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Towards Automatic Diagram Chasing

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Context of the internship

This paper presents the work i did during the Research internship of the M1 Jacques Hadamard from ENS Paris-Saclay. It took place from April 19 to July 22, 2022 at the Mathematisches Institut from Albert-Ludwigs-Universität in Freiburg (Germany). The mathematic context of this internship is the informal use by mathematicians of diagram. Especially in fields like commutative algebra, algebraic geometry, algebraic topology, and category theory, many proofs rely on commutative diagrams and diagram chases. But this kind of proof are not really adapted to linear writing in a proof assistant. There is thus a desire of algorithms able to perform these proofs automatically and eventually able to give a proof certificate of the conclusion.

The internship was composed of back and forth between the general theory to identify methods for diagram chasing and actual implementation of an actual algorithm. There was the need to understand the tools from category theory that allows to do diagram chasing and then to actually apply them. For more flexibility the implementation was done in python, however (even if time didn't permit it) the goal was to provide an algorithm in the proof assistant LEAN. A lot of things didn't work as expected and then prevented to reach the step of the implementation in LEAN.

The Covid situation allowed me to spent the whole internship in Germany. I had a confortable space for working in the office of my supervisor. I also had the opportunity to attend to a Differential Galois theory seminar, the aim was to introduce the topic to the people of the lab, and it was a good opportunity to try different things. The very dynamic city of Freiburg was the opportunity to meet a lot of really nice people, and the proximity with the Black Forest was the occasion to do very nice hikes during the week-end.

Acknowledgements

I would like to warmly thank Johan Commelin for offering me to work on this topic and for welcoming me in Freiburg. He was always available to answer my questions and helpful as soon as i needed. He allways did his best so that everything went fine for me.

I would like to thank the people from the fourth floor for being really kind with me.

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Résumé

Le but de ce mémoire est d'étudier des moyens algorithmiques pour déduire à partir d'un graphe des informations sur tout étiquetage (compatible avec la structure) à valeur dans certaines catégories. Ainsi que de présenter les remarques empiriques obtenues suite à diverses implémentations (ou tentative d'implémentations) pratiques.

Les informations que l'on aimerait extraire dans le cas général sont l'exactitude de la composition de deux morphismes mais ce problème est indécidable (au sens algorithmique) ce que amène à se poser des questions plus simples tel que le cas des zéros (qui est décidable mais moins intéressant en pratique) de l'analogue (en théorie des catégories) de l'injectivité et de la surjectivité ou sur la question des graphes avec de cycles.

Dans le cas d'un graphe général, la structure de donnée représenté par le graphe ne permet pas de tenir compte de la structure algébrique de l'opération de composition. A contrario une implémentation naive qui la prend en compte conduit à une explosion combinatoire qui ne permet pas d'utiliser l'algorithme pour quoi que ce soit en pratique.

Cela conduit donc à considérer le cas particulier des doubles complexes de chaines. Sa structure plus simple permet d'éviter les problèmes du cas général, de plus en pratique cela permet de traiter (bien que de façon plus détournée) les cas les plus courants.

Abstract

The purpose of this thesis is to study algorithmic means to deduce, from a graph, information on any labeling (compatible with the structure) with value in certain categories. As well as presenting the empirical remarks obtained following various practical implementations (or attempted implementations).

The information that one would like to extract in the general case is the exactness of the composition of two morphisms but this problem is undecidable (in the algorithmic sense) which leads to ask simpler questions such as the case of zeros (which is decidable but less interesting in practice) , the analogue (in category theory) of injectivity and surjectivity or the question of graphs with cycles.

In the case of a general graph, the data structure represented by the graph does not take into account the algebraic structure of the composition operation. Conversely, a naive implementation that takes it into account leads to a combinatorial explosion that does not allow the algorithm to be used for anything in practice.

This therefore leads to consider the special case of double chain complexes. Its simpler structure makes it possible to avoid the problems of the general case, moreover in practice it makes it possible to treat (although in a more diverted way) the most current cases.

MATHEMATICS SUBJECT CLASSIFICATION 2020: 18-04,18-08,18A10, 18E10

Contents

Introduction

The section 1 is the the place to introduce all notion of category theory that will be used in the rest of the paper.

One want's to package in a diagram the fact that we have some objects, some morphisme between them and some relations. In practice, we take an oriented graph, we label the vertices with objects of a category, the edges with morphism between the label of the source and the label of the target. We want that diagram to be un-ambiguous, that is to say if we follow two path with the same source and the same target we want to have obtain the same morphisme by taking the composition. The section 2 is the place to give formal meaning to that. Elements on how to check that the labeling is un-ambiguous are also presented.

In most of applications, the labels take values in an abelian category. They are the analogous of R-modules for R a ring, and are indeed an adapted set-up for applying the techniques of diagram chasing (for which the best way to understand them is to look at a proof of diagram chasing). They are presented with the tools that allows to treat them as R-modules in the section 3. Mainly the goal of this section is to express the usual set-up of linear algebra in terms of category

In the section 4 are explored some examples of decision problem that can be solved (or not) with some algorithm. It's also explained how it's possible to assume without loss of generality that the graphs are acyclic. That is interesting mainly because this is a standard hypothesis for applying graph algorithm, including the one checking that a labeling is un-ambiguous

The section 5 is a place to give more detail on the work of implementation that was done (and that can be found at `https://github.com/ymonbru/Diagram-chasing`) and to comment on what didn't worked and the empirical results obtained.

In A are presented some example of results one wants to obtain automatically (they all were in the end). Some example of actual diagram chasing are also given. In B is presented an algorithm that compute if a set of relation is enough to get a commutative diagram. In C is presented a theorem that allows one to think an abelian category as being made of modules (and give precise meaning to this sentence). In D are listed some details of the algorithm i implemented. In E is given (as an example) a proof of the snake lemma that was auto-generated by my program. Finally, in F are given some proofs that were too long to fit in the main part of the text.

1 Category

The definitions and the arguments used in the proof came from [1] and [5]

Definition 1.1

A category $\mathcal{C}$ is the following data:

1. *A collection $Ob(\mathcal{C})$ whose elements are called objects of $\mathcal{C}$*
2. *For any X and Y two objects of $\mathcal{C}$ a set $Hom(X, Y)$ whose elements are called morphisme between X and Y*
3. *For any X an object of $\mathcal{C}$ a special element id_X of $Hom(X, X)$*
4. *For any X, Y and Z three objects of $\mathcal{C}$ a function*

$$\circ_{X,Y,Z} : Hom(Y, Z) \times Hom(X, Y) \to Hom(X, Z)$$

such that the following relation holds: if $f : X \to Y, g : Y \to Z$ and $h : Z \to W$ then $f \circ id_X = id_Y \circ f = f$ and $(h \circ g) \circ f = h \circ (g \circ f)$

Example : Set with maps and the usual composition

Example : Any algebraic structure with their morphisms and the usual composition

Example : Topological spaces with continuous maps and the usual composition

Example : Partialy ordered set with the set as objects and $\mathrm{Hom}(x, y) = \{\bullet\}$ if and only if $x \leq y$ (thus there is no choice for the composition-map). Then the axioms of a relation of order are equivalent to the one of a category.

Example : *Category generated by a graph* (V, E, o, t) (see 2.1) Conversely if $\mathcal{C}$ is a category, whose collection of objects is a set, then one get a graph by having an edge between x and y for each element of $\mathrm{Hom}(x, y)$ (it can be empty).

Example : Let $\mathcal{C}$ be a category, we obtain a new category noted $\mathcal{C}^{op}$ by having:

1. $Ob(\mathcal{C}^{op}) = Ob(\mathcal{C})$
2. $\mathrm{Hom}_{\mathcal{C}^{op}}(A, B) = \mathrm{Hom}_{\mathcal{C}}(B, A)$
3. $f \circ_{op} g = g \circ f$

Informally the category is obtained by *reversing the arrows*. We will see a lot of examples where applying a theorem proven in $\mathcal{C}$ to the category $\mathcal{C}^{op}$ provides a new theorem (called *dual theorem*) once interpreted in terms of $\mathcal{C}$

Definition 1.2

If $\mathcal{C}$ and $\mathcal{D}$ are two categories, a $functor$ between $\mathcal{C}$ and $\mathcal{D}$ is the following data:

1. *For each object X of $\mathcal{C}$, an object $F(X)$ of $\mathcal{D}$ (it's noted as un function but if the underling collection are not sets it could possibly not be a function)*
2. *For each $f : X \to Y$ an element $F(f) : F(X) \to F(Y)$*
3. *Such that the following relations holds $F(id_X) = id_{F(X)}$ and $F(g \circ_{\mathcal{C}} f) = F(g) \circ_{\mathcal{D}} F(f)$.*

1.1 Products and coproducts

Definition 1.3

Let A and B be two objects of a category $\mathcal{C}$.

1. *A product of A and B is a object C with two morphisms: $\pi_A : C \to A$ and $\pi_B : C \to B$ of $\mathcal{C}$, satisfying the universal propriety: for every maps $f : X \to A$ and $g : X \to B$, there is a unique morphism $\phi : X \to C$ (denoted $f \times g$) such that the following diagram commutes :*

$$\begin{tikzcd} & & B \\ X \arrow[r, dotted, "\exists!\phi"] \arrow[urr, bend left, "g"] \arrow[drr, bend right, "f"'] & C \arrow[ur, "\pi_B"'] \arrow[dr, "\pi_A"] & \\ & & A \end{tikzcd}$$

2. *A coproduct of A and B is a object C with two morphisms: $i_A : A \to C$ and $i_B : B \to C$ of $\mathcal{C}$, satisfying the universal propriety: for every maps $f : A \to X$ and $g : B \to X$, there is a unique morphism $\phi : C \to X$ (denoted $f \coprod g$) such that the following diagram commutes :*

$$\begin{tikzcd} B \arrow[dr, "i_B"] \arrow[drr, bend left, "g"] & & \\ & C \arrow[r, dotted, "\exists!\phi"] & X \\ A \arrow[ur, "i_A"'] \arrow[urr, bend right, "f"'] & & \end{tikzcd}$$

Remark : The definition does not assumes the existence of products and coproducts and indeed there are general category where they don't exists. It won't be the case in the example considered.

Remark : The two notion are dual: (C, π_A, π_B) is a product of A and B in $\mathcal{C}$ if and only if (C, π_A, π_B) is a coproduct of A and B in $\mathcal{C}^{op}$

Lemma 1.1

A product (resp. a coproduct) is unique up to isomorphism, hence one abusively speak of the product (resp the coproduct) and writes $A \times B$ (resp $A \coprod B$)

Proof : The two statment are dual, therefore it is enough to prove the first one. Let (C, π_A, π_B) and (C', π'_A, π'_B) be two products of A and B. Let ϕ be the morphism of the universal propriety of C applied to π'_A and π'_B:

$$\begin{tikzcd} & & B \\ C' \arrow[r, dotted, "\phi"] \arrow[urr, bend left, "\pi'_B"] \arrow[drr, bend right, "\pi'_A"'] & C \arrow[ur, "\pi_B"] \arrow[dr, "\pi_A"'] & \\ & & A \end{tikzcd}$$

and ψ be the morphism of the universal propriety of C' applied to π_A and π_B:

$$\begin{tikzcd} & & B \\ C \arrow[r, dotted, "\psi"] \arrow[urr, bend left, "\pi_B"] \arrow[drr, bend right, "\pi_A"'] & C' \arrow[ur, "\pi'_B"] \arrow[dr, "\pi'_A"'] & \\ & & A \end{tikzcd}$$

Then it is straightforward to check that the following diagram is commutative:

$$\begin{tikzcd} & & & B \\ C \arrow[r, dotted, "\psi"] \arrow[urrr, bend left, "\pi_B"] \arrow[drrr, bend right, "\pi_A"'] & C' \arrow[r, dotted, "\phi"] \arrow[urr, "\pi'_B"] \arrow[drr, "\pi'_A"'] & C \arrow[ur, "\pi_B"] \arrow[dr, "\pi_A"'] & \\ & & & A \end{tikzcd}$$

in particular $\phi \circ \psi$ is the map given by the universal propriety of C applied to itself. However it is straightforward that id_C is also solution, then by uniqueness $\phi \circ \psi = \mathrm{id}_C$ and by the same argument with C' we get $\psi \circ \phi = \mathrm{id}_{C'}$, hence C and C' are isomorphic. ■

Remark : This technique is general (apply the universal propriety to each other and conclude by uniqueness that the compositions must be the identities) and allow to prove that any object defined by a universal propriety ("there is a unique map such that some diagram commutes") is unique up to isomorphism

Lemma 1.2

The product is associative in the sense that if the products exists $(A \times B) \times C$ and $A \times (B \times C)$ are isomorphic. Then $(f \times g) \times h$ and $f \times (g \times h)$ can be identified. By duality the same result is true for the coproduct.

Proof : A proof may be found in F ■

1.2 Monomorphism and Epimorphism

Definition 1.4

Let $\mathcal{C}$ be a category and $f : X \to Y$ a morphism between two objects of $\mathcal{C}$

1. *f is called isomorphism is there is a morphism $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. g is called and inverse of f.*

2. *f is called a monomorphism if for any morphism a and b with the same domain and with codomain X such that $f \circ a = f \circ b$ then $a = b$. It's denoted $f : X \hookrightarrow Y$.*

3. *f is called an epimorphism if for any morphism a and b with the same codomain with domain Y such that $a \circ f = b \circ f$ then $a = b$. It's denoted $f : X \twoheadrightarrow Y$.*

Remark : A morphism is a monomorphism in $\mathcal{C}$ if and only if it is an epimorphism in $\mathcal{C}^{op}$

Remark : In the category of set the monomorphism are exactly the injection and the epimorphisme are exactly the surjection however if the morphism are depicted as maps, the converse may be false, for example in the category of Rings, $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is an epimorphism but is not surjective.

Remark : $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is also a monomorphism, then a morphism that is a monomorphism and an epimorphism must not be an isomorphism. Although this statement is true if the category is abelian (3.1).

Remark : It is straightforward that if $f : X \to Y$ and $g : Y \to Z$ are monomorphisms (resp. epimorphisms) the $g \circ f$ is a monomorphism (resp. an epimorphism) and that if $g \circ f$ is a monomorphism (resp. an epimorphism) then f is a monomorphism (resp g is an epimorphism).

Remark : If g is an isomorphism and $g \circ f$ is an epimorphism then so does f. Indeed if $a \circ f = b \circ f$ then $a \circ g^{-1} \circ g \circ f = b \circ g^{-1} \circ g \circ f$ then because $g \circ f$ is an epimorphism, $a \circ g^{-1} = b \circ g^{-1}$ then by composing with g one gets $a = b$.

By the same argument, if f is an isomorphism, and $g \circ f$ is a monomorphism, then so does g.

Example : If f is an isomorphisme, then f is a monomorphism and an epimorphism. Indeed, let g be an inverse, if $f \circ a = f \circ b$ then $a = (g \circ f) \circ a = g \circ (f \circ a) = g \circ (f \circ b) = b$ and if $a \circ f = b \circ f$ then $a = a \circ (f \circ g) = (a \circ f) \circ g = (b \circ f) \circ g = b$.

In particular the inverse of f must be unique and is denoted f^{-1}

Definition 1.5

Let B be an object of a category $\mathcal{C}$

1. *The $category of subobjects$ of B is defined by the following:*
   (a) *The objects are the monomorphism of $\mathcal{C}$ with codomain B*
   (b) *The morphisms between $i_1 : A_1 \hookrightarrow B$ and $i_2 : A_2 \hookrightarrow B$ are the morphism $f : A_1 \to A_2$ (of $\mathcal{C}$) such that the diagram commute:*
$$\begin{tikzcd} A_1 \arrow[d, "f"'] \arrow[dr, hook, "i_1"] & \\ A_2 \arrow[r, hook, "i_2"'] & B \end{tikzcd}$$
   (c) *The composition is given by the restriction of $\circ_{\mathcal{C}}$*
2. *Two subobjects are said to be equal (as subobjects of B) if they are isomorphic in the category of subobjects*
3. *The $category of quotient$ of B is defined by the following:*
   (a) *The objects are the epimorphism of $\mathcal{C}$ with domain B*
   (b) *The morphisms between $\pi_1 : B \twoheadrightarrow A_1$ and $\pi_2 : B \twoheadrightarrow A_2$ are the morphism $f : A_1 \to A_2$ (of $\mathcal{C}$) such that the diagram commute:*
$$\begin{tikzcd} & A_1 \arrow[d, "f"] \\ B \arrow[ur, two heads, "\pi_1"] \arrow[r, two heads, "\pi_2"'] & A_2 \end{tikzcd}$$
   (c) *The composition is given by the restriction of $\circ_{\mathcal{C}}$*
4. *Two quotients are said to be equal (as quotients of B) if they are isomorphic in the category of quotients*

Remark : Once again, the two notion are dual: the subobject of B in $\mathcal{C}$ are exactly the quotient of B in $\mathcal{C}^{op}$ and conversely.

Remark : If $g \circ f$ is a monomorphism, then f is also one, in particular a morphism between two subobjects is a monomorphism in the original category, and by duality a morphism between quotients is an epimorphism.

Remark : In particular if two objects are equal as subobjects their domain are isomorphic in $\mathcal{C}$ and if if two objects are equal as quotient objects their codomain are isomorphic in $\mathcal{C}$

Definition 1.6

Let $i : A \hookrightarrow C$ and $j : B \hookrightarrow C$ be two subobjects of an object C. $(I, i_A : I \to A, i_B : I \to B)$ is said to be the intersection of the subobjects if $i \circ i_A = j \circ i_B$ and $I \to C$ is a subobject that satisfy the following universal propriety: For every $X \hookrightarrow C$ subobject of C that factors through i and j, there is a unique map $\phi : X \to I$ such that the following diagram commutes:

$$\begin{tikzcd}
 & & A \arrow[dr, "i"] & \\
X \arrow[urr, bend left, "f"] \arrow[r, dashed, "\exists!\ \phi"] \arrow[drr, bend right, "g"'] & I \arrow[ur, "i_A"] \arrow[dr, "i_B"'] & & C \\
 & & B \arrow[ur, "j"'] &
\end{tikzcd}$$

Example : In the category of modules, if the monomorphism are seen as inclusions, the intersection is $A \cap B$ with the inclusions maps.

1.3 Zero object and zero-map

Definition 1.7

Let $\mathcal{C}$ be a category

1. *An object I of $\mathcal{C}$ is sait to be an* initial object *if for every object X of $\mathcal{C}$ there is a unique morphism from I to X*

2. *An object T of $\mathcal{C}$ is sait to be a* terminal object *if for every object X of $\mathcal{C}$ there is a unique morphism from X to T*

3. *An object 0 of $\mathcal{C}$ is sait to be a* zero-object *if it is both initial and terminal*

4. *If $\mathcal{C}$ has a zero-object 0, the zero-map (denoted $0_{A\to B}$) between A and B two objects of $\mathcal{C}$ is the map*
$$(0 \to B) \circ (A \to 0)$$

Remark : By definition X is initial in $\mathcal{C}$ if and only if X is terminal in $\mathcal{C}^{op}$. Thus X is a zero object in $\mathcal{C}$ if and only if it is a zero object in $\mathcal{C}^{op}$

Remark : Two initial objects are isomorphic one then talk of the initial object of a category. And by the preceding remark, it is also the case for terminals objects.

Indeed if I_1 and I_2 are two initial objects, let f be the unique morphism $I_1 \to I_2$ and g be the unique $I_2 \to I_1$, then $g \circ f$ is a morphisme $I_1 \to I_1$ then by uniqueness $g \circ f = \mathrm{id}_1$, in the same way (as I_2 is initial) $f \circ g = \mathrm{id}_2$ thus f is an isomorphism.

Remark : If $F : \mathcal{C} \Rightarrow \mathcal{D}$ is a functor between two categories with zero object such that $F(0_{\mathcal{C}}) = 0_{\mathcal{D}}$ then F send zero-maps over zero-maps. Indeed, $F(A \to 0) : F(A) \to 0_{\mathcal{D}}$ is by uniqueness $0_{F(A)\to 0_{\mathcal{D}}}$, in the same way (or by duality) $F(0 \to B) = 0_{0_{\mathcal{D}}\to F(B)}$, hence by functoriality $F(0_{A\to B}) = F(0 \to B) \circ F(A \to 0) = 0_{0\to F(B)} \circ 0_{F(A)\to 0} = 0_{F(A)\to F(B)}$

Lemma 1.3

> *If a morphism is obtained from a zero map by right and left composition then it's the zero morphism*

Proof : By induction on the number of composition (the case with zero composition is straightforward) it is enough to proof that $g \circ 0 = 0$ and $0 \circ f = 0$.

1.
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[drr, dashed, "0_{A\to 0}\circ f"'] & A \arrow[rr, dashed, "0_{A\to B}"] \arrow[dr, "0_{A\to 0}"] & & B \\ & & 0 \arrow[ur, "0_{0\to B}"'] & \end{tikzcd}$$

   then $0_{A\to 0} \circ f$ is a map from X to 0, then by uniqueness it's equal to $0_{X\to 0}$. Thus $0_{X\to B} = 0_{0\to B} \circ 0_{X\to 0} = 0_{0\to B} \circ (0_{A\to 0} \circ f) = (0_{0\to B} \circ 0_{A\to 0}) \circ f = 0_{A\to B} \circ f$.

2. The first proof would be by duality : the zero map of $\mathcal{C}$ and the zero map of $\mathcal{C}^{op}$ are the same and the second statement is exactly the first stated in $\mathcal{C}^{op}$. But we can also draw a diagram:

$$\begin{tikzcd} A \arrow[rr, dashed, "0_{A\to B}"] \arrow[dr, "0_{A\to 0}"'] & & B \arrow[r, "g"] & X \\ & 0 \arrow[ur, "0_{0\to B}"] \arrow[urr, dashed, "g\circ 0_{0\to B}"'] & & \end{tikzcd}$$

   $g \circ 0_{0\to B}$ is a map from 0 to X, then by uniqueness it's equal to $0_{0\to X}$. Thus $0_{A\to X} = 0_{0\to X} \circ 0_{A\to 0} = (g \circ 0_{0\to B}) \circ 0_{A\to 0} = g \circ (0_{0\to B} \circ 0_{A\to 0}) = g \circ 0_{A\to B}$. ■

1.4 Kernels and Cokernels

Definition 1.8

> *Let $f : A \to B$ and $g : A \to B$ be two morphism in $\mathcal{C}$*
>
> 1. *A morphism $i : E \to A$ is called equalizer of f and g (and noted $Eq(f,g)$)) if it satisfies the universal propriety: For every $j : X \to A$ such that $f \circ j = g \circ j$ there is a unique map $\varphi : X \to E$ such that the following diagram commutes:*
>
> $$\begin{tikzcd} X \arrow[d, dotted, "\exists!\ \varphi"'] \arrow[dr, "j"] & & \\ E \arrow[r, "i"'] & A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & B \end{tikzcd}$$
>
> 2. *If $\mathcal{C}$ has a zero-object, a kernel of f is an equalizer of f and $0_{A\to B}$. It is noted* $\ker(f)$.
>
> 3. *A morphism $\pi : B \to CE$ is called coequalizer of f and g (and noted $coEq(f,g)$) if it satisfies the universal propriety: For every $j : B \to Y$ such that $j \circ f = j \circ g$ there is a unique map $\psi : CE \to Y$ such that the following diagram commutes:*
>
> $$\begin{tikzcd} & & Y \\ A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & B \arrow[ur, "j"] \arrow[r, "\pi"] & CE \arrow[u, dotted, "\exists!\ \psi"'] \end{tikzcd}$$
>
> 4. *If $\mathcal{C}$ has a zero-object, a cokernel of f is an coequalizer of f and $0_{A\to B}$. It is noted coker(f).*

Remark : It is straightforward that equalizer and coequalizers are duals notions: a morphism is an equalizer in $\mathcal{C}$ if and only if it's a coequalizer in $\mathcal{C}^{op}$

Remark : If $i : E \to A$ is the equalizer of f and g, then i is a monomorphism. Indeed if $u : X \to E$ and $v : X \to E$ are such that $i \circ u = i \circ v$ then $f \circ i \circ u = g \circ i \circ u$ thus by the uniqueness in the factorisation of $i \circ u$ through i, $u = v$. By duality a coequalizer is an epimorphism.

Remark : Because of the universal propriety the equalizers and coequalizers are unique (as subobjects or quotient objects) up to unique isomorphism. For example with the kernel: if (K, i) and (K', i') are two solution, we get $\phi : K \to K'$ by applying the universal propriety of K' to K and $\psi : K' \to K$ by applying the universal propriety of K to K'. It's straightforward that $\phi \circ \psi$ satisfy the universal propriety of K' to K' hence by uniqueness, $\phi \circ \psi = \mathrm{id}_{K'}$. By the same argument $\psi \circ \phi = \mathrm{id}_K$

Example : If $\mathcal{C}$ is the category of R-modules (for R a ring) then the usual kernel with the inclusion map is a kernel:
If $j : X \to A$ is such that $f \circ j = 0$ then if ϕ exists, $\forall x \in X$, $i \circ \varphi(x) = j(x)$ but i is injective (an inclusion) therefore if φ exists it is unique. Conversely, because $f \circ j = 0$, $\mathrm{im}(j) \subset \{x \in A / \ f(x) = 0\} = i(\ker(f))$. i is injective so bijective onto it's image then $\phi = \mathrm{i}^{-1} \circ j$ is a well defined morphism, and it's straightforward that the diagram commutes.

In the same way the usual cokernel $(B/_{\mathrm{im}(f : A \to B)})$ with the canonical projection is a cokernel: If $j : B \to Y$ is such that $j \circ f = 0$ then if ψ exists $\forall b \in B$, $\psi \circ \pi(b) = j(b)$, then (because π is surjective) $\psi : \pi(b) \mapsto j(b)$ is uniquely defined. Conversely let's have $\psi : \pi(b) \mapsto j(b)$, it's well defined because if $\pi(b) = \pi(b')$ then $b - b' \in \mathrm{im}(f)$ and because $j \circ f = 0$ we get $j(b) - j(b') = j(b - b') = 0$ and it's straightforward that the diagram commutes.

Lemma 1.4

Let $f : A \to B$ and g be two morphism in $\mathcal{C}$ a category with a zero-object.

1. *If g is a monomorphism, then f has a kernel if and only if $g \circ f$ has a kernel and in that case:* $\ker(g \circ f) = \ker(f)$
2. *If f is an epimorphism, then then g has a cokernel if and only if $g \circ f$ has a cokernel and in that case: coker$(g \circ f) =$ coker(g)*

Proof : The two statements are dual therefore it is enough to prove the first one.

If f has a kernel, let $j : X \to A$ be a morphism such that $g \circ f \circ j = 0$. If there is a ϕ such that $j = \ker(f) \circ \phi$, because $\ker(f)$ is a monomorphism, ϕ must be unique. Conversely $g \circ f \circ j = 0 = g \circ 0$. But g is a monomorphism, then $f \circ j = 0$, then by the universal propriety of the kernel of f, there is a map $\phi : X \to K(f)$ such that $j = \ker(f) \circ \phi$. Then by definition $g \circ f$ has a kernel and it's $\ker(f)$.

If $g \circ f$ has a kernel let $j : X \to A$ be a morphism such that $f \circ j = 0$. If there is a ϕ such that $j = \ker(g \circ f) \circ \phi$, because $\ker(g \circ f)$ is a monomorphism, ϕ must be unique. Conversely $g \circ f \circ j = 0 = g \circ 0 = 0$. Then by the universal propriety of the kernel of $g \circ f$, there is a map $\phi : X \to K(g \circ f)$ such that $j = \ker(g \circ f) \circ \phi$. Then by definition f has a kernel and it's $\ker(g \circ f)$. ∎

Definition 1.9

If $f : A \to B$ has a kernel $i : K \to A$ and i has a cokernel, then it is called the image of f. Dualy if f has a cokernel π and π has a kernel, it is called the coimage of f

Remark : Then a kernel (in particular an image) is always a subobject, and a cokernel (in particular a coimage) is always a quotient object.

Remark : In particular if $A \hookrightarrow B$ is a subobject of B then the quotient object $\mathrm{coker}(A \hookrightarrow B)$ is denoted (if there is no ambiguity over $A \hookrightarrow B$) $B \twoheadrightarrow B/_A$

1.5 Pullback

Definition 1.10

1. *Let $f : A \to C$ and $g : B \to C$ be two morphism, then $(X, h : X \to A, k : X \to B)$ is said to be the pullback of f and g if $f \circ h = g \circ k$ and if the following universal propriety holds: For every $u : Y \to A$ and $v : Y \to B$ such that $f \circ u = g \circ v$ there is a unique map $\phi : Y \to X$*

such that the following diagram commutes:

$$\begin{tikzcd}
 & & A \arrow[dr, "f"] & \\
Y \arrow[r, dashed, "\exists!\ \phi"] \arrow[urr, bend left, "u"] \arrow[drr, bend right, "v"'] & X \arrow[ur, "h"] \arrow[dr, "k"'] & & C \\
 & & B \arrow[ur, "g"'] &
\end{tikzcd}$$

Remark : The dual notion is called $puschforward$

Example : In an abelian category, according to F.1, the intersections of two subobjects is their pullbacks

2 Graphs and Diagrams

2.1 Graphs

Definition 2.1

1. *A graph is the following data:*
   (a) *A set V and a set E whose elements are called vertex and edges of the graph*
   (b) *Two functions o and t :$E \to V$ called origin map and tail map*
2. *A path in G between x and y to vertex of G is a finite sequence $a_1, \ldots, a_n$ of edges such that $o(a_1) = x$, $t(a_n) = y$ and $\forall i \in [\![1, n-1]\!]$ $t(a_i) = o(a_{i+1})$*
3. *If (V, E, o, t) is a graph D, then the graph (V, E, t, o) is called the opposite graph and noted D^{op}*

Remark : It is not one of the most usual definitions but it is quite flexible and it allows to have an easier implementation

Definition 2.2

Let D be a graph and x a vertex of D

1. *$Anc(x)$ is the set of $y \in D$ such that there is a (non-trivial) path from y to x in D. The elements are called the ancestors of y.*
2. *$Des(x)$ is the set of $y \in D$ such that there is a (non-trivial) path from x to y in D. The elements are called the descendants of y.*

Remark : If we see $\mathcal{C}(D)$ (see 2.3) as a graph, it's straightforward that two vertex are connected in D if and only if they are in $\mathcal{C}(D)$, the only thing that changes is the length of the path. therefore the two sets are the same for the two graphs.

Remark : By definition, if x is a vertex of D, then $Anc_D(x)$ is the set $Des_{D^{op}}(x)$, that allows us to compute the two set with the same function by using the opposite graph.

Remark : Those two sets can be easily computed in $O(\#V + \#E)$ time by using a DFS algorithm

Remark : If D is a graph and x a vertex of D, the the strongly connected component of x (set of y such that there is a path from x to y and from y to x) in D is $\{x\} \cup (Anc(x) \cap Des(x))$.

Definition 2.3

If $G = (V, E, o, t)$ is a graph, lets call $\mathcal{C}(G)$ the category generated by the following data:

1. $Ob(\mathcal{C}) = V$

2. *If A is a vertex of G such that there is a path from A to A in G:*

$$Hom(A, A) = \{e \in E / o(e) = A,\ t(e) = B\} \uplus \{\bullet_{A,A}\} \uplus \{id_A\}$$

with id_A and $\bullet_{A,A}$ two new symbols. Otherwise

$$Hom(A, A) = \{e \in E / o(e) = A,\ t(e) = B\} \uplus \{id_A\}$$

3. *If A and B are two vertex of G and that are path connected (with length at least one) then*

$$Hom(A, B) = \{e \in E / o(e) = A,\ t(e) = B\} \uplus \{\bullet_{A,B}\}$$

with $\bullet_{A,B}$ a new symbol

4. *If A and B are two vertex of G not path connected then $Hom(A, B) = \varnothing$*

5. *If $f \in Hom(A, B)$ then $f \circ id_A = f$ and $id_B \circ f = f$*

6. *If $g \in Hom(B, C)$ and $f \in Hom(A, B)$ then (in particular there is a path from A to C) $g \circ f = \bullet_{A,C}$. Otherwise $\circ$ is the empty map (the unique possibility)*

Remark : It is possible to verify by induction over the length of paths that $\mathcal{C}(D)^{op} = \mathcal{C}(D^{op})$

2.2 Diagrams

DEFINITION 2.4

1. *A graph G is sait to be acyclic if there is no path with the same origin and end-point.*

2. *A path in G will be said to be simple if it does not contain twice the same arrow.*

Remark : In the case of oriented graphs, this notion is not the notion of Tree: the following graph is acyclic but not a tree

$$\begin{tikzcd}
 & \bullet \arrow[dl] \arrow[dr] & \\
\bullet \arrow[dr] & & \bullet \arrow[dl] \\
 & \bullet &
\end{tikzcd}$$

DEFINITION 2.5

1. *A commutative diagram over a category $\mathcal{C}$, if a functor from the category induced by a graph to the category $\mathcal{C}$*

2. *Let D is a graph, a pre-diagram $f : D \Rightarrow \mathcal{C}$ is a map that send vertex of D into a category $\mathcal{C}$ and the edges $x \to y$ of D into $Hom(f(x), f(y))$*

LEMMA 2.1

Let $f : D \Rightarrow \mathcal{C}$ be a pre-diagram. If for any $x_1 \to \ldots \to x_n$ and $y_1 \to \ldots \to y_n$, two simple path between u and v any two vertex of D the relation $f(x_{n-1} \to x_n) \circ \ldots \circ f(x_1 \to x_2) = f(y_{m-1} \to y_m) \to \ldots \to f(y_1 \to y_2)$ holds then f extend in a unique way into a diagram $F : \mathcal{C}(D) \Rightarrow \mathcal{C}$

Proof: If F is a functor that extend f, then $\forall x \in D = Ob(\mathcal{C})$ $F(\mathrm{id}_x) = \mathrm{id}_{F(x)}$. If there is a path from A to B: $x_1 \to \ldots \to x_n$ then by functoriality $F(\bullet_{A,B}) = F(x_{n-1} \to x_n) \circ \ldots \circ F(x_1 \to x_2) = f(x_{n-1} \to x_n) \circ \ldots \circ f(x_1 \to x_2)$. All the others images, are defined by f, then the values of F are uniquely determined by f, thus F is unique.

The condition is in fact valid for non-simple path:lLet $x^0 \to \ldots \to x^k$ be a simple path in D, then by construction $f(x^{k-1} \to x^k) \circ \ldots \circ f(x^0 \to x^1)$ only depends on x^0 ans x^k. In particular if $x^0 \to \ldots \to x^k$ is a simple cycle $f(x^{k-1} \to x^k) \circ \ldots \circ f(x^0 \to x^1) = \mathrm{id}$, and because a cycle is a composition of simple cycles, it's also true for cycles. Then if $x^0 \to \ldots \to x^k$ is a path in D and $x^0 = y^0 \to \ldots \to y^l = x^k$ is the path $x^0 \to \ldots \to x^k$ without the cycles: $f(x^{k-1} \to x^k) \circ \ldots \circ f(x^0 \to x^1) =$f(y$^{l-1} \to y^l) \circ \ldots \circ f(y^0 \to y^1)$. However a path without cycle is necessarily simple, then the composition only depend on $y^0 = x^0$ and $y^l = x^k$.

Let F be defined by the following:

1. $\forall x \in D$ $F(x) = f(x)$
2. $\forall x \in D$ $F(\mathrm{id}_x) = \mathrm{id}_{F(x)}$
3. $\forall x \to y \in E$ $F(x \to y) = f(x \to y)$
4. If $x_1 \to \ldots \to x_n$ is a in D then $F(\bullet_{x_1,x_2}) = F(x_{n-1} \to x_n) \circ \ldots \circ F(x_1 \to x_2)$ (it is well defined because the value does not depend of the path)

To conclude that F is a functor, we need to check the composition rule. Let $a : x \to y$ and $b : y \to z$ be two composable map of $\mathcal{C}(D)$. Then $b \circ a = \bullet_{x,z}$. Let $x = u_1 \to \ldots \to u_n = y$ and $y = v_1 \to \ldots \to v_m = z$ be two path in D (that exists by construction of $\mathcal{C}(D)$), then $u_1 \to \ldots \to u_n = v_1 \to \ldots \to v_m$ is a path from x to z, then by definition $F(\bullet_{x,z}) = f(v_{m-1} \to v_m) \circ \ldots \circ f(v_1 \to v_2) \circ f(u_{n-1} \to u_n) \circ \ldots \circ f(u_1 \to u_2) = F(b) \circ F(a)$ by definition, thus F is a functor. ■

Remark: It is even possible to only consider the case of path with non common vertex (except u and v)

A diagram with a cycle has an infinite number of distinct path, and more generally, diagram chasing in a diagram with a cycle is more subtil. Following an idea of G.Kuperberg in [2] (the construction proposed was with $n = 2$) it's possible to map any graph to an acyclic one that will encode the informations about the original diagram.

Definition 2.6

Let G be a finite graph, and n be the maximal length of a simple path in G. Let's define a graph $AC(G)$ by the following:

1. *The vertex of $AC(G)$ are of the form A_i with A a vertex of G and $i \in [\![0, n]\!]$*
2. *The edges of $AC(G)$, are of the form $\phi_{A,i} : A_i \to A_{i+1}$ or $f_i : A_i \to B_i$ with $f : A \to B$ an edge of G*

Let $F : D \Rightarrow \mathcal{C}$ be a pre-diagram, the pre-diagram $AC(F) : AC(D) \Rightarrow \mathcal{C}$ is defined by the following data:

1. *If A is a vertex in D and $i \in [\![0, n]\!]$, then $AC(F)(A_i) = F(A)$*
2. *If A is a vertex in D and $i \in [\![0, n]\!]$, then $AC(F)(\phi_{A,i}) = \mathrm{id}_{F(A)}$*
3. *If f is an edge of D and $i \in [\![0, n]\!]$, then $AC(F)(f_i) = F(f)$*

Remark: The tail of any arrow in $AC(G)$ has one index bigger than the one of its origin. Then by straightforward induction, it's also true for any path. It's therefore impossible to have a path from a vertex to itself, then $AC(G)$ is acyclic.

Remark: If G is acyclic, then $AC(G)$ is not G, however by using a DFS algorithm, it's possible to detect in linear time if a graph is acyclic or not and then to use the AC construction only in case of need.

Example : The graph $\begin{tikzcd} A \arrow[r, bend left, "f"] & B \arrow[l, bend left, "g"] \end{tikzcd}$ is send to : $\begin{tikzcd} A \arrow[d, "\phi_A"'] \arrow[dr, "f"] & B \arrow[d, "\phi_B"] \arrow[dl, "g"'] \\ A' \arrow[d, "\phi'_A"'] \arrow[dr, "f'"] & B' \arrow[d, "\phi'_B"] \arrow[dl, "g'"'] \\ A'' & B'' \end{tikzcd}$

Lemma 2.2

Let $f : D \Rightarrow \mathcal{C}$ be a pre-diagram over a finite graph. Then f satisfy the hypothesis of 2.1 if and only if it's the case for $AC(f)$

Proof : Let's assume that f satisfy the hypothesis of 2.1, let F be the corresponding diagram. Let $x_i^0 \to \ldots \to x_{i+k}^k$ be a path in $AC(D)$. To conclude, it's enough to show that $AC(f)(x_{i+k-1}^{k-1} \to x_{i+k}^k) \circ \ldots \circ AC(f)(x_i^0 \to x_{i+1}^1) = F(x^0 \to x^k)$. With the convention that in D $F(x \to x) = \mathrm{id}_{F(x)}$, the construction gives $AC(f)(x_{i+j-1}^{j-1} \to x_{i+j}^j) = F(x^{j-1} \to x^j)$, then by functoriality of F: $AC(f)(x_{i+k-1}^{k-1} \to x_{i+k}^k) \circ \ldots \circ AC(f)(x_i^0 \to x_{i+1}^1) = F(x^{k-1} \to x^k) \circ \ldots \circ F(x^0 \to x^1) = F(x^0 \to x^k)$.

Conversely, let $x^0 \to \ldots \to x^k$ be a simple path in D, then ($k \leq n$) by construction $x_0^0 \to \ldots \to x_k^k$ is a valid path in $AC(D)$, then $AC(f)(x_{k-1}^{k-1} \to x_k^k) \circ \ldots \circ AC(f)(x_0^0 \to x_1^1)$ only depends of x_0^0 and x_k^k. But $AC(f)(x_{k-1}^{k-1} \to x_k^k) \circ \ldots \circ AC(f)(x_0^0 \to x_1^1) = f(x^{k-1} \to x^k) \circ \ldots \circ f(x^0 \to x^1)$ then $f(x^{k-1} \to x^k) \circ \ldots \circ f(x^0 \to x^1)$ only depends of $x0$ and x^k ■

Remark : To prove this theorem, it is just necessary to have n such that if the hypothesis of 2.1 is true for all path of length smaller than n then it's true for all path. Then one can just take $n = \#E$ and there is no need to compute it, but also look for an optimal value.

Corollary 2.1

It's possible to check in polynomial time the condition of 2.1, and thus to check if a finite diagram is commutative

Proof : In [3] the author proves that it can be donne for acyclic diagrams in $O(C_F \#V^2 \#E)$, with C_F the complexity of the computation of composition. For an acyclic diagram the construction multiply by n the number of vertex, the new number of edges is $(n-1)\#V + n\#E$. Because $n \leq \#V$ the complexity remains polynomial. ■

Remark : This is not fully satisfying for a proof assistent, because one would like to give to the computer the proof of some compositions relations, and then use an algorithm that provide a proof that the diagram commutes. However the algorithm in [3] compute a set (minimal) of relations that it's enough to check, but this set is not necessarily unique:

$$\begin{tikzcd} \bullet \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, "x"] & \bullet \arrow[d, "g"] \\ \bullet \arrow[r, "b"'] & \bullet \end{tikzcd}$$

To conclude that the diagram commutes it's enough to prove one of the conditions: $\{g \circ f = x, b \circ a = x\}$ or $\{g \circ f = x, b \circ a = g \circ f\}$. By rewriting the proof of theorem 2.2 in [4],(and rephrasing some definitions) a straightforward corollary is that a family of graph are connected if and only if verifying the condition of lemma 2.1 on the set of pair of path R is enough to get it for all path. Moreover all the computation can be donne in polynomial time (in $O(\#V^2 \#E)$). The algorithm is detailed in B

3 Abelian category

The goal of abelian category is to express in the language of category what is needed to do homological algebra. The definitions, and arguments used in the proof came from [5] and [6]

Definition 3.1

A category $\mathcal{C}$ is said to be abelian if:

1. *$\mathcal{C}$ has a zero-object.*
2. *The products and coproducts of two object exists in $\mathcal{C}$*
3. *Every morphism has a kernel and a cokernel*
4. *Every monomorphism is a kernel and every epimorphism is a cokernel*

Remark : All the axioms are self-duals, therefore $\mathcal{C}$ is abelian if and only if $\mathcal{C}^{op}$ is abelian.

Remark : In particular all the images of morphisms exists.

Example : If R is a ring, then the category of R-modules is abelian: the zero-object is the zero-module $\{0\}$, the kernels and cokernels are the usual ones, every monomorphism is the kernek of it's cokernel and every epimorphism is the cokernel of it's kernel.

3.1 Theorem in Abelian category

Let $\mathcal{C}$ be an abelian category.

Theorem 3.1

A morphism $f : A \to B$ in $\mathcal{C}$ is an isomorphism if and only if f is a monomorphism and an epimorphism.

Proof : the direct sense is discussed in 1.2. Conversely, if f is a monomorphism and an epimorphism, because $\mathcal{C}$ is abelian, f is the cokernel of some map $a : A' \to A$ and the kernel of some map $b : B \to B'$.

Then the following diagram is commutative (because $b \circ 0 = 0 \circ 0$):
$$\begin{tikzcd}
A' \arrow[d, bend right, "0"'] \arrow[d, bend left, "a"] \arrow[dr, dotted, "f\circ a=0"] & & \\
A \arrow[r, hook, two heads, "f"] & B \arrow[r, "b"] & B'
\end{tikzcd}$$
then by the uniqueness (in the universal propriety of $\ker(b)$), of the factorisation of $f \circ a$ we get $a = 0$. Then $\mathrm{id}_A \circ a = \mathrm{id}_A \circ 0 = 0_{A' \to A}$, thus the universal propriety of $\mathrm{coker}(a)$ gives us a morphism ψ such that $\psi \circ f = \mathrm{id}_A$:
$$\begin{tikzcd}
 & & A \\
A' \arrow[r, bend left, "a=0"] \arrow[r, bend right, "0"'] & A \arrow[ur, "id"] \arrow[r, hook, two heads, "f"] & B \arrow[u, "\psi"']
\end{tikzcd}$$
On the other hand, the following diagram is commutative (because $b \circ 0 = 0 \circ 0$):
$$\begin{tikzcd}
 & & B' \\
A' \arrow[r, bend left, "a"] \arrow[r, bend right, "0"'] & A \arrow[ur, dotted, "b\circ f=0"] \arrow[r, hook, two heads, "f"] & B \arrow[u, bend left, "b"] \arrow[u, bend right, "0"']
\end{tikzcd}$$
then by the uniqueness (in the universal propriety of $\mathrm{coker}(a)$), of the factorisation of $b \circ f$ we get $b = 0$. Then $b \circ \mathrm{id}_B = 0 \circ \mathrm{id}_B = 0_{B \to B'}$, thus the universal propriety of $\ker(b)$ gives us a morphism ϕ such that $f \circ \phi = \mathrm{id}_B$:
$$\begin{tikzcd}
B \arrow[d, dotted, "\phi"] \arrow[dr, "id"] & & \\
A \arrow[r, hook, two heads, "f"] & B \arrow[r, bend left, "b"] \arrow[r, bend right, "0"'] & B'
\end{tikzcd}$$
.

To conclude that f is an isomorphisme it's therefore enough to have $\phi = \psi$. The relation holds because: $\phi = \mathrm{id}_A \circ \phi = (\psi \circ f) \circ \phi = \psi \circ (f \circ \phi) = \psi \circ \mathrm{id}_B = \psi$. ■

Lemma 3.1

Let $f : A \to B$ be a morphism in $\mathcal{C}$

1. *f is a monomorphism if and only if it's kernel is* $0 : 0 \to A$

2. *f is an epimorphism if and only if it's cokernel is* $0 : B \to 0$

Proof: The two statements are dual therefore it is enough to prove the first one. If f is a monomorphism, and j is a morphism such that $f \circ j = 0 = f \circ 0$, then $j = 0$ thus by definition of a zero map, j factors through $0_{0 \to A}$ in a unique way, thus by definition $0_{0 \to A}$ is the kernel of f.

Conversely if the kernel of f is $0_{0 \to A}$, let u and v be two morphism with the same domain X and codomain A such that $f \circ u = f \circ v$, then w be the coequalizer of u and v, and ϕ the universal map given by f.

$$\begin{tikzcd}
 & & B \\
X \arrow[r, bend left, "v"] \arrow[r, bend right, "u"'] & A \arrow[ur, "f"] \arrow[r, two heads, "w"] & CE \arrow[u, dashed, "\phi"]
\end{tikzcd}$$

But w is an epimorphism and then the cokernel of some morphism g, $f \circ g = \phi \circ w \circ g = \phi \circ 0 = 0$ then g factors through $\ker(f) = 0$ thus (by 1.3) g is a zero-map.

Then $id \circ g = 0$ thus id factors through w, let ψ be the universal map such that $\mathrm{id} = \psi \circ w$, but id is an epimorphism, then w is also an epimorphism. By definition of the coequalizer $w \circ u = w \circ v$, thus $u = v$. ■

Lemma 3.2

1. *If i is a monomorphism, then* $im(i) = \ker(coker(i)) = i$

2. *If p is an epimorphism, then* $coim(p) = coker(\ker(p)) = p$

Proof: The two statement are dual, then it is enough to prove the first one. Let $i : A \to B$ be a monomorphism, let $f : B \to C$ be it's cokernel then by definition $f \circ i = 0$.

Let $j : X \to B$ such that $f \circ j = 0$. i is a monomorphism then the kernel of some morphism $g : B \to D$, then $g \circ i$, thus there is a unique map $\phi : C \to D$ such that $\phi \circ f = g$.

Then $g \circ j = \phi \circ 0 = 0$ thus j factors through $ker(g) = i$, and because i is a monomorphism, the factorisation must be unique. Then by definition i is the kernel of f. ■

Remark: A kernel is a monomorphism an a cokernel an epimorphism thus the reciprocal statement are also true.

Remark: Then the image of a monomorphism is itself and the coimage of an epimorphism is itself.

Lemma 3.3

Let $f : A \to B$ be a morphism in $\mathcal{C}$. Let $CI(f)$ be the codomain of it's coimage and $I(f)$ the domain of it's image. Then there is a unique isomorphism $\phi : CI(f) \to I(f)$ such thta the diagram commutes:

$$\begin{tikzcd}
K(f) \arrow[r, hook, "ker(f)"] & A \arrow[r, "f"] \arrow[d, two heads, "coim(f)"] & B \arrow[r, two heads, "coker(f)"] & CK(f) \\
 & CI(f) \arrow[r, "\phi"] & I(f) \arrow[u, "im(f)"'] &
\end{tikzcd}$$

In particular, $CI(f)$ and $I(f)$ can be identified.

Proof: The proof can be found in F ■

Remark: In particular, for every morphism: $f = \mathrm{im}(f) \circ \mathrm{coim}(f)$. *i.e.* any map can be writen as a composition of a monomorphism with an epimorphism. Moreover, this decomposition is unique up to isomorphism according to 3.4.

Lemma 3.4

> *Let $i : I \to B$ be a monomorphism, $p : A \to I$ an epimorphism. Let $f = i \circ p$, then i is an image of f and p a coimage of f.*

Proof: By duality, it is enough to prove that i is the image of f. p is an epimorphism, then by 1.4 $\mathrm{coker}(f) = \mathrm{coker}(i)$. But i is a monomorphism, then by 3.2 $i = \ker(\mathrm{coker}(i)) = \ker(\mathrm{coker}(f)) = \mathrm{im}(f)$. ■

Theorem 3.2

> *If A and B are two objects of $\mathcal{C}$ then $A \times B$ and $A \coprod B$ are isomorphic, hence one identifies then (and denote $A \bigoplus B$) the object*

Proof: The proof can be found in F ■

Remark: Then $\pi_1 \circ i_1$ is seen as $\pi_1 \circ \phi \circ i_1 = \pi_1 \circ (\phi \circ i_1) = \pi_1 \circ (\mathrm{id}_A \times 0) = \mathrm{id}_A$, and with the same argument (using the composition rule with i and π from the universal propriety) one gets that $\pi_1 \circ i_2 = 0_{B \to A}$, $\pi_2 \circ i_1 = 0_{A \to B}$ and $\pi_2 \circ i_2 = \mathrm{id}_B$.

Theorem 3.3

> *An abelian category $\mathcal{C}$ is pre-additive in the sense that:*
>
> 1. *$\mathcal{C}$ has a zero object.*
> 2. *For any A and B two objects of $\mathcal{C}$, there is an abelian group structure:$+_{A,B}$ on Hom(A, B)*
> 3. *$0_{A \to B}$ is the neutral element for $+_{A,B}$*
> 4. *$\circ$ is $\mathbb{Z}$-bilinear with respect to $+$*

Proof: Let f and g be two elements of $\mathrm{Hom}(A, B)$, then $f+g$ is defined by being the map $(f \coprod g) \circ (\mathrm{id}_A \times \mathrm{id}_A)$ (the composition is well defined because of the identification of $A \times A$ and $A \coprod A$ in abelian category)

$$\begin{tikzcd}
 & A & & A \arrow[dl, "i_2"'] \arrow[dr, "g"] & \\
A \arrow[ur, "id"] \arrow[dr, "id"'] \arrow[rr, "id \times id"'] & & A \oplus A \arrow[ul, "\pi_2"'] \arrow[dl, "\pi_1"] \arrow[rr, "f \coprod g"] & & B \\
 & A & & A \arrow[ul, "i_1"] \arrow[ur, "f"'] &
\end{tikzcd}$$

The proof of the claim can be found in F ■

Remark: If $\mathcal{C}$ is a category of R-modules then $f + g$ is just the linear map that send x over $f(x) + g(x)$

Remark: If we assume that $+$ is an additive structure over an abelian category, then the proof (in F) of the relation $(f \coprod g) \circ (u \times v) = f \circ u + g \circ v$ is still valid and then $+$ must be equal to the structure defined in the proposition. In particular the additive structure given by an abelian category and it's opposite category are the same.

Lemma 3.5

> *Let $f, g : A \to B$ be two morphism in an abelian category C then :*
>
> 1. $\ker(f - g) = Eq(f, g)$
> 2. *coker*$(f - g) = coEq(f, g)$

Proof: Because $+$ is the same in $\mathcal{C}$ and in $\mathcal{C}^{op}$, the two statements are dual, therefore it's enough to prove the first one.

Let $j : X \to A$ be such that $(f - g) \circ j = 0$. If there is $i : X \to EQ$ such that $j = Eq(f, g) \circ i$ then because $Eq(f, g)$ is a monomorphism, i must be unique.

Conversely $\circ$ is bilinear then $f \circ j - g \circ j = 0$ then $f \circ j = g \circ j$, then by the universal propriety of the equalizer there is i such that $j = Eq(f,g) \circ i$. Then by definition $Eq(f,g)$ is a kernel of $f - g$. ■

Lemma 3.6

An abelian category admits all the pullbacks and the push-forward

Proof: By duality it's enough to construct the pullbacks. Let $f : A \to C$ and $g : B \to C$ be two morphism with the same codomain. By $F.2$, let $k : EQ \to A \bigoplus B$ be the equalizer of $f \circ \pi_A : A \bigoplus B \to C$ and $g \circ \pi_B : A \bigoplus B \to C$. Then $(EQ, \pi_A \circ k, \pi_B \circ k)$ is a pullback.

By construction of k: $(f \circ \pi_A) \circ k = (g \circ \pi_B) \circ k$ then by associativity: $f \circ (\pi_A \circ k) = g \circ (\pi_B \circ k)$. Let (X, u, v) be such that $f \circ u = g \circ v$. If there is a map z such that $u = \pi_A \circ k \circ z$ and $v = \pi_B \circ k \circ z$, then $k \circ z = u \times v$. However k is a monomorphism, then z must be unique.

Conversely let w be the map $u \times v$. $(f \circ \pi_A) \circ w = f \circ (\pi_A \circ w) = f \circ u = g \circ v = (g \circ \pi_B) \circ w$ then by the universal propriety of the equalizer, there is a map z such that $w = k \circ z$, thus $u = \pi_A \circ (k \circ z) = (\pi_A \circ k) \circ z$ and $v = (\pi_B \circ k) \circ z$. ■

Remark: With the notation of the previous proof, if f is an epimorphism then $\pi_B \circ k$ is also an epimorphism (and by the same argument if g is an epimorphism then $\pi_A \circ k$ is also an epimorphism).

First, $f \circ \pi_A - g \circ \pi_B$ is an epimorphism because $f = (f \circ \pi_A - g \circ \pi_B) \circ i_A$ is. Let $b : B \to Y$ be a maps such that $b \circ \pi_B \circ k = 0$. By 3.5 $k = Eq(f \circ \pi_A, g \circ \pi_B) = \ker(f \circ \pi_A - g \circ \pi_B).Thus\text{coker(k)=coker}(\ker(f \circ \pi_A - g \circ \pi_B)) = f \circ \pi_A - g \circ \pi_B$ because it's an epimorphism (3.2).

Then by the universal propriety there is a map j such that $b \circ \pi_B = j \circ (f \circ \pi_A - g \circ \pi_B)$. Thus $j \circ f = j \circ (f \circ \pi_A - g \circ \pi_B) \circ i_A = b \circ \pi_B \circ i_A = 0$, but f is an epimorphism then $j = 0$ then $b \circ \pi_B = 0$ but π_B is an epimorphism then $b = 0$.

3.2 Diagram Chase

In this section are presented the useful results for doing diagram chase without using elements, and thus to approach the automatisation. The arguments in the proofs came from [7]

In the first place it is not the usual method (and $C.1$ allows to diagram chase with elements) its more suitable for automatic diagram chase as one (and thus maybe a computer) can work with the graph structure of the diagram and avoid any arbitrary choice. In particular the only objects involved are maps, whereas there are usually elements and maps.

Definition 3.2

1. *A composition $f : A \to B$ and $g : B \to C$ is said to be exact if $\text{im}(f)$ is the kernel of g. (In particular they are equal as subobjects of B)*
2. *A sequence (finite or infinite) $(f_i : A_i \to A_{i+1})_{i \in I}$ of composition is said to be exact if all the composition of the sequence are exact*

Lemma 3.7

A composition $f : A \to B$ and $g : B \to C$ is exact in $\mathcal{C}$ if and only if $g : C \to B$ and $f : B \to A$ is exact in $\mathcal{C}^{op}$

Proof: By duality it is enough to prove one implication. If $f : A \to B$ and $g : B \to C$ is exact in $\mathcal{C}$, then $\text{im}_{\mathcal{C}}(f) = \ker_{\mathcal{C}}(\text{coker}_{\mathcal{C}}(f)) = \ker_{\mathcal{C}}(g)$, thus by duality: $\text{coker}_{\mathcal{C}^{op}}(\ker_{\mathcal{C}^{op}}(f)) = \text{coker}_{\mathcal{C}^{op}}(g)$. Then $\ker_{\mathcal{C}^{op}}(\text{coker}_{\mathcal{C}^{op}}(\ker_{\mathcal{C}^{op}}(f))) = \ker_{\mathcal{C}^{op}}(\text{coker}_{\mathcal{C}^{op}}(g)) = \text{im}_{\mathcal{C}^{op}}(g)$. But $\ker_{\mathcal{C}^{op}}(f)$ is a monomorphism, then by 3.2, $\ker_{\mathcal{C}^{op}}(\text{coker}_{\mathcal{C}^{op}}(\ker_{\mathcal{C}^{op}}(f))) = \ker_{\mathcal{C}^{op}}(f)$. Thus the composition is exact in $\mathcal{C}^{op}$. ■

Remark: In particular $\text{im}(f)$ is a kernel of g if and only if $\text{coim}(g)$ is a cokernel of f.

Lemma 3.8

1. *A morphism $f : A \to B$ in $\mathcal{C}$ is a monomorphism if and only if $0 \to A \xrightarrow{f} B$ is an exact sequence.*
2. *A morphism $f : A \to B$ in $\mathcal{C}$ is an epimorphism if and only if $A \xrightarrow{f} B \to 0$ is an exact sequence.*

Proof: By duality it is enough to prove the first point. If f is a monomorphism, $\ker(f)$ is $0 \to A$. $0 \to A$ is a monomorphism, then it is it's image, thus $0 \to A = \mathrm{im}(0 \to A)$ and $\ker(f)$ are the same map. Conversely, if is the sequence is exact, then $0 \to A = \mathrm{im}(0 \to A)$ and $\ker(f)$ then their domain are isomorphic, thus by uniqueness of the map from the initial object $\ker(f)$ is $0 \to A$. ■

Remark: Then if $0 \to A \to 0$ is exact, then A is 0. Indeed, the map $0_{A \to 0}$ is a monomorphism. It's cokernel has 0 as domain, then it's a zero-map, thus $0_{A \to 0}$ is an epimorphism. And by 3.1 it is an isomorphism.

Proposition 3.1

If $f : A \to B$ and $g : B \to C$ are two composable morphism in $\mathcal{C}$ then there is an exact sequence

$$0 \to K(f) \to K(g \circ f) \to K(g) \to CK(f) \to CK(g \circ f) \to CK(g) \to 0$$

such that the following diagram is commutative:

$$\begin{tikzcd}
 & K(g \circ f) \arrow[r] \arrow[d, hook] & K(g) \arrow[d, hook] \arrow[dr] & \\
K(f) \arrow[r, hook] \arrow[ur] & A \arrow[r, "f"] \arrow[ddr, "g \circ f"'] & B \arrow[r, two heads] \arrow[dd, "g"] & CK(f) \arrow[dd] \\
0 \arrow[u] & & & \\
 & & C \arrow[r, two heads] \arrow[d, two heads] & CK(g \circ f) \arrow[dl] \\
 & 0 & CK(g) \arrow[l] &
\end{tikzcd}$$

where $K(\ )$ denotes the domain of a kernel and $CK(\ )$ the codomain of a cokernel.

Remark: This exact sequence (and the previous remark) is a way to encode in a diagram rules such that if f and g are monomorphism then $g \circ f$ is a monomorphism. Then the use of those rules can be implemented by extending the diagram, computing the zero maps and then using the exact sequence on this diagram.

Proof: Because the construction of the morphism $K(g) \to CK(f)$ will not depend of being in $\mathcal{C}$ or in $\mathcal{C}^{op}$, by duality it is enough of to construct and proove that the sequence $0 \to K(f) \to K(g \circ f) \to K(g) \to CK(f)$ is exact.

1. $0 \to \ker(f)$ is the unique zero-map.
2. $a : K(f) \to K(g \circ f)$ is the map from the universal propriety of $\ker(g \circ f)$ applied to $\ker(f)$ (because $(g \circ f) \circ \ker(f) = g \circ (f \circ \ker(f)) = g \circ 0 = 0$).
3. $b : K(g \circ f) \to K(g)$ is the map from the universal propriety of $\ker(g)$ applied to $f \circ \ker(g \circ f)$ (because $g \circ (f \circ \ker(g \circ f)) = (g \circ f) \circ \ker(g \circ f) = 0$).
4. $c : K(g) \to CK(f)$ is the map $\mathrm{coker}(f) \circ \ker(g)$.

Let's prove that the composition are all exacts

1. By definition $\ker(g \circ f) \circ a = \ker(f)$ then (it's a kernel) it's a monomorphism, thus a must be a monomorphism, then by a previous lemma $a \circ 0$ is an exact composition.
2. $b \circ a$: a is a monomorphism, then (by 3.2) $\mathrm{im}(a) = \ker(\mathrm{coker}(a)) = a$. Thus to conclude, it's enough to show that a is a kernel of b. Let $j : X \to K(g \circ f)$ be a map such that $b \circ j = 0$. If there is a $\phi : X \to K(f)$ such that $j = a \circ \phi$ then $\ker(f) \circ \phi = \ker(g \circ f) \circ a \circ \phi = \ker(g \circ f) \circ j$. Then if

ϕ exists, ϕ is unique by the uniqueness in the universal propriety of f.

Conversely, $f \circ \ker(g \circ f) \circ j = g \circ b \circ j = g \circ 0 = 0$ thus there is a unique map ϕ such that $\ker(f) \circ \phi = \ker(g \circ f) \circ j$. Then $\ker(g \circ f) \circ a \circ \phi = \ker(f) \circ \phi = \ker(g \circ f) \circ j$ but $\ker(g \circ f)$ is a monomorphism, then $a \circ \phi = j$. Then by the universal propriety a is the kernel of b.

3. $c \circ b$: Let $j : X \to K(g)$ be a morphism such that $\mathrm{coker}(b) \circ j = 0$. If there is a ϕ such that $j = \ker(c) \circ \phi$, then because $\ker(c)$ is a monomorphism, ϕ must be unique.

   Conversely, $c \circ b = \mathrm{coker}(f) \circ \ker(g) \circ b = \mathrm{coker}(f) \circ f \circ \ker(g \circ f) = 0 \circ \ker(g \circ f) = 0$. Then c factors through the cokernel of b by ψ, then $c \circ j = \psi \circ \mathrm{coker}(b) \circ j = \psi \circ 0 = 0$. Then by the universal propriety of $\ker(c)$, there is a ϕ such that $j = \ker(c) \circ \phi$. Then $\ker(c)$ is the kernel of $\mathrm{coker}(b)$, *i.e.* it's image. ■

Lemma 3.9

Let $f : A \to B$ and $g : B \to C$ be two composable morphism in $\mathcal{C}$ then:

1. *$g \circ \mathrm{im}(f)$ factors through $\mathrm{im}(g \circ f)$: there is an epimorphism θ such that $g \circ \mathrm{im}(f) = \mathrm{im}(g \circ f) \circ \theta$.*
2. *If g is a monomorphism, then θ is an isomorphism, then $g \circ \mathrm{im}(f)$ and $\mathrm{im}(g \circ f)$ can be identified.*
3. *$\mathrm{im}(g \circ f)$ factors through $\mathrm{im}(g)$: there is a monomorphism χ such that $\mathrm{im}(g \circ f) = \mathrm{im}(g) \circ \chi$*

Proof: Let's consider the following commutative diagram:

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, two heads, "\mathrm{coim}(f)"'] & & B \arrow[rr, "g"] \arrow[dr, two heads, "\mathrm{coim}(g)"] & & C \\
& I(f) \arrow[ur, hook, "\mathrm{im}(f)"] \arrow[rr, "h=\mathrm{coim}(g)\circ\mathrm{im}(f)"] \arrow[dr, two heads, "\mathrm{coim}(h)"'] & & I(g) \arrow[ur, hook, "\mathrm{im}(g)"'] & \\
& & I(h) \arrow[ur, hook, "\mathrm{im}(h)"'] & &
\end{tikzcd}$$

By composition $a = \mathrm{coim}(h) \circ \mathrm{coim}(f)$ is an epimorphism and b=im(g)∘im(h) is a monomorphism. Moreover, because the diagram is commutative, $b \circ a = g \circ f$. Then by 3.4 b is the image of $g \circ f$. Thus with $\theta = \mathrm{coim}(h)$ one get $g \circ \mathrm{im}(f) = \mathrm{im}(g \circ f) \circ \theta$. With $\chi = \mathrm{im}(h)$, one get $\mathrm{im}(g) \circ \chi = b = \mathrm{im}(g \circ f)$. ■

Lemma 3.10

Let $f : A \to B$ be a morphism in an abelian category.

1. *If f is of the form $\mathrm{im}(a) \circ j$ then there is an epimorphism θ such that $f \circ \theta$ is of the form $a \circ j'$.*
2. *If f is of the form $j \circ \mathrm{coim}(a)$ then there is a monomorphism θ such that $\circ$ is of the form $j' \circ a$.*

Proof: The two statements are dual, then it's enough to prove the first one.

Let (j', θ) be the pullback of $(j, \mathrm{coim}(a))$. $\mathrm{coim}(a)$ is an epimorphism then θ is an epimorphism and $f \circ \theta = \mathrm{im}(a) \circ j \circ \theta = \mathrm{im}(a) \circ \mathrm{coim}(a) \circ j' = a \circ j'$ ■

Example: As example one can diagram chase the four-lemma: A.1

3.3 Double Complex and Salamander lemma

Let $\mathcal{C}$ be an abelian category.

Definition 3.3

1. *The graph DC is defined by the following:*
   (a) *The set of vertex is* $\mathbb{Z}^2$
   (b) *The set of edges is* $\{dh_{i,j} : (i,j) \to (i,j+1), dv_{i,j} : (i,j) \to (i+1,j) /\ (i,j) \in \mathbb{Z}^2\}$
2. *A double chain complex with value in* $\mathcal{C}$ *is a diagram* F *over the graph DC such that*

$$\forall (i,j) \in \mathbb{Z}^2 \begin{cases} F(dh_{i,j+1}) \circ F(dh_i, j) = 0 \\ F(dv_{i+1,j}) \circ F(dv_i, j) = 0 \end{cases}$$

3. *A finite double chain complex is a double chain complex is a double chain complex where all but finitely many objects are* $0_{\mathcal{C}}$.

Remark : Informally it's just a diagram over a grid such that if you move fo more than 1 step in any direction you get 0.

Definition 3.4

Let A *be in object part of a double chain complex,*

$$\begin{tikzcd}
\bullet \arrow[r, "a"] \arrow[d, "b"] \arrow[dr, "p"] & \bullet \arrow[d, "c"] & \\
\bullet \arrow[r, "d"] & A \arrow[r, "e"] \arrow[d, "f"] \arrow[dr, "q"] & \bullet \arrow[d, "g"] \\
& \bullet \arrow[r, "h"] & \bullet
\end{tikzcd}$$

Let's define the following objects:

1. *The horizontal homology at A:* A^h, *the codomain of* $\ker(e)/_{\mathrm{im}(d)}$
2. *The vertical homology at A:* A^v, *the codomain of* $\ker(f)/_{\mathrm{im}(c)}$
3. *The donor at A:* $A_\Box$, *the codomain of* $\ker(e \times f)/_{\mathrm{im}(p)}$
4. *The receptor at A:* ${}^\Box A$, *the codomain of* $\ker(q)/_{\mathrm{im}(c \coprod d)}$

Remark : These objects are well defined because the images are subobjects of A, and by the propriety of double chain complex one get *zero* by composing the morphism in the kernel with the one in the image, then the subobject factors throuhg the kernel which is a monomorphism, then the factorized morphism is also a monomorphism, and then a subobject of the kernel.

Remark : In the case of R-modules $\mathrm{im}(c \coprod d)$ is the inclusion of $I(c) + I(d)$ and $\ker(e \times f)$ is the inclusion of $K(e) \cap K(f)$.

Lemma 3.11

Let $A \to B \to 0$ *and* $0 \to C \to D$ *be two horizontal or vertical lines in a double chain complex, then there is a map (called connection morphism)* $\partial : B \to C$ *such that* $A \to B \to C \to D$ *is exact if and only if the corresponding homology* HB *and* HC *are isomorphic.*

Proof : The result is stated in [8]. Because of the 0 in the line. $\ker(B \to 0)/_{\mathrm{im}(A \to B)} = \mathrm{coker}(\mathrm{im}(A \to B)) = \mathrm{coker}(\ker(\mathrm{coker}(A \to B)) = \mathrm{coker}(A \to B)$ by 3.2. In particular $HB = CK(A \to B)$ and by duality $HC = K(C \to D)$

If the morphism exists, and the induced sequence is exact, then by the exact condition and their dual $\begin{cases} \operatorname{coker}(A \to B) = \operatorname{coim}(B \to C) \\ \operatorname{im}(B \to C) = \ker(C \to D) \end{cases}$, thus $CK(A \to B) = I(B \to C) = K(C \to D)$, then HB and HC are isomorphic.

Conversely, let $\phi : HB \to HC$ be an isomorphism, because $CK(A \to B) = HB$ and $HC = K(C \to D)$ the morphism $\partial = \ker(C \to D) \circ \phi \circ \operatorname{coker}(A \to B)$ is well defined.

$\ker(C \to D) \circ \phi$ is a monomorphism then by using the exact sequence of 3.1, one get $\ker(\partial) = \ker(\operatorname{coker}(A \to B) = \operatorname{im}(A \to B)$. $\phi \circ \operatorname{coker}(A \to B)$ is an epimorphism, then by using the exact sequence of 3.1, one get $\operatorname{coker}(\partial) = \operatorname{coker}(\ker(C \to D)) = \operatorname{coim}(C \to D)$ then by duality of the exact condition $\operatorname{im}(\partial) = \ker(C \to D)$ ■

Remark : The connection morphism is not necessarily unique. If ∂ is a connection morphism then $-\partial$ is also one.

Remark : This lemma give a simple criteria to test if there is a connection morphism, in particular it allows to prove the snake lemma A.4

Lemma 3.12

1. *In a double chain complex, the* $\mathrm{id} : A \to A$ *induce (by taking the quotient) a commutative square:*

$$\begin{tikzcd} {}^{\square}A \arrow[r] \arrow[d] \arrow[dr] & A^h \arrow[d] \\ A^v \arrow[r] & A_{\square} \end{tikzcd}$$

2. *In a double chain complex any map* $A \to B$ *induce a map* $A_{\square} \to^{\square} B$

The maps of this lemma are called the intramural maps

Proof : The result came from [8], and is left as an exercise. It is indeed just diagram chasing, a proof can be found in F. ■

Lemma 3.13 (*Salamander lemma*)

1. *If the following maps are part of a double chain complex* $\begin{tikzcd} C \arrow[d] & \\ A \arrow[r] & B \arrow[d] \\ & D \end{tikzcd}$ *then there is an exact sequence given by the intramural maps:*

$$\begin{tikzcd} {}^{\square}A \arrow[dr] & & & & & \\ C_{\square} \arrow[u] \arrow[r] & A^h \arrow[r] & A_{\square} \arrow[r] & {}^{\square}B \arrow[r] & B^h \arrow[r] \arrow[dr] & {}^{\square}D \\ & & & & & B_{\square} \arrow[u] \end{tikzcd}$$

2. *If the following maps are part of a double chain complex* $\begin{tikzcd} C \arrow[r] & A \arrow[d] & \\ & B \arrow[r] & D \end{tikzcd}$ *then there is an exact sequence given by the intramural maps:*

$$\begin{tikzcd} {}^{\square}A \arrow[dr] & & & & & \\ C_{\square} \arrow[u] \arrow[r] & A^v \arrow[r] & A_{\square} \arrow[r] & {}^{\square}B \arrow[r] & B^v \arrow[r] \arrow[dr] & {}^{\square}D \\ & & & & & B_{\square} \arrow[u] \end{tikzcd}$$

Proof: The result came from [8] ■

4 Decision Problems over diagrams

Definition 4.1

1. *A decision problem is two disjoint set: Ω^+ and Ω^-, the elements of $\Omega = \Omega^+ \uplus \Omega^-$ are called the instances of the problem, the elements of Ω^+ the positives and the elements of Ω^- the negatives.*
2. *A decision problem is said to be solvable if there is an algorithm (a Turing Machine) defined over Ω that terminate, return **True** over Ω^+ and **False** over Ω^-*
3. *Otherwise it is said to be unsolvable*

Remark: By using the function *not* the switch **True** and **False**, it's straightforward that $\Omega^+ \uplus \Omega^-$ is solvable if and only if $\Omega^- \uplus \Omega^+$ is solvable

4.1 The commutative exactness problem

Definition 4.2

Let D be a finite graph

1. *An exactitude condition over D is a pair (a, b) of composable arrow $(t(a) = o(b))$ in D. We say that a diagram $F : \mathcal{C}(D) \Rightarrow \mathcal{C}$ with value in an abelian category satisfy the exactitude condition if the sequence $F(o(a)) \xrightarrow{F(a)} F(t(a)) = F(o(b)) \xrightarrow{F(b)} F(t(b))$ is exact.*
2. *An instance of the commutative exactness problem is the data of a graph D a set E of exactitudes conditions over D and a pair (a, b) of composable arrow. The instance is positive if for all diagram $F : \mathcal{C}(D) \Rightarrow \mathcal{C}$ with value in an abelian category such that F satisfy all the condition of E, F satisfy the exactness condition (a, b)*

Remark: In the same way, it is straightforward to define a zero condition, an epimorphism condition, a monomorphism condition or a zero-object condition.

In the paper [9] it is proven the following theorem

Theorem 4.1

The commutative exactness problems is unsolvable

Hence there is no Algorithm able to compute on a given instance of the problem if the instance is positive or not. Therefore if we want to provide some algorithm of diagram chasing we need to restrict ourselves either the possible sources (for example diagram chase only in double chain complexes) , the possible targets of the diagrams. (for example only vector spaces) or to restrict the question asked (for example monomorphism instead of exact condition)

4.2 The commutative zero problem

The proof of the result presented in this section came from personnel work.

Definition 4.3

1. *A zero condition over a finite graph D is an edge e of D. We say that a diagram $F : \mathcal{C}(D) \Rightarrow \mathcal{C}$*

with value in a category with a zero-object satisfy the zero condition if the map $F(o(e)) \xrightarrow{F(e)} F(t(e))$ is the zero-map.

2. *An instance of the commutative zero problem is the data of a finite graph D a set Z of zero conditions over D and an edge f of $\mathcal{C}(D)$. The instance is positive if for all diagram $F : \mathcal{C}(D) \Rightarrow \mathcal{C}$ with value in a category with a zero object such that F satisfy all the condition of Z, F satisfy the zero condition f*

Theorem 4.2

The commutative zero problem is solvable (in polynomial time)

Lemma 4.1

Let $f : a \to b$ be an arrow in $\mathcal{C}(D)$, then the instance (D, Z, f) is positive if and only if there is an edge e in Z such that $a \in Ans(o(e)) \cup \{o(e)\}$ and $b \in Des(t(e)) \cup \{t(e)\}$

Proof: If the condition is satisfied, let then $a = x_1 \to \ldots \to x_n = o(e)$ be a path from a to $o(e)$ (by assumption n can be 1) and $t(e) = y_1 \to \ldots \to y_m = b$. Thus we have two path from a to b, because the diagram is commutative we therefore have $F(a \to b) = F(a = x_1 \to \ldots o(e) \to t(e) \to \ldots y_m = b) = F(a = x_1 \to x_2) \circ \ldots \circ F(o(e) \to t(e)) \circ \ldots \circ F(y_{m-1} \to y_m = b)$ as F is a functor, but $F(o(e) \to t(e))$ is the zero-map by assumption hence by 1.3 $F(a \to b)$ is the zero-map.

Conversely, if the condition is not satisfied, it's enough to exhibit a case where $F(a \to b)$ is a non-zero map. The construction and the proof are given in F

■

Proof of the theorem: It is enough to compute the sets Anc and Des for every $e \in Z$ and to test the condition given by the lemma 4.1:

Algorithm 4.1

Input: *An instance (D, Z, f) of the commutative zero problem*
Output: ***True** if the instance is positive, **False** otherwise*

for $e \in Z$ **do**
- *Compute $Anc(o(e))$ and $Des(t(e))$ by DFS*
- **if** $(o(f) = o(e) \vee o(f) \in Anc(o(e))) \wedge (t(f) = t(e) \vee t(f) \in Des(t(e)))$ **then**
  - *return **True***

- *return **False***

Because the complexity of DFS is in $O(\#V + \#E)$ the complexity of 4.1 is $O(\#Z(\#V + \#E))$ which is at most quadratic in the size of the graph. ■

Definition 4.4

1. *An instance of the commutative zero problem with zero-object is the data of a finite graph D a set Z of zero conditions over D, a set OZ of vertex of D and an edge f of $\mathcal{C}(D)$.*

2. *The instance is positive if for all diagram $F : \mathcal{C}(D) \Rightarrow \mathcal{C}$ with value in a category with a zero object, such that F satisfy all the condition of Z and $\forall d \in OZ$ $F(d) = 0_{\mathcal{C}}$, F satisfy the zero condition f*

Proposition 4.1

The commutative zero problem with zero object is solvable

Proof: By using the algorithm 4.1, it is enough to prove that (D, Z, OZ, f) is positive if and only if the result of algorithm 4.2 :(D', Z', f) is positive.

ALGORITHM 4.2

Input: *An instance* (D, Z, OZ, f) *of the commutative zero problem with zero-obect*
Output: *An instance* (D, Z', f) *of the commutative zero problem*

• $Z' = Z$
for $e \in E(D)$ **do**
  if $o(e) \in OZ \vee t(e) \in OZ$ **then**
    • $Z' = Z' \cup \{e\}$
• *return* (D, Z', f)

If (D, Z', f) is positive, let $F : \mathcal{C}(D) \Rightarrow \mathcal{C}$ be a diagram with value in a category with a zero object, such that F satisfy all the condition of Z and $\forall d \in OZ\ F(d) = 0_{\mathcal{C}}$. For $e \in Z'$ there are two cases:

1. If $e \in Z$, then by assumption $F(e) = 0_{F(o(e)) \to F(t(e))}$
2. If $e \in Z' \backslash Z$, then $o(e) \in OZ$ or $t(e) \in OZ$ thus $F(e) : F(o(e)) \to F(t(e))$ has either it's domain or it's codomain which is $0_{\mathcal{C}}$. $F(e)$ is therefore a zero-map.

Then F satisfy all the condition Z', thus by definition, because (D, Z', f) is positive, $F(f)$ is a zero map. Then by definition (D, Z, OZ, f) is positive.

Conversely if (D, Z', f) is not positive let F be the related counter example constructed in the proof of 4.1. In F there is the proof that F is also a counter example for this case. ■

4.3 Acyclic diagrams

The construction AC allows to build equivalent decisions problems dealing only with acyclic diagrams, then it will be possible to assume that all the diagrams are acyclic.

DEFINITION 4.5

Let (D, Z, E, OZ, M, Ep) be a diagram with zero-condition, exactness condition, zero-object conditions, monomorphism condition an epimorphism condition. Let's define a new set of conditions over $AC(D)$:

1. $Z_{AC} = \{f_0 /\ f \in Z\}$
2. $E_{AC} = \{(f_0, g_1) /\ (f, g) \in E\}$
3. $OZ_{AC} = \{A_0 /\ A \in OZ\}$
4. $M_{AC} = \{f_0 /\ f \in M\} \cup \{\phi_{A,i} /\ A \text{ a vertex}, i \in [\![0, n]\!]\}$
5. $Ep_{AC} = \{f_0 /\ f \in Ep\} \cup \{\phi_{A,i} /\ A \text{ a vertex}, i \in [\![0, n]\!]\}$

Remark: It would be possible to avoid using monomorphism and epimorphism condition and to express it with zero and exact conditions, but that would require to modify the graph $AC(D)$. All the diagrams considered are assumed to be with values in an abelian category (even if for some situation it's not the minimal hypothesis)

LEMMA 4.2

(D, Z, E, OZ, M, Ep) be a diagram with zero-condition, exactness condition, zero-object conditions, monomorphism condition an epimorphism condition. Let A be a vertex of D,f and g two

edge of D.

1. *(D, Z, E, OZ, M, Ep, A) is a positive instance of the zero-object problem if and only if $(AC(D), Z_{AC}, E_{AC}, OZ_{AC}, M_{AC}, Ep_{AC}, A_0)$ is positive*
2. *(D, Z, E, OZ, M, Ep, f) is a positive instance of the zero (resp mono, epi) problem if and only if $(AC(D), Z_{AC}, E_{AC}, OZ_{AC}, M_{AC}, Ep_{AC}, f_0)$ is positive*
3. *$(D, Z, E, OZ, M, Ep, (f, g))$ is a positive instance of the exactness problem if and only if $(AC(D), Z_{AC}, E_{AC}, OZ_{AC}, M_{AC}, Ep_{AC}, (f_0, g_1))$ is positive*

Proof: The proof can be found in F. ■

5 Elements of Implementation

During the internship a large part of time was dedicated to develop and implement algorithm able to perform automatic diagram chasing. A lot of things were found to be inefficient and more subtile than expected. The goal of the internship was to reach implementation in LEAN, however in the first place it was simpler to try with python in order to see what could work. The working part of the code, together with a user-friendly notebook can be found in `https://github.com/ymonbru/Diagram-chasing`.

The general idea was to have a data structure behaving like a graph (representing the diagram) with label on edges and vertices to give them special propriety, and then to compute new label. However it turned out to be more efficient to compute and build at the same time. Then as soon as a function is used to add an edge (or a vertex or any label) , then the function check if any label can be added to the graph. Thus when the data structure is build, all the computation are performed and it's enough to display the results.

In practice a graph will be a tuple with orgin and tail functions but also labels that can be on vertex (*labl* for a name , or zobj for being a zero object) on edges (*zero, epi, mono*) or on pair of composable edges (*exact*). For example (a simple version of) the function that add an edge is reproduced there:

```
def add_edge(g,o,t,is_zero=False,is_mono=False,is_epi=False):
        new_g=g
        (og,tg,labl,exact,zero,mono,epi,zobj)=g
        e=numb_e(g) #the number of edges
        tg.append(t)
        og.append(o)
        zero.append(False)
        mono.append(False)
        epi.append(False)
        new_g=(og,tg,labl,exact,zero,mono,epi,zobj)

        new_g=propagate_info(new_g,e)

        if is_zero:
                new_g=add_zero(new_g,e)
        if is_mono:
                new_g=add_mono(new_g,e)
        if is_epi:
                new_g=add_epi(new_g,e)
        return (new_g,e)
```

The first part is really just adding the edge to the data structure, then one uses the function *propagate-info* to compute if the information present in the graph forces the edge to have any label, for example *zero*. Finally is any information (a zero morphism, a monomorphism, an epimorphism) are imposed to this edge, it's added with the corresponding function that first add the information to the data structure and then propagate the possible consequences of this new label.

The propagation of information step uses the propriety discussed in 3.2 (and the basic propriety of zero, epimorphism and monomorphism. The exact list of deduction rules used in the program is given in D.1. In order to avoid infinite deduction loop, it is necessary to test wether the conclusion is known before applying a rule.

The goal of this section is to explain elements that worked or not in the implementation and to draw conclusion from that.

5.1 Vector spaces does not simplify the thing

In [2] Daniel Litt suggest to apply informal algorithm 5.1 in the case of diagram chasing in diagram labeled by objects and maps in the category of finite dimensional vector spaces over a field.

Algorithm 5.1

- *add all the composition in the diagram*
- *add all the kernel, the cokernels and natural maps induced by universal propriety*
- *Compute the exact paths* $0 \to A_1 \to \ldots \to A_n \to 0$ *that gives the equation* $\sum_{i=1}^{n} (-1)^i \dim(A_i) = 0$
- *Solve the resulting linear system*

It turns out that the step of computing all the exact path is really long and difficult to implement. However the equation $\sum_{i=1}^{n} (-1)^i \dim(A_i) = 0$ is proven (by induction) with the help of rank theorem on each map of the sequence. Then the algorithm can be implemented in python by using *sympy* to solve the system of equation.

First of all the algorithm was not able to claim the lemma given in example. In addition the information provided by this algorithm were either of the form $\sum_{i=1}^{n} (-1)^i \dim(A_i) = 0$ or translated consequences (in the language of dimension, for example f is a monomorphism if and only if $\dim(\ker(f)) = 0$) of what can be deduced without the use of dimension. Then the use of dimension was not helpful to simplify the problem.

5.2 The augmented diagram is not commutative

The algorithm 5.1 can easily be adapted to the general case by following the first two step and then applying the deduction rules.

One of the key assumption was that the new diagram obtained in 5.1 was commutative and thus that the result of a composition only depends on the origin and the end of a path. This hypothesis is very convenient because it allows the algorithm to propagate the information with usual algorithm over graphs. This turned out to be false: for example in the diagram

$$A \xrightarrow{\quad f \quad} B \xrightarrow{\quad g=0 \quad} C$$

without any exact condition, the algorithm computes that f is an epimorphism (which is false as soon as a category contain a map that is not an epimorphism in general). It happen because the constructions allows to construct a zero map from B to $CK(f)$ and because this map is not $\mathrm{coker}(f)$ the hypothesis leads to a contradiction.

The map is constructed as follow: $\mathrm{coker}(f)\circ f = 0$ then $\mathrm{coker}(f)\circ f \circ \ker(g\circ f) = 0$. Thus $\mathrm{coker}(f)\circ f$ factors through the cokernel of $\ker(g \circ f)$ which is $\mathrm{coim}(g \circ f)$: there is a $\varphi : I(g \circ f) \to CK(f)$ such that $\text{coker}(f) \circ f = \phi \circ \text{coim}(g \circ f)$. On the other hand there is a natural map $I(g) \to I(g \circ f)$ (it's true because g is a zero map and thus it's the zero map) thus $\phi \circ 0_{I(g)\to I(g\circ f)} \circ \text{coim}(g)$ is a zero map from B to $CK(f)$.

5.3 Compute with associativity

The consequence of the previous issue is that it is necessary to use a graph with different edges with the same origin and end. The definition of graph given in 2.1 is adapted to this situation (and thus the implementation). However in this situation in order to get the composition of two edges it is no more possible to follow a path. Then it's necessary to remember the results of compositions, the issue is thus that in oder to perform computation the associativity (which play a key role) of composition who was hidden in the "path" has now to be encoded. Unfortunately it gives rise to too much relations to remember, and perform effective computation. If one add one edge $f : A \to B$ to the diagram the there is the need to ad one edge from A to any edge that can be reached starting from B, the edges that fill the triangle induced, and all the composition relation induced.

There too much is in the sense that the maximal recursion depth of python was hit and when not all the relation are added, the time of computation was too high (one our for a toy example, and without claiming the lemma).

It's not enough to remember the relation of the form $h \circ (g \circ f) = (h \circ g) \circ f$ over the original edges of the graph, indeed remembering this relation creates new edges (because the two maps are *a priori* different), and associativity relations induced must also be remembered.

More concretely if $f_n \circ \ldots \circ f_1$ is a path of length n, in order to check if the zero-propagation rule does apply one has to check if any map of the form $f_i \circ f_{i-1} \circ \ldots \circ f_j$ is 0 and then to add all these (there are $\binom{n+1}{2}$ choices because on has to choose the beginning and the end of the sequence) arrow and the composition relation induced. And this go on recursively (even if it eventually end).

5.4 Differences or merging?

This step was not implemented because of the issue of the last point, but it's still a problem.

At some step the hypothesis monomorphism needs to be applied in the form $f \circ a = f \circ b \Rightarrow a = b$ which is stronger than $f \circ a = 0 \Rightarrow a = 0$. Then the algorithm need to be able to either deal with differences morphisms (in that case the two statement are equivalent) or to deal with equality. In the first case that leads to add an exponential number of arrow, in the second case that leads to completely rebuild the data structure in order to merge two arrows.

According to empirical experiments, the second option does not seems to be efficient enough (especially when a lot of composition relation are involved)

5.5 Useless hypothesis

A slogan for diagram chasing is "do the only possible thing to do". And indeed with experience it's the feeling one can get. Then a nice algorithm of diagram chasing would be able to perform the same computation as a human being does. However when useless hypothesis are given there are way to be stuck. Then an algorithm has to take into account more general things than just apply the basic steps of diagram chase.

For example in the proof of A.1, the hypothesis that the composition $g \circ f$ is exact is useless for the first statement. And one could start the proof by doing the following: $\gamma \circ j = 0$ then $g \circ \gamma \circ j = 0$ then $\gamma \circ j$ factors through the kernel of g which is the image of f, and then be stuck.

5.6 In the case of double chain complex

By assuming that the diagram was a finite double-chain complex, it is possible to get better results. Indeed because of the particular shape of the diagram, there is no need to store anymore the edges, in particular there is no more the need to add composition maps and to increase the number of edges in the graph. Because any path with length 3 or more is 0, the propagation step can be improved by just considering the propagation on each square. It requires a re-implementation but the ideas remain the same than in the general case:

1. Build the graph the objects of homology add the intramural maps
2. Add the exact sequences of salamander lemma
3. Propagate information by using the rules of D.2
4. Use 3.11 to find the connecting morphism

The data structure is now composed of two graphs: the grid and a graph containing the homology object and the intramural maps (3.3). As detailed in 3.3, if one take into account the only intramural maps and the salamander lemme, then there is (at most) one map between any two objects. It's then possible to avoid the issue of the general cas detailed in 5.3.

Then the actual implementation was able to claim the usual lemma listed in A, (and even more, for example lemmas in [5]). Even if some elements were implemented in a naive way, the algorithm turned out to be qui efficient: for example it compute the 10×10 (100 vertex) lemma in less than $30s$.

However, it turns out that this algorithm is not complete in the sense that it's not able to compute any consequence of a diagram, for example with the following finite double complex :

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

If the two lines, the first column and the third column are exact then (by A.2), the second column is also exact. The algorithm is not able to claim that. In practice it's not a problem because one can consider the following double chain complex:

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & 0 \arrow[d] & K \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & 0 \\
 & 0 & CK \arrow[d] & 0 & \\
 & & 0 & &
\end{tikzcd}$$

In which all lines and column are exact and then get the conclusion (K and CK stand for kernel and cokernel). Similar modifications can be made in all cases i encountered, and the feature can be implemented by adding kernels and cokernels to form an exact sequence as soon as possible.

5.7 Extracting proofs

The algorithm detailed in the previous subsection was able to compute the results, but the goal is to extract proof. In order to do that i used a dictionary that at each new information found in the computation stores the rule and the statement used to deduce this information. Then it's possible to extract a proof by considering this dictionary as a graph, and to extract from it the tree of a statement and it's descendants in the tree.

In practice it was more technical than a simple DFS (depth first-search) algorithm because of the back and forth of the two data structures (the grid and the graph of homology objects) involved, but the idea remains to implement this algorithm. An example of a proof generated by the program can be found in E.

As expected the proof is longer than a human made, but most of the time it's for stating things that an human would consider as "obvious" or "straightforward".

On the other hand the "systematic" approach of the computer allows to find particular things. For example the computer found that the exact sequence of the snake lemma A.4 can be extended (with two other connecting morphism) into a long exact sequence:

$$0 \to K(a_1) \to K(f) \to K(g) \to K(h) \to CK(f) \to CK(g) \to CK(h) \to CK(b_2) \to 0$$

Conclusion

This internship was for me the occasion to study in detail the structure of abelian category, and more particularly the way of doing diagram chase in them. Some theoretical problems were considered like restricting the problem to acyclic graphs or solving the diagram chase question in case of zero morphism.

However solve an other problem (in the sense of 4) remain to be done. It's easy to compute some consequence, but harder to prove that an algorithm has computed all the possible consequences.

This internship was then the occasion to circumvent some issues that would need to be fixed (or bypassed) in order to go further into the automatisation of diagram chasing. As the case of double chain complex showed, in practice it's possible to do what is needed for the applications, *i.e.* there is a way to translate the situation into a suitable double chain complex in such a way that the salamander lemma is enough to get the expected conclusion. Nevertheless being able to compute in a more general case would still be interesting. For example the user would not have to translate his graph into a double chain complex (that includes adding kernel,cokernel, exact sequences and give give coordinate on the grig to the vertices).

Time did not permit to implement the algorithm in LEAN, (or at least produce a proof that can be verified in LEAN) that would not be really different of producing the pen-and-paper proof found in E, but that would require to implement in LEAN the salamander lemma (and more generally lemmas from 3.3)

Bilan Personnel

Ce stage à été pour moi l'occasion de découvrir la vie d'un laboratoire de recherche (mon stage de $L3$ ayant été fait en distanciel) et de me rendre compte à quel point les échanges entres les uns et les autres jouent un rôle fondamental dans le travail de recherche. En effet discuter avec divers interlocuteurs, leur expliquer ce que je faisais, ce qui marchait et ce qui ne marchait pas était souvent l'occasion d'avoir une nouvelle idée ou une nouvelle compréhension de ce que j'étais en train de faire.

J'ai également le sentiment d'une meilleure compréhension du travail de recherche. Le plus souvent la question que je posais n'était pas "quelle est la réponse" mais "qu'elle est la question". Le fait d'être confronté à des questions ou je n'étais pas sur que la réponse existe (que ce soit en théorie ou en tant qu'algorithme implémenté) était pour moi une façon très nourrissante d'aborder les mathématiques.Tout cela couplé avec une grande satisfaction quand ce travail aboutit et une frustration naturelle quand ce n'est pas le cas.

Le travail de rédaction m'a également beaucoup apporté dans la mesure ou cella permet d'avoir les idées claires sur ce qui se passe. Ainsi c'est souvent lors de la rédaction (ou programmation) que je me suis rendus compte que telle idée ne marchait pas ou que telle autre pouvait être exploitée pour faire autre chose.

Je conclurais ce bilan personnel, en disant que le séjour en Allemagne fut pour moi une aventure pleine d'enrichissement . En effet vivre pendant trois mois dans un pays étranger où l'on ne parle (presque pas) la langue est une grande source d'expériences et de rencontres qui font voir les choses différemment et grandir tout simplement.

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Annexes

A Diagram chasing lemmas

Lemma A.1 (*Four-lemma*)

Let's have a commutative diagram with value in an abelian category:

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "\alpha"] & B \arrow[r, "b"] \arrow[d, "\beta"] & C \arrow[r, "c"] \arrow[d, "\gamma"] & D \arrow[d, "\delta"] \\ E \arrow[r, "e"] & F \arrow[r, "f"] & G \arrow[r, "g"] & H \end{tikzcd}$$

If all the rows of the diagram above are exact, α is an epimorphism and δ is a monomorphism then :

1. *If β is a monomorphism then γ is a monomorphism*
2. *if γ is an epimorphism then β is an epimorphism.*

Proof with morphism : The two statements are dual, then it's enough to prove the first one. Let $j : X \to C$ be a morphism such that $\gamma \circ j = 0$. Then $\delta \circ c \circ j = g \circ \gamma \circ j = g \circ 0 = 0$, but δ is a monomorphism, thus $c \circ j = 0$ then there is j_1 such that $j = \ker(c) \circ j_1 = \operatorname{im}(b) \circ j_1$. Then by lemma 3.10 there is an epimorphism θ such that $j \circ \theta = b \circ j_2$. Then $0 = \gamma \circ j \circ \theta = \gamma \circ b \circ j_2 = f \circ \beta \circ j_2$. Then $\beta \circ j_2$ factors through $\ker(f) = \operatorname{im}(e)$, but α is an epimorphism then $\operatorname{im}(e) = \operatorname{im}(e \circ \alpha) = \operatorname{im}(\beta \circ \alpha)$ then by lemma 3.10 there is an epimorphism χ such that $\beta \circ j_2 \circ \chi = \beta \circ a \circ j_3$.

But β is a monomorphism, then $j_2 \circ \chi = a \circ j_3$, then $j \circ \theta \circ \chi = b \circ a \circ j_3 = 0$ and because $\theta \circ \chi$ is an epimorphism $j = 0$, then γ is a monomorphism. ■

Proof with elements :

1. Let x be in $\ker(\gamma)$. $0_H = g(0_G) = g \circ \gamma(x) = \delta \circ c(x)$ but δ is injective so $c(x) = 0$, i.e. $x \in \ker(c) = \operatorname{im}(b)$. So there is $y \in B$ such that $x = b(y)$. Thus $0_G = \gamma(x) = \gamma \circ b(y) = f \circ \beta(y)$ so $\beta(y) \in \ker(f) = \operatorname{im}(e)$. But α is surjective so $\operatorname{im}(e) = \operatorname{im}(e \circ \alpha)$ so let z be in A such that $\beta(y) = e \circ \alpha(z) = \beta \circ a(z)$. Then by linearity $y - a(z)$ is in $\ker(\beta) = 0$, in addition $b(y - a(z)) = b(y) - b \circ a(z) = x - 0 = x$ (because $a(z) \in \operatorname{im}(a) = \ker(b)$) i.e. $x = b(0) = 0$.
2. Let y be in F and $z \in C$ such that $f(y) = \gamma(z)$ then $\delta \circ c(z) = g \circ \gamma(z) = g \circ f(y) = 0$. But δ is injective then $c(z) = 0$ i.e. $z \in \ker(c) = \operatorname{im}(b)$, so let's have $x \in B$ such that $z = b(x)$. Then $f \circ \beta(x) = \gamma \circ b(x) = \gamma(z) = f(y)$ so $\beta(x) - y \in \ker(f) = \operatorname{im}(e) = \operatorname{im}(e \circ \alpha)$, because α is surjective, so let's have $u \in A$ such that $\beta(x) - y = e \circ \alpha(u)$ thus $y = \beta(x) - \beta \circ a(u) = \beta(x - a(u))$ then $y \in \operatorname{im}(\beta)$ then $F \subset \operatorname{im}(\beta)$, *i.e.* β is surjective. ■

Remark : The two proof are quite similar and indeed they contain the same ideas, however the first one avoid making choice, and can be interpreted as adding some arrows to a graph an using basic rules to compute. In addition diagram chasing with morphism allows to use the full power of duality in abelian category.

Lemma A.2 (*Five-lemma*)

Let's have a commutative diagram with value in an abelian category:

$$\begin{tikzcd} A_1 \arrow[r, "a_1"] \arrow[d, "f_1"] & A_2 \arrow[r, "a_2"] \arrow[d, "f_2"] & A_3 \arrow[r, "a_3"] \arrow[d, "f_3"] & A_4 \arrow[r, "a_4"] \arrow[d, "f_4"] & A_5 \arrow[d, "f_5"] \\ B_1 \arrow[r, "b_1"] & B_2 \arrow[r, "b_2"] & B_3 \arrow[r, "b_3"] & B_4 \arrow[r, "b_4"] & B_5 \end{tikzcd}$$

If all the rows are exact then:

1. *If f_2 and f_4 are epimorphism and f_5 is a monomorphism then f_3 is an epimorphism*
2. *If f_2 and f_4 are monomorphism and f_1 is an epimorphism then f_3 is a monomorphism*

3. *In particular if f_1, f_2, f_4 and f_5 are isomorphisms then f_3 is also an isomorphism.*

Proof: It is possible to do the same (but longer) kind of proof than for the four-lemma, but we can simply apply the four lemma: The first point is the four lemma applied to

$$\begin{tikzcd} A_2 \arrow[r, "a_2"] \arrow[d, "f_2"] & A_3 \arrow[r, "a_3"] \arrow[d, "f_3"] & A_4 \arrow[r, "a_4"] \arrow[d, "f_4"] & A_5 \arrow[d, "f_5"] \\ B_2 \arrow[r, "b_2"] & B_3 \arrow[r, "b_3"] & B_4 \arrow[r, "b_4"] & B_5 \end{tikzcd}$$

And the second point is the four-lemma applied to

$$\begin{tikzcd} A_1 \arrow[r, "a_1"] \arrow[d, "f_1"] & A_2 \arrow[r, "a_2"] \arrow[d, "f_2"] & A_3 \arrow[r, "a_3"] \arrow[d, "f_3"] & A_4 \arrow[d, "f_4"] \\ B_1 \arrow[r, "b_1"] & B_2 \arrow[r, "b_2"] & B_3 \arrow[r, "b_3"] & B_4 \end{tikzcd}$$

. For the third point we can apply the first and the second one to get that f_3 is an epimorphism and a monomorphism hence by 3.1 an isomorphism. ■

Lemma A.3 (***n × n lemma***)

Let's have a double chain complex with value in abelian category:

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & \dots & 0 \arrow[d] \\ 0 \arrow[r] & A_{1,1} \arrow[r] \arrow[d] & A_{1,2} \arrow[r] \arrow[d] & \dots \arrow[r] & A_{1,n} \arrow[d] \\ 0 \arrow[r] & A_{2,1} \arrow[r] \arrow[d] & A_{2,2} \arrow[r] \arrow[d] & \dots \arrow[r] & A_{2,n} \arrow[d] \\ \vdots & \vdots \arrow[d] & \vdots \arrow[d] & \ddots & \vdots \arrow[d] \\ 0 \arrow[r] & A_{n,1} \arrow[r] & A_{n,2} \arrow[r] & \dots \arrow[r] & A_{n,n} \end{tikzcd}$$

Then if all the columns and all the line but the first one are exact then the first line is exact.

Proof: A proof may be found in [8] as a consequence of the salamander lemma. ■

Lemma A.4 (***Snake-lemma***)

Let's have a commutative diagram with value in an abelian category:

$$\begin{tikzcd} & A_1 \arrow[r, "a_1"] \arrow[d, "f"] & A_2 \arrow[r, "a_2"] \arrow[d, "g"] & A_3 \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & B_1 \arrow[r, "b_1"] & B_2 \arrow[r, "b_2"] & B_3 & \end{tikzcd}$$

If all the rows are exact then there is a morphism $\partial : K(h) \to CK(f)$ such that the sequence

$$K(f) \to K(g) \to K(h) \to CK(f) \to CK(g) \to CK(h)$$

is exact. Where the arrows between the domain of kernels are the restrictions of a_i and the arrows between the codomain of the cokernels are the quotient maps obtained from the b_i. Moreover:

1. *if a_1 is a monomorphism $K(f) \to K(g)$ is also a monomorphism*

2. *if b_2 is an epimorphism then $CK(g) \to CK(h)$ is also an epimorphism.*

Proof : A proof may be found in [8] as a consequence of the salamander lemma. A version of this proof autogenerated by my program can be found in E ■

The name is due to the fact that with some imagination the sequence (in red) looks like a snake:

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & \mathrm{K}(f) \arrow[r] \arrow[d] & \mathrm{K}(g) \arrow[r] \arrow[d] & \mathrm{K}(h) \arrow[d] \arrow[dddll, bend left] & \\
 & A_1 \arrow[r, "a_1"] \arrow[d, "f"] & A_2 \arrow[r, "a_2"] \arrow[d, "g"] & A_3 \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & B_1 \arrow[r, "b_1"] \arrow[d] & B_2 \arrow[r, "b_2"] \arrow[d] & B_3 \arrow[d] & \\
 & \mathrm{CK}(f) \arrow[r] \arrow[d] & \mathrm{CK}(g) \arrow[r] \arrow[d] & \mathrm{CK}(h) \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

B Check if a diagram is commutative

Definition B.1

Let D be a diagram and $f : D \Rightarrow \mathcal{C}$ a pre-diagram. A set R of pair of parallel path is said to be generator if it is enough to check the hypothesis of 2.1 over the pair of path in R to get that f extend in a unique way into a functor $F : \mathcal{C}(D) \Rightarrow \mathcal{C}$

Remark : this notion does not depend on the pre-diagram f

Definition B.2

Let G be a graph and u and v be two distinct vertex of G.

1. *Let $G[u, v]$ be the sub graph of G generated by all path from u to v in G (it is void if there is no path)*

2. *The decomposition of $G[u, v]$ is the family of graphs $(G_i)_{i\in I}$ such that each G_i contain u and v and if we remove then we get the confected components of $G[u, v]$ without u and v.*

3. *Let R be a set of pair of parallel path in G. The graph associated with $G[u, v]$ and R: $\tilde{G}([u, v], R)$ is defined by the following:*

   (a) *The vertex are the G_i of the decomposition of $G[u, v]$*

   (b) *There is an edge (non oriented) between G_i and G_j.if and only if there is (p, q) in R such that their origin is u their end is v, p is a path in G_i and q is a path in G_j.*

Theorem B.1

Let D be a diagram. A set R of pair of parallel path is generator if and only if for any two vertex u and v the non-oriented graph $\tilde{G}([u, v], R)$ is connected (with the convention that the void-graph is connected)

Proof : It's a consequence of the proof of theorem 2.2 in [4] ■

Remark : The theorem 2.2 in [4] states that R is generator and minimal if and only if all the graph $\tilde{G}([u,v],R)$ are trees which is equivalent to being connected and having one vertex more than edges.

In practice the user could have more relations than the minimum needed, and want the proof assistant conclude that it's enough to have a commutative diagram then it's necessary to characterize the fact of being generator and not only minimal

Remark : By applying this theorem it's thus straightforward to deduce an algorithm able to tell the user if a set of equality statement is enough to prove the commutativity of a diagram. This algorithm run in polynomial time (according to [4])

The algorithm could also be used to compute the possible generator set for the diagram (but this time in exponential time)

Example : In particular as soon as there is only two path between two vertex of a diagram, the set R must contain the pair of this two path.

For the diagram D:

$$\begin{tikzcd}
& A \arrow[dl] \arrow[dr] & \\
B \arrow[rr] \arrow[dr] & & C \arrow[dl] \\
& D &
\end{tikzcd}$$

All the $G(u,v)$ except $G(A,C)$ and $G(B,D)$ are empty or with a single element then they are connected. The two graphs are triangle, then there is only two path in them R must contain them.

Thus a pre-diagram f over D is commutative if and only if $f(A \to C) = f(B \to C) \circ f(A \to B)$ and $f(B \to D) = f(C \to D) \circ f(B \to C)$

C Freyd–Mitchell Embeding

The proof of the statement of this section can be found in [5]

Definition C.1

If $\mathcal{C}$ and $\mathcal{D}$ are two categories, a functor F between $\mathcal{C}$ and $\mathcal{D}$ is said to be:

1. *An embedding if for any X and Y two objects of $\mathcal{C}$, the map*

$$F : Hom(X,Y) \to Hom(F(X),F(Y))$$

is an injection.

2. *Full if if for any X and Y two objects of $\mathcal{C}$, the map*

$$F : Hom(X,Y) \to Hom(F(X),F(Y))$$

is surjective.

Definition C.2

If $\mathcal{C}$ and $\mathcal{D}$ are two abelian categories, then a functor $F : \mathcal{C} \Rightarrow \mathcal{D}$ is said to be additive if for any A and B two objects of $\mathcal{C}$, the induced map $F : (Hom(A,B),+) \to (Hom(F(A),F(B)),+)$ is a group homomorphism.

Proposition C.1

If $\mathcal{C}$ and $\mathcal{D}$ are two abelian categories, then a functor $F : \mathcal{C} \Rightarrow \mathcal{D}$ is additive if and only if F for any A and B two objects of $\mathcal{C}$, $F(A \bigoplus B) = F(A) \bigoplus F(B)$

Definition C.3

If $\mathcal{C}$ and $\mathcal{D}$ are two abelian categories then a functor $F : \mathcal{C} \Rightarrow \mathcal{D}$ is said to be exact if for any *morphism f in $\mathcal{C}$ $F(\ker(f)) = \ker(F(f))$ and $F(coker(f)) = coker(F(f))$*

Lemma C.1

An exact functor $F : \mathcal{C} \Rightarrow \mathcal{D}$ between two abelians categories is aditive.

Theorem C.1 (*Weak Freyd–Mitchell Embeding*)

If $\mathcal{C}$ is an abelian category then there is an exact embeding of $\mathcal{C}$ into the category of abelian groups

Remark : In particular if a decision problem is satisfied on every instance of diagrams with values in abelian groups then it is satisfied in every diagram.

Theorem C.2 (*Strong Freyd–Mitchell Embeding*)

If $\mathcal{C}$ is an abelian category then there is an ring R (not necessarily commutative) and exact full embeding of $\mathcal{C}$ into the category of R-modules

Remark : The fact that the exact embeding is full allows one to construct morphisms (like in the snake lemma) to prove proprieties on them on the embeded diagram and to deduce that this morphism also exists (with the same propriety of exactness) in the original diagram.

Proposition C.2

The commutativity and exactness conditions of a diagram is equivalent to the exactness and commutativity of it's image by an exact embeding

D Elementary deduction rules of diagram chasing

D.1 In general

In this section are presented the rules implemented in order to deduce automatically information from diagram chasing. Some of the maps or rules discussed in 3.2 are not implemented, it's because the set bellow was in practice enough to deduce them.

In the following $f : A \to B$ and $g : B \to C$ are two morphism in some abelian category.

1. If $f = 0$ then $g \circ f = 0$
2. If $g = 0$ then $g \circ f = 0$
3. If f and g are monomorphisms then so does $g \circ f$
4. If $g \circ f$ is a monomorphism then so does f
5. If f and g are epimorphisms then so does $g \circ f$
6. If $g \circ f$ is an epimorphism then so does f
7. If g is a monomorphism and $g \circ f = 0$ then $f = 0$
8. If f is an epimorphism and $g \circ f = 0$ then $g = 0$
9. If $g \circ f = 0$ there is a ϕ such that $f = \ker(g) \circ \phi$
10. If $g \circ f = 0$ there is a ψ such that $g = \psi \circ \mathrm{coker}(f)$
11. If (f, g) is exact and $f = 0$ then g is a monomorphism
12. If (f, g) is exact and $g = 0$ then f is an epimorphism
13. If (f, g) is exact and g is a monomorphism then $f = 0$
14. If (f, g) is exact and f is an epimorphism then $g = 0$
15. If $A = 0$ then $f = 0$
16. If $B = 0$ then $f = 0$
17. If $f = 0$ and f is a monomorphism then $A = 0$
18. If $f = 0$ and f is an epimorphism then $B = 0$

19. If $B = 0$ then f is an epimorphism
20. If $A = 0$ then f is a monomorphism
21. If f is a monomorphism then $\ker(f) = 0$
22. If f is an epimorphism then $\text{coker}(f) = 0$
23. If $\ker(f) = 0$ then f is a monomorphism
24. If $\text{coker}(f) = 0$ then f is an epimorphism
25. If (f, g) is exact then $g \circ f = 0$
26. If f is of the form $\text{im}(b) \circ j$ then there is an epimorphism θ and a map j' such that $f \circ \theta = b \circ j'$
27. If f is of the form $j \circ \text{coim}(b)$ then there is a monomorphism θ and a map j' such that $\theta \circ f = j' \circ b$

D.2 In the case of the double chain complex

The rules of the general case not involving the kernel and cokernel are still used (even if from a practical point of view their implementation is different due to the particular shape of the double chain complex. If the following diagram is part of the double chain complex:

$$\begin{tikzcd}
\bullet \arrow[r, "a"] \arrow[d, "b"'] \arrow[dr, "p"] & \bullet \arrow[d, "c"] & \\
\bullet \arrow[r, "d"] & A \arrow[r, "e"] \arrow[d, "f"'] \arrow[dr, "q"] & \bullet \arrow[d, "g"] \\
& \bullet \arrow[r, "h"] & \bullet
\end{tikzcd}$$

then the following rules are added (their proof is straightforward by construction of the objects) :

1. If e is a monomorphism then ${}^{\square}A = 0$ and $A^h = 0$
2. If f is a monomorphism then ${}^{\square}A = 0$ and $A^v = 0$
3. If q is a monomorphism then $A_{\square} = 0$
4. If c is an epimorphism then $A_{\square} = 0$ and $A^v = 0$
5. If d is an epimorphism then $A_{\square} = 0$ and $A^h = 0$
6. If p is an epimorphism then ${}^{\square}A = 0$
7. If $A \to B$ is zero then $A_{\square} \to^{\square} B$ is zero

E Automatic proof of snake-lemma

In this section is given (as an example) the proof of the snake lemma A.4, generated by my program and using the salamander lemma (3.3). It's the proof that there is a connection morphism and that the sequence is exact at the origin and the end of this morphism.

$ZERO$ is zero obj by assumption

$donor$ $ZERO$ is zero obj because: if an object is 0 then it's donor is 0 applied to:
* $ZERO$ is zero obj

$donor\ ZERO \to receptor\ CK(f)$ is zero because: if $X = 0$ then $f : X \to Y = 0$ applied to :
* $donor\ ZERO$ is zero obj

$donor\ ZERO \to receptor\ CK(f) \to h\_hom\ CK(f)$ is exact by horizontal Salamander lemma at $ZERO \to CK(f)$

$receptor\ CK(f) \to h\_homCK(f)$ is mono because: if f, g is exact and $f = 0$ then g is mono applied to :
* $donor\ ZERO \to receptor\ CK(f)$ is zero
* $donor\ ZERO \to receptor\ CK(f) \to h\_homCK(f)$ is exact

$receptor\ ZERO$ is zero obj because: if an object is 0 then it's receptor is 0 applied to :
* $ZERO$ is zero obj

$h\_hom\ CK(f) \to receptor\ ZERO$ is zero because: if $Y = 0$ then $f : X \to Y = 0$ applied to :
* $receptor\ ZERO$ is zero obj

$receptor\ CK(f) \to h\_homCK(f) \to receptor\ ZERO$ is exact by horizontal Salamander lemma at $ZERO \to CK(f)$

$receptor\ CK(f) \to h\_homCK(f)$ is epi because: if f, g is exact and $g = 0$ then f is epi applied to :
* $h\_homCK(f) \to receptor\ ZERO$ is zero
* $receptorCK(f) \to h\_homCK(f) \to receptor\ ZERO$ is exact

$receptor\ CK(f) \to h\_homCK(f)$ is iso because: if f is a mono and an epi then f is an iso applied to :
* $receptor\ CK(f) \to h\_homCK(f)$ is mono
* $receptor\ CK(f) \to h\_hom\ CK(f)$ is epi

$A_1 \to B_1 \to CK(f)$ is exact by assumption

$v\_hom\ B_1$ is zero obj because: if the vertical composition is exact

then the v_hom is 0 applied to :
* $A_1 \to B_1 \to CK(f)$ is exact

$v\_hom\ B_1 \to donor\ B_1$ is zero because: if $X = 0$ then $f : X \to Y = 0$ applied to :
* $v\_hom\ B_1$ is zero obj

$v\_hom\ B_1 \to donor\ B_1 \to receptor\ CK(f)$ is exact by vertical Salamander lemma at $B_1 \to CK(f)$

$donor\ B_1 \to receptor\ CK(f)$ is mono because: if f, g is exact and $f = 0$ then g is mono applied to :
* $v\_hom\ B_1 \to donor\ B_1$ is zero
* $v\_hom\ B_1 \to donor\ B_1 \to receptor\ CK(f)$ is exact

$B_1 \to CK(f) \to ZERO$ is exact by assumption

$v\_hom\ CK(f)$ is zero obj because: if the vertical composition is exact then the v_hom is 0 applied to :
* $B_1 \to CK(f) \to ZERO$ is exact

$receptor\ CK(f) \to v\_hom\ CK(f)$ is zero because: if $Y = 0$ then $f : X \to Y$ applied to :
* $v\_hom\ CK(f)$ is zero obj

$donor\ B_1 \to receptor\ CK(f) \to v\_hom\ CK(f)$ is exact by vertical Salamander lemma at $B_1 \to CK(f)$

$donor\ B_1 \to receptor\ CK(f)$ is epi because: if f, g is exact and $g = 0$ then f is epi applied to :
* $receptor\ CK(f) \to v\_hom\ CK(f)$ is zero
* $donor\ B_1 \to receptor\ CK(f) \to v\_hom\ CK(f)$ is exact

$donor\ B_1 \to receptor\ CK(f)$ is iso because: if f is a mono and an epi then f is an iso applied to :
* $donor\ B_1 \to receptor\ CK(f)$ is mono
* $donor\ B_1 \to receptor\ CK(f)$ is epi

$ZERO \to B_1 \to B_2$ is exact by assumption

$h\_hom\ B_1$ is zero obj because: if the horizontal composition is exact then the h_hom is 0 applied to :
* $ZERO \to B_1 \to B_2$ is exact

$receptor\ B_1 \to h\_hom\ B_1$ is zero because: if $Y = 0$ then $f : X \to Y$ applied to :
* $h\_hom\ B_1$ is zero obj

$donor\ ZERO \to receptor\ B_1$ is zero because: if $X = 0$ then $f : X \to Y = 0$ applied to :
* $donor\ ZERO$ is zero obj

$donor\ ZERO \to receptor\ B_1 \to h\_hom\ B_1$ is exact by horizontal Salamander lemma at $ZERO \to B_1$

$receptor\ B_1 \to h\_hom\ B_1$ is mono because: if f, g is exact and $f = 0$ then g is mono applied to :
* $donor\ ZERO \to receptor\ B_1$ is zero
* $donor\ ZERO \to receptor\ B_1 \to h\_hom\ B_1$ is exact

$receptor\ B_1$ is zero obj because: if $f : X \to Y$ is 0 and mono then $X = 0$ applied to :
* $receptor\ B_1 \to h\_hom\ B_1$ is zero
* $receptor\ B_1 \to h\_hom\ B_1$ is mono

$receptor\ B_1 \to v\_hom\ B_1$ is zero because: if $X = 0$ then $f : X \to Y = 0$ applied to :
*$receptor\ B_1$ is zero obj

$receptor\ B_1 \to donor\ B_1$ is zero because: if $f = 0$ then $gf = 0$ applied to :
* $receptor\ B_1 \to v\_hom\ B_1$ is zero

$receptor\ B_1 \to h\_hom\ B_1$ is epi because: if $Y = 0$ then $f : X \to Y$ is epi applied to :
* $h\_hom\ B_1$ is zero obj

$h\_hom\ B_1 \to donor\ B_1$ is zero because: if f is epi and $gf = 0$ then $g = 0$ applied to :
* $receptor\ B_1 \to donor\ B_1$ is zero
* $receptor\ B_1 \to h\_hom\ B_1$ is epi

$h\_hom\ B_1 \to donor\ B_1 \to receptor\ B_2$ is exact by horizontal Salamander lemma at $B_1 \to B_2$

$donor\ B_1 \to receptor\ B_2$ is mono because: if f, g is exact and $f = 0$ then g is mono applied to :
* $h\_hom\ B_1 \to donor\ B_1$ is zero
* $h\_hom\ B_1 \to donor\ B_1 \to receptor\ B_2$ is exact

$B_1 \to B_2 \to B_3$ is exact by assumption

$h\_hom\ B_2$ is zero obj because: if the horizontal composition is exact then the h_hom is 0 applied to :
* $B_1 \to B_2 \to B_3$ is exact

$receptor\ B_2 \to h\_hom\ B_2$ is zero because: if $Y = 0$ then $f : X \to Y$ applied to :
* $h\_hom\ B_2$ is zero obj

$donor\ B_1 \to receptor\ B_2 \to h\_hom\ B_2$ is exact by horizontal Salamander lemma at $B_1 \to B_2$

$donor\ B_1 \to receptor\ B_2$ is epi because: if f, g is exact and $g = 0$ then f is epi applied to :
* $receptor\ B_2 \to h\_hom\ B_2$ is zero
* $donor\ B_1 \to receptor\ B_2 \to h\_hom\ B_2$ is exact

$donor\ B_1 \to receptor\ B_2$ is iso because: if f is a mono and an epi then f is an iso applied to :
* $donor\ B_1 \to receptor\ B_2$ is mono
* $donor\ B_1 \to receptor\ B_2$ is epi

$K(g) \to A_2 \to B_2$ is exact by assumption

$v\_hom\ A_2$ is zero obj because: if the vertical composition is exact then the v_hom is 0 applied to :
* K$(g) \to A_2 \to B_2$ is exact

$v\_hom\ A_2 \to donor\ A_2$ is zero because: if $X = 0$ then $f : X \to Y = 0$ applied to :
* $v\_hom\ A_2$ is zero obj

$v\_hom\ A_2 \to donor\ A_2 \to receptor\ B_2$ is exact by vertical Salamander lemma at $A_2 \to B_2$

$donor\ A_2 \to receptor\ B_2$ is mono because: if f, g is exact and $f = 0$ then g is mono applied to :
* $v\_hom\ A_2 \to donor\ A_2$ is zero
* $v\_hom\ A_2 \to donor\ A_2 \to receptor\ B_2$ is exact

$A_2 \to B_2 \to CK(g)$ is exact by assumption

$v\_hom\ B_2$ is zero obj because: if the vertical composition is exact then the v_hom is 0 applied to :
* $A_2 \to B_2 \to CK(g)$ is exact

$receptor\ B_2 \to v\_hom\ B_2$ is zero because: if $Y = 0$ then $f : X \to Y$ applied to :
* $v\_hom\ B_2$ is zero obj

$donor\ A_2 \to receptor\ B_2 \to v\_hom\ B_2$ is exact by vertical Salamander lemma at $A_2 \to B_2$

$donor\ A_2 \to receptor\ B_2$ is epi because: if f, g is exact and $g = 0$ then f is epi applied to :
* $receptor\ B_2 \to v\_hom\ B_2$ is zero
* $donor\ A_2 \to receptor\ B_2 \to v\_hom\ B_2$ is exact

$donor\ A_2 \to receptor\ B_2$ is iso because: if f is a mono and an epi then f is an iso applied to :
* $donor\ A_2 \to receptor\ B_2$ is mono
* $donor\ A_2 \to receptor\ B_2$ is epi

$A_1 \to A_2 \to A_3$ is exact by assumption

$h\_hom\ A_2$ is zero obj because: if the horizontal composition is exact then the h_hom is 0 applied to :
* $A_1 \to A_2 \to A_3$ is exact

$receptor\ A_2 \to h\_hom\ A_2$ is zero because: if $Y = 0$ then $f : X \to Y$ applied to :
* $h\_hom\ A_2$ is zero obj

$receptor\ A_2 \to donor\ A_2$ is zero because: if $f = 0$ then $gf = 0$ applied to :
* $receptor\ A_2 \to h\_hom\ A_2$ is zero

$receptor\ A_2 \to h\_hom\ A_2$ is epi because: if $Y = 0$ then $f : X \to Y$ is epi applied to :
* $h\_hom\ A_2$ is zero obj

$h\_hom\ A_2 \to donor\ A_2$ is zero because: if f is epi and $gf = 0$ then $g = 0$ applied to :
* $receptor\ A_2 \to donor\ A_2$ is zero
* $receptor\ A_2 \to h\_hom\ A_2$ is epi

$h\_hom\ A_2 \to donor\ A_2 \to receptor\ A_3$ is exact by horizontal Salamander lemma at $A_2 \to A_3$

$donor\ A_2 \to receptor\ A_3$ is mono because: if f, g is exact and $f = 0$ then g is mono applied to :
* $h\_hom\ A_2 \to donor\ A_2$ is zero
* $h\_hom\ A_2 \to donor\ A_2 \to receptor\ A_3$ is exact

$A_2 \to A_3 \to ZERO$ is exact by assumption

$h\_hom\ A_3$ is zero obj because: if the horizontal composition is exact then the h_hom is 0 applied to :
* $A_2 \to A_3 \to ZERO$ is exact

$receptor\ A_3 \to h\_hom\ A_3$ is zero because: if $Y = 0$ then $f : X \to Y$ applied to :
* $h\_hom\ A_3$ is zero obj

$donor\ A_2 \to receptor\ A_3 \to h\_hom\ A_3$ is exact by horizontal Salamander lemma at $A_2 \to A_3$

$donor\ A_2 \to receptor\ A_3$ is epi because: if f, g is exact and $g = 0$ then f is epi applied to :
* $receptor\ A_3 \to h\_hom\ A_3$ is zero
* $donor\ A_2 \to receptor\ A_3 \to h\_hom\ A_3$ is exact

$donor\ A_2 \to receptor\ A_3$ is iso because: if f is a mono and an epi then f is an iso applied to :
* $donor\ A_2 \to receptor\ A_3$ is mono
* $donor\ A_2 \to receptor\ A_3$ is epi

$ZERO \to K(h) \to A_3$ is exact by assumption

$v\_hom\ K(h)$ is zero obj because: if the vertical composition is exact then the v_hom is 0 applied to :
* $ZERO \to K(h) \to A_3$ is exact

$receptor\ K(h) \to v\_hom\ K(h)$ is zero because: if $Y = 0$ then $f : X \to Y$ applied to :
* $v\_hom\ K(h)$ is zero obj

$donor\ ZERO \to receptor\ K(h)$ is zero because: if $X = 0$ then $f : X \to Y = 0$ applied to :
* $donor\ ZEROiszeroobj$

$donor\ ZERO \to receptor\ K(h) \to v\_hom\ K(h)$ is exact by vertical Salamander lemma at $ZERO \to K(h)$

$receptor\ K(h) \to v\_hom\ K(h)$ is mono because: if f, g is exact and $f = 0$ then g is mono applied to :
* $donor\ ZERO \to receptor\ K(h)$ is zero
* $donor\ ZERO \to receptor\ K(h) \to v\_hom\ K(h)$ is exact

$receptor\ K(h)$ is zero obj because: if $f : X \to Y is 0$ and mono then $X = 0$ applied to :
* $receptor\ K(h) \to v\_hom\ K(h)$ is zero
* $receptor\ K(h) \to v\_hom\ K(h)$ is mono

$receptor\ K(h) \to h\_hom\ K(h)$ is zero because: if $X = 0$ then $f : X \to Y = 0$ applied to :
* $receptor\ K(h)$ is zero obj

$receptor\ K(h) \to donor\ K(h)$ is zero because: if $f = 0$ then $gf = 0$ applied to :
* $receptor\ K(h) \to h\_hom\ K(h)$ is zero

$v\_hom\ K(h) \to receptor\ ZERO$ is zero because: if $Y = 0$ then $f : X \to Y = 0$ applied to :
* $receptor\ ZERO$ is zero obj

$receptor\ K(h) \to v\_hom\ K(h) \to receptor\ ZERO$ is exact by vertical Salamander lemma at $ZERO \to K(h)$

$receptor\ K(h) \to v\_hom\ K(h)$ is epi because: if f, g is exact and $g = 0$ then f is epi applied to :
* $v\_hom\ K(h) \to receptor\ ZERO$ is zero
* $receptor\ K(h) \to v\_hom\ K(h) \to receptor\ ZERO$ is exact

$v\_hom\ K(h) \to donor\ K(h)$ is zero because: if f is epi and $gf = 0$ then $g = 0$ applied to :
* $receptor\ K(h) \to donor\ K(h)$ is zero
* $receptor\ K(h) \to v\_hom\ K(h)$ is epi

$v\_hom\ K(h) \to donor\ K(h) \to receptor\ A_3$ is exact by vertical Salamander lemma at $K(h) \to A_3$

$donor\ K(h) \to receptor\ A_3$ is mono because: if f, g is exact and $f = 0$ then g is mono applied to :
* $v\_hom\ K(h) \to donor\ K(h)$ is zero
* $v\_hom\ K(h) \to donor\ K(h) \to receptor\ A_3$ is exact

$K(h) \to A_3 \to B_3$ is exact by assumption

$v\_hom\ A_3$ is zero obj because: if the vertical composition is exact then the v_hom is 0 applied to :
* $K(h) \to A_3 \to B_3$ is exact

$receptor\ A_3 \to v\_hom\ A_3$ is zero because: if $Y = 0$ then f: $X \to Y$ applied to :
* $v\_hom\ A_3$ is zero obj

$donor\ K(h) \to receptor\ A_3 \to v\_hom\ A_3$ is exact by vertical Salamander lemma at $K(h) \to A_3$

$donor\ K(h) \to receptor\ A_3$ is epi because: if f, g is exact and $g = 0$ then f is epi applied to :
* $receptor\ A_3 \to v\_hom\ A_3$ is zero
* $donor\ K(h) \to receptor\ A_3 \to v\_hom\ A_3$ is exact

$donor\ K(h) \to receptor\ A_3$ is iso because: if f is a mono and an epi then f is an iso applied to :
* $donor\ K(h) \to receptor\ A_3$ is mono
* $donor\ K(h) \to receptor\ A_3$ is epi

$donor\ ZERO \to h\_hom\ K(h)$ is zero because: if $X = 0$ then $f : X \to Y = 0$ applied to :
* $donor\ ZERO$ is zero obj

$donor\ ZERO \to h\_hom\ K(h) \to donor\ K(h)$ is exact by horizontal Salamander lemma at $K(h) \to ZERO$

$h\_hom\ K(h) \to donor\ K(h)$ is mono because: if f, g is exact and $f = 0$ then g is mono applied to :
* $donor\ ZERO \to h\_hom\ K(h)$ is zero
* $donor\ ZERO \to h\_hom\ K(h) \to donor\ K(h)$ is exact

$donor\ K(h) \to receptor\ ZERO$ is zero because: if $Y = 0$ then $f : X \to Y = 0$ applied to :
* $receptor\ ZERO$ is zero obj

$h\_hom\ K(h) \to donor\ K(h) \to receptor\ ZERO$ is exact by horizontal Salamander lemma at $K(h) \to ZERO$

$h\_hom\ K(h) \to donor\ K(h)$ is epi because: if f, g is exact and $g = 0$ then f is epi applied to :
* $donor\ K(h) \to receptor\ ZERO$ is zero
* $h\_hom\ K(h) \to donor\ K(h) \to receptor\ ZERO$ is exact

$h\_hom\ K(h) \to donor\ K(h)$ is iso because: if f is a mono and an epi then f is an iso applied to :
* $h\_hom\ K(h) \to donor\ K(h)$ is mono
* $h\_hom\ K(h) \to donor\ K(h)$ is epi

$h\_hom\ K(h)$ and $h\_hom\ CK(f)$ are iso obj because: if there is a chain of isomorphism between X and Y then they are isomorphic applied to :
* $receptor\ CK(f) \to h\_hom\ CK(f)$ is iso
* $donor\ B_1 \to receptor\ CK(f)$ is iso
* $donor\ B_1 \to receptor\ B_2$ is iso
* $donor\ A_2 \to receptor\ B_2$ is iso
* $donor\ A_2 \to receptor\ A_3$ is iso
* $donor\ K(h) \to receptor\ A_3$ is iso
* $h\_hom\ K(h) \to donor\ K(h)$ is iso

$K(g) \to K(h) \to CK(f) \to CK(g)$ is exact(connected) because: if $A \to B \to 0$ and $0 \to C \to D$ are part of the complex and homology at B iso homology at C then there is connecting morphism such that the sequence is exact applied to :
* $h\_hom\ K(h)$ and $h\_hom\ CK(f)$ are iso obj

F Proof of some results

Proof of 1.2

Let $(A \times B, \pi_A, \pi_{B,A})$ be the product of A and B, let $(B \times C, \pi_{B,C}, \pi_C)$ be the product of B and C, let $((A \times B) \times C, \pi_1, \pi_2)$ be the product of $A \times B$ and C.

To conclude by 1.1 it is enough to prove that $((A \times B) \times C, \pi_A \circ \pi_1, (\pi_{B,A} \circ \pi_1) \times \pi_2)$ satisfy the universal propriety of $(A \times (B \times C), \rho_1, \rho_2)$. Let X be an object of $\mathcal{C}$, $f : X \to A$ and $g : B \times C$. By uniqueness in the universal propriety of $B \times C$, one gets that $g = (\pi_B \circ g) \times (\pi_C \circ g) = g_B \times g_C$.

If there is a $\phi : X \to$(A$\times B) \times C$ such that $\pi_A \circ \pi_1 \circ \phi = f$ and $((\pi_{B,A} \circ \pi_1) \times \pi_2) \circ \phi = g_B \times g_C$.

By uniqueness in the universal proprety of $B \times C$, $((\pi_{B,A} \circ \pi_1) \times \pi_2) \circ \phi = (\pi_{B,A} \circ \pi_1 \circ \phi) \times (\pi_2 \circ \phi)$.

$$\begin{tikzcd}
 & & & B \\
X \arrow[r, "\phi"] \arrow[rrru, bend left, "\pi_{B,A}\circ\pi_1\circ\phi"] \arrow[rrrd, bend right, "\pi_2\circ\phi"'] & (A \times B) \times C \arrow[rru, "\pi_{B,A}\circ\pi_1"] \arrow[rrd, "\pi_2"'] \arrow[r, "(\pi_{B,A}\circ\pi_1)\times\pi_2"] & B \times C \arrow[ru] \arrow[rd] & \\
 & & & C
\end{tikzcd}$$

Therefore (still by uniqueness in the universal propriety of $B\times C$) $g_B = \pi_{B,A}\circ\pi_1\circ\phi$ and $g_C = \pi_2\circ\phi$. Then the following diagram is commutative:

$$\begin{tikzcd}
 & & A \\
X \arrow[r, "\pi_1\circ\phi"] \arrow[rru, bend left, "f"] \arrow[rrd, bend right, "g_B"'] & A \times B \arrow[ru, "\pi_A"] \arrow[rd, "\pi_{B,A}"] & \\
 & & B
\end{tikzcd}$$

Thus by uniqueness in the universal propriety $\pi_1 \circ \phi$ is equal to $f \times g_B$. But by uniqueness in the universal propriety of $(A\times B)\times C$, $\phi = (\pi_1\circ\phi)\times(\pi_2\circ\phi) = (f\times g_B)\times g_C$. Hence if ϕ exists ϕ is unique.

Let ϕ be $(f \times g_B) \times g_C$, $\pi_A \circ \pi_1 \circ \phi = \pi_A \circ (f \times g_B) = f$ and $((\pi_{B,A} \circ \pi_1) \times \pi_2) \circ \phi = (\pi_{B,A} \circ \pi_1 \circ \phi) \times (\pi_2 \circ \phi) = (\pi_{B,A} \circ (f \times g_B)) \times g_C = g_B \times g_C$. Then ϕ is solution of the universal problem.

Proof of 3.3

Lemma F.1

An abelian category admits all the intersections of sub-objects

Proof: Let $i_A : A \hookrightarrow C$ and $i_B : B \hookrightarrow C$ be two subobjects of C. $\mathcal{C}$ is abelian then i_A is the kernel of some morphism $f : C \to D$ and i_B is the kernel of some morphism $g : C \to E$. Let $i : K \to C$ be the kernel of $f \times g : C \to D \times E$. $f \circ i = \pi_1 \circ (f \times g) \circ i = \pi_1 0 = 0$ thus i factors through i_A (by definition of the kernel of f). $g \circ i = \pi_2 \circ (f \times g) \circ i = \pi_2 0 = 0$ thus i factors through i_B (by definition of the kernel of g). Let u and v be the morphisms such that, $i = i_A \circ u = i_B \circ v$. To conclude it is enough to show that (K, u, v) satisfy the universal propriety of the intersection 1.6, in fact one can show better (and it will be useful) and don't assume that $X \to C$ is a subobject.

$$\begin{tikzcd}
 & A \arrow[rd, "i_A"] & \\
X \arrow[ru, bend left, "u_A"] \arrow[rr, "u"] \arrow[rd, bend right, "u_B"'] & & C \\
 & B \arrow[ru, "j_B"'] &
\end{tikzcd}$$

$(f \times g) \circ u = (f \circ u) \times (g \circ u) = (f \circ i_A \circ u) \times (g \circ i_B \circ u_B) = 0 \times 0 = 0$, then by definition of the kernel of $f \times g$, u factors through i. ■

Lemma F.2

An abelian category admits all the equalizers and coequalizers

Proof: By duality it is enough to construct all the equalizers. Let $u, v : A \to B$ be two morphisms. Then $(\mathrm{id}_A \times u) : A \to A \times B$ and $(\mathrm{id}_A \times v) : A \to A \times B$ are two subobjects (composed by π_1 they equals id_A which is a monomorphism) of $A \times B$, thus let $(i : K \to A \times B, i_u, i_v)$ be their intersection.

Then $i_u = \mathrm{id}_A \circ i_u = \pi_1 \circ (\mathrm{id}_A \times u) \circ i_u = \pi_1 \circ (\mathrm{id}_A \times v) \circ i_v = i_v$. Let's show that $i = i_u = i_v$ is the equalizer of u and v. $u \circ i = \pi_2 \circ (\mathrm{id}_A \times u) \circ i = \pi_2 \circ (\mathrm{id}_A \times v) \circ i = v \circ i$.

Let $j : X \to A$ be a map such that $u \circ j = v \circ j$, then $(\mathrm{id}_A \times u) \circ j = (j \times u \circ j) = (j \times v \circ j) = (\mathrm{id}_A \times v) \circ j$ is a map $X \to A \times B$ that factors through the two subobjects, then (by the construction of intersection in abelian category F.1) there is a unique map $\phi : X \to K$ such that $j = i \circ \phi$. Thus i is solution of the universal problem of the kernel. ■

1. $+$ is associative. Let h be a third element of $\mathrm{Hom}(A, B)$, by 1.2 the products and coproducts are associative then (by uniqueness of the map) $h \coprod (f \coprod g) = (h \coprod f) \coprod g$ and $\mathrm{id}_A \times (\mathrm{id}_A \times \mathrm{id}_A) = (\mathrm{id}_A \times \mathrm{id}_A) \times \mathrm{id}_A$, by composition of those two relation one gets $h + (f + g) = (h + f) + g$.

2. $+$ is commutative, $(A \bigoplus A, i_1, i_2)$ and $(A \bigoplus A, i_2, i_1)$ are two coproduct of A and A thus the two universal maps $f \coprod g$ given from f and g and the one given by g and f: $g \coprod f$ are identified thus by composition with $\mathrm{id}_A \times \mathrm{id}_A$ one get $f + g = g + f$.

3. $0_{A \to B}$ is the neutral element, indeed by commutativity it's enough to prove that $f + 0_{A \to B} = f$, indeed with the remark 3.1, one gets that $f \circ \pi_1 \circ i_1 = f$ and $f \circ \pi_1 \circ i_2 = 0$. Thus by uniqueness in the universal propriety $f0 = f\pi_1$, then $f + 0 = f \circ \pi_1 \circ (\mathrm{id} \times \mathrm{id}) = f \circ \mathrm{id} = f$.

4. If h is a morphism of $\mathrm{Hom}(B, C)$ then $h \circ (f + g) = h \circ f + h \circ g$. Indeed the diagram:

$$\begin{tikzcd}
A \arrow[dr, "i_2"'] \arrow[drr, "g"] \arrow[drrr, bend left, "h\circ g"] & & & \\
& A \oplus A \arrow[r, "f\coprod g"] & B \arrow[r, "h"] & C \\
A \arrow[ur, "i_1"] \arrow[urr, "f"'] \arrow[urrr, bend right, "h\circ f"'] & & &
\end{tikzcd}$$

is commutative, then by uniqueness of the map given by universal propriety $(h \circ f) \coprod (h \circ g) = h \circ (f \coprod g)$ it gives the formula by composition with $\mathrm{id}_A \times \mathrm{id}_A$.

5. If h is a morphism of $\mathrm{Hom}(Z, A)$ then $(f + g) \circ h = f \circ h + g \circ h$, indeed $\mathcal{C}^{op}$ is an abelian category, then one can apply the previous point in $\mathcal{C}^{op}$, wich is exactly the relation expected.

Let ϕ be the map $(\mathrm{id}_A \coprod 0_{A \to B}) \times (f \coprod \mathrm{id}_B) : A \bigoplus B \to A \bigoplus B$. Let $u, v : X \to A \bigoplus B$, be two morphisms such that $\phi \circ u = \phi \circ v$. By uniqueness in the universal propriety of $A \times B$, $u = (\pi_A \circ u) \times (\pi_B \circ u) = u_A \times u_B$, and $v = (\pi_A \circ v) \times (\pi_B \circ v) = v_A \times v_B$, in the previous point one showed that $\mathrm{id}_A \coprod 0_{A \to B} = \mathrm{id}_A \circ \pi_A = \pi_A$ thus $\pi_1 \circ \phi \circ u = (\mathrm{id}_A \coprod 0_{A \to B}) \circ u = \pi_A \circ u = u_A$ thus by the same calculation on v (and the hypothesis) $u_A = v_A$.

The dual formula of $f \coprod 0 = f \circ \pi_A$ is $0 \times f = i_B \circ f$, thus $\pi_B \circ \phi \circ u = (f \coprod \mathrm{id}_B) \circ (0 \times u_B) = (f \coprod \mathrm{id}_B) \circ i_B \circ u_B = \mathrm{id}_B \circ u_B = u_B$. Then $u_B = v_B$, thus $u = v$, then ϕ is a monomorphism. By duality it is also an epimorphism, then according to 3.1 it is an isomorphism. Let ψ be it's inverse, $\pi_B \circ \psi \circ i_A$ is noted $-f$.

Let's remark that $(f \circ \pi_A + g \circ \pi_B) \circ i_A = f \circ \pi_A \circ i_A + g \circ \pi_B \circ i_A = f \circ \mathrm{id}_A + g \circ 0 = f$ and $(f \circ \pi_A + g \circ \pi_B) \circ i_B = f \circ \pi_A \circ i_B + g \circ \pi_B \circ i_B = f \circ 0 + g \circ \mathrm{id}_B = g$ thus by uniqueness in the universal propriety $(f \circ \pi_A + g \circ \pi_B) = f \coprod g$.

Then $(f \coprod g) \circ (u \times v) = (f \circ \pi_A + g \circ \pi_B) \circ (u \times v) = f \circ \pi_A \circ (u \times v) + g \circ \pi_B \circ (u \times v) = f \circ u + g \circ v$.

$-f$ is the inverse element of f for $+$, indeed, by commutativity it is enough to prove that $f + (-f) = 0_{A \to B}$. It is straightforward that $f = \pi_B \circ \phi \circ i_A$, and by uniqueness in the universal propriety: $\psi = (\pi_A \circ \psi) \times (\pi_B \circ \psi)$, then $0 = \pi_B \circ i_A = \pi_B \circ \mathrm{id} \circ i_A = \pi_B \circ \phi \circ \psi \circ i_A = (f \coprod \mathrm{id}_B) \circ (\pi_A \circ \psi \circ i_A) \times (\pi_B \circ \psi \circ i_A) = (f \coprod \mathrm{id}_B) \circ (\pi_A \circ \psi \circ i_A \times -f) = f \circ \pi_A \circ \psi \circ i_A + \mathrm{id}_B \circ (-f)$.

However $\mathrm{id}_A = \pi_A \circ i_A = \pi_A \circ \mathrm{id} \circ i_A = \pi_A \circ \phi \circ \psi \circ i_A = (\mathrm{id}_A \coprod 0) \circ (\pi_A \circ \psi \times \pi_B \circ \psi) \circ i_A = (\mathrm{id}_A \coprod 0) \circ (\pi_A \circ \psi \circ i_A \times \pi_B \circ \psi \circ i_A) = \mathrm{id}_A \circ \pi_A \circ \psi \circ i_A + 0 \circ \pi_B \circ \psi \circ i_A = \pi_A \circ \psi \circ i_A$, thus $f \circ \pi_A \circ \psi \circ i_A + \mathrm{id}_B \circ (-f) = f + (-f)$ and then $f + (-f) = 0$.

Remark : In the category of R-Modules, the isomorphism ϕ would be $(x, y) \mapsto (x, f(x)+y)$ and then ψ would be $(x, y) \mapsto (x, y - f(x))$.

Proof of 3.3

Because $\operatorname{im}(f)$ is a monomorphism and $\operatorname{coim}(f)$ is an epimorphism, the uniqueness is straightforward. By construction $\operatorname{coker}(f) \circ f = 0$, then f factors through the kernel of $\operatorname{coker}(f)$ (*i.e.* $\operatorname{im}(f)$)by some morphism ψ : $f = \operatorname{im}(f) \circ \psi$. Then $\operatorname{im}(f) \circ \psi \circ \ker(f) = f \circ \ker(f) = 0$. But $\operatorname{im}(f)$ is a monomorphism, then $\psi \circ \ker(f) = 0$, thus ψ factors through the cokernel of $\ker(f)$ (*i.e.* $\operatorname{coim}(f)$)by some morphism ϕ: $\psi = \phi \circ \operatorname{coim}(f)$. Then $\operatorname{im}(f) \circ \phi \circ \operatorname{coim}(f) = f$. By 3.1 it is enough to show that ϕ is a monomorphism, and then that it's kernel is 0.

Let $j : X \to CI(f)$ such that $\phi \circ j = 0$. then $\operatorname{im}(f) \circ \phi \circ j = 0$, then $\operatorname{im}(f) \circ \phi$ factors through $\operatorname{coker}(j)$, there is a map θ such that $\operatorname{im}(f) \circ \phi = \theta \circ \operatorname{coker}(j)$. By composition $\operatorname{coker}(j) \circ \operatorname{coim}(f)$ is an epimorphism, then of the form $\operatorname{coker}(v)$. Then $f \circ v = \operatorname{im}(f) \circ \phi \circ \operatorname{coim}(f) \circ v = \theta \circ \operatorname{coker}(j) \circ \operatorname{coim}(f) \circ v = \theta \circ 0 = 0$. Then v factors through $\ker(f)$, *i.e.* is of the form $\ker(f) \circ w$. In particular $\operatorname{coim}(f) \circ v = 0 \circ w = 0$. Then $\operatorname{coim}(f)$ factors through $\operatorname{coker}(v) = \operatorname{coker}(j) \circ \operatorname{coim}(f)$: there is a map ρ such that $\operatorname{coim}(f) = \rho \circ \operatorname{coker}(j) \circ \operatorname{coim}(f)$, but $\operatorname{coim}(f)$ is an epimorphism, then $\rho \circ \operatorname{coker}(j) = \mathrm{id}$ thus $\operatorname{coker}(j)$ is a monomorphism, then because $\operatorname{coker}(j) \circ j = 0 = \operatorname{coker}(j) \circ 0$, j must be a zero-map, then by definition factors in a unique way through $0_{0 \to CI(f)}$, thus ϕ is a monomorphism.

Proof of 3.2

Let Let ϕ be the morphisme $(\mathrm{id}_A \coprod 0_{B \to A}) \times (0_{A \to B} \coprod \mathrm{id}_B) : A \coprod B \to A \times B$. By 3.1 it is enough to prove that ϕ is a monomorphism and an epimorphism, and by duality it is enough to prove that ϕ is a monomorphism, and thus to show that $\ker(\phi)$ is $0_{0 \to A \coprod B}$.

Let's show that $\operatorname{coker}(i_A) = 0_{A \to B} \coprod \mathrm{id}_B$, let $f : A \coprod B \to X$ be a morphism such that $f \circ i_A = 0$, then (by uniqueness in the universal propriety of $A \coprod B$) $f = (f \circ i_A) \coprod (f \circ i_B) = 0 \coprod (f \circ i_B) = (f \circ i_B \circ 0) \coprod (f \circ i_B \circ \mathrm{id}_B) = (f \circ i_B) \circ (0 \coprod \mathrm{id}_B)$. Then f factors through $0_{A \to B} \coprod \mathrm{id}_B$. If ϕ is such that $f = \phi \circ (0_{A \to B} \coprod \mathrm{id}_B)$, then $f = (\phi \circ 0_{A \to B} \coprod \phi \circ \mathrm{id}_B) = (0_{A \to B} \coprod \phi)$, thus by the uniqueness in the universal propriety of $A \coprod B$, ϕ must be $f \circ i_B$. By symmetry $\operatorname{coker}(i_B) = \mathrm{id}_A \coprod 0_{A \to B}$

Thus by 3.2, $i_A = \ker(0_{A \to B} \coprod \mathrm{id}_B)$ and $i_B = \ker(\mathrm{id}_A \coprod 0_{A \to B})$, then by the construction of intersection in abelian categories: F.1 $\ker(\phi) = A \cap B$.

To conclude let's show that $A \cap B$ is zero. Let $f : X \to A$ and $g : X \to B$ be two maps such that $i_A \circ f = i_B \circ g$ is a subobject of $A \coprod B$, then $f = (\mathrm{id}_A \coprod 0) \circ i_A \circ f = (\mathrm{id}_A \coprod 0) \circ i_B \circ g = 0 \circ g = 0$ and by symmetry $g = 0$. Thus f and g factors through $0 : 0 \to A \coprod B$. Thus by definition $0 : 0 \to A \coprod B$ is the intersection.

Proof of 3.12

1. First of all if one consider the transposed double chain complex, then it's straightforward that the donor and receptor of an object remain the same, and that the horizontal and vertical homology are switched, then it's enough to built ${}^{\square}A \to A^h$, ${}^{\square}A \to A_{\square}$ and $A^h \to A_{\square}$ and to prove that the triangle commutes.
   $e \circ \mathrm{id}_A \circ \ker(e \times f) = e \circ \ker(e \times f) = \pi_1 \circ (e \times f) \circ \ker(e \times f) = \pi_1 \circ 0 = 0$, then (by universal propriety), there is a map $\phi : K(e \times f) \to K(e)$ such that, $\ker(e \times f) = \ker(e) \circ \phi$.

   $\operatorname{coker}(I(d) \hookrightarrow K(e)) \circ \phi$ gives a map $K(e \times f) \to K(e) \to A^h$, and $\ker(e) \circ \phi \circ (I(p) \hookrightarrow K(e \times f)) = \ker(e \times f) \circ (I(p) \hookrightarrow K(e \times f)) = \operatorname{im}(p) = \operatorname{im}(d \circ b)$, then by 3.9, there is a monomorphism $\chi : I(p) \to I(d)$ such that $\ker(e) \circ \phi \circ (I(p) \hookrightarrow K(e \times f)) = \operatorname{im}(d) \circ \chi = \ker(e) \circ (I(d) \hookrightarrow K(e)) \circ \chi$, then because $\ker(e)$ is a monomorphism, $\phi \circ (I(p) \hookrightarrow K(e \times f)) = \circ (I(d) \hookrightarrow K(e)) \circ \chi$, then $\operatorname{coker}(I(d) \hookrightarrow K(e)) \circ (I(p) \hookrightarrow K(e \times f)) = \operatorname{coker}(I(d) \hookrightarrow K(e)) \circ (I(d) \hookrightarrow K(e)) \circ \chi = 0 \circ \chi = 0$. Then by the universal propriety there is a $\psi_1 : {}^{\square}A \to A^h$ such that $\operatorname{coker}(I(d) \hookrightarrow K(e)) \circ \phi = \psi_1 \circ \operatorname{coker}(I(p) \hookrightarrow K(e \times f))$.

$q \circ \mathrm{id}_A \circ \ker(e \times f) = g \circ e \circ \ker(e \times f) = g \circ 0 = 0$, (by a previous computation) then (by universal propriety), there is a unique map $\phi' : K(e \times f) \to K(q)$ such that, $\ker(e \times f) = \ker(q) \circ \phi'$. $\mathrm{coker}(I(cd) \hookrightarrow K(q)) \circ \phi'$ gives a map $K(e \times f) \to K(q) \to A_{\square}$, and $\ker(q) \circ \phi' \circ (I(p) \hookrightarrow K(e \times f)) = \ker(e \times f) \circ (I(p) \hookrightarrow K(e \times f)) = \mathrm{im}(p) = \mathrm{im}(d \circ b)$, then by 3.9, there is a monomorphism $\chi : I(p) \to I(d)$ such that $\ker(q) \circ \phi' \circ (I(p) \hookrightarrow K(e \times f)) = \mathrm{im}(d) \circ \chi = \ker(q) \circ (I(d) \hookrightarrow K(q)) \circ \chi$, then because $\ker(q)$ is a monomorphism, $\phi' \circ (I(p) \hookrightarrow K(e \times f)) = \circ(I(d) \hookrightarrow K(q)) \circ \chi$, then $\mathrm{coker}(I(d) \hookrightarrow K(q)) \circ (I(p) \hookrightarrow K(e \times f)) = \mathrm{coker}(I(d) \hookrightarrow K(q)) \circ (I(d) \hookrightarrow K(q)) \circ \chi = 0 \circ \chi = 0$. Then by the universal propriety there is a unique $\psi_2 : {}^{\square}A \to A_{\square}$ such that $\mathrm{coker}(I(d) \hookrightarrow K(q)) \circ \phi' = \psi_2 \circ \mathrm{coker}(I(p) \hookrightarrow K(e \times f))$.

By 3.1, there is a $\phi" : K(e) \to K(q)$ such that $\ker(e) = \ker(q) \circ \phi"$. Then $\mathrm{coker}(I(c \coprod d) \hookrightarrow K(q)) \circ \phi"$ is a map from A to $A_{\square}$. $\ker(q) \circ \phi" \circ (I(d) \hookrightarrow K(e)) = \ker(e) \circ (I(d) \hookrightarrow K(e)) = \mathrm{im}(d) = \mathrm{im}((c \coprod d) \circ i_2)$, then by 3.9 there is map $\chi : I(d) \to I(c \coprod d)$ such that $\ker(q) \circ \phi" \circ (I(d) \hookrightarrow K(e)) = \mathrm{im}(c \coprod d) \circ \chi = \ker(q) \circ (I(c \coprod d) \hookrightarrow K(q)) \circ \chi$. Then because $\ker(q)$ is a monomorphism $\phi" \circ (I(d) \hookrightarrow K(e)) = (I(c \coprod d) \hookrightarrow K(q)) \circ \chi$, then $\mathrm{coker}(I(c \coprod d) \hookrightarrow K(q)) \circ \phi" \circ (I(d) \hookrightarrow K(e)) = \mathrm{coker}(I(c \coprod d) \hookrightarrow K(q)) \circ (I(c \coprod d) \hookrightarrow K(q)) \circ \chi = 0$. Then by the universal propriety there is a map $\psi_3 : A^h \to A_{\square}$ such that $\mathrm{coker}(I(c \coprod d) \hookrightarrow K(q)) \circ \phi" = \psi_3 \circ \mathrm{coker}(I(d) \hookrightarrow K(e))$.

Then it remains to justify that $\psi_2 = \psi_3 \circ \psi_1$. By the uniqueness in the universal propriety it's enough to check that $\mathrm{coker}(I(d) \hookrightarrow K(q)) \circ \phi' = (\psi_3 \circ \psi_1) \circ \mathrm{coker}(I(p) \hookrightarrow K(e \times f))$. $(\psi_3 \circ \psi_1) \circ \mathrm{coker}(I(p) \hookrightarrow K(e \times f)) = \psi_3 \circ \mathrm{coker}(I(d) \hookrightarrow K(e)) \circ \phi = \mathrm{coker}(I(c \coprod d) \hookrightarrow K(q)) \circ \phi" \circ \phi$. Then, to conclude it's enough to show that $\phi' = \phi" \circ \phi$. And it's the case by uniqueness in the universal propriety (of ϕ') because $\ker(q) \circ \phi" \circ \phi = \ker(e) \circ \phi = \ker(e \times f)$.

2. Let w be a map $A \to B$ in a double chain complex, by considering the transposed double chain complex it's possible to assume that w is horizontal. Let's name the edges in the following way:

$$\begin{tikzcd}
\bullet \arrow[dr, "p"] & \bullet \arrow[d, "a"] & & \\
\bullet \arrow[r, "b"] & A \arrow[r, "f"] & B \arrow[r, "c"] \arrow[d, "d"] \arrow[dr, "q"] & \bullet \\
& & \bullet & \bullet
\end{tikzcd}$$

Then $A_{\square}$ is the codomain of $\ker(d \circ f)/_{\mathrm{im}(a \coprod b)}$ and ${}^{\square}B$ is the codomain of $\ker(c \times d)/_{\mathrm{im}(f \circ a)}$. $(c \times d) \circ f \circ \ker(d \circ f) = (c \circ f \circ \ker(d \circ f)) \times (d \circ f \circ \ker(d \circ f)) = (0 \circ \ker(d \circ f)) \times 0 = 0 \times 0 = 0$, then (by universal propriety), there is a map $\psi : K(d \circ f) \to K(c \times d)$ such that $f \circ \ker(d \circ f) = \ker(c \times d) \circ \psi$

Then $\mathrm{coker}(I(f \circ a) \hookrightarrow K(c \times d)) \circ \psi$ is a map $K(d \circ f) \to^{\square} B$. The intramural map is going to be the quotient map of this one.

Then $\ker(c \times d) \circ \psi \circ (I(a \coprod b) \hookrightarrow K(d \circ f)) = f \circ \ker(d \circ f) \circ (I(a \coprod b) \hookrightarrow K(d \circ f)) = f \circ \mathrm{im}(a \coprod b)$. By 3.10 there is an epimorphism θ and a morphism j' such that $\mathrm{im}(a \coprod b) \circ \theta = (a \coprod b) \circ j'$, then $f \circ \mathrm{im}(a \coprod b) \circ \theta = f \circ (a \coprod b) \circ j' = ((f \circ a) \coprod (f \circ b)) \circ j' = ((f \circ a) \coprod 0) \circ j' = (\mathrm{im}(f \circ a) \circ \mathrm{coim}(f \circ a) \coprod (\mathrm{im}(f \circ a) \circ 0)) \circ j' = ((\mathrm{im}(f \circ a) \circ \mathrm{coim}(f \circ a)) \coprod 0) \circ j' = \ker(c \times d) \circ (I(f \circ a) \hookrightarrow K(c \times d)) \circ j''$. But $\ker(c \times d)$ is a monomorphism, then $\psi \circ (I(a \coprod b) \hookrightarrow K(d \circ f)) \circ \theta = (I(f \circ a) \hookrightarrow K(c \times d)) \circ j''$, then $\mathrm{coker}(I(f \circ a) \hookrightarrow K(c \times d)) \circ \psi \circ (I(a \coprod b) \hookrightarrow K(d \circ f)) \circ \theta = \mathrm{coker}(I(f \circ a) \hookrightarrow K(c \times d)) \circ (I(f \circ a) \hookrightarrow K(c \times d)) \circ j'' = 0 \circ j'' = 0$, but theta is an epimorphism, then $\mathrm{coker}(I(f \circ a) \hookrightarrow K(c \times d)) \circ \psi \circ (I(a \coprod b) \hookrightarrow K(d \circ f)) = 0$. Then by universal propriety, there is a map $\tilde{f}$ such that $\mathrm{coker}(I(f \circ a) \hookrightarrow K(c \times d)) \circ \psi = \tilde{f} \circ \mathrm{coker}(I(a \coprod b) \hookrightarrow K(d \circ f))$.

Proof of 4.1

Definition F.1

Let D be a graph, the graph of connected components of D: $SC(D)$ is defined by the following:

1. *$V_C = \{$strongly connected components of $D\}$*
2. *E_C the quotient of E by the relation $(x \to y) \sim (a \to b) \Leftrightarrow SC(x) = S(a)$ and $SC(y) = SC(b)$*
3. *$o : \overline{x \to y} \mapsto SC(x)$ and $t : \overline{x \to y} \mapsto SC(y)$*

Remark : In $SC(D)$ there is no cycle, except maybe with an edge from a vertex to itself, indeed if there is a path from $SC(a)$ to $SC(b)$ and conversely, then there is a path in D from a to b and from b to a thus a and b are in the same strongly connected component.

If a and b are in the same strongly connected component SC, then let $\mathcal{C}$ be the category:

$$\begin{tikzcd} 1 \arrow[d, bend left, "0_{1\to 0}"] \\ 0 \arrow[u, bend left, "0_{0\to 1}"] \end{tikzcd}$$

All the maps id, $0_{0\to 0}$ and $0_{1\to 1}$ are implicit.The composition are all zero (according to 1.3) except $\mathrm{id}_1 \circ \mathrm{id}_1 = \mathrm{id}_1$. In $\mathcal{C}$, 0 is a zero-object (there is a unique arrow from every object ($\mathrm{id}_0 = 0_{0\to 0}$) and to every object).

Let F be defined by the following:

1. $\forall x \in V(D) \backslash SC\ F(x) = 0$ and $\forall x \in SC\ F(x) = 1$
2. F send the id_x to the corresponding $\mathrm{id}_{F(x)}$
3. If x and y are in SC then $\forall f \in \mathrm{Hom}(x, y)\ F(f) = \mathrm{id}_1$ and otherwise $F(x \to y) = 0_{F(x)\to F(y)}$

In particular if x or y is not in SC then $F(\bullet_{x,y}) = 0$ thus if a path not included in SC has more than one non-identity arrow it is sent to the zero-map.

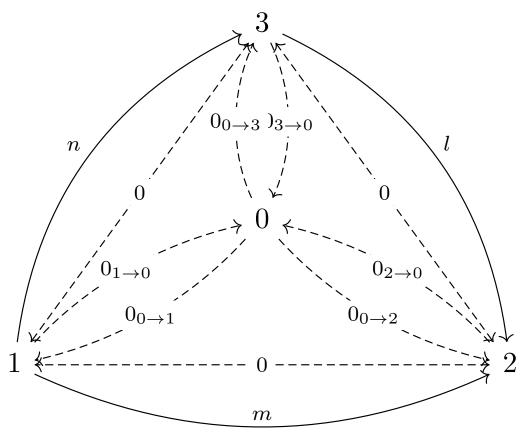
To conclude that F extend in a unique way into a functor, according to 2.1, it is enough to prove that if $x_1 \to \ldots \to x_n$ and $y_1 \to \ldots \to y_m$ are two path between u and v two vertex of $\mathcal{C}(D)$ then $F(x_{n-1} \to x_n) \circ \ldots \circ F(x_1 \to x_2) = F(y_{m-1} \to y_m) \circ \ldots \circ F(y_1 \to y_2)$. There are several cases:

1. If u and v are in SC then by composition of path: we get $\left\{\begin{array}{l} x_i \to \ldots \to x_n = v \to \ldots \to a \\ a \to \ldots \to b \\ b \to \ldots \to u = x_1 \to \ldots \to x_i \end{array}\right.$ and $\left\{\begin{array}{l} y_i \to \ldots \to y_m = v \to \ldots \to a \\ a \to \ldots \to b \\ b \to \ldots \to u = y_1 \to \ldots \to y_i \end{array}\right.$. Then all x_i and y_i are in SC so the relation is $\mathrm{id}_1^n = \mathrm{id}_1^m$ wich is straightforward by definition of $\circ$.
2. If $u = v$ and $u \notin SC$. Then the two path are cycles at u hence their vertex are in $SC(u)$, but because $u \notin SC$, SC and $SC(u)$ are disjoint so by definition all the arrows are sent to 0 then the relation is $0_{F(u)\to F(u)} = 0_{F(u)\to F(v)}$ which is true.
3. If $u \neq v$ is not in SC then there is at least one non-identity morphism on each path. Because $F(id_u) = id_{F(u)}$ and the composition with id vanishes we can assume that $u = x_1 \to x_2$ and $u = y_1 \to y_2$ are not the identity map then they must be zero by definition of F, then by 1.3 the relation is $0_{F(u)\to F(v)} = 0_{F(u)\to F(v)}$ which is true.

4. If v is not in SC then there is at least one non-identity morphism on each path. Because $F(id_v) = id_{F(v)}$ and the composition with id vanishes we can assume that $x_{n-1} \to x_n = v$ and $y_{m-1} \to y$ are not the identity then they must be zero by definition of F, map then by 1.3 the relation is $0_{F(u)\to F(v)} = 0_{F(u)\to F(v)}$ which is true.

Hence F is a functor. Let $e : x \to y \in Z$, if $x \in SC$ then by definition a is x or is in the ancestors of x. If $y \in SC$ then by definition b is y or is in the descendent of y. But the condition is satisfied then x or y is not in SC, then by construction of F, $F(e)$ is a zero map. By construction $F(f : a \to b)$ is not the zero map, thus (D, Z, f) is not a positive instance.

If a and b are not in the same strongly connected components $SC(a)$ and $SC(b)$, (in particular they are disjoint). Let S be the set of vertex v that are not in $SC(a)$, not in $SC(b)$ such that there is a path from a to v and a path from v to b. Let $\mathcal{C}$ be the category:



All the maps id, $0_{0\to0}$, $0_{1\to1}$, $0_{2\to2}$ and $0_{3\to3}$ are implicit. The composition are all zero (according to 1.3) except $l \circ n = m$. In $\mathcal{C}$, 0 is a zero-object (there is a unique arrow to 0 from every object and to every object from 0).

Let F be defined by the following:

1. $\forall x \in S\ F(x) = 3$, $\forall x \in SC(a)\ F(x) = 1$ and $\forall x \in SC(b)\ F(x) = 2$
2. Otherwise $F(x) = 0$
3. F send id_x to $\mathrm{id}_{F(x)}$
4. If x and y are in $SC(a)$ then: $\forall f \in \mathrm{Hom}(x, y)\ F(f) = \mathrm{id}_1$
5. If x and y are in $SC(b)$ then: $\forall f \in \mathrm{Hom}(x, y)\ F(f) = \mathrm{id}_2$
6. If x and y are in S then $\forall f \in \mathrm{Hom}(x, y)\ F(f) = \mathrm{id}_3$
7. If x is in S and y is in $SC(b)$ then $\forall f \in \mathrm{Hom}(x, y)\ F(f) = l$
8. If x is in $SC(a)$ and y is in $SC(b)$ then $\forall f \in \mathrm{Hom}(x, y)\ F(f) = m$
9. If x is in $SC(a)$ and y is in S then $\forall f \in \mathrm{Hom}(x, y)\ F(f) = n$
10. Otherwise $F(x \to y) = 0_{F(x)\to F(y)}$

The functor label things like this:

$$\begin{tikzcd} & S \arrow[d] \arrow[ddr] & \\ & 0 & \\ SC(a) \arrow[uur] \arrow[ur] \arrow[rr] & & SC(b) \arrow[ul] \end{tikzcd}$$

To conclude that F extend in a unique way into a functor, according to 2.1, it is enough to prove that if $x_1 \to \ldots \to x_n$ and $y_1 \to \ldots \to y_m$ are two path between u and v two vertex of $\mathcal{C}(D)$ then $F(x_{n-1} \to x_n) \circ \ldots \circ F(x_1 \to x_2) = F(y_{m-1} \to y_m) \circ \ldots \circ F(y_1 \to y_2)$. There are several cases:

1. If v is not in $S \cup S(a) \cup S(b)$ and $v \neq u$, then there is at least one non-identity morphism in each

path, thus a zero map in the image thus by 1.3 the relation is $0_{F(u)\to F(v)} = 0_{F(u)\to F(v)}$ which is true.

2. If u is not in $S \cup S(a) \cup S(b)$ and $u \neq v$ then there is at least one non-identity morphism in each path, thus a zero map in the image thus by 1.3 the relation is $0_{F(u)\to F(v)} = 0_{F(u)\to F(v)}$ which is true.

3. If $u = v$ is not in $S \cup S(a) \cup S(b)$ then $F(u) = 0$ and the two maps are maps from $F(u)$ to $F(u)$, but there is only one: $\mathrm{id}_0 = 0_{0\to 0}$ (as 0 is a zero-object in $\mathcal{C}$) so the two maps must be equal.

4. If u is in $SC(a)$ and v is $SC(a)$, then all the vertex of the two path are in $SC(a)$ by composition of path, and by definition the relation is $\mathrm{id}_1^n = \mathrm{id}_1^m$ which is true.

5. If u is in $SC(a)$ and v is in $SC(b)$, let x_i and y_j be the first vertex of each path not in $SC(a)$. All the arrow before on the path are then send to id_1 we can thus assume that $i = j = 2$.

   If x_2 is in $SC(b)$, then $F(x_1 \to x_2) = m$ and by composing path (from v to b and from b to x_2) all the other x_i (except x_1) are in $SC(b)$ then the other arrow are sent to id_2, thus the map $F(x_{n-1} \to x_n) \circ \ldots \circ F(x_1 \to x_2) = m$, the same is true if $y_2 \in SC(b)$.

   If x_2 is in S then $F(x_1 \to x_2) = n$. Let x_i be the first x_i ($i \geq 2$) not in S, as there is a path from x_i to $v \in SC(b)$ and then a path to b x_i (and a path from a: $a \to \ldots \to u = x_1 \to x_2$) x_i must be in $SC(b) \cup SC(a)$. It can't be in $SC(a)$ because then x_2 would be in $SC(a)$. By composing path, all the x are in S so all the arrow between x_2 and x_{i-1} are sent to id_3. By construction of F, $F(x_{i-1} \to x_i) = l$. By composition of path all the vertex after x_i are in $SC(b)$ then the arrows are sent to id_2. Thus the map $F(x_{n-1} \to x_n) \circ \ldots \circ F(x_1 \to x_2) = \mathrm{id}_2^c \circ l \circ \mathrm{id}_3^k \circ n = m$, the same is true if $y_2 \in S$.

   In any case the maps are equal to m, thus the relation holds.

6. If u is in $SC(a)$ and v is in S, then by composition of path there is a path from a to every vertex of the path, thus they are all in $SC(a)$ or in S. Let x_i, be the first x not in $SC(a)$. All the arrow before on the path are then send to id_1 we can thus assume that $i = 2$. By construction $F(x_1 \to x_2) = n$. If one of the next vertex was in $SC(a)$ then x_i would be in too, by composition of path, then all the other arrow are sent to id_3. Thus the map $F(x_{n-1} \to x_n) \circ \ldots \circ F(x_1 \to x_2) = n$. The same is true for y, then the relation holds.

7. If u is in $SC(b)$ and v is $SC(a)$, then there would be a path from b to a and (because there is a path $f : a \to b$) then a and b would be in the same strongly connected component which is false.

8. If u is in $SC(b)$ and v is in $SC(b)$ then by composition of path all the vertex are in $SC(b)$ then all the arrow are sent to id_2, thus the relation is $\mathrm{id}_2^n = \mathrm{id}_2^m$ which is true.

9. If u is in $SC(b)$ and v is in S, then there would be a path from b to u to v and thus v would be in $SC(b)$ (there is a path from the elements of S to b) which is false.

10. If u is in S and v is $SC(a)$, then there would be a path from u to v to a and thus u would be in $SC(a)$ (there is a path from a to the elements S) which is false.

11. If u is in S and v is in $SC(b)$, then by composition of path there is a path from every vertex of the path to b, thus they are all in $SC(b)$ or in S. Let x_i, be the first x not in S. All the arrow before on the path are then send to id_3 we can thus assume that $i = 2$. By construction $F(x_1 \to x_2) = l$. By composition of path all the other vertex are in $SC(b)$ then all the other arrow are sent to id_2. Thus the map $F(x_{n-1} \to x_n) \circ \ldots \circ F(x_1 \to x_2) = l$. The same is true for y, then the relation holds.

12. If u is in S and v is in S, then by composition of path all the vertex are in S then all the arrow are sent to id_3, thus the relation is $\mathrm{id}_3^n = \mathrm{id}_3^m$ which is true.

Hence F is a functor. Let be $e : x \to y \in Z$.

If x is in $SC(a)$ then by definition a is an ancestor of x (or is x) , if x is in S then by definition a is an ancestor of x, if x is in $SC(b)$ then by composition of $f : a \to b$ and a path from b to x, a is in the ancestors of x. Therefore, if $x \in SC(a) \cup SC(b) \cup S$ then $a \in Ans(x) \cup \{x\}$.

If y is in $SC(b)$ then by definition b is a descendent of y (or is y) , if y is in S then by definition b is a descendent of y, if y is in $SC(a)$ then by composition of a path from y to a and $f : a \to b$, b is in the descendent of y. Therefore, if $y \in SC(a) \cup SC(b) \cup S$ then $b \in Dec(y) \cup \{y\}$.

But the condition is satisfied then either x or y is not in $SC(a) \cup SC(b) \cup S$, thus by definition of F, $F(e)$ is a zero-map, so F satisfy all the condition of Z. By construction $F(f : a \to b)$ is not the zero map, thus (D, Z, f) is not a positive instance.

Proof of 4.2

In particular $F(f)$ is not a zero-map, and F satisfy all the condition of $Z' \supseteq Z$, thus to conclude that F is also a counter example for (D, Z, OZ, f) it is enough to prove the $\forall d \in OZ\ F(d) = 0$. Let d be a vertex in OZ, we perform the same case analysis about the strongly connected component of $a = o(f)$ and $b = t(f)$ than in the proof of 4.1:

1. If $SC(a) = SC(b)$.
   (a) If $d \notin SC(a)$ then by construction $F(d) = 0$.
   (b) If $d = a$ then by definition of Z' f is in Z' thus $F(f)$ can't be a non-zero map.
   (c) Then there is a nontrivial path: $a \to \ldots \to x \to d$ and a path from d to b (because $d \in SC(a) = SC(b)$). Then by functoriality $F(f)$ factors through $F(x \to d)$. However $x \to d$ is by definition in Z' then $F(x \to d)$ is a zero map, then by lemma 1.3 $F(f)$ is a zero-map which is a contradiction.
2. If $SC(a) \neq SC(b)$.
   (a) If $d \notin SC(a) \cup SC(b) \cup S$ then by construction $F(d) = 0$
   (b) If $d \in S$ then by construction there is a non trivial path from a to d to b, by the same argument as before $F(f)$ must then be a zero-map, which is a contradiction.
   (c) If $d \in SC(a)$ then by the same argument as before $d \neq a$. Then there is a non trivial path $a \to \ldots \to x \to d$ and a path from d to $a \to b$. Then by functoriality $F(f)$ factors through $F(x \to d)$. However $x \to d$ is by definition in Z' then $F(x \to d)$ is a zero map, then by lemma 1.3 $F(f)$ is a zero-map which is a contradiction.
   (d) If $d \in SC(b)$ then by the same argument as before $d \neq b$. Then there is a non trivial path $a \to b \to \ldots \to x \to d$ and a path from d to b. Then by functoriality $F(f)$ factors through $F(x \to d)$. However $x \to d$ is by definition in Z' then $F(x \to d)$ is a zero map, then by lemma 1.3 $F(f)$ is a zero-map which is a contradiction.

Then in any cases $F(d) = 0$. Then (D, Z, OZ, f) is not positive.

Proof of 4.2

Let's assume that (D, Z, E, OZ, M, Ep, A) is a positive instance. Let F_{AC} be a diagram over $AC(D)$ that satisfy the conditions $(AC(D), Z_{AC}, E_{AC}, OZ_{AC}, M_{AC}, Ep_{AC})$. Let's define a pre-diagram F over D by the following:

1. If A is a vertex of D, then $F(A) = F_{AC}(A_0)$
2. If $f : A \to B$ is an edge of D, then $F(f) = F_{AC}(\phi_{B,0})^{-1} \circ F_{AC}(f_0)$.

It's well define because the ϕ are in the monomorphism and epimorphism condition then their images are isomorphism.

Let $f : A \to B$ and $g : B \to C$ be two composable arrow in D, By commutativity of the diagram F_{AC}, $\mathrm{F}_{AC}(\phi_{C,0})^{-1} \circ F_{AC}(g_0) = F_{AC}(\phi_{C,0})^{-1} \circ F_{AC}(\phi_{C,1})^{-1} \circ F_{AC}(g_1) \circ F_{AC}(\phi_{B,0})$ then: $F(g) \circ F(f) = F_{AC}(\phi_{C,0})^{-1} \circ F_{AC}(g_0) \circ F_{AC}(\phi_{B,0})^{-1} \circ F_{AC}(f_0) = F_{AC}(\phi_{C,1} \circ \phi_{C,0})^{-1} \circ F_{AC}(g_1) \circ F_{AC}(f_0) \circ F_{AC}(\phi_{B,0}) = F_{AC}(\phi_{C,1} \circ \phi_{C,0})^{-1} \circ F_{AC}(g_1 \circ f_0) \circ F_{AC}(\phi_{B,0})$. In particular, it depends only on A_0 and B_1. Then by a straightforward induction the composition along a simple path in D only depends of it's origin and it's end. then by 2.1 F extends in a diagram.

F is an instance of the problem. The maps ϕ are isomorphism then $F_{AC}(\phi_{B,0})^{-1} \circ F_{AC}(f_0)$ is a zero-map (resp mono, epi) if and only if $F_{AC}(f_0)$ is one. Then F satisfy the condition Z, M and Ep. Because F_{AC} satisfy the conditions of OZ_{AC}, by construction F satisfy the conditions of OZ.

Let $(f, g) \in E$, then $(f_0, g_1) \in E_{AC}$ then $\ker(F_{AC}(g_1)) = \mathrm{im}(F_{AC}(f_0))$ but then composition by isomorphism does not changes the kernel and the images then $\ker(F(g)) = \ker(F_{AC}(\phi_{C,0})^{-1} \circ F_{AC}(g_0)) = \ker(F_{AC}(g_0)) = \ker(F_{AC}(\phi_{C,0})^{-1} \circ F_{AC}(g_1) \circ F_{AC}(\phi_{B,0})) = \ker(F_{AC}(g_1)) = \mathrm{im}(F_{AC}(f_0)) = \mathrm{im}(F_{AC}(\phi_{B,0})^{-1} \circ F_{AC}(f_0)) = \mathrm{im}(F(f))$.

Then F satisfy all the conditions of (D, Z, E, OZ, M, Ep), then by assumption $F(A)$ is a zero object, thus $F_{AC}(A_0)$ is a zero-object.

Conversely let $F : \mathcal{C}(D) \Rightarrow \mathcal{C}$ be a diagram that satisfy the conditions (D, Z, E, OZ, M, Ep). Then by construction $AC(F)$ satisfy the conditions $(AC(D), Z_{AC}, E_{AC}, OZ_{AC}, M_{AC}, Ep_{AC})$, thus $AC(F)(A_0)$ is a zero-object. Then by definition $F(A) = AC(F)(A_0)$ is a zero object. Thus (D, Z, E, OZ, M, Ep, A) is a positive instance.

Let's assume that (D, Z, E, OZ, M, Ep, f) is a positive instance. Let F_{AC} be a diagram over $AC(D)$ that satisfy the conditions $(AC(D), Z_{AC}, E_{AC}, OZ_{AC}, M_{AC}, Ep_{AC})$. Let F be the diagram constructed in the previous point (which satisfy the condition of (D, Z, E, OZ, M, Ep, f)).

By assumption $F(f)$ is a zero-map (resp mono, epi). Then because composition by an isomorphism does not changes this point $F_{AC}(f_0)$ is a zero-map (resp mono, epi).

Conversely, let $F : \mathcal{C}(D) \Rightarrow \mathcal{C}$ be a diagram that satisfy the conditions (D, Z, E, OZ, M, Ep). Then by construction $AC(F)$ satisfy the conditions $(AC(D), Z_{AC}, E_{AC}, OZ_{AC}, M_{AC}, Ep_{AC})$, then $F(f) = AC(F)(f_0)$ is a monomorphism.

Let's assume that $(D, Z, E, OZ, M, Ep, (f, g))$ is a positive instance. Let F_{AC} be a diagram over $AC(D)$ that satisfy the conditions $(AC(D), Z_{AC}, E_{AC}, OZ_{AC}, M_{AC}, Ep_{AC})$. Let F be the diagram constructed in the previous point.

In the proof that F satisfy E it's proven that $\ker(F_{AC}(g_1)) = \ker(F(g))$ and $\mathrm{im}(F_{AC}(f_0)) = \mathrm{im}(F(f))$, then because $(D, Z, E, OZ, M, Ep, (f, g))$ is positive, the composition $F(f), F(g)$ is exact then the composition $F_{AC}(g_1), F_{AC}(f_0)$ is exact.

Conversely let $F : \mathcal{C}(D) \Rightarrow \mathcal{C}$ be a diagram that satisfy the conditions (D, Z, E, OZ, M, Ep). Then by construction $AC(F)$ satisfy the conditions $(AC(D), Z_{AC}, E_{AC}, OZ_{AC}, M_{AC}, Ep_{AC})$. Then the composition $(AC(F)(f_0), AC(F)(g_1))$ is exact but by definition of AC this composition if $(F(f), F(g))$.