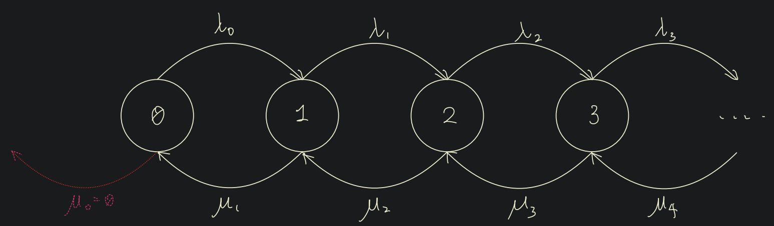
3.1 Birth - Death Process

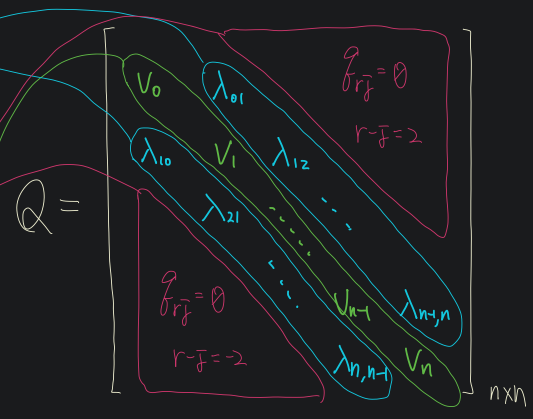


$\lambda_n = q_{n,n+1}$

$\mu_n = q_{n,n-1}$

$q_{rj} = 0$, if $|r-j| \ge 2$

$v_n = \lambda_n + \mu_n$, $\mu_0 = 0$



< Stationary Eq >

$$P_n = P_0 \prod_{i=1}^{n} \frac{\lambda_{i-1}}{\mu_i} \quad , \quad P_0 = \left(1 + \sum_{n=1}^{\infty} \prod_{i=1}^{n} \frac{\lambda_{i-1}}{\mu_i}\right)^{-1}$$

<pf>

(step 1) Find $P_n = P_0 \prod_{i=1}^{n} \frac{\lambda_{i-1}}{\mu_i}$

Known : $\frac{dP_j\cancel{(t)}}{dt} = -v_j\cancel{(t)}\, P_j\cancel{(t)} + \sum_{\forall r \ne j} P_r\cancel{(t)}\, q_{rj}\cancel{(t)}$

let $P_j(t) = P_j$

let $j = n \Rightarrow$ $0 = -v_n P_n + \sum_{\forall r \ne n} P_r q_{rn}$

$v_n = \lambda_n + \mu_n$

Since $q_{rn} = 0$, $\forall |r-n| \ge 2$

$\sum_{\forall r \ne n} \equiv r = n-1 , r = n+1$

$$= -(\lambda_n+\mu_n)\,p_n + p_{n-1}\,\underbrace{q_{n-1,n}}_{}^{\lambda_{n-1}} + p_{n+1}\,\underbrace{q_{n+1,n}}_{}^{\mu_{n+1}}$$

$$\Rightarrow \begin{cases} 0 = -(\lambda_n+\mu_n)\,p_n + p_{n-1}\lambda_{n-1} + p_{n+1}\mu_{n+1}, & n \ge 1 \\ 0 = -(\lambda_0+0)\,p_0 + 0 + p_1\mu_1, & n = 0 \end{cases} \Rightarrow \mu_0 = 0$$

$\div\,\mu_{n+1})$
$\div\,\mu_1)$

$$\Rightarrow \begin{cases} p_{n+1} = \dfrac{(\lambda_n+\mu_n)\,p_n}{\mu_{n+1}} - \dfrac{\lambda_{n-1}\,p_{n-1}}{\mu_{n+1}} \\ p_1 = \dfrac{\lambda_0 p_0}{\mu_1} \end{cases}$$

$$p_1 = \frac{\lambda_0 p_0}{\mu_1}$$

$$p_2 = \frac{(\lambda_1+\mu_1)p_1 - \lambda_0 p_0}{\mu_2} = \frac{(\lambda_1+\mu_1)\frac{\lambda_0 p_0}{\mu_1} - \lambda_0 p_0}{\mu_2} = \frac{\frac{\lambda_1\lambda_0 p_0}{\mu_1} + \cancel{\lambda_0 p_0} - \cancel{\lambda_0 p_0}}{\mu_2} = \frac{\lambda_0\,\lambda_1}{\mu_1\,\mu_2}\,p_0$$

$$\Rightarrow p_n = p_0 \cdot \prod_{i=1}^{n} \frac{\lambda_{i-1}}{\mu_i}$$

(step 2) Find $p_0 = \left(1+\sum_{n=1}^{\infty}\prod_{i=1}^{n}\frac{\lambda_{i-1}}{\mu_i}\right)^{-1}$

Known : $\sum_{\forall j} p_j = 1$

$$1 = \sum_{n=0}^{\infty} p_n$$

$$= p_0 + \sum_{n=1}^{\infty} p_n$$

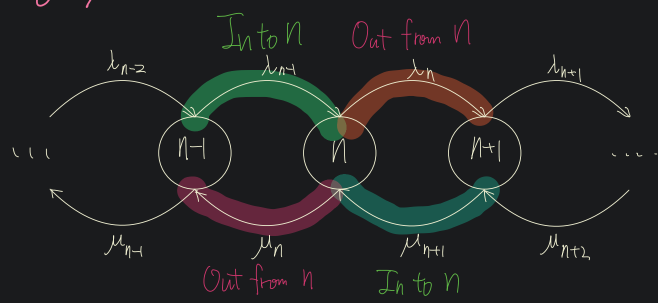
$$= p_0 + \sum_{n=1}^{\infty} p_0 \prod_{i=1}^{n} \frac{\lambda_{i-1}}{\mu_i}$$

$$= \left(1 + \sum_{n=1}^{\infty}\prod_{i=1}^{n}\frac{\lambda_{i-1}}{\mu_i}\right) p_0$$

$$p_0 = \underbrace{\left(1 + \sum_{n=1}^{\infty}\prod_{i=1}^{n}\frac{\lambda_{i-1}}{\mu_i}\right)^{-1}}_{}$$

p_0 has sol. when $\frac{1}{\square} < \infty$!

⟨ Balance Eq ⟩



1. Local Balance

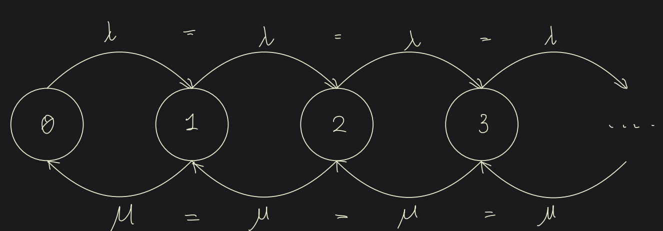
$$0 = \lambda_{n-1} p_{n-1} - \lambda_n p_n$$

$$0 = -\mu_n p_n + \mu_{n+1} p_{n+1}$$

2. Global Balance

$$0 = \underbrace{-(\lambda_n + \mu_n) p_n}_{\text{Out from } n} + \underbrace{(\lambda_{n-1} p_{n-1} + \mu_{n+1} p_{n+1})}_{\text{In to } n}$$

3.2 Single-Server Queues (M/M/1)



$$\rho \triangleq \frac{\lambda_{\leftarrow \text{in}}}{\mu_{\leftarrow \text{out}}}$$

Busy rate of server

if $\rho = 1$, Full loading

< Stationary Eq of M/M/1 >

If $\rho > 1$, Queue would explode! without Steady-State.

$$p_n = \rho^n (1-\rho) \quad , \quad p_0 = 1-\rho$$

<pf>

Known Balance Eq :

$$\begin{cases} 0 = -(\lambda+\mu)p_n + \lambda p_{n-1} + \mu p_{n+1} , \ n \geq 1 & -① \\ 0 = -\lambda p_0 + \mu p_1 , \ n = 0 & -② \end{cases}$$

$$② \Rightarrow \quad p_1 = \frac{\lambda}{\mu} p_0$$

let $n=1$: $① \Rightarrow \quad 0 = -(\lambda+\mu)p_1 + \lambda p_0 + \mu p_2$

$$p_2 = \frac{(\lambda+\mu)p_1 - \lambda p_0}{\mu}$$

$$= \frac{\frac{\lambda^2}{\mu} p_0 + \cancel{\lambda p_0} - \cancel{\lambda p_0}}{\mu} = \left(\frac{\lambda}{\mu}\right)^2 p_0$$

By MI :

$$p_n = \rho^n p_0$$

Known : $\sum_{\forall j} p_j = 1$

$$\sum_{\forall n} p_n = 1 \Rightarrow \sum_{n=0}^{\infty} \rho^n p_0$$

$$= \frac{1}{1-\rho} p_0 \quad \text{when } \rho < 1$$

$$\Rightarrow p_0 = 1-\rho$$

#

< By Generating Function >

Want to know α, where $p_n = \alpha$

$$P(z) = \sum_{n=0}^{\infty} z^n p_n, \quad |z| \le 1$$

<pf>

Known $\begin{cases} 0 = -(\lambda+\mu)p_n + \lambda p_{n-1} + \mu p_{n+1}, & n \ge 1 \\ 0 = -\lambda p_0 + \mu p_1, & n = 0 \end{cases}$

$$\div \mu \Rightarrow \begin{cases} (\rho+1)p_n = \rho p_{n-1} + p_{n+1} \\ \rho p_0 = p_1 \end{cases}$$

$\times z^n$)
$\times z^0$)

$$\Rightarrow \begin{cases} (\rho+1) z^n p_n = \rho z^n p_{n-1} + z^n p_{n+1}, & n \ge 1 \\ \rho z^0 p_0 = z^0 p_1, & n = 0 \end{cases}$$

+)

$$(\rho+1)\sum_{n=1}^{\infty} z^n p_n + \rho z^0 p_0 = \rho \sum_{n=1}^{\infty} z^n p_{n-1} + \sum_{n=1}^{\infty} z^n p_{n+1} + z^0 p_1$$

Combine $\Rightarrow \sum_{n=0}^{\infty} z^n p_{n+1}$

$$\Rightarrow (\rho+1)\sum_{n=1}^{\infty} z^n p_n + \rho z^0 p_0 = \rho \sum_{n=1}^{\infty} z^n p_{n-1} + \sum_{n=0}^{\infty} z^n p_{n+1}$$

$\sum_{n=0}^{\infty} z^n p_n - z^0 p_0$ $\quad$ $z\sum_{n=1}^{\infty} z^{n-1} p_{n-1}$ $\quad$ $\frac{1}{z}\sum_{n=0}^{\infty} z^{n+1} p_{n+1}$

* $P(z) = \sum_{n=0}^{\infty} z^n p_n$

$$\Rightarrow (\rho+1)P(z) - \underset{1}{z^0} p_0 - \cancel{\rho z^0 p_0} + \cancel{\rho z^0 p_0} = \rho z \sum_{n=1}^{\infty} z^{n-1} p_{n-1} + \frac{1}{z}\sum_{n=0}^{\infty} z^{n+1} p_{n+1}$$

$\sum_{n=0}^{\infty} z^n p^n$ $\quad$ $\sum_{n=1}^{\infty} z^{n+1} p_{n+1} - z^0 p_0$ $\quad$ $\sum_{n=0}^{\infty} z^n p_n - 1 \cdot p_0$

$$\Rightarrow (\rho+1)P(z) - p_0 = \rho z P(z) + \frac{1}{z}\left(P(z) - p_0\right)$$

$$\left(\rho+1 - \rho z - \frac{1}{z}\right)P(z) = p_0 - \frac{1}{z}p_0$$

$\times z) \Rightarrow (z\rho + z - \rho z^2 - 1) P(z) = (z-1) p_0$

$$\Rightarrow P(z) = \frac{\cancel{(1-z)} p_0}{\rho z^2 - \rho z - z + 1} \quad = \cancel{(1-z)}(1-\rho z)$$

$$= \frac{p_0}{1-\rho z}$$

Known : $\sum_{\forall j} p_j = 1$

$$\Rightarrow P(1) = \sum_{n=0}^{\infty} 1^n p_n = 1$$

$$\Rightarrow 1 = \frac{p_0}{1-\rho \cdot 1}$$

$$\Rightarrow p_0 = 1-\rho \quad !!$$

$$P(z) = \frac{1-\rho}{1-\rho z} \Rightarrow \sum_{n=0}^{\infty} z^n p_n$$

$$= (1-\rho) \frac{1}{1-\rho z} \quad \sum_{n=0}^{\infty} (\rho z)^n$$

$$= (1-\rho) \sum_{n=0}^{\infty} (\rho z)^n, \quad (\rho < 1)$$

$$= \sum_{n=0}^{\infty} z^n (1-\rho)\rho^n \Leftrightarrow \sum_{n=0}^{\infty} z^n p_n$$

$$\Rightarrow p_n = (1-\rho)\rho^n, \quad (\rho < 1)$$

< By Operators >

Using Linear Difference Eq. to solve $\{p_n\}$

$$\frac{\lambda : \text{in}}{\mu : \text{out}}$$

$$p_{n+1} = (1+\rho)p_n - \rho\, p_{n-1}$$

Linear Operator D on seq. $\{a_0, a_1, \cdots\}$

$$\begin{cases} D \cdot a_n = a_{n+1}, & \forall n \\ D^m \cdot a_n = a_{n+m}, & \forall n, m \end{cases} \qquad a_n = \sum_{i=1}^{k} d_i r_i^{\,n}, \quad \begin{cases} d_i \in \text{const.} \\ r_i \in \text{root} \end{cases}$$

<pf>

$$0 = -(\lambda+\mu)p_n + \lambda p_{n-1} + \mu p_{n+1}, \quad n \geq 1$$

$$= -(\lambda+\mu)p_{n+1} + \lambda p_n + \mu p_{n+2}, \quad n \geq 0$$

D-Operator $= -(\lambda+\mu) D p_n + \lambda p_n + \mu D^2 p_n$

$$= (\mu D^2 - (\lambda+\mu) D + \lambda)\, p_n$$

$$= \underbrace{(D-1)(\mu D - \lambda)\, p_n}_{} = 0$$

$$D = 1 \text{ or } \frac{\lambda}{\mu}$$

Known: $a_n = \sum_{i=1}^{k} d_i \underset{\text{roots}}{r_i^{\,n}} \Rightarrow a_n = d_1 \cdot 1^n + d_2 \left(\frac{\lambda}{\mu}\right)^n$

$$\Rightarrow p_n = d_1 + d_2 \left(\frac{\lambda}{\mu}\right)^n$$

let n=1

Known: $p_1 = \rho\, p_0$

$$\Rightarrow p_1 = d_1 + d_2 \rho^1 = 0 + \rho\, p_0$$

$\Rightarrow \begin{cases} d_1 = 0 \\ d_2 = p_0 \end{cases} \Rightarrow \underbrace{p_n = \left(\frac{\lambda}{\mu}\right)^n p_0}$ get p_n #

Known : $\sum_{\forall j} p_j = 1$

$\Rightarrow \sum_{n=0}^{\infty} \rho^n p_0 = 1$

$* \sum_{n=0}^{\infty} x^n \Rightarrow \frac{a_1(1-r^n)}{1-r} = \frac{1}{1-x}$

$\Rightarrow \underbrace{p_0 = \left(\sum_{n=0}^{\infty} \rho^n\right)^{-1} = 1-\rho}$ get p_0 #

3.2.4 Measures of Effectiveness - M/M/1

< Mean System Size >

$$L = \sum_{n=0}^{\infty} n \, p_n = \frac{\rho}{1-\rho}$$

n: n people in sys.

p_n: prob. of n people in sys

ρ: Service rate

$1-\rho$: Not in Service's rate

<pf>

Known : $p_n = (1-\rho)\rho^n = p_0 \rho^n$

$$L = \sum_{n=0}^{\infty} n(1-\rho)\rho^n$$

$$= (1-\rho)\underbrace{\sum_{n=0}^{\infty} n\rho^n}$$

$$= \frac{\rho}{1-\rho}$$ #

$$\sum_{n=0}^{\infty} n\rho^n$$

$$\Rightarrow \rho \sum_{n=0}^{\infty} n\rho^{n-1}$$

$$\Rightarrow \rho \frac{d}{d\rho} \sum_{n=0}^{\infty} \rho^n$$

$$\Rightarrow \rho \frac{d}{d\rho} \frac{1}{1-\rho}$$

$$\Rightarrow \rho \cdot \frac{-1}{(1-\rho)^2} = \frac{\rho}{(1-\rho)^2}$$

< Mean Queue Size >

$$L_q = \sum_{n=1}^{\infty} (n-1) p_n = \boxed{\sum_{n=0}^{\infty} n \, p_n} - \sum_{n=1}^{\infty} p_n$$

n: In Sys.

1: In Server

$n-1$: In Queue

$\boxed{\sum_{n=0}^{\infty} n \, p_n} = L$

< Mean Queue Size of Nonempty Queues >

$$L_q' = E[N_q \mid N_q \neq 0] = \sum_{n=2}^{\infty} (n-1) P_n' = \frac{1}{1-\rho}$$

$$P_n' = \Pr\{n \text{ in sys} \mid n \geq 2\} = \frac{\Pr\{n \text{ in sys},\ n \geq 2\}}{\Pr\{n \geq 2\}} = \frac{P_n}{\underset{All}{1} - \underset{n=0}{P_0} - \underset{n=1}{P_1}} = \frac{P_n}{\rho^2}$$

$1 - P_0 - P_0\rho$

$1 - (1+\rho)P_0$

$1 - (1+\rho)(1-\rho)$

$\cancel{1} - \cancel{1} + \rho^2$

<pf>

$$\sum_{n=2}^{\infty} (n-1)\frac{P_n}{\rho^2} \Rightarrow \frac{1}{\rho^2}\sum_{n=2}^{\infty}(n-1)P_n$$

Known $P_n = (1-\rho)\rho^n$

$$\Rightarrow \frac{1-\rho}{\rho^2}\sum_{n=2}^{\infty}(n-1)\rho^n$$

< Waiting - Time Dist >

T_q : R.V. of waiting time in Queue

$W_q(t)$: CDF of T_q

q_n : $\Pr\{n$ customs in sys. at an arrival$\}$ $\Rightarrow q_n = P_n$ in M/M/1

1. $W_q(t) = 1 - \rho e^{-\mu(1-\rho)t}$ $\Rightarrow \lambda e^{-\mu\lambda t}$

2. $W_q = \dfrac{\rho}{\mu(1-\rho)}$

<pf>

$$W_q(t) = P_r\{T_q \le t\}$$

(need waiting) $T_q > 0$ (Don't need waiting) $T_q = 0$

$$= \underbrace{\sum_{n=1}^{\infty} P_r\{T_q \le t \mid n \text{ customs in sys at an arrival}\} \cdot P_r\{n \text{ customs in sys at an arrival}\}}_{T_q>0} + \underbrace{P_r\{T_q = 0\}}_{T_q=0}$$

time for n complete (Other customers in front) → "Erlang Type-n"

$q_n = P_n = (1-\rho)\rho^n$

$W_q(0) = q_0 = P_0 \Rightarrow 1-\rho$

$$= \sum_{n=1}^{\infty} \int_0^t \frac{\mu(\mu x)^{n-1}}{(n-1)!} e^{-\mu x}\, dx \cdot (1-\rho)\rho^n + (1-\rho)$$

$\rho \cdot \rho^{n-1}$

$$= (1-\rho)\rho \int_0^t \mu e^{-\mu x} \underbrace{\sum_{n=1}^{\infty} \frac{(\mu x \rho)^{n-1}}{(n-1)!}}_{\Rightarrow e^{-\mu \rho x}} dx + (1-\rho)$$

$$* \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^{-x}$$

$$= (1-\rho)\rho \int_0^t \mu \underbrace{e^{-\mu x} e^{-\mu \rho x}}_{e^{-(1+\rho)\mu x}} dx + (1-\rho)$$

$$= \rho \int_0^t (1-\rho)\mu \; e^{-(1-\rho)\mu x} dx + (1-\rho)$$

$*$ let $(1-\rho)\mu = \lambda$

$\Rightarrow \int_0^t \lambda e^{-\lambda x} dx$: CDF of Exp → $1 - e^{-\lambda t}$

$$= \rho\left[1 - e^{-(1-\rho)\mu t}\right] + (1-\rho)$$

$$W_q(t) = 1 - \rho \cdot e^{-(1-\rho)\mu t}$$

!!!

$$W_q = E[T_q] = \int_0^{\infty} t \; dW_q(t)$$

$W_q(t) = 1 - \rho e^{-(1-\rho)\mu t}$

$dW_q(t) = (1-\rho)\mu \cdot \rho e^{-(1-\rho)\mu t}$

$$= \int_0^{\infty} t \; \rho\mu(1-\rho)e^{-\mu(1-\rho)t} dt$$

$$= \rho \int_0^\infty \underbrace{\mu(1-\rho)}_{\lambda} e^{\overset{\lambda}{-\mu(1-\rho)}t} \, dt$$

$\int_0^\infty \lambda e^{-\lambda t} dt \Rightarrow$ Exp dist

Mean of Exp $= \frac{1}{\lambda}$

$$= \rho \frac{1}{\mu(1-\rho)}$$

< System Time Dist >

T : R.V. of staying in System

$W(t)$: CDF of T_q

q_n : $P_r\{n$ customs in sys. at an arrival$\} \Rightarrow q_n = p_n$ in M/M/1

1. $W(t) = 1 - e^{-\mu(1-\rho)t} \iff W_q(t) = 1 - \rho e^{-\mu(1-\rho)t}$

2. $W = \dfrac{1}{\mu(1-\rho)} = \dfrac{W_q}{\rho}$

<pf>

wait for n people

Erlang Type-$(n+1)$ wait for self

$$W(t) = P_r\{T \le t\}$$

$$= \sum_{n=0}^{\infty} P_r\{T \le t \mid n \text{ customers in Sys at an arr}\} \cdot p_n$$

$$= \sum_{n=0}^{\infty} \int_0^t \frac{\mu(\mu x)^n}{n!} e^{-\mu x} dx \cdot (1-\rho)\rho^n$$

$$= (1-\rho) \int_0^t \mu e^{-\mu x} \boxed{\sum_{n=0}^{\infty} \frac{(\mu x \rho)^n}{n!}} dx$$

$e^{\mu x \rho}$

$$= \int_0^t (1-\rho)\,\mu\, \underbrace{e^{-\mu x} e^{\mu x \rho}}_{e^{-\mu x + \mu x \rho} = e^{(\rho-1)\mu x} = e^{-(1-\rho)\mu x}} \, dx$$

$$= \int_0^t \boxed{(1-\rho)\mu} \cdot e^{\boxed{-(1-\rho)\mu} x} \, dx \longrightarrow \text{Exp CDF} \Rightarrow 1-e^{-\lambda t}$$

$$\Rightarrow W(t) = 1 - e^{-(1-\rho)\mu t}$$

$$\Rightarrow W = \underbrace{E[W(t)]}_{\text{Exp Mean} = \frac{1}{\lambda}} = \frac{1}{(1-\rho)\mu}$$

3.3 Multiserver Queues (M/M/c)

$$r = \frac{\lambda}{\mu}, \quad \rho = \frac{r}{c}$$

1. $\lambda_n = \lambda$

2. $$\mu_n = \begin{cases} n\mu, & (1 \le n < c) \\ c\mu, & (n \ge c) \end{cases}$$

3. $$p_n = \begin{cases} ① \ \dfrac{1}{n!} r^n, & (1 \le n < c) \\ ② \ \dfrac{1}{c^{n-c}\, c!} r^n p_0, & (n \ge c) \end{cases}$$

4. $$p_0 = \left[\sum_{n=0}^{c-1} p_n^{①} + \sum_{n=c}^{\infty} p_n^{②} \right]^{-1} = \left[\sum_{n=0}^{c-1} \frac{r^n}{n!} + \frac{r^c}{c!(1-\rho)} \right]^{-1}$$

*

$$\sum_{n=c}^{\infty} \frac{r^n}{c^{n-c}\, c!} \qquad r^n = r^c r^{n-c}$$

$$\frac{r^c}{c!} \underbrace{\sum_{n=0}^{\infty} \frac{r^{n-c}}{c^{n-c}}}_{} \Rightarrow \frac{1}{1-\underbrace{\frac{r}{c}}_{=\rho}}$$

$$\frac{r^c}{c!} \frac{1}{1-\rho}$$

$$L_q = \frac{r^c \rho}{c!\,(1-\rho)^2}\, P_0$$

$$W_q = \frac{L_q}{\lambda}$$

$$W = \frac{1}{\mu} + W_q$$

$$L = r + L_q$$

By Little Law :

$$\begin{tikzcd}
L \arrow[r, dash, "L = \lambda W"] \arrow[d, dash, "L = L_q + r"'] & W \arrow[d, dash, "W = W_q + \frac{1}{\mu}"] \\
L_q \arrow[r, dash, "L_q = \lambda W_q"] & W_q
\end{tikzcd}$$

<pf>

$$P_0 = \left[\sum_{n=0}^{c-1} \frac{r^n}{n!} + \underbrace{\sum_{n=c}^{\infty} \frac{r^n}{c^{n-c}\,c!}} \right]^{-1}$$

$$\sum_{n=c}^{\infty} \frac{r^n}{c^{n-c}\,c!} = \frac{r^c}{c!} \sum_{n=c}^{\infty} \frac{r^{n-c}}{c^{n-c}}$$

$$= \underbrace{\left[\sum_{n=0}^{c-1} \frac{r^n}{n!} + \frac{r^c}{c!\,(1-\rho)} \right]^{-1}}$$!!!

Known $\frac{r}{c} = \rho$

$$\Rightarrow \frac{r^c}{c!} \sum_{n=c}^{\infty} \rho^{n-c}$$

$$\Rightarrow \frac{r^c}{c!} \underbrace{\sum_{n=0}^{\infty} \rho^{n}}$$

$$\Rightarrow \frac{r^c}{c!} \cdot \frac{1}{1-\rho}$$

#Customers #Servers

$$L_q = \sum_{n=c+1}^{\infty} \underbrace{(n-c)}_{\#\text{InQ}} P_n$$

Only occ. when #Customers > #Servers

Known $P_n = \frac{r^n}{c^{n-c}\,c!} P_0$

$$\Rightarrow \sum_{n=c+1}^{\infty} (n-c) \frac{r^n}{c^{n-c}\,c!} P_0$$

$r^n = r^c r^{n-c}$

$$\Rightarrow \frac{r^c}{c!} P_0 \sum_{n=c+1}^{\infty} (n-c) \underbrace{\frac{r^{n-c}}{c^{n-c}}}$$

$\frac{r}{c} = \rho$

$$\sum_{n=1}^{\infty} n x^n = \frac{x}{(1-x)^2}$$

let $m = n-c$ $\Rightarrow \frac{r^c}{c!} P_0 \underbrace{\sum_{m=1}^{\infty} m \rho^m} \Rightarrow \frac{r^c \rho}{c!\,(1-\rho)^2} P_0$

$$W_q(0) = P_r\{T_q = 0\} = 1 - \frac{r^c \, p_0}{c!\,(1-\rho)}$$

$$W_q(t) = P_r\{T_q \le t\} = 1 - P_r\{T_q > t\}$$

$\mu c - \lambda, \ \because r = \frac{\lambda}{\mu}$

$$P_r\{T_q > t\} = \frac{r^c p_0}{c!\,(1-\rho)} \, e^{-\mu(c-r)t}$$

<pf>

$$W_q(0) = P_r\{\underbrace{T_q = 0}_{\text{without waiting}}\}$$

$$\Rightarrow P_r\{\overset{\#\text{Customers}}{\boxed{n}} \le \overset{\#\text{Server}}{\boxed{c-1}} \text{ in sys.}\}$$

at least remain 1 server

$$\Rightarrow \sum_{n=0}^{c-1} p_n$$

Known $p_n = \frac{r^n}{n!} p_0$, $p_0 = \left[\frac{r^c}{c!\,(1-\rho)} + \sum_{n=0}^{c-1} \frac{r^n}{n!}\right]^{-1}$

$$\Rightarrow p_0 \sum_{n=0}^{c-1} \frac{r^n}{n!}$$

$$\Rightarrow \sum_{n=0}^{c-1} \frac{r^n}{n!} = p_0^{-1} - \frac{r^c}{c!\,(1-\rho)}$$

$$W_q(0) = 1 - \frac{r^c}{c!\,(1-\rho)} p_0$$

!!!

$$W_q(t) = P_r\{T_q \le t\}$$

All Exclude itself

Erlang Type $(n-c+1)$

$$\Rightarrow \sum_{n=c}^{\infty} P_r\{n-(c-1) \text{ complete } \le t \mid n \text{ customers in sys. at an arr.}\} \cdot p_n + W_q(0)$$

$$\Rightarrow \sum_{n=c}^{\infty} \int_0^t \frac{c\mu\,(c\mu x)^{n-c}}{(n-c)!} e^{-c\mu x} \, dx \; \frac{r^n}{\cancel{c^{n-c}\,c!}} p_0 + W_q(0)$$

$(c-1)!$

$$\Rightarrow \sum_{n=c}^{\infty} \int_0^t \frac{\mu(\mu x)^{n-c}}{(n-c)!} e^{-c\mu x}\, dx \; \boxed{\frac{r^n}{(c-1)!}}\, p_0 + W_q(0)$$

$r^n = r^c r^{n-c}$

$*\ \sum_{n=c}^{\infty} \frac{(\mu r x)^{n-c}}{(n-c)!}$

$\sum_{m=0}^{\infty} \frac{(\mu r x)^{m}}{m!}$

$\Rightarrow e^{\mu r x}$

$$\Rightarrow \frac{r^c p_0}{(c-1)!} \int_0^t \mu e^{-c\mu x} \boxed{\sum_{n=c}^{\infty} \frac{(r\mu x)^{n-c}}{(n-c)!}\, dx} + W_q(0)$$

$$\Rightarrow \frac{r^c p_0}{(c-1)!} \int_0^t \mu \underbrace{e^{-c\mu x} e^{\mu r x}}_{e^{-\mu x(c-r)}}\, dx + W_q(0)$$

$$\Rightarrow \frac{r^c p_0}{(c-1)!\,(c-r)} \int_0^t \mu(c-r)\, e^{-\mu(c-r)x}\, dx + W_q(0)$$

Exp CDF

with $\mu(c-r)$

$\Rightarrow 1 - e^{-\lambda t}$

$$\Rightarrow \frac{r^c p_0}{(c-1)!\,(c-r)} \left(1 - e^{-\mu(c-r)t}\right) + W_q(0)$$

$\rho = \frac{r}{c} \therefore c = r\rho$

$$\Rightarrow \frac{r^c p_0}{(c-1)!\, c(1-\rho)} \left(\cancel{1} - e^{-\mu(c-r)t}\right) + \cancel{1 - \frac{r^c p_0}{c!\,(1-\rho)}}$$

Known : $W_q(t) = P_r\{T_q \le t\} = 1 - P_r\{T_q > t\}$

$$\Rightarrow 1 - \frac{r^c p_0}{c!\,(1-\rho)} e^{-\mu(c-r)t}$$

$$P_r\{T_q > t\} = \frac{r^c p_0}{c!\,(1-\rho)} e^{-\mu(c-r)t} \quad !!!$$

$$W(t) = P_r\{T \le t\} = W_q(0)\left(1 - e^{-\mu t}\right) + \left(1 + W_q(0)\right) \cdot W'(t)$$

$$W'(t) = \int_0^t w'(\tau)\, d\tau$$

$$w'(t) = A_1(1-e^{-\mu t}) + A_2(1-e^{-(c\mu-\lambda)t})$$

{ Laplace Transform }

$$\mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\,dt$$

$$f(t) = \mathcal{L}^{-1}(F)$$

if $f(t) = e^{at} \Rightarrow \mathcal{L}(f) = \frac{1}{sa}$

* Linearity: $\mathcal{L}(\alpha f(t) + \beta g(t)) = \alpha \mathcal{L}(f) + \beta(g)$

<pf>

$$W(t) = P_r\{T \le t\}$$

with Condition

prob. without waiting — Service Time ~ Exp CDF = $1-e^{-\lambda t}$ — prob. with waiting — Queue Time

$$\Rightarrow W_q(0)(1-e^{-\mu t}) + (1-W_q(0))\cdot W'(t)$$

let $W'(t) = P_r\{T \le t \mid T_q > 0\}$

Service Time ~ Exp PDF $\Rightarrow f(x) = \mu e^{-\mu x}$

$$\Rightarrow \int_0^t \int_0^\tau f(x)\, g(\tau - x)\, dx\, d\tau$$

Conditional PDF with $T_q > 0 \Rightarrow g(x) = (c\mu-\lambda)e^{-(c\mu-\lambda)x}$

let $w'(t) = f * g \Rightarrow \mathcal{L}(f)\cdot\mathcal{L}(g)$

$$\begin{cases} f(x) = \mu e^{-\mu x} \\ g(x) = (c\mu-\lambda) e^{-(c\mu-\lambda)x} \end{cases}$$

$$\mathcal{L} \Rightarrow \begin{cases} \mathcal{L}(f) = \mu \cdot \frac{1}{s-(-\mu)} \\ \mathcal{L}(g) = (c\mu-\lambda) \cdot \frac{1}{s-(-(c\mu-\lambda))} \end{cases}$$

$$\mathcal{L}(f)\mathcal{L}(g) = A_1 \frac{1}{s+\mu} + A_2 \frac{c\mu-\lambda}{s+c\mu-\lambda}$$

$$w'(t) = \mathcal{L}^{-1}(\mathcal{L}(f)\mathcal{L}(g))$$

$$= A_1(1-e^{-\mu t}) + A_2(1-e^{-(c\mu-\lambda)t}) \quad \#$$

3.4 Choosing the Number of Servers

<Quality Domain (QD)>

Choose c to maintain a constant traffic intensity ρ

* $r \to \infty$, Conjestion : $1-W_q(0) \to \infty$

<e.g.>

$$\rho_{new} = \frac{r_{new}}{c_{new}} = \rho \Rightarrow \frac{r_{new}}{c_{old} + \Delta}$$ ↗ New Stervers

<Quality & Efficiency Domain (QED)>

maintain a constant measure of "Congestion"

*

$r \to \infty$, $\mathbb{C}$: $1 - W_q(0)$ keep constant.

<e.g.>

let Congestion $\triangleq \mathbb{C}(c,r) \equiv 1 - W_q(0) = \dfrac{r^c}{c!\,(1-\rho)} \Big/ \left(\dfrac{r^c}{c!\,(1-\rho)} + \sum_{n=0}^{c-1} \dfrac{c^n}{n!} \right)$

$r_{old} = 9$, $c_{old} = 32$, $\mathbb{C}(c_{old}, r_{old}) = 0.266$

$r_{new} = 36$, $c_{new} = 42$, $\mathbb{C}(c_{new}, r_{new}) = 0.246$

<Efficiency Domain (ED)>

maintain a constant "padding Δ" of servers

*

$r \to \infty$, $\mathbb{C}$: $1 - W_q(0) \to 1$

<e.g.>

$r_{new} = 36$, $c_{new} = r_{new} + \overset{3}{\cancel{\Delta}} = 39$