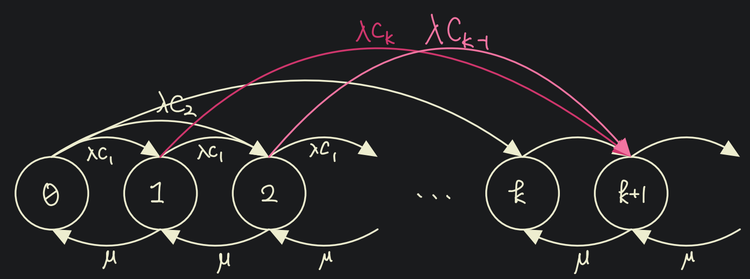
4.1 Bulk Input ($M^{[x]}/M/1$)

C_k : $P_r\{\text{Batch size} = k\}$, $k \in (1, \infty)$, $\sum_k C_k = 1$



$$\begin{cases} 0 = -(\lambda+\mu)\,p_n + \mu\, p_{n+1} + \lambda \sum_{k=1}^{n} p_{n-k} C_k, & n \ge 1 \\ 0 = -\lambda p_0 + \mu p_1 \end{cases}$$

By Generating Function : $P(z) \triangleq \sum_{n=0}^{\infty} z^n p_n$, $|z| \le 1$

$$z^n \cdot \begin{cases} 0 = -(\lambda+\mu)\,p_n + \mu\, p_{n+1} + \lambda \sum_{k=1}^{n} p_{n-k} C_k, & n \ge 1 \\ \end{cases}$$

$$+)\ z^0 \cdot \quad 0 = -\lambda p_0 + \mu p_1$$

$$0 = -\lambda \sum_{n=0}^{\infty} z^n p_n - \mu \sum_{n=1}^{\infty} z^n p_n + \mu \frac{1}{z} \sum_{n=0}^{\infty} z^{n+1} p_{n+1} + \lambda \overset{A}{\boxed{\sum_{n=1}^{\infty} \sum_{k=1}^{n} z^n p_{n-k} C_k}}$$

$$= -\lambda P(z) - \mu\big(P(z) - p_0\big) + \frac{\mu}{z}\big(P(z) - p_0\big) + \lambda A$$

A :

$$\sum_{n=1}^{\infty} \sum_{k=1}^{n} \overset{z^{n-k}\ z^k}{z^n} p_{n-k} C_k = \boxed{\sum_{k=1}^{\infty}} \underbrace{\boxed{\sum_{n=k}^{\infty} z^{n-k} p_{n-k}}}_{P(z)} \cdot \underbrace{\boxed{z^k C_k}}_{C(z)}$$

$$\Rightarrow P(z) = \left.\frac{\mu p_0 (1-z)}{\mu(1-z) - \lambda z(1-C(z))}\right|_{z=1}$$

L'Hopital's

$$\Rightarrow P(z)\Big|_{z=1} = \left.\frac{-\mu p_0}{-\mu + \lambda \boxed{\frac{d}{dz}C(z)}}\right|_{z=1} = 1$$

$$C(z) = \sum_{k=1}^{\infty} z^k c_k$$

$$\frac{d}{dz}C(z)\Big|_{z=1} = \sum_{k=1}^{\infty} k z^{k-1} c_k \Big|_{z=1} = \sum_{k=1}^{\infty} k c_k = E[X]$$

$$\Rightarrow p_0 = 1 - \frac{\lambda}{\mu} E[X] = 1 - \rho$$

$$\Rightarrow \frac{\lambda}{\mu} = \frac{\rho}{E[X]}$$

Eg. $X \sim Geo$, $c_n = (1-\alpha)\alpha^{n-1}$, $0 < \alpha < 1$ $\Rightarrow E[X] = \frac{1}{1-\alpha}$

Known: $P(z) = \frac{\mu p_0 (1-z)}{\mu(1-z) - \lambda z(1-C(z))}$

$$C(z) = (1-\alpha)\sum_{n=1}^{\infty} z^n \alpha^{n-1}$$

$$= z(1-\alpha)\boxed{\sum_{n=0}^{\infty}(z\alpha)^n} = \frac{1}{1-\alpha z}$$

$$= \frac{z(1-\alpha)}{1-\alpha z}$$

Known: $p_0 = 1 - \rho = 1 - \frac{\lambda}{\mu}E[X]$

$$\Rightarrow P(z) = \frac{\mu p_0 \cancel{(1-z)}}{\underbrace{\mu(1-z) - \lambda z\underbrace{\left(1 - \frac{z(1-\alpha)}{1-\alpha z}\right)}_{A}}_{B}}$$

$$= \frac{\mu p_0}{\mu - \frac{\lambda z}{1-\alpha z}}$$

A:

$$\lambda z\left(\frac{1-\cancel{\alpha z} - z + \cancel{\alpha z}}{1-\alpha z}\right)$$

$$= \lambda z \frac{1-z}{1-\alpha z}$$

B:

$$\mu(1-z) - \lambda z \frac{1-z}{1-\alpha z}$$

$$\cancel{(1-z)}\left(\mu - \frac{\lambda z}{1-\alpha z}\right)$$

$$= \frac{\cancel{\mu} p_0 (1-\alpha z)}{\cancel{\mu}(1-\alpha z) - \frac{\lambda z \cancel{\mu}}{\mu}}$$

Known : $\frac{\lambda}{\mu} = \frac{t}{E[X]} = \rho(1-\alpha)$

$$= \frac{(1-\rho)(1-\alpha z)}{\boxed{1-\alpha z - \rho(1-\alpha) z}} \quad C$$

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$$

C :

$$\frac{(1-\alpha z)\cdot 1}{1-(\alpha+(1-\alpha)\rho) z}$$

$$= 1 \cdot \sum_{n=0}^{\infty}((\alpha+(1-\alpha)\rho) z)^n - \alpha z \sum_{n=0}^{\infty}((\alpha+(1-\alpha)\rho) z)^n$$

Wanted : Back from Gen. func.

$$P(z) = \sum_{n=0}^{\infty} \{\cdots\} z^n \quad p_n$$

$$\Rightarrow P(z) = (1-\rho)\sum_{n=0}^{\infty}(\alpha+(1-\alpha)\rho)^n z^n - \alpha \boxed{z} \sum_{n=0}^{\infty}(\alpha+(1-\alpha)\rho)^n z^{n+1}$$

$$\therefore\ p_n = (1-\rho)\left[\underbrace{(\alpha+(1-\alpha)\rho)^n - \alpha(\alpha+(1-\alpha)\rho)^{n-1}}_{D}\right]$$

D :

$$\$\ (1-\alpha)\rho = \beta$$

$$(\alpha+\beta)^n - \alpha(\alpha+\beta)^{n-1} = (\alpha+\beta)^{n-1}(\cancel{\alpha}+\beta-\cancel{\alpha})$$

$$= (\alpha+(1-\alpha)\rho)^{n-1}((1-\alpha)\rho)$$

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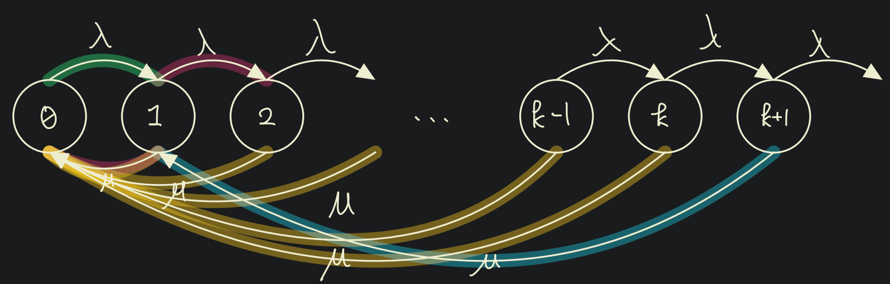
4.2 Bulk Service ($M/M^{[Y]}/1$)

Partial - Batch



$$
\begin{cases}
0 = \boxed{-(\lambda+\mu)\,p_n} + \boxed{\lambda\, p_{n-1}} + \boxed{\mu\, p_{n+k}}, & n \geq 1 \\
0 = -\lambda P_0 + \boxed{\mu P_1 + \cdots + \mu P_k}
\end{cases}
$$

By Operator D

1. Index Shifting

$$
\begin{cases}
① - \quad 0 = -(\lambda+\mu) P_{n+\boxed{1}} + \lambda P_{n+0} + \mu\, P_{n+\boxed{k+1}}, & n \geq 0 \\
② - \quad 0 = -\lambda P_0 + \mu P_1 + \cdots + \mu P_k
\end{cases}
$$

①: $0 = \left(-(\lambda+\mu)\, D + \lambda + \mu\, D^{k+1}\right) p_n$

Known : $\exists!$ root $\in (0,1)$ denoted by r_0

$$\Rightarrow p_n = \sum_{i=1}^{k+1} c_i r_i^n$$

$$= c_0 r_0^n, \quad n \geq 0, \quad 0 < r_0 < 1$$