overset{\alpha(x)}{\lambda} p_n(t) = \overset{g(x)}{\lambda\, p_{n-1}(t)}$$

By 1-ODE

$$\Rightarrow p_n(t) = Ce^{-\lambda t} + e^{-\lambda t} \int e^{\lambda t}\, \lambda\, p_{n-1}(t)\, dt$$

When $n=1$:

$$\Rightarrow P_1(t) = ce^{-\lambda t} + e^{-\lambda t} \int e^{\lambda t} \lambda\, p_0(t)\, dt$$

$$= ce^{-\lambda t} + e^{-\lambda t} \int \lambda\, \cancel{e^{\lambda t}}\, \cancel{e^{-\lambda t}}\, dt$$

$$= ce^{-\lambda t} + \lambda t e^{-\lambda t}$$

By $P_n(0) = 0$, $t \in \mathbb{N}$

$$\Rightarrow P_1(0) = 0 = C + 0 = 0 \quad \therefore C = 0$$

$$\Rightarrow P_1(t) = \lambda t e^{-\lambda t}$$

Using MI :

Known : $P_1(t) = \lambda t e^{-\lambda t}$

Wanted : $P_n(t) = \dfrac{(\lambda t)^n}{n!} e^{-\lambda t}$

step 1) OK !

step 2) Assume $P_k(t) = \dfrac{(\lambda t)^k}{k!} e^{-\lambda t}$ is established,

step 3) Prove $P_{k+1}(t) = \dfrac{(\lambda t)^{k+1}}{(k+1)!} e^{-\lambda t}$

By 1-ODE :

$$P_{k+1}(t) = ce^{-\lambda t} + e^{-\lambda t}\int e^{\lambda t}\cdot \lambda P_k(t)\, dt$$

$$= ce^{-\lambda t} + e^{-\lambda t}\int \cancel{e^{\lambda t}}\cdot \lambda \cdot \frac{(\lambda t)^k}{k!}\cancel{e^{-\lambda t}}\, dt$$

$$= ce^{-\lambda t} + e^{-\lambda t}\cdot \frac{\lambda}{k!}\lambda^k \cdot \int t^k\, dt$$

$$= ce^{-\lambda t} + e^{-\lambda t}\frac{\lambda^{k+1}}{k!}\cdot\frac{t^{k+1}}{k+1} + C_2$$

$$= e^{-\lambda t}\left(c + \frac{(\lambda t)^{k+1}}{(k+1)!} + C_2\right)$$

By $P_n(0) = 0$:

$$P_{k+1}(0) = 0 = C + C_2$$

$$\Rightarrow P_{n+1}(t) = e^{-\lambda t}\frac{(\lambda t)^{n+1}}{(n+1)!}$$

By MI

~~#~~

[Thm 2.7] Poi is Stationary Incr.

$$Pr\{N(t+h) - N(s+h) \leq x\} = Pr\{N(t) - N(s) \leq x\}$$

"Same length intervals would have same # happen"

(case 1)

(case 2)

<pf>

$$\Pr\{N(t+h)-N(s+h)\le x\} = \Pr\{N(t)-N(s)\le x\}$$

(case 1)

By Indep. Incr.

$(s, s+h]$, $(s+h, t]$, $(t, t+h]$ are indep.

$$(s+h, t+h] \overset{\text{split}}{\Longrightarrow} (s+h, t],\ (t, t+h]$$

$$LHS = \sum_{i=0}^{n} \Pr\{N(t)-N(s+h)=i\}\cdot \Pr\{N(t+h)-N(t)=n-i\}$$

$$= \sum_{i=0}^{n} P_i(t-s-h)\, P_{n-i}(h)$$

$$(s, t] \overset{\text{split}}{\Longrightarrow} (s, s+h],\ (s+h, t]$$

$$RHS = \sum_{i=0}^{n} \Pr\{N(s+h)-N(s)=i\}\cdot \Pr\{N(t)-N(s+h)=n-i\}$$

$$= \sum_{i=0}^{n} P_i(h)\cdot P_{n-i}(t-s-h)$$

equiv

$\#$

(case 2)

$$LHS = \Pr\{N(t+h)-N(s+h)=n\}$$

$$= P_n(t-s)$$

$$= \Pr\{N(t)-N(s)=n\} = RHS$$

$\#$

[Thm 2.8]

In a Poi with rate λ

$\Longleftrightarrow$ All Interarrival times $\in$ same Exp(λ) and $\perp$

<pf>

(step1) $L \rightarrow R$

let T, $a(t)$, $A(t)$ are RV, PDF, CDF of T

CCDF :

$\Pr\{T \geq t\} = \Pr\{\text{No happen in time } t\}$

$= P_0(t) \Rightarrow P_n(t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$

So CDF $= A(t) = 1 - \text{CCDF}$

$= 1 - e^{-\lambda t}$

Then PDF $= a(t) = \frac{d}{dt} A(t)$ $\Rightarrow$ Exp. Dist.

$= \lambda e^{-\lambda t}$

* Exp :

$f(t) = \lambda e^{-\lambda t}$

$F(t) = 1 - e^{-\lambda t}$

$\bar{F}(t) = e^{-\lambda t}$

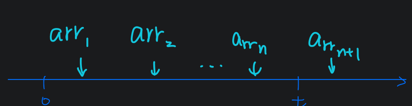
$E[X] = \frac{1}{\lambda}$

$Var[X] = \frac{1}{\lambda^2}$

step2) $R \rightarrow L$

$p_n(t) = \Pr\{N(t) = n\}$

Def : $P_n(t) \triangleq \Pr\{N(t) \leq n\}$



Erlang Type (n+1)

$P_n(t) = \Pr\{\text{Sum of } (n+1) \text{ interval times} > t\}$

large than $t \Rightarrow (t, \infty)$

* Erlang-Type n :

$\frac{\lambda(\lambda x)^{n-1} e^{-\lambda x}}{(n-1)!}$

Erlang Type (n+1) $\Rightarrow \int_t^{\infty} \frac{\lambda(\lambda x)^n}{n!} e^{-\lambda x} \, dx$

Change $X = u+t \Rightarrow \int_0^\infty \frac{\lambda^{n+1}(u+t)^n}{n!} e^{-\lambda(u+t)}\,du$

$u \in (t, \infty)$

$dx \Rightarrow du$

* Binomial Thm : $(a+b)^n$

$\sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$

$\sum_{i=0}^{n} \frac{n!}{(n-i)!\,i!} a^{n-i} b^i$

$\Rightarrow \int_0^\infty \frac{\lambda^{n+1}}{n!} e^{-\lambda(u+t)} \sum_{i=0}^{n} \frac{n!}{(n-i)!\,i!} u^{n-i} t^i \, du$

$\Rightarrow \sum_{i=0}^{n} \frac{\lambda^{n+1} e^{-\lambda t}}{(n-i)!\,i!} t^i \int_0^\infty e^{-\lambda u} u^{n-i} \, du$

Wanted let $u \Rightarrow \lambda u$ from $\lambda^{n+1} \Rightarrow \lambda^i \cdot \lambda^{n-i} \cdot \lambda^1$

$\Rightarrow \sum_{i=0}^{n} \frac{e^{-\lambda t} t^i}{(n-i)!\,i!} \lambda^i \boxed{\int_0^\infty e^{-\lambda u} (\lambda u)^{n-i} \lambda\, du}$ $d(\lambda u)$

$\Gamma(n-i+1) = (n-i)!$

$\Rightarrow \sum_{i=0}^{n} \frac{e^{-\lambda t} t^i}{\cancel{(n-i)!}\, i!} \lambda^i \cancel{(n-i)!}$

* Gamma func.

$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx$

$= (n-1)!$

$\Rightarrow \sum_{i=0}^{n} \boxed{\frac{(\lambda t)^i}{i!} e^{-\lambda t}}$ Poi

$\Rightarrow Pr\{N(t) \leq n\} = P_n(t)$

$p_n(t) = P_n(t) - P_{n-1}(t)$

$= \frac{(\lambda t)^n}{n!} e^{-\lambda t}$ ∦

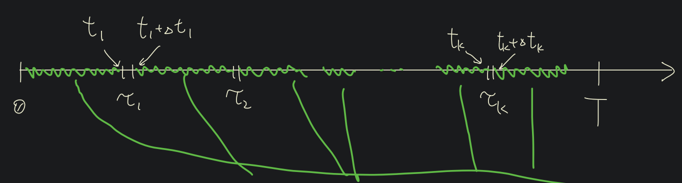
[Thm 2.9] Dist of Arrivals in $N(t) \sim Poi$ is Unif.

let $N(t) \sim Poi$

given k arrivals $\tau_1 < \tau_2 < \cdots < \tau_k$ in time $[0, T]$

then Order Statistics of k Indep. Uni. RV on $[0, T]$

<pf>



vector index

$$f_{\vec{\tau}}(\vec{t} \mid k) \triangleq f(t_1, t_2, \cdots t_k \mid k \text{ arr in } [0,T])\, dt_1 dt_2 \cdots dt_k$$

$$\approx \Pr\{t_1 \le \tau_1 < t_1+dt_1, \cdots t_k \le \tau_k \le t_k+dt_k \mid k \text{ arr in } [0,T]\}$$

$f_{exp}(dt_1) = \lambda dt_1 e^{-\lambda dt_1}$

$$\Rightarrow \frac{\Pr\{t_1 \le \tau_1 \le t_1+dt_1, \cdots t_k \le \tau_k \le t_k+dt_k, k \text{ arr in } [0,T]\}}{\Pr\{k \text{ arr in } [0,T]\}}$$

Poi(T

$$\Rightarrow \frac{1}{\frac{(\lambda T)^k}{k!}\cancel{e^{-\lambda T}}} \lambda dt_1 \cancel{e^{-\lambda dt_1}} \cdot \lambda dt_2 \cancel{e^{-\lambda dt_2}} \cdot \cdots \cdot \lambda dt_n \cancel{e^{-\lambda dt}} \cdot e^{-\lambda(T-dt_1-dt_2-\cdots dt_k)}$$

$$\Downarrow$$

$$\cancel{e^{-\lambda T}}\, \cancel{e^{\lambda dt_1}}\, \cancel{e^{\lambda dt_2}} \cdots \cancel{e^{\lambda dt_k}}$$

$$\Rightarrow \frac{1}{\frac{\cancel{\lambda^k} T^k}{k!}} \cancel{\lambda^k}\, dt_1 dt_2 \cdots dt$$

$$\Rightarrow \frac{k!}{T^k} dt_1 \cdots dt_k$$

Permutation

Total time

A RV

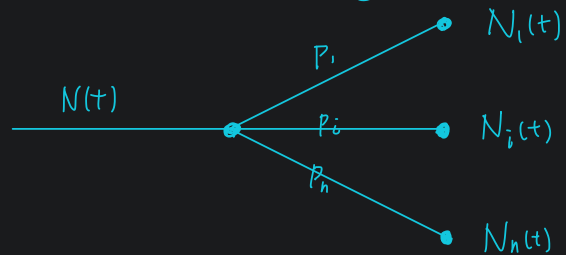
#

<PASTA>

"Poi Arrivals See Time Average"

Happens $\in$ Uni on interval $[0,T]$

[Thm 2.10] Splitting are Poi



let $N(t) \sim$ Poi with λ

p_i : Prob of each Event be labed type$_i$

$N_i(t)$: # Event of labed to type$_i$

$\Rightarrow$ ① $N_i(t)$: Poi with rate λp_i

② $N_i(t) \perp N_j(t)$, $\forall i \neq j$

<pf>

Assume $n=2$ ( ) $\Rightarrow$ $X(t) = X_1(t) + X_2(t)$

$Pr\{X_1(t) = k,\ X_2(t) = m\}$

For all prob. $\Rightarrow \sum_{n=0}^{\infty} P_r\{X_1(t)=k,\ X_2(t)=m \mid X(t)=n\} \cdot P_r\{X(t)=n\}$

must be $k+m$

$Po_i(k+m)\ :\ \frac{(\lambda t)^{k+m}}{(k+m)!}e^{-\lambda t}$

let $n=k+m$ $\Rightarrow P_r\{X_1(t)=k,\ X_2(t)=m \mid X(t)=k+m\} \cdot P_r\{X(t)=k+m\}$

let $q=1-p$

Def of Binomial Thm

$$\Rightarrow \binom{k+m}{k} p^k q^m \cdot Po_i(k+m)$$

$$e^{-\lambda t} = e^{(-\lambda t)(1-p+p)}$$

$$= e^{-\lambda t(p+q)}$$

$$= e^{-\lambda tp} \cdot e^{-\lambda tq}$$

$$\Rightarrow \frac{\cancel{(k+m)!}}{m!\,k!} p^k q^m \frac{(\lambda t)^{k+m}}{\cancel{(k+m)!}} e^{-\lambda t}$$

$$P_r\{X_1(t)=k,\ X_2(t)=m\} \Rightarrow \frac{(\lambda t q)^m}{m!} e^{-\lambda t q} \cdot \frac{(\lambda t p)^k}{k!} e^{-\lambda t p}$$

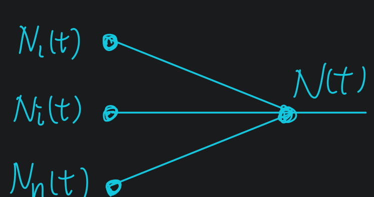
So $P_r\{X_1(t)=k\} = \sum_{m=0}^{\infty} P_r\{X_1(t)=k,\ X_2(t)=m\}$

$$= \frac{(\lambda t p)^k}{k!} e^{-\lambda t p} \cdot \cancel{e^{-\lambda t q}} \sum_{m=0}^{\infty} \frac{(\lambda q t)^m}{m!}$$

$\cancel{e^{\lambda q t}}$

$$= \frac{(\lambda t p)^k}{k!} e^{-\lambda t p} \sim Po_i$$

[Thm 2.11] Superposition are Poi



<pf>

$$\Pr\{X(t) = X_1(t) + X_2(t) = n\}$$

$$\Rightarrow \sum_{k=0}^{n} \Pr\{X_1(t)=k\} \cdot \Pr\{X_2(t) = n-k\}$$

$$\Rightarrow \sum_{k=0}^{n} \frac{(\lambda_1 t)^k}{k!} e^{-\lambda_1 t} \frac{(\lambda_2 t)^{n-k}}{(n-k)!} e^{-\lambda_2 t}$$

$$\Rightarrow \sum_{k=0}^{n} \frac{(\lambda_1 t)^k (\lambda_2 t)^{n-k}}{k!\,(n-k)!} e^{-(\lambda_1+\lambda_2)t}$$

Binomial Thm

$$\frac{n!}{n!} \Rightarrow \frac{1}{n!} \boxed{\sum_{k=0}^{n} \overbrace{\frac{n!}{k!\,(n-k)!}}^{\binom{n}{k}} (\lambda_1 t)^k (\lambda_2 t)^{n-k}} e^{-(\lambda_1+\lambda_2)t}$$

$$\Rightarrow \frac{1}{n!} (\lambda_1 t + \lambda_2 t)^n e^{-(\lambda_1+\lambda_2)t}$$

$$\Rightarrow \frac{\left((\lambda_1+\lambda_2)t\right)^n}{n!} e^{-(\lambda_1+\lambda_2)t}$$

//

2.2.1 Generalization of Poi

<Def> NHPP (NonHomogeneous Poisson Process)

from λ change to $\lambda(t)$

$$\begin{cases} N(0) = 0 \end{cases}$$

$$\begin{cases} Pr\{1 \text{ Event on } (t, t+\Delta t]\} = \lambda(t)\Delta t + o(\Delta t) \\ Pr\{\geq 2 \text{ Event on } (t, t+\Delta t]\} = o(\Delta t) \\ \text{Indep. Incr.} \end{cases}$$

[Thm 2.12] Mean # Event $_{t \to s}$ = $m(t) - m(s)$, $m(t) = \int_0^t \lambda(u)\, dt$

$N(t)$: NHPP

$\lambda(t)$: Mean Event Rate

$m(t) - m(s)$: # Event happen on $(s, t]$

$\lambda t \Leftrightarrow m(t) = \int_0^t \lambda(u)\, dt \Rightarrow$ Mean Value Func. (cumulated expected numbers)

$$Pr\{N(t) - N(s) = n\} = \frac{[m(t) - m(s)]^n}{n!} e^{-[m(t)-m(s)]^n}$$

<pf>

Known : $\lambda(t) \Rightarrow \begin{cases} P_n'(t) = -\lambda(t) P_n(t) + \lambda(t) P_{n+1}(t) \ , \ n \geq 1 \\ P_0'(t) = -\lambda(t) P_0(t) \end{cases}$

1. (solve $P_0(t)$)

$$P_0'(t) = -\underset{\alpha(x)}{\lambda(t)} \underset{f(x)}{P_0(t)} + \underset{r(x)}{0}$$

1ODE $\Rightarrow P_0(t) = Ce^{-h} + e^{-h} \int e^{h}\, 0\, dt$

* 1-ODE

$f'(x) + \alpha(x) f(x) = r(x)$

$\Rightarrow f(x) = ce^{-h} + e^{-h} \int e^{h} r(x)\, dx$,

$h = \int \alpha(x)\, dx$

Known : $m(t) - m(s) = \int_s^t \lambda(u)\, du$

$$\boxed{h = \int_0^t \lambda(u)\, du = m(t) - m(0)}$$

$\Rightarrow P_0(t) = C e^{-[m(t) - m(0)]}$

Known : $P_0(0) = 1$

$\Rightarrow P_0(0) = 1 = C \quad \therefore C = 1$

$\Rightarrow P_0(t) = e^{-[m(t) - m(0)]}$ ☆ $m(t) - m(0) \Rightarrow h$

2. (solve $P_1(t)$)

$$\underbrace{P_1'(t)}_{f'(x)} = -\underbrace{\lambda(t)}_{\alpha(x)} \underbrace{P_1(t)}_{f(x)} + \underbrace{\lambda(t) P_0(t)}_{r(x)}$$

By 1-ODE $\Rightarrow$ $P_1(t) = Ce^{-h} - e^{-h} \int \cancel{e^{h}}\, \lambda(t)\, \cancel{e^{-h}}\, dt$

$\int \lambda(t)\, dt \Rightarrow \int_0^t \lambda(u)\, du \Rightarrow m(t) - m(0)$

$P_1(t) = Ce^{-h} - e^{-h} [m(t) - m(0)]$

Known $P_1(0) = 0$: $\Rightarrow h = m(0) - m(0) = 0$

$P_1(0) = 0 = C$

$\Rightarrow P_1(t) = [m(t) - m(0)]\ e^{-(m(t) - m(0))}$ ☆

By MI

(step 2) Assume $P_k(t) = \dfrac{[m(t) - m(0)]^n}{n!} e^{-[m(t) - m(0)]^n}$ is est.

(step 3) $\underbrace{P_{k+1}'(t)}_{f'(x)} = -\lambda(t) P_{k+1}(t) + \lambda(t) P_k(t)$

$$\alpha(x) \quad f(x) \qquad \qquad \qquad \gamma(x)$$

$$= -\lambda(t) P_{k+1}(t) + \lambda(t) \frac{[m(t)-m(0)]^n}{n!} e^{-[m(t)-m(0)]^n}$$

1-ODE

$$P_{k+1}(t) = ce^{-h} + e^{-h} \int e^{h} \lambda(t) \frac{[m(t)-m(0)]^k}{k!} e^{-[m(t)-m(0)]^k} dt$$

$$h = \int \alpha(x)\,dx = \int_0^t \lambda(u)\,du = m(t)-m(0)$$

$$P_{k+1}(t) = ce^{-h} + e^{-h} \int \cancel{e^{h}} \lambda(t) \frac{h^k}{k!} \cancel{e^{-h}} dt$$

$$= ce^{-h} + \frac{e^{-h}}{k!} \int \lambda(t) h^k \, dt$$

$$= ce^{-h} + \frac{e^{-h}}{k!} \int_0^t \lambda(u) \int_0^t [\lambda(u)]^k du\, du$$

$$= ce^{-h} + \frac{e^{-h}}{k!} \int_0^t \lambda(u) \frac{[m(t)-m(0)]^k}{k+1} du$$

$m(t)-m(0)$

$$= ce^{-h} + \frac{e^{-[m(t)-m(0)]}}{(k+1)!} [m(t)-m(0)]^k \int_0^t \lambda(u)\,du$$

$$P_{k+1}(t) = ce^{-[m(t)-m(0)]} + \frac{[m(t)-m(0)]^{k+1}}{(k+1)!} e^{-[m(t)-m(0)]}$$

Known $P_n(0)=0$, $n \geq 1$

$$P_{k+1}(0) = 0 = C$$

$$\Rightarrow P_{k+1}(t) = \frac{[m(t)-m(0)]^{k+1}}{(k+1)!} e^{-[m(t)-m(0)]}$$

$\#$

2.3 DTMC (Discrete-Time Markov Chains)

< Markov Property >

$$Pr\{X_{n+1}=j \mid X_0=i_0, \cdots X_n=i_n\} = Pr\{X_{n+1}=j \mid X_n=i_n\}$$

If Homogeneous → $p_{ij} \triangleq Pr\{X_{n+1}=j \mid X_n=i\}$

$P \triangleq [p_{ij}]_{n\times n}$: The transition matrix.

1. $p_{ij}^{(m)} \equiv Pr\{X_{n+m}=j \mid X_n=i\} \Rightarrow P^{(m)} \triangleq [p_{ij}^{(m)}]_{n\times n} = P\cdot P\cdots P = P^m$

prob. after m steps

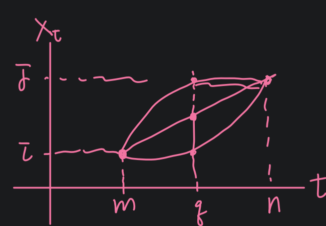
2. $\boxed{\pi}_j^{(m)} \equiv Pr\{X_m=j\} \equiv \sum_i \pi_i^{(m-1)} p_{ij}$

stay at j

3. $\pi^{(m)} = \pi^{(m-1)} P = \pi^{(0)} P^{(m)}$

graph init state

< C-K Eq. > $\sum_{\forall k\in X_t} p_{ik}(m,q)\, p_{kj}(q,n)\}$



<pf>

$$Pr\{X_n=j \mid X_m=i\} = \sum_{\forall k\in X_t} Pr\{X_n=j, X_q=k \mid X_m=i\}, \quad m<q<n$$

redundant

$$= \sum_{\forall k\in X_t} Pr\{X_n=j \mid \cancel{X_m=i}, X_q=k\}\cdot Pr\{X_q=k \mid X_m=i\}$$

$$= \sum_{\forall k\in X_t} Pr\{X_n=j \mid X_q=k\}\cdot Pr\{X_q=k \mid X_m=i\}$$

$$= \sum_{\forall k \in X_t} P_{ik}(m, q) \cdot P_{kj}(q, n) \quad \#$$

⟨ Accessible ⟩ $i \to j$

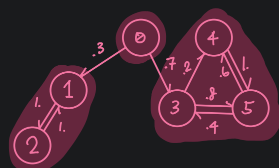
$\exists\, n \geq 0$ s.t. $p_{ij}^{(n)} > 0$

⟨ Communicate ⟩ $i \leftrightarrow j$

$i \to j \;\wedge\; j \to i$

⟨ Commnicate Classes ⟩

1. Mut. Ex.
2. Same Properties.



⟨ Irreducible ⟩

$i \leftrightarrow j \;,\; \forall i, j \in \text{Graph}$

⟨ Period ⟩

$\gcd(m)$ s.t. $p_{jj}^{(m)} > 0$

* Aperiod : period $= 1$

<Def>

first In n steps

$f_{jj}^{(n)}$: Prob. of First time back to self in n transitions.

Back to self

without steps limit

$f_{jj} : \sum_{n=1}^{\infty} f_{jj}^{(n)}$

< Recurrent & Transient State >

R: $f_{jj} = 1$

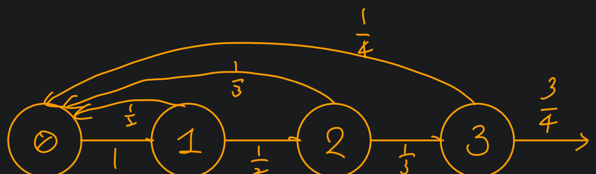
T: $f_{jj} < 1$

< Mean Recurrence Time > m_{jj}

$$m_{jj} = \sum_{n=1}^{\infty} n \cdot f_{jj}^{(n)}$$

$$m_{jj} \begin{cases} < \infty : \text{Pos. Reccurent} \\ = \infty : \text{Null Reccurent} \end{cases}$$

(Eg. 2.3.1) State 0



$$f_{00} = \sum_{n=1}^{\infty} f_{00}^{(n)}$$

$$= 0 + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\cdot\frac{1}{3}\right) + \cdots + \left(\frac{1}{2}\cdots\frac{1}{n}\right)$$

$$= 0 + \frac{1}{2} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} + \cdots + \cancel{\frac{1}{n-1}} - \underbrace{\frac{1}{n}}_{\to 0}$$

$$= 1$$

Recurrent
(Would back to self)

$$f_{00}^{(1)} = 0$$

$$f_{00}^{(2)} = 1\cdot\frac{1}{2} = 1 - \frac{1}{2}$$

$$f_{00}^{(3)} = 1\cdot\frac{1}{2}\cdot\frac{1}{3} = \frac{1}{2} - \frac{1}{3}$$

$$\vdots$$

$$f_{00}^{(n)} = 1\cdot\frac{1}{2}\cdot\cdots\cdot\frac{1}{n} = \frac{1}{n-1} - \frac{1}{n}$$

Known $m_{ij} = \sum_{n=1}^{\infty} n \cdot f_{ij}^{(n)}$

$$m_{00} = \sum_{n=1}^{\infty} n f_{00}^{(n)}$$

$$= \sum_{n=1}^{\infty} \cancel{n} \frac{1}{n-1} \cancel{\frac{1}{n}} = \infty$$

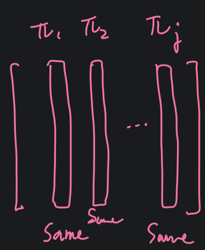
Null Recurrent

2.3.2 Long-Run Behavior

< Steady-State Dist >

1. $\lim_{n\to\infty} P_{ij}^{(n)} = \pi_j$

2. $\lim_{n\to\infty} \pi_i^{(n)} = \pi_j$



<pf>

(case 1) $\pi_j = \lim_{n\to\infty} \pi_j^{(n)}$

$$= \lim_{n\to\infty} \sum_{\forall i\in G} \pi_i^{(0)} P_{ij}^{(n)}$$

$$= \sum_{i\in G} \pi_i^{(0)} \lim_{n\to\infty} P_{ij}^{(n)}$$

let $\pi^{(0)} = (0, 0, \cdots, \underset{\text{index}=k}{\overset{\pi_k^{(0)}}{1}}, 0, \cdots, 0)$

$$= \pi_k^{(0)} \lim_{n\to\infty} p_{kj}^{(n)}$$

$$= \lim_{n\to\infty} p_{kj}^{(n)} = \lim_{n\to\infty} p_{ij}^{(n)}$$

(case 2)

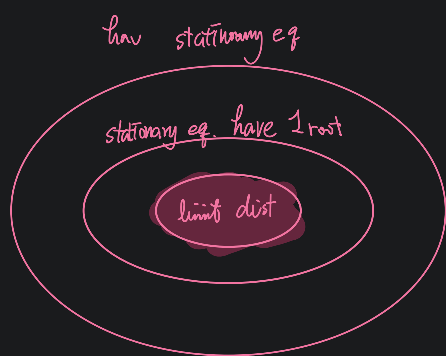
$$\lim_{n\to\infty} \pi_j^{(n)} = \lim_{n\to\infty} \sum_{\forall i \in G} \pi_i^{(0)} p_{ij}^{(n)}$$

$$= \underbrace{\sum_{\forall i \in G} \pi_i^{(0)}}_{1} \cdot \lim_{n\to\infty} \sum_{\forall i \in G} p_{ij}^{(n)}$$

$$= \lim_{n\to\infty} \pi_j^{(n)} = \pi_j$$

⟨ Stationary Eq ⟩

$$\begin{cases} \pi = \pi P \\ \sum_{\forall j} \pi_j = 1 \end{cases}$$



⟨ Limiting Dist ⟩

Limiting Dist $\underset{\not\Longleftarrow}{\Longrightarrow}$ stat. eq. have and only have one root.

[Thm 2.13]

① + ② $\Rightarrow$ $\exists$ stat. dist $\wedge$ $\pi_j = \dfrac{1}{m_{jj}}$

①+②+③ $\Rightarrow$ $\exists$ limiting dist = stationary dist

* m_{jj} : Mean Recurrence Time

$$m_{jj} = \sum_{n=1}^{\infty} n \cdot f_{jj}^{(n)}$$

$$m_{jj} \begin{cases} < \infty, & \text{Pos Rec} \\ = \infty, & \text{Null Re} \end{cases}$$

① Irreducible ⇒ ∀ state are communicate

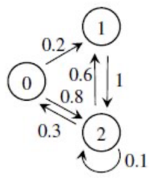
② Pos Rec ⇒ $m_{jj} < \infty$

③ Aperiodic ⇒ period

(contd.)

▶ Example 2.8

$$P = \begin{array}{ccc} \pi_0 & \pi_1 & \pi_2 \\ \end{array}$$
$$P = \begin{pmatrix} 0 & .2 & .8 \\ 0 & 0 & 1 \\ .3 & .6 & .1 \end{pmatrix}.$$



$$\begin{aligned} \pi_0 &= 0.3\pi_2, \\ \pi_1 &= 0.2\pi_0 + 0.6\pi_2, \qquad \pi_0 + \pi_1 + \pi_2 = 1, \\ \pi_2 &= 0.8\pi_0 + \pi_1 + 0.1\pi_2. \end{aligned}$$

limiting dist

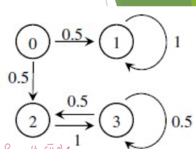
$$P^{10} \doteq \begin{pmatrix} 0.1713 & 0.3689 & 0.4598 \\ 0.1858 & 0.3943 & 0.4199 \\ 0.1260 & 0.2891 & 0.5849 \end{pmatrix}, \quad P^{40} \doteq \begin{pmatrix} 0.1531 & 0.3368 & 0.5100 \\ 0.1532 & 0.3369 & 0.5099 \\ 0.1530 & 0.3366 & 0.5105 \end{pmatrix}$$

53

(contd.)

▶ Example 2.9

$$P = \begin{pmatrix} 0 & 0.5 & 0.5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0.5 & 0.5 \end{pmatrix}$$



haven't single

have root, but root ⇒ Not limiting dist

$$\begin{aligned} \pi_0 &= 0, \\ \pi_1 &= 0.5\pi_0 + \pi_1, \\ \pi_2 &= 0.5\pi_0 + 0.5\pi_3, \qquad \pi_0 + \pi_1 + \pi_2 + \pi_3 = 1, \qquad \boxed{\pi_0 = 0 \text{ and } \pi_2 = 0.5\pi_3.} \\ \pi_3 &= \pi_2 + 0.5\pi_3. \end{aligned}$$

$$\boxed{\begin{aligned} \pi &= (0, 0.7, 0.1, 0.2) \\ \pi &= (0, 0.1, 0.3, 0.6) \end{aligned}}$$

Not limiting dist

$$\lim_{n\to\infty} P^n = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} \end{pmatrix}.$$

54

▶ Example 2.11

have 1 single root

$\frac{1}{2}$ $\frac{1}{2}$

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \qquad (\pi_0, \pi_1) = (\pi_0, \pi_1)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

No limiting dist

$$P^{(m)} = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & m \text{ even}, \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & m \text{ odd}. \end{cases}$$

2.3.3 Ergodicity

< Time Average >

* $X_i(t)$: i^{th} experiments of $\{X(t),\ t \geq 0\}$

k^{th} moment

$$\overline{X_T^k} = \frac{1}{T}\int_0^T [X_0(t)]^k \, dt$$

Time Interval

< Ensemble Average >

split into n parts, then average

$$E[X(t)]^k = m_k(t) = \boxed{\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}}[X_i(t)]^k$$

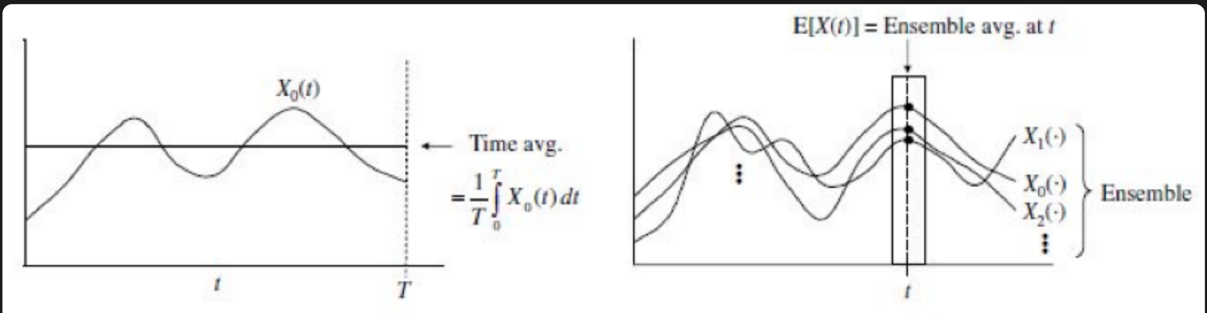


Figure 2.4 Time average versus ensemble average.

< Ergodicity >

$X(t)$ at k^{th} moment is Egodic

$$\Rightarrow \underbrace{\lim_{T\to\infty}\overline{X_T^k}}_{\text{Long Interval}} = \underbrace{\lim_{t\to\infty} m_k(t)}_{\text{Long Run States}} = \overline{X^k} < \infty$$

< Stationary > if $\pi^{(0)} = \pi$

Table 2.1 Long-run behavior concepts for DTMC

Condition	Resulting Properties	Comment
Irreducible, positive recurrent	Unique stationary distribution	π_j is the long-run fraction of time in state j
Irreducible, positive recurrent, $\pi^{(0)} = \pi$	Unique stationary distribution, process is stationary and ergodic	Ensemble averages = time averages $m_k(t) = \overline{X^k_T}$
Irreducible, positive recurrent, aperiodic	Unique stationary distribution, process is ergodic, limiting distribution exists (equal to stationary distribution)	Process independent of starting state in the limit

2.3.3 Ergodicity (contd.)

- Example 2.13

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \qquad \underbrace{(\pi_0, \pi_1)}_{\pi} = \underbrace{(\pi_0, \pi_1) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{= \pi P},$$

- $\Rightarrow \bar{x} = \frac{1}{2}$ if $T \to \infty$
- $E[X_n] = \begin{cases} 0 & (n \text{ even}), \\ 1 & (n \text{ odd}). \end{cases}$ if $X_0 = 0$
  - $E[X_n]$ does not converge as $n \to \infty$.
    - Not ergodic

2.3.3 Ergodicity (contd.)

- If $\pi^{(0)}$ is set equal to the stationary distribution $\left(\frac{1}{2}, \frac{1}{2}\right)$, then

$$X_n = \begin{cases} 0 & \text{w.p. } \frac{1}{2} \\ 1 & \text{w.p. } \frac{1}{2} \end{cases} \text{ for all } n$$

$$E[X_n] = \frac{1}{2} \text{ independent of } n$$

  - Ergodic

2.4 CTMC (Continuous-Time Markov Chains)

<CTMC>

1. Stochastic Proc. $\{X(t), t \geq 0\}$

2. with countable state space.

S.t. $\Rightarrow$ $\nu_i \sim$ Exp. : Rate of remain in state i for a period of time

[Forward C-K Eq]

$$\frac{\partial P_{ij}(u,t)}{\partial t} = -\nu_j(t) \cdot P_{ij}(u,t) + \sum_{\forall r \neq j} P_{ir}(u,t)\, q_{rj}(t)$$

<pf>

$$P_{ij}(u,s) = \sum_{\forall r} P_{ir}(u,t) \cdot P_{rj}(t,s), \quad 0 \leq u < t < s$$

let $s = t + \Delta t$ $\quad \Rightarrow \sum_{\forall r} P_{ir}(u,t) \cdot P_{rj}(t, t+\Delta t)$

$-P_{ij}(u,t)$ $\Rightarrow P_{ij}(u,s) - P_{ij}(u,t) = \sum_{\forall r} P_{ir}(u,t) \cdot P_{rj}(t, t+\Delta t) - P_{ij}(u,t)$

Filter out when $r=j$

$$= \sum_{\forall r\neq j} P_{ir}(u,t)\cdot P_{rj}(t,t+\Delta t) + P_{ij}(u,t)\cdot P_{jj}(t,t+\Delta t) - P_{ij}(u,t)$$

$$= -\big(1-P_{jj}(t,t+\Delta t)\big)P_{ij}(u,t) + \sum_{\forall r\neq j} P_{ir}(u,t)\cdot P_{rj}(t,t+\Delta t)$$

Not stay in j

from others move in j

$\div \Delta t)\ \lim_{\Delta t\to 0})\ \Rightarrow$

$$\frac{\partial P_{ij}(u,t)}{\partial t} = \left(\lim_{\Delta t\to 0}\frac{-\big(1-P_{jj}(t,t+\Delta t)\big)}{\Delta t}\right)P_{ij}(u,t) + \lim_{\Delta t\to 0}\sum_{\forall r\neq j}\frac{P_{rj}(t,t+\Delta t)}{\Delta t}\,P_{ir}(u,t)$$

move out rate

move in rate

$$= \big(-\nu_j(t)\big)P_{ij}(u,t) + \sum_{\forall r\neq j} q_{rj}(t)\,P_{ij}(u,t)$$

#

[Backward C-K Eq.]

$$\frac{\partial P_{ij}(u,t)}{\partial u} = \nu_i(u)\,P_{ij}(u,t) - \sum_{r\neq i} q_{ir}(u)\cdot P_{rj}(u,t)$$

<pf>

$$P_{ij}(u,s) = \sum_{\forall r} P_{ir}(u,t)\cdot P_{rj}(t,s)\ ,\quad 0\le u<t<s$$

let $u=t-\Delta t$

$$\Rightarrow P_{ij}(t-\Delta t,s) = \sum_{\forall r} P_{ir}(t-\Delta t,t)\cdot P_{rj}(t,s)$$

Filter out $r=i$

$-P_{ij}(t,s)$

$$\Rightarrow P_{ij}(t-\Delta t,s)-P_{ij}(t,s) = \sum_{\forall r\neq i} P_{ir}(t-\Delta t,t)\cdot P_{rj}(t,s) + P_{ii}(t-\Delta t,t)\cdot P_{ij}(t,s) - P_{ij}(t,s)$$

$\cdot(-1)$

$$\Rightarrow P_{ij}(t,s)-P_{ij}(t-\Delta t,s) = \big[1-P_{ii}(t-\Delta t,t)\big]\cdot P_{ij}(t,s) - \sum_{\forall r\neq i} P_{ir}(t-\Delta t,t)\cdot P_{rj}(t,s)$$

$\div \Delta t) \lim_{\Delta t \to 0})$ $\Rightarrow$ $$\frac{\partial P_{ij}(t,s)}{\partial t} = \boxed{\lim_{\Delta t \to 0} \frac{1 - P_{ii}(t-\Delta t, t)}{\Delta t}} P_{ij}(t,s) - \sum_{\forall r \neq i} \boxed{\frac{\lim_{\Delta t \to 0} P_{ir}(t-\Delta t, t)}{\Delta t}} \cdot P_{rj}(t,s)$$

$$= \nu_i(t)\, P_{ij}(t,s) - \sum_{\forall r \neq i} q_{ir}(t) \cdot P_{rj}(t,s)$$

let $t=u$ $\Rightarrow$ $$\frac{\partial P_{ij}(u,s)}{\partial t} = \nu_i(u)\, P_{ij}(u,s) - \sum_{\forall r \neq i} q_{ir}(u)\, P_{rj}(u,s)$$

<CTMC>

$q_{ij}(t)$: move from i to j

$\nu_i(t)$: move out from i

Start time from 0:

$$\frac{d\, p_j(t)}{dt} = -\nu_j(t)\, p_j(t) + \sum_{r \neq j} p_r(t)\, q_{rj}(t)$$

<pf>

Forward CK Eq.:

$$\frac{\partial P_{ij}(u,t)}{\partial t} = -\nu_j(t)\, P_{ij}(u,t) + \sum_{r \neq j} P_{ir}(u,t)\, q_{rj}(t)$$

let $u=0$ $\Rightarrow$ $$\frac{d\, P_{ij}(0,t)}{dt} = -\nu_j(t)\, P_{ij}(0,t) + \sum_{r \neq j} P_{ir}(0,t)\, q_{rj}(t)$$

$\times p_i(0)) \sum_{\forall i})$ $\Rightarrow$ $$\underbrace{\sum_{\forall i} \frac{d\, P_{ij}(0,t)\, P_i(0)}{dt}}_{p_j(t)} = -\nu_j(t) \underbrace{\sum_{\forall i} P_i(0)\, P_{ij}(0,t)}_{p_j(t)} + \sum_{r \neq j} \underbrace{\sum_{\forall i} P_i(0)\, P_{ir}(0,t)}_{p_r(t)}\, q_{rj}(t)$$

$$\Rightarrow \quad \frac{d\,p_j(t)}{dt} = -v_j(t)\,p_j(t) + \sum_{r \neq j} p_r(t)\,q_{rj}(t)$$

$$\Rightarrow \quad p'(t) = p(t)\,Q(t), \quad \begin{cases} Q(t) = \begin{bmatrix} -v_0(t) & q_{01}(t) & \cdots & q_{0n}(t) \\ q_{10}(t) & -v_1(t) & & \vdots \\ \vdots & & \ddots & \vdots \\ q_{n0}(t) & \cdots & & -v_n(t) \end{bmatrix}_{n \times} \\ p(t) = \begin{bmatrix} p_0(t) & p_1(t) & \cdots & p_n(t) \end{bmatrix}_{1 \times n} \end{cases}$$

< Homo. CTMC >

$$Q(t) \Rightarrow Q \in \text{Const}, \quad p'(t) = p(t)\,Q$$

[Thm 2.16] $0 = pQ \;\wedge\; \sum_j p_j = 1$ (sum of row)

if Imbedded DTC is

1. Irreducible
2. Pos. Reccurent

then $\exists!$ root of $0 = pQ \;\wedge\; \sum_j p_j = 1$

and if $\forall$ states $\in$ bounded ($v_i > 0 \ \forall i$)

then Limiting Prob. Dist $\equiv$ Stationary Dist.