Logic and Metalogic

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Main article

Logic

Logic (from the Greek *λογική* logikē)[1] refers to both the study of modes of reasoning (which are valid and which are fallacious)[2] and the use of valid reasoning. In the latter sense, logic is used in most intellectual activities, including philosophy and science, but in the first sense is studied primarily in the disciplines of philosophy, mathematics, semantics, and computer science. It examines general forms that arguments may take. In mathematics, it is the study of valid inferences within some formal language.[3] Logic is also studied in argumentation theory.[4]

Logic was studied in several ancient civilizations, including India,[5] China,[6] and Greece. In the West, logic was established as a formal discipline by Aristotle, who gave it a fundamental place in philosophy. The study of logic was part of the classical trivium, which also included grammar and rhetoric.

Logic is often divided into three parts, inductive reasoning, abductive reasoning, and deductive reasoning.

The study of logic

> Upon this first, and in one sense this sole, rule of reason, that in order to learn you must desire to learn, and in so desiring not be satisfied with what you already incline to think, there follows one corollary which itself deserves to be inscribed upon every wall of the city of philosophy: Do not block the way of inquiry.
>
> —Charles Sanders Peirce, "First Rule of Logic"

The concept of logical form is central to logic, it being held that the validity of an argument is determined by its logical form, not by its content. Traditional Aristotelian syllogistic logic and modern symbolic logic are examples of formal logics.

- **Informal logic** is the study of natural language arguments. The study of fallacies is an especially important branch of informal logic. The dialogues of Plato[7] are good examples of informal logic.
- **Formal logic** is the study of inference with purely formal content. An inference possesses a *purely formal content* if it can be expressed as a particular application of a wholly abstract rule, that is, a rule that is not about any particular thing or property. The works of Aristotle contain the earliest known formal study of logic. Modern formal logic follows and expands on Aristotle.[8] In many definitions of logic, logical inference and inference with purely formal content are the same. This does not render the notion of informal logic vacuous, because no formal logic captures all of the nuance of natural language.
- **Symbolic logic** is the study of symbolic abstractions that capture the formal features of logical inference.[9][10] Symbolic logic is often divided into two branches: propositional logic and predicate logic.
- **Mathematical logic** is an extension of symbolic logic into other areas, in particular to the study of model theory, proof theory, set theory, and recursion theory.

Logical form

Logic is generally accepted to be **formal**, in that it aims to analyze and represent the *form* (or logical form) of any valid argument type. The form of an argument is displayed by representing its sentences in the formal grammar and symbolism of a logical language to make its content usable in formal inference. If one considers the notion of form to be too philosophically loaded, one could say that formalizing is nothing else than translating English sentences into the language of logic.

This is known as showing the *logical form* of the argument. It is necessary because indicative sentences of ordinary language show a considerable variety of form and complexity that makes their use in inference impractical. It requires, first, ignoring those grammatical features which are irrelevant to logic (such as gender and declension if the argument is in Latin), replacing conjunctions which are not relevant to logic (such as 'but') with logical conjunctions like 'and' and replacing ambiguous or alternative logical expressions ('any', 'every', etc.) with expressions of a standard type (such as 'all', or the universal quantifier ∀).

Second, certain parts of the sentence must be replaced with schematic letters. Thus, for example, the expression 'all As are Bs' shows the logical form which is common to the sentences 'all men are mortals', 'all cats are carnivores', 'all Greeks are philosophers' and so on.

That the concept of form is fundamental to logic was already recognized in ancient times. Aristotle uses variable letters to represent valid inferences in *Prior Analytics*, leading Jan Łukasiewicz to say that the introduction of variables was 'one of Aristotle's greatest inventions'.[11] According to the followers of Aristotle (such as Ammonius), only the logical principles stated in schematic terms belong to logic, and not those given in concrete terms. The concrete terms 'man', 'mortal', etc., are analogous to the substitution values of the schematic placeholders 'A', 'B', 'C', which were called the 'matter' (Greek 'hyle') of the inference.

The fundamental difference between modern formal logic and traditional or Aristotelian logic lies in their differing analysis of the logical form of the sentences they treat.

- In the traditional view, the form of the sentence consists of (1) a subject (e.g. 'man') plus a sign of quantity ('all' or 'some' or 'no'); (2) the copula which is of the form 'is' or 'is not'; (3) a predicate (e.g. 'mortal'). Thus: all men are mortal. The logical constants such as 'all', 'no' and so on, plus sentential connectives such as 'and' and 'or' were called 'syncategorematic' terms (from the Greek 'kategorei' – to predicate, and 'syn' – together with). This is a fixed scheme, where each judgement has an identified quantity and copula, determining the logical form of the sentence.
- According to the modern view, the fundamental form of a simple sentence is given by a recursive schema, involving logical connectives, such as a quantifier with its bound variable, which are joined to by juxtaposition to other sentences, which in turn may have logical structure.
- The modern view is more complex, since a single judgement of Aristotle's system will involve two or more logical connectives. For example, the sentence "All men are mortal" involves in term logic two non-logical terms "is a man" (here *M*) and "is mortal" (here *D*): the sentence is given by the judgement A(M,D). In predicate logic the sentence involves the same two non-logical concepts, here analyzed as $m(x)$ and $d(x)$, and the sentence is given by $\forall x.(m(x) \rightarrow d(x))$, involving the logical connectives for universal quantification and implication.
- But equally, the modern view is more powerful: medieval logicians recognized the problem of multiple generality, where Aristotelean logic is unable to satisfactorily render such sentences as "Some guys have all the luck", because both quantities "all" and "some" may be relevant in an inference, but the fixed scheme that Aristotle used allows only one to govern the inference. Just as linguists recognize recursive structure in natural languages, it appears that logic needs recursive structure.

Deductive and inductive reasoning, and retroductive inference

Deductive reasoning concerns what follows necessarily from given premises (if a, then b). However, inductive reasoning—the process of deriving a reliable generalization from observations—has sometimes been included in the study of logic. Similarly, it is important to distinguish deductive validity and inductive validity (called "cogency"). An inference is deductively valid if and only if there is no possible situation in which all the premises are true but the conclusion false. An inductive argument can be neither valid nor invalid; its premises give only some degree of probability, but not certainty, to its conclusion.

The notion of deductive validity can be rigorously stated for systems of formal logic in terms of the well-understood notions of semantics. Inductive validity on the other hand requires us to define a reliable generalization of some set of observations. The task of providing this definition may be approached in various ways, some less formal than others; some of these definitions may use mathematical models of probability. For the most part this discussion of logic deals only with deductive logic.

Retroductive inference is a mode of reasoning that Peirce proposed as operating over and above induction and deduction to "open up new ground" in processes of theorizing (1911, p. 2). He defines retroduction as a logical inference that allows us to "render comprehensible" some observations/events which we perceive, by relating these back to a posited state of affairs that would help to shed light on the observations (Peirce, 1911, p. 2). He remarks that the "characteristic formula" of reasoning that he calls retroduction is that it involves reasoning from a consequent (any observed/experienced phenomena that confront us) to an antecedent (that is, a posited state of things that helps us to render comprehensible the observed phenomena). Or, as he otherwise puts it, it can be considered as "regressing from a consequent to a hypothetical antecedent" (1911, p. 4). See for instance, the discussion at: http://www.helsinki.fi/science/commens/dictionary.html

Some authors have suggested that this mode of inference can be used within social theorizing to postulate social structures/mechanisms that explain the way that social outcomes arise in social life and that in turn also indicate that these structures/mechanisms are alterable with sufficient social will (and visioning of alternatives). In other words, this logic is specifically liberative in that it can be used to point to transformative potential in our way of organizing our social existence by our re-examining/exploring the deep structures that generate outcomes (and life chances for people). In her book on New Racism (2010) Norma Romm offers an account of various interpretations of what can be said to be involved in retroduction as a form of inference and how this can also be seen to be linked to a style of theorizing (and caring) where processes of knowing (which she sees as dialogically rooted) are linked to social justice projects (http://www.springer.com/978-90-481-8727-0)

Consistency, validity, soundness, and completeness

Among the important properties that logical systems can have:

- **Consistency**, which means that no theorem of the system contradicts another.[12]
- **Validity**, which means that the system's rules of proof will never allow a false inference from true premises. A logical system has the property of soundness when the logical system has the property of validity and only uses premises that prove true (or, in the case of axioms, are true by definition).[12]
- **Completeness**, of a logical system, which means that if a formula is true, it can be proven (if it is true, it is a theorem of the system).
- **Soundness**, the term soundness has multiple separate meanings, which creates a bit of confusion throughout the literature. Most commonly, soundness refers to logical systems, which means that if some formula can be proven in a system, then it is true in the relevant model/structure (if A is a theorem, it is true). This is the converse of completeness. A distinct, peripheral use of soundness refers to arguments, which means that the premises of a valid argument are true in the actual world.

Some logical systems do not have all four properties. As an example, Kurt Gödel's incompleteness theorems show that sufficiently complex formal systems of arithmetic cannot be consistent and complete;[10] however, first-order predicate logics not extended by specific axioms to be arithmetic formal systems with equality can be complete and consistent.[13]

Rival conceptions of logic

Logic arose (see below) from a concern with correctness of argumentation. Modern logicians usually wish to ensure that logic studies just those arguments that arise from appropriately general forms of inference. For example, Thomas Hofweber writes in the Stanford Encyclopedia of Philosophy that logic "does not, however, cover good reasoning as a whole. That is the job of the theory of rationality. Rather it deals with inferences whose validity can be traced back to the formal features of the representations that are involved in that inference, be they linguistic, mental, or other representations".[3]

By contrast, Immanuel Kant argued that logic should be conceived as the science of judgment, an idea taken up in Gottlob Frege's logical and philosophical work, where thought (German: *Gedanke*) is substituted for judgment (German: *Urteil*). On this conception, the valid inferences of logic follow from the structural features of judgments or thoughts.

History

The earliest sustained work on the subject of logic is that of Aristotle.[14] Aristotelian logic became widely accepted in science and mathematics and remained in wide use in the West until the early 19th century.[15] Aristotle's system of logic was responsible for the introduction of hypothetical syllogism,[16] temporal modal logic,[17][18] and inductive logic.[19] In Europe during the later medieval period, major efforts were made to show that Aristotle's ideas were compatible with Christian faith. During the High Middle Ages, logic became a main focus of philosophers, who would engage in critical logical analyses of philosophical arguments.


Aristotle, 384–322 BC.

The Chinese logical philosopher Gongsun Long (ca. 325–250 BC) proposed the paradox "One and one cannot become two, since neither becomes two."[20] In China, the tradition of scholarly investigation into logic, however, was repressed by the Qin dynasty following the legalist philosophy of Han Feizi.

In India, innovations in the scholastic school, called Nyaya, continued from ancient times into the early 18th century with the Navya-Nyaya school. By the 16th century, it developed theories resembling modern logic, such as Gottlob Frege's "distinction between sense and reference of proper names" and his "definition of number," as well as the theory of "restrictive conditions for universals" anticipating some of the developments in modern set theory.[21] Since 1824, Indian logic attracted the attention of many Western scholars, and has had an influence on important 19th-century logicians such as Charles Babbage, Augustus De Morgan, and George Boole.[22] In the 20th century, Western philosophers like Stanislaw Schayer and Klaus Glashoff have explored Indian logic more extensively.

The syllogistic logic developed by Aristotle predominated in the West until the mid-19th century, when interest in the foundations of mathematics stimulated the development of symbolic logic (now called mathematical logic). In 1854, George Boole published *An Investigation of the Laws of Thought on Which are Founded the Mathematical Theories of Logic and Probabilities*, introducing symbolic logic and the principles of what is now known as Boolean logic. In 1879, Gottlob Frege published *Begriffsschrift* which inaugurated modern logic with the invention of quantifier notation. From 1910 to 1913, Alfred North Whitehead and Bertrand Russell published *Principia Mathematica*[9] on the foundations of mathematics, attempting to derive mathematical truths from axioms and inference rules in symbolic logic. In 1931, Gödel raised serious problems with the foundationalist program and logic ceased to focus on such issues.

The development of logic since Frege, Russell, and Wittgenstein had a profound influence on the practice of philosophy and the perceived nature of philosophical problems (see Analytic philosophy), and Philosophy of mathematics. Logic, especially sentential logic, is implemented in computer logic circuits and is fundamental to computer science. Logic is commonly taught by university philosophy departments, often as a compulsory discipline.

Topics in logic

Syllogistic logic

The *Organon* was Aristotle's body of work on logic, with the *Prior Analytics* constituting the first explicit work in formal logic, introducing the syllogistic.[23] The parts of syllogistic logic, also known by the name term logic, are the analysis of the judgements into propositions consisting of two terms that are related by one of a fixed number of relations, and the expression of inferences by means of syllogisms that consist of two propositions sharing a common term as premise, and a conclusion which is a proposition involving the two unrelated terms from the premises.

Aristotle's work was regarded in classical times and from medieval times in Europe and the Middle East as the very picture of a fully worked out system. However, it was not alone: the Stoics proposed a system of propositional logic that was studied by medieval logicians. Also, the problem of multiple generality was recognised in medieval times. Nonetheless, problems with syllogistic logic were not seen as being in need of revolutionary solutions.

Today, some academics claim that Aristotle's system is generally seen as having little more than historical value (though there is some current interest in extending term logics), regarded as made obsolete by the advent of propositional logic and the predicate calculus. Others use Aristotle in argumentation theory to help develop and critically question argumentation schemes that are used in artificial intelligence and legal arguments.

Propositional logic (sentential logic)

A propositional calculus or logic (also a sentential calculus) is a formal system in which formulae representing propositions can be formed by combining atomic propositions using logical connectives, and in which a system of formal proof rules allows certain formulae to be established as "theorems".

Predicate logic

Predicate logic is the generic term for symbolic formal systems such as first-order logic, second-order logic, many-sorted logic, and infinitary logic.

Predicate logic provides an account of quantifiers general enough to express a wide set of arguments occurring in natural language. Aristotelian syllogistic logic specifies a small number of forms that the relevant part of the involved judgements may take. Predicate logic allows sentences to be analysed into subject and argument in several additional ways, thus allowing predicate logic to solve the problem of multiple generality that had perplexed medieval logicians.

The development of predicate logic is usually attributed to Gottlob Frege, who is also credited as one of the founders of analytical philosophy, but the formulation of predicate logic most often used today is the first-order logic presented in Principles of Mathematical Logic by David Hilbert and Wilhelm Ackermann in 1928. The analytical generality of predicate logic allowed the formalisation of mathematics, drove the investigation of set theory, and allowed the development of Alfred Tarski's approach to model theory. It provides the foundation of modern mathematical logic.

Frege's original system of predicate logic was second-order, rather than first-order. Second-order logic is most prominently defended (against the criticism of Willard Van Orman Quine and others) by George Boolos and Stewart Shapiro.

Modal logic

In languages, modality deals with the phenomenon that sub-parts of a sentence may have their semantics modified by special verbs or modal particles. For example, "*We go to the games*" can be modified to give "*We should go to the games*", and "*We can go to the games*"" and perhaps "*We will go to the games*". More abstractly, we might say that modality affects the circumstances in which we take an assertion to be satisfied.

The logical study of modality dates back to Aristotle,[24] who was concerned with the alethic modalities of necessity and possibility, which he observed to be dual in the sense of De Morgan duality. While the study of necessity and possibility remained important to philosophers, little logical innovation happened until the landmark investigations of Clarence Irving Lewis in 1918, who formulated a family of rival axiomatizations of the alethic modalities. His work unleashed a torrent of new work on the topic, expanding the kinds of modality treated to include deontic logic and epistemic logic. The seminal work of Arthur Prior applied the same formal language to treat temporal logic and paved the way for the marriage of the two subjects. Saul Kripke discovered (contemporaneously with rivals) his theory of frame semantics which revolutionised the formal technology available to modal logicians and gave a new graph-theoretic way of looking at modality that has driven many applications in computational linguistics and computer science, such as dynamic logic.

Informal reasoning

The motivation for the study of logic in ancient times was clear: it is so that one may learn to distinguish good from bad arguments, and so become more effective in argument and oratory, and perhaps also to become a better person. Half of the works of Aristotle's Organon treat inference as it occurs in an informal setting, side by side with the development of the syllogistic, and in the Aristotelian school, these informal works on logic were seen as complementary to Aristotle's treatment of rhetoric.

This ancient motivation is still alive, although it no longer takes centre stage in the picture of logic; typically dialectical logic will form the heart of a course in critical thinking, a compulsory course at many universities.

Argumentation theory is the study and research of informal logic, fallacies, and critical questions as they relate to every day and practical situations. Specific types of dialogue can be analyzed and questioned to reveal premises, conclusions, and fallacies. Argumentation theory is now applied in artificial intelligence and law.

Mathematical logic

Mathematical logic really refers to two distinct areas of research: the first is the application of the techniques of formal logic to mathematics and mathematical reasoning, and the second, in the other direction, the application of mathematical techniques to the representation and analysis of formal logic.[25]

The earliest use of mathematics and geometry in relation to logic and philosophy goes back to the ancient Greeks such as Euclid, Plato, and Aristotle.[26] Many other ancient and medieval philosophers applied mathematical ideas and methods to their philosophical claims.[27]


Kurt Gödel

One of the boldest attempts to apply logic to mathematics was undoubtedly the logicism pioneered by philosopher-logicians such as Gottlob Frege and Bertrand Russell: the idea was that mathematical theories were logical tautologies, and the programme was to show this by means to a reduction of mathematics to logic.[9] The various attempts to carry this out met with a series of failures, from the crippling of Frege's project in his *Grundgesetze* by Russell's paradox, to the defeat of Hilbert's program by Gödel's incompleteness theorems.

Both the statement of Hilbert's program and its refutation by Gödel depended upon their work establishing the second area of mathematical logic, the application of mathematics to logic in the form of proof theory.[28] Despite the negative nature of the incompleteness theorems, Gödel's completeness theorem, a result in model theory and another application of mathematics to logic, can be understood as showing how close logicism came to being true: every rigorously defined mathematical theory can be exactly captured by a first-order logical theory; Frege's proof calculus is enough to *describe* the whole of mathematics, though not *equivalent* to it. Thus we see how complementary the two areas of mathematical logic have been.

If proof theory and model theory have been the foundation of mathematical logic, they have been but two of the four pillars of the subject. Set theory originated in the study of the infinite by Georg Cantor, and it has been the source of many of the most challenging and important issues in mathematical logic, from Cantor's theorem, through the status of the Axiom of Choice and the question of the independence of the continuum hypothesis, to the modern debate on large cardinal axioms.

Recursion theory captures the idea of computation in logical and arithmetic terms; its most classical achievements are the undecidability of the Entscheidungsproblem by Alan Turing, and his presentation of the Church-Turing thesis.[29] Today recursion theory is mostly concerned with the more refined problem of complexity classes — when is a problem efficiently solvable? — and the classification of degrees of unsolvability.[30]

Philosophical logic

Philosophical logic deals with formal descriptions of natural language. Most philosophers assume that the bulk of "normal" proper reasoning can be captured by logic, if one can find the right method for translating ordinary language into that logic. Philosophical logic is essentially a continuation of the traditional discipline that was called "Logic" before the invention of mathematical logic. Philosophical logic has a much greater concern with the connection between natural language and logic. As a result, philosophical logicians have contributed a great deal to the development of non-standard logics (e.g., free logics, tense logics) as well as various extensions of classical logic (e.g., modal logics), and non-standard semantics for such logics (e.g., Kripke's technique of supervaluations in the semantics of logic).

Logic and the philosophy of language are closely related. Philosophy of language has to do with the study of how our language engages and interacts with our thinking. Logic has an immediate impact on other areas of study. Studying logic and the relationship between logic and ordinary speech can help a person better structure his own arguments and critique the arguments of others. Many popular arguments are filled with errors because so many people are untrained in logic and unaware of how to formulate an argument correctly.

Computational Logic

Logic cut to the heart of computer science as it emerged as a discipline: Alan Turing's work on the Entscheidungsproblem followed from Kurt Gödel's work on the incompleteness theorems, and the notion of general purpose computers that came from this work was of fundamental importance to the designers of the computer machinery in the 1940s.

In the 1950s and 1960s, researchers predicted that when human knowledge could be expressed using logic with mathematical notation, it would be possible to create a machine that reasons, or artificial intelligence. This turned out to be more difficult than expected because of the complexity of human reasoning. In logic programming, a program consists of a set of axioms and rules. Logic programming systems such as Prolog compute the consequences of the axioms and rules in order to answer a query.

Today, logic is extensively applied in the fields of Artificial Intelligence, and Computer Science, and these fields provide a rich source of problems in formal and informal logic. Argumentation theory is one good example of how logic is being applied to artificial intelligence. The ACM Computing Classification System in particular regards:

- Section F.3 on Logics and meanings of programs and F.4 on Mathematical logic and formal languages as part of the theory of computer science: this work covers formal semantics of programming languages, as well as work of formal methods such as Hoare logic
- Boolean logic as fundamental to computer hardware: particularly, the system's section B.2 on Arithmetic and logic structures, relating to operatives AND, NOT, and OR;
- Many fundamental logical formalisms are essential to section I.2 on artificial intelligence, for example modal logic and default logic in Knowledge representation formalisms and methods, Horn clauses in logic programming, and description logic.

Furthermore, computers can be used as tools for logicians. For example, in symbolic logic and mathematical logic, proofs by humans can be computer-assisted. Using automated theorem proving the machines can find and check proofs, as well as work with proofs too lengthy to be written out by hand.

Bivalence and the law of the excluded middle

The logics discussed above are all "bivalent" or "two-valued"; that is, they are most naturally understood as dividing propositions into true and false propositions. Non-classical logics are those systems which reject bivalence.

Hegel developed his own dialectic logic that extended Kant's transcendental logic but also brought it back to ground by assuring us that "neither in heaven nor in earth, neither in the world of mind nor of nature, is there anywhere such an abstract 'either–or' as the understanding maintains. Whatever exists is concrete, with difference and opposition in itself".[31]

In 1910 Nicolai A. Vasiliev extended the law of excluded middle and the law of contradiction and proposed the law of excluded fourth and logic tolerant to contradiction.[32] In the early 20th century Jan Łukasiewicz investigated the extension of the traditional true/false values to include a third value, "possible", so inventing ternary logic, the first multi-valued logic.

Logics such as fuzzy logic have since been devised with an infinite number of "degrees of truth", represented by a real number between 0 and 1.[33]

Intuitionistic logic was proposed by L.E.J. Brouwer as the correct logic for reasoning about mathematics, based upon his rejection of the law of the excluded middle as part of his intuitionism. Brouwer rejected formalisation in mathematics, but his student Arend Heyting studied intuitionistic logic formally, as did Gerhard Gentzen. Intuitionistic logic has come to be of great interest to computer scientists, as it is a constructive logic and can be applied for extracting verified programs from proofs.

Modal logic is not truth conditional, and so it has often been proposed as a non-classical logic. However, modal logic is normally formalised with the principle of the excluded middle, and its relational semantics is bivalent, so this inclusion is disputable.

"Is logic empirical?"

What is the epistemological status of the laws of logic? What sort of argument is appropriate for criticizing purported principles of logic? In an influential paper entitled "Is logic empirical?"[34] Hilary Putnam, building on a suggestion of W.V. Quine, argued that in general the facts of propositional logic have a similar epistemological status as facts about the physical universe, for example as the laws of mechanics or of general relativity, and in particular that what physicists have learned about quantum mechanics provides a compelling case for abandoning certain familiar principles of classical logic: if we want to be realists about the physical phenomena described by quantum theory, then we should abandon the principle of distributivity, substituting for classical logic the quantum logic proposed by Garrett Birkhoff and John von Neumann.[35]

Another paper by the same name by Sir Michael Dummett argues that Putnam's desire for realism mandates the law of distributivity.[36] Distributivity of logic is essential for the realist's understanding of how propositions are true of the world in just the same way as he has argued the principle of bivalence is. In this way, the question, "Is logic empirical?" can be seen to lead naturally into the fundamental controversy in metaphysics on realism versus anti-realism.

Implication: strict or material?

It is obvious that the notion of implication formalised in classical logic does not comfortably translate into natural language by means of "if… then…", due to a number of problems called the *paradoxes of material implication*.

The first class of paradoxes involves counterfactuals, such as "If the moon is made of green cheese, then 2+2=5", which are puzzling because natural language does not support the principle of explosion. Eliminating this class of paradoxes was the reason for C. I. Lewis's formulation of strict implication, which eventually led to more radically revisionist logics such as relevance logic.

The second class of paradoxes involves redundant premises, falsely suggesting that we know the succedent because of the antecedent: thus "if that man gets elected, granny will die" is materially true since granny is mortal, regardless of the man's election prospects. Such sentences violate the Gricean maxim of relevance, and can be modelled by logics that reject the principle of monotonicity of entailment, such as relevance logic.

Tolerating the impossible

Hegel was deeply critical of any simplified notion of the Law of Non-Contradiction. It was based on Leibniz's idea that this law of logic also requires a sufficient ground to specify from what point of view (or time) one says that something cannot contradict itself. A building, for example, both moves and does not move; the ground for the first is our solar system and for the second the earth. In Hegelian dialectic, the law of non-contradiction, of identity, itself relies upon difference and so is not independently assertable.

Closely related to questions arising from the paradoxes of implication comes the suggestion that logic ought to tolerate inconsistency. Relevance logic and paraconsistent logic are the most important approaches here, though the concerns are different: a key consequence of classical logic and some of its rivals, such as intuitionistic logic, is that they respect the principle of explosion, which means that the logic collapses if it is capable of deriving a contradiction. Graham Priest, the main proponent of dialetheism, has argued for paraconsistency on the grounds that there are in fact, true contradictions.[37]

Rejection of logical truth

The philosophical vein of various kinds of skepticism contains many kinds of doubt and rejection of the various bases upon which logic rests, such as the idea of logical form, correct inference, or meaning, typically leading to the conclusion that there are no logical truths. Observe that this is opposite to the usual views in philosophical skepticism, where logic directs skeptical enquiry to doubt received wisdoms, as in the work of Sextus Empiricus.

Friedrich Nietzsche provides a strong example of the rejection of the usual basis of logic: his radical rejection of idealisation led him to reject truth as a "mobile army of metaphors, metonyms, and anthropomorphisms—in short ... metaphors which are worn out and without sensuous power; coins which have lost their pictures and now matter only as metal, no longer as coins".[38] His rejection of truth did not lead him to reject the idea of either inference or logic completely, but rather suggested that "logic [came] into existence in man's head [out] of illogic, whose realm originally must have been immense. Innumerable beings who made inferences in a way different from ours perished".[39] Thus there is the idea that logical inference has a use as a tool for human survival, but that its existence does not support the existence of truth, nor does it have a reality beyond the instrumental: "Logic, too, also rests on assumptions that do not correspond to anything in the real world".[40]

This position held by Nietzsche however, has come under extreme scrutiny for several reasons. He fails to demonstrate the validity of his claims and merely asserts them rhetorically. Furthermore, his position has been claimed to be self-refuting by philosophers, such as Jürgen Habermas, who have accused Nietszche of not even having a coherent perspective let alone a theory of knowledge.[41] George Lukacs in his book *The Destruction of Reason* has asserted that "Were we to study Nietzsche's statements in this area from a logico-philosophical angle, we would be confronted by a dizzy chaos of the most lurid assertions, arbitrary and violently incompatible".[42] Extreme skepticism such as that displayed by Nietzsche has not been met with much seriousness by analytic philosophers in the 20th century. Bertrand Russell famously referred to Nietzsche's claims as "empty words" in his book *A History of Western Philosophy*.[43]

Notes

[1] "possessed. of reason, intellectual, dialectical, argumentative", also related to wiktionary:λόγος (logos), "word, thought, idea, argument, account, reason, or principle" (Liddell & Scott 1999; Online Etymology Dictionary 2001).

[2] Richard Henry Popkin; Avrum Stroll (1 July 1993). *Philosophy Made Simple* (http://books.google.com/books?id=TWNo-4euyesC&pg=PR7). Random House Digital, Inc. p. 238. ISBN 978-0-385-42533-9. . Retrieved 5 March 2012.

[3] Hofweber, T. (2004). "Logic and Ontology" (http://plato.stanford.edu/entries/logic-ontology). In Zalta, Edward N. *Stanford Encyclopedia of Philosophy*. .

[4] Cox, J. Robert; Willard, Charles Arthur, eds. (1983). *Advances in Argumentation Theory and Research*. Southern Illinois University Press. ISBN 978-0-8093-1050-0.

[5] For example, Nyaya (syllogistic recursion) dates back 1900 years.

[6] Mohists and the school of Names date back at 2200 years.

[7] Plato (1976). Buchanan, Scott. ed. *The Portable Plato*. Penguin. ISBN 0-14-015040-4.

[8] Aristotle (2001). "Posterior Analytics". In Mckeon, Richard. *The Basic Works*. Modern Library. ISBN 0-375-75799-6.

[9] Whitehead, Alfred North; Russell, Bertrand (1967). *Principia Mathematica to *56*. Cambridge University Press. ISBN 0-521-62606-4.

[10] For a more modern treatment, see Hamilton, A. G. (1980). *Logic for Mathematicians*. Cambridge University Press. ISBN 0-521-29291-3.

[11] Łukasiewicz, Jan (1957). *Aristotle's syllogistic from the standpoint of modern formal logic* (2nd ed.). Oxford University Press. p. 7. ISBN 978-0-19-824144-7.

[12] Bergmann, Merrie, James Moor, and Jack Nelson. The Logic Book fifth edition. New York, NY: McGraw-Hill, 2009.

[13] Mendelson, Elliott (1964). "Quantification Theory: Completeness Theorems". *Introduction to Mathematical Logic*. Van Nostrand. ISBN 0-412-80830-7.

[14] E.g., Kline (1972, p.53) wrote "A major achievement of Aristotle was the founding of the science of logic".

[15] " Aristotle (http://chemistry.mtu.edu/~pcharles/SCIHISTORY/aristotle.html)", MTU Department of Chemistry.

[16] Jonathan Lear (1986). " *Aristotle and Logical Theory* (http://books.google.com/books?id=lXI7AAAAIAAJ&pg=PA34&dq&hl=en#v=onepage&q=&f=false)". Cambridge University Press. p.34. ISBN 0-521-31178-0

[17] Simo Knuuttila (1981). " *Reforging the great chain of being: studies of the history of modal theories* (http://books.google.com/books?id=iCCUF_OtA8AC&pg=PA71&dq&hl=en#v=onepage&q=&f=false)". Springer Science & Business. p.71. ISBN 90-277-1125-9

[18] Michael Fisher, Dov M. Gabbay, Lluís Vila (2005). " *Handbook of temporal reasoning in artificial intelligence* (http://books.google.com/books?id=Ajsvo6jWNhgC&pg=PA119&dq&hl=en#v=onepage&q=&f=false)". Elsevier. p.119. ISBN 0-444-51493-7
[19] Harold Joseph Berman (1983). " *Law and revolution: the formation of the Western legal tradition* (http://books.google.com/books?id=9-8fIBVgCQYC&pg=PA133&dq&hl=en#v=onepage&q=&f=false)". Harvard University Press. p.133. ISBN 0-674-51776-8
[20] The four Catuṣkoṭi logical divisions are formally very close to the four opposed propositions of the Greek tetralemma, which in turn are analogous to the four truth values of modern relevance logic Cf. Belnap (1977); Jayatilleke, K. N., (1967, The logic of four alternatives, in *Philosophy East and West*, University of Hawaii Press).
[21] Kisor Kumar Chakrabarti (June 1976). "Some Comparisons Between Frege's Logic and Navya-Nyaya Logic". *Philosophy and Phenomenological Research* (International Phenomenological Society) **36** (4): 554–563. doi:10.2307/2106873. JSTOR 2106873. "This paper consists of three parts. The first part deals with Frege's distinction between sense and reference of proper names and a similar distinction in Navya-Nyaya logic. In the second part we have compared Frege's definition of number to the Navya-Nyaya definition of number. In the third part we have shown how the study of the so-called 'restrictive conditions for universals' in Navya-Nyaya logic anticipated some of the developments of modern set theory."
[22] Jonardon Ganeri (2001). *Indian logic: a reader*. Routledge. pp. vii, 5, 7. ISBN 0-7007-1306-9.
[23] "Aristotle" (http://www.britannica.com/EBchecked/topic/346217/history-of-logic/65920/Aristotle). *Encyclopædia Britannica*. .
[24] Peter Thomas Geach (1980). " *Logic Matters* (http://books.google.com/books?id=XCUrx6WGgrwC&pg=PA316&dq&hl=en#v=onepage&q=&f=false)". University of California Press. p.316. ISBN 0-520-03847-9
[25] Stolyar, Abram A. (1983). *Introduction to Elementary Mathematical Logic*. Dover Publications. p. 3. ISBN 0-486-64561-4.
[26] Barnes, Jonathan (1995). *The Cambridge Companion to Aristotle*. Cambridge University Press. p. 27. ISBN 0-521-42294-9.
[27] Aristotle (1989). *Prior Analytics*. Hackett Publishing Co.. p. 115. ISBN 978-0-87220-064-7.
[28] Mendelson, Elliott (1964). "Formal Number Theory: Gödel's Incompleteness Theorem". *Introduction to Mathematical Logic*. Monterey, Calif.: Wadsworth & Brooks/Cole Advanced Books & Software. OCLC 13580200.
[29] Brookshear, J. Glenn (1989). "Computability: Foundations of Recursive Function Theory". *Theory of computation: formal languages, automata, and complexity*. Redwood City, Calif.: Benjamin/Cummings Pub. Co.. ISBN 0-8053-0143-7.
[30] Brookshear, J. Glenn (1989). "Complexity". *Theory of computation: formal languages, automata, and complexity*. Redwood City, Calif.: Benjamin/Cummings Pub. Co.. ISBN 0-8053-0143-7.
[31] Hegel, G. W. F (1971) [1817]. *Philosophy of Mind*. Encyclopedia of the Philosophical Sciences. trans. William Wallace. Oxford: Clarendon Press. p. 174. ISBN 0-19-875014-5.
[32] Joseph E. Brenner (3 August 2008). *Logic in Reality* (http://books.google.com/books?id=Jnj5E6C9UwsC&pg=PA28). Springer. pp. 28–30. ISBN 978-1-4020-8374-7. . Retrieved 9 April 2012.
[33] Hájek, Petr (2006). "Fuzzy Logic" (http://plato.stanford.edu/entries/logic-fuzzy/). In Zalta, Edward N.. *Stanford Encyclopedia of Philosophy*. .
[34] Putnam, H. (1969). "Is Logic Empirical?". *Boston Studies in the Philosophy of Science* **5**.
[35] Birkhoff, G.; von Neumann, J. (1936). "The Logic of Quantum Mechanics". *Annals of Mathematics* (Annals of Mathematics) **37** (4): 823–843. doi:10.2307/1968621. JSTOR 1968621.
[36] Dummett, M. (1978). "Is Logic Empirical?". *Truth and Other Enigmas*. ISBN 0-674-91076-1.
[37] Priest, Graham (2008). "Dialetheism" (http://plato.stanford.edu/entries/dialetheism). In Zalta, Edward N.. *Stanford Encyclopedia of Philosophy*. .
[38] Nietzsche, 1873, On Truth and Lies in a Nonmoral Sense.
[39] Nietzsche, 1882, *The Gay Science*.
[40] Nietzsche, 1878, *Human, All Too Human*
[41] Babette Babich, Habermas, Nietzsche, and Critical Theory
[42] http://www.marxists.org/archive/lukacs/works/destruction-reason/ch03.htm
[43] Bertrand Russell, A History of Western Philosophy http://www.amazon.com/History-Western-Philosophy-Bertrand-Russell/dp/0671201581

References

- Nuel Belnap, (1977). "A useful four-valued logic". In Dunn & Eppstein, *Modern uses of multiple-valued logic*. Reidel: Boston.
- Józef Maria Bocheński (1959). *A précis of mathematical logic*. Translated from the French and German editions by Otto Bird. D. Reidel, Dordrecht, South Holland.
- Józef Maria Bocheński, (1970). *A history of formal logic*. 2nd Edition. Translated and edited from the German edition by Ivo Thomas. Chelsea Publishing, New York.
- Brookshear, J. Glenn (1989). *Theory of computation: formal languages, automata, and complexity*. Redwood City, Calif.: Benjamin/Cummings Pub. Co.. ISBN 0-8053-0143-7.

- Cohen, R.S, and Wartofsky, M.W. (1974). *Logical and Epistemological Studies in Contemporary Physics*. Boston Studies in the Philosophy of Science. D. Reidel Publishing Company: Dordrecht, Netherlands. ISBN 90-277-0377-9.
- Finkelstein, D. (1969). "Matter, Space, and Logic". in R.S. Cohen and M.W. Wartofsky (eds. 1974).
- Gabbay, D.M., and Guenthner, F. (eds., 2001–2005). *Handbook of Philosophical Logic*. 13 vols., 2nd edition. Kluwer Publishers: Dordrecht.
- Hilbert, D., and Ackermann, W, (1928). *Grundzüge der theoretischen Logik* (*Principles of Mathematical Logic*). Springer-Verlag. OCLC 2085765 (http://worldcat.org/oclc/2085765)
- Susan Haack, (1996). *Deviant Logic, Fuzzy Logic: Beyond the Formalism*, University of Chicago Press.
- Hodges, W., (2001). *Logic. An introduction to Elementary Logic*, Penguin Books.
- Hofweber, T., (2004), Logic and Ontology (http://plato.stanford.edu/entries/logic-ontology/). *Stanford Encyclopedia of Philosophy*. Edward N. Zalta (ed.).
- Hughes, R.I.G., (1993, ed.). *A Philosophical Companion to First-Order Logic*. Hackett Publishing.
- Kline, Morris (1972). *Mathematical Thought From Ancient to Modern Times*. Oxford University Press. ISBN 0-19-506135-7.
- Kneale, William, and Kneale, Martha, (1962). *The Development of Logic*. Oxford University Press, London, UK.
- Liddell, Henry George; Scott, Robert. "Logikos" (http://www.perseus.tufts.edu/cgi-bin/ptext?doc=Perseus:text:1999.04.0057:entry=#63716). *A Greek-English Lexicon*. Perseus Project. Retrieved 8 May 2009.
- Mendelson, Elliott, (1964). *Introduction to Mathematical Logic*. Wadsworth & Brooks/Cole Advanced Books & Software: Monterey, Calif. OCLC 13580200 (http://worldcat.org/oclc/13580200)
- Harper, Robert (2001). "Logic" (http://www.etymonline.com/index.php?term=logic). *Online Etymology Dictionary*. Retrieved 8 May 2009.
- Smith, B., (1989). "Logic and the Sachverhalt". *The Monist* 72(1):52–69.
- Whitehead, Alfred North and Bertrand Russell, (1910). *Principia Mathematica*. Cambridge University Press: Cambridge, England. OCLC 1041146 (http://worldcat.org/oclc/1041146)

External links and further readings

- Introductions and tutorials
  - An Introduction to Philosophical Logic (http://www.galilean-library.org/manuscript.php?postid=43782), by Paul Newall, aimed at beginners.
  - forall x: an introduction to formal logic (http://www.fecundity.com/logic/), by P.D. Magnus, covers sentential and quantified logic.
  - Logic Self-Taught: A Workbook (http://www.filozofia.uw.edu.pl/kpaprzycka/Publ/xLogicSelfTaught.html) (originally prepared for on-line logic instruction).
    - Nicholas Rescher. (1964). *Introduction to Logic*, St. Martin's Press.
- Essays
  - "Symbolic Logic" (http://durendal.org:8080/lcsl/) and "The Game of Logic" (http://www.gutenberg.org/etext/4763), Lewis Carroll, 1896.
  - Math & Logic: The history of formal mathematical, logical, linguistic and methodological ideas. (http://etext.lib.virginia.edu/DicHist/analytic/anaVII.html) In *The Dictionary of the History of Ideas.*
- Online Tools
  - Interactive Syllogistic Machine (http://thefirstscience.org/syllogistic-machine/) A web based syllogistic machine for exploring fallacies, figures, terms, and modes of syllogisms.
- Reference material

  - Translation Tips (http://www.earlham.edu/~peters/courses/log/transtip.htm), by Peter Suber, for translating from English into logical notation.
  - Ontology and History of Logic. An Introduction (http://www.ontology.co/history-of-logic.htm) with an annotated bibliography.
- Reading lists
  - The London Philosophy Study Guide (http://www.ucl.ac.uk/philosophy/LPSG/) offers many suggestions on what to read, depending on the student's familiarity with the subject:
    - Logic & Metaphysics (http://www.ucl.ac.uk/philosophy/LPSG/L&M.htm)
    - Set Theory and Further Logic (http://www.ucl.ac.uk/philosophy/LPSG/SetTheory.htm)
    - Mathematical Logic (http://www.ucl.ac.uk/philosophy/LPSG/MathLogic.htm)

History

History of logic

The **history of logic** is the study of the development of the science of valid inference (logic). Formal logic was developed in ancient times in China, India, and Greece. Greek logic, particularly Aristotelian logic, found wide application and acceptance in science and mathematics.

Aristotle's logic was further developed by Islamic and Christian philosophers in the Middle Ages, reaching a high point in the mid-fourteenth century. The period between the fourteenth century and the beginning of the nineteenth century was largely one of decline and neglect, and is regarded as barren by at least one historian of logic.[1]

Logic was revived in the mid-nineteenth century, at the beginning of a revolutionary period when the subject developed into a rigorous and formalistic discipline whose exemplar was the exact method of proof used in mathematics. The development of the modern so-called "symbolic" or "mathematical" logic during this period is the most significant in the two-thousand-year history of logic, and is arguably one of the most important and remarkable events in human intellectual history.[2]

Progress in mathematical logic in the first few decades of the twentieth century, particularly arising from the work of Gödel and Tarski, had a significant impact on analytic philosophy and philosophical logic, particularly from the 1950s onwards, in subjects such as modal logic, temporal logic, deontic logic, and relevance logic.

Prehistory of logic


The pyramids of Egypt were built using geometry

Valid reasoning has been employed in all periods of human history. However, logic studies the *principles* of valid reasoning, inference and demonstration. It is probable that the idea of demonstrating a conclusion first arose in connection with geometry, which originally meant the same as "land measurement".[3] In particular, the ancient Egyptians had empirically discovered some truths of geometry, such as the formula for the volume of a truncated pyramid.[4]

Another origin can be seen in Babylonia. Esagil-kin-apli's medical *Diagnostic Handbook* in the 11th century BC was based on a logical set of axioms and assumptions,[5] while Babylonian astronomers in the 8th and 7th centuries BC employed an internal logic within their predictive planetary systems, an important contribution to the philosophy of science.[6]

Logic in Greek philosophy

Before Plato

While the ancient Egyptians empirically discovered some truths of geometry, the great achievement of the ancient Greeks was to replace empirical methods by demonstrative science.[4] The systematic study of this seems to have begun with the school of Pythagoras in the late sixth century BC.[4] The three basic principles of geometry are that certain propositions must be accepted as true without demonstration, that all other propositions of the system are derived from these, and that the derivation must be *formal*, that is, independent of the particular subject matter in question.[4] Fragments of early proofs are preserved in the works of Plato and Aristotle,[7] and the idea of a deductive system was probably known in the Pythagorean school and the Platonic Academy.[4]

Separately from geometry, the idea of a standard argument pattern is found in the *Reductio ad absurdum* used by Zeno of Elea, a pre-Socratic philosopher of the fifth century BC. This is the technique of drawing an obviously false, absurd or impossible conclusion from an assumption, thus demonstrating that the assumption is false.[8] Plato's Parmenides portrays Zeno as claiming to have written a book defending the monism of Parmenides by demonstrating the absurd consequence of assuming that there is plurality. Other philosophers who practised such *dialectic* reasoning were the so-called minor Socratics, including Euclid of Megara, who were probably followers of Parmenides and Zeno. The members of this school were called "dialecticians" (from a Greek word meaning "to discuss").

Further evidence that pre-Aristotelian thinkers were concerned with the principles of reasoning is found in the fragment called *Dissoi Logoi*, probably written at the beginning of the fourth century BC. This is part of a protracted debate about truth and falsity.[9]

Plato's logic


Plato's academy

None of the surviving works of the great fourth-century philosopher Plato (428–347) include any formal logic,[10] but they include important contributions to the field of philosophical logic. Plato raises three questions:

- What is it that can properly be called true or false?
- What is the nature of the connection between the assumptions of a valid argument and its conclusion?
- What is the nature of definition?

The first question arises in the dialogue Theaetetus, where Plato identifies thought or opinion with talk or discourse (*logos*).[11] The second question is a result of Plato's theory of Forms. Forms are not things in the ordinary sense, nor strictly ideas in the mind, but they correspond to what philosophers later called universals, namely an abstract entity common to each set of things that have the same name. In both The Republic and The Sophist, Plato suggests that the necessary connection between the premisses and the conclusion of an argument corresponds to a necessary connection between "forms".[12] The third question is about definition. Many of Plato's dialogues concern the search for a definition of some important concept (justice, truth, the Good), and it is likely that Plato was impressed by the importance of definition in mathematics.[13] What underlies every definition is a Platonic Form, the common nature present in different particular things. Thus a definition reflects the ultimate object of our understanding, and is the foundation of all valid inference. This had a great influence on Aristotle, in particular Aristotle's notion of the essence of a thing, the "what it is to be" a particular thing of a certain kind.[14]

Aristotle's logic

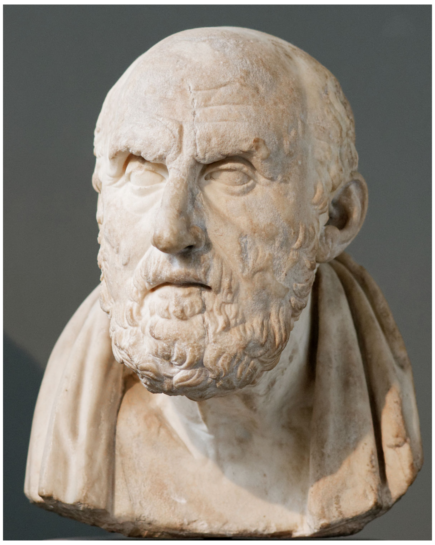
Aristotle's logic was still influential in the Renaissance

The logic of Aristotle, and particularly his theory of the syllogism, has had an enormous influence in Western thought.[15] His logical works, called the *Organon*, are the earliest formal study of logic that have come down to modern times. Though it is difficult to determine the dates, the probable order of writing of Aristotle's logical works is:

- *The Categories*, a study of the ten kinds of primitive term.
- *The Topics* (with an appendix called *On Sophistical Refutations*), a discussion of dialectics.
- *On Interpretation*, an analysis of simple categorical propositions, into simple terms, negation, and signs of quantity; and a comprehensive treatment of the notions of opposition and conversion.
- *The Prior Analytics*, a formal analysis of valid argument or "syllogism".
- *The Posterior Analytics*, a study of scientific demonstration, containing Aristotle's mature views on logic.

These works are of outstanding importance in the history of logic. Aristotle was the first logician to attempt a systematic analysis of logical syntax, into noun (or *term*), and verb. In the *Categories*, he attempted to classify all the possible things that a term can refer to. This idea underpins his philosophical work, the *Metaphysics*, which also had a profound influence on Western thought. He was the first to deal with the principles of contradiction and excluded middle in a systematic way. He was the first *formal logician* (i.e. he gave the principles of reasoning using variables to show the underlying logical form of arguments). He was looking for relations of dependence which characterise necessary inference, and distinguished the validity of these relations, from the truth of the premises (the soundness of the argument). The *Prior Analytics* contains his exposition of the "syllogistic", where three important principles are applied for the first time in history: the use of variables, a purely formal treatment, and the use of an axiomatic system. In the *Topics* and *Sophistical Refutations* he also developed a theory of non-formal logic (e.g. the theory of fallacies).[16]

Stoic logic



Chrysippus of Soli

The other great school of Greek logic is that of the Stoics.[17] Stoic logic traces its roots back to the late 5th century BC philosopher, Euclid of Megara, a pupil of Socrates and slightly older contemporary of Plato. His pupils and successors were called "Megarians", or "Eristics", and later the "Dialecticians". The two most important dialecticians of the Megarian school were Diodorus Cronus and Philo who were active in the late 4th century BC. The Stoics adopted the Megarian logic and systemized it. The most important member of the school was Chrysippus (c. 278–c. 206 BC), who was its third head, and who formalized much of Stoic doctrine. He is supposed to have written over 700 works, including at least 300 on logic, almost none of which survive.[18][19] Unlike with Aristotle, we have no complete works by the Megarians or the early Stoics, and have to rely mostly on accounts (sometimes hostile) by later sources, including prominently Diogenes Laertius, Sextus Empiricus, Galen, Aulus Gellius, Alexander of Aphrodisias and Cicero.[20]

Three significant contributions of the Stoic school were (i) their account of modality, (ii) their theory of the Material conditional, and (iii) their account of meaning and truth.[21]

- *Modality*. According to Aristotle, the Megarians of his day claimed there was no distinction between potentiality and actuality.[22] Diodorus Cronus defined the possible as that which either is or will be, the impossible as what will not be true, and the contingent as that which either is already, or will be false.[23] Diodorus is also famous for his so-called Master argument, that the three propositions "everything that is past is true and necessary", "the impossible does not follow from the possible", and "What neither is nor will be is possible" are inconsistent. Diodorus used the plausibility of the first two to prove that nothing is possible if it neither is nor will be true.[24] Chrysippus, by contrast, denied the second premiss and said that the impossible could follow from the possible.[25]
- *Conditional statements*. The first logicians to debate conditional statements were Diodorus and his pupil Philo of Megara. Sextus Empiricus refers three times to a debate between Diodorus and Philo. Philo argued that a true conditional is one that does not begin with a truth and end with a falsehood. such as "if it is day, then I am talking". But Diodorus argued that a true conditional is what could not possibly begin with a truth and end with falsehood – thus the conditional quoted above could be false if it were day and I became silent. Philo's criterion of truth is what would now be called a truth-functional definition of "if ... then". In a second reference, Sextus says "According to him there are three ways in which a conditional may be true, and one in which it may be false."[26]
- *Meaning and truth*. The most important and striking difference between Megarian-Stoic logic and Aristotelian logic is that it concerns propositions, not terms, and is thus closer to modern propositional logic.[27] The Stoics distinguished between utterance (*phone*), which may be noise, speech (*lexis*), which is articulate but which may be meaningless, and discourse (*logos*), which is meaningful utterance. The most original part of their theory is the idea that what is expressed by a sentence, called a *lekton*, is something real. This corresponds to what is now called a *proposition*. Sextus says that according to the Stoics, three things are linked together, that which is signified, that which signifies, and the object. For example, what signifies is the word *Dion*, what is signified is what Greeks understand but barbarians do not, and the object is Dion himself.[28]

Logic in Asia

Logic in India

Formal logic began independently in ancient India and continued to develop through to early modern times, without any known influence from Greek logic.[29] Medhatithi Gautama (c. 6th century BCE) founded the *anviksiki* school of logic.[30] The *Mahabharata* (12.173.45), around the 5th century BCE, refers to the *anviksiki* and *tarka* schools of logic. Pāṇini (c. 5th century BCE) developed a form of logic (to which Boolean logic has some similarities) for his formulation of Sanskrit grammar. Logic is described by Chanakya (c. 350-283 BCE) in his *Arthashastra* as an independent field of inquiry *anviksiki*.[31]

Two of the six Indian schools of thought deal with logic: Nyaya and Vaisheshika. The Nyaya Sutras of Aksapada Gautama (c. 2nd century CE) constitute the core texts of the Nyaya school, one of the six orthodox schools of Hindu philosophy. This realist school developed a rigid five-member schema of inference involving an initial premise, a reason, an example, an application and a conclusion.[32] The idealist Buddhist philosophy became the chief opponent to the Naiyayikas. Nagarjuna (c. 150-250 CE), the founder of the Madhyamika ("Middle Way") developed an analysis known as the catuskoti (Sanskrit). This four-cornered argumentation systematically examined and rejected the affirmation of a proposition, its denial, the joint affirmation and denial, and finally, the rejection of its affirmation and denial. But it was with Dignaga (c 480-540 CE), who developed a formal syllogistic,[33] and his successor Dharmakirti that Buddhist logic reached its height. Their analysis centered on the definition of necessary logical entailment, "vyapti", also known as invariable concomitance or pervasion.[34] To this end a doctrine known as "apoha" or differentiation was developed.[35] This involved what might be called inclusion and exclusion of defining properties.

The difficulties involved in this enterprise, in part, stimulated the neo-scholastic school of Navya-Nyāya, which developed a formal analysis of inference in the sixteenth century. This later school began around eastern India and Bengal, and developed theories resembling modern logic, such as Gottlob Frege's "distinction between sense and reference of proper names" and his "definition of number," as well as the Navya-Nyaya theory of "restrictive conditions for universals" anticipating some of the developments in modern set theory.[36] Since 1824, Indian logic attracted the attention of many Western scholars, and has had an influence on important 19th-century logicians such as Charles Babbage, Augustus De Morgan, and particularly George Boole, as confirmed by his wife Mary Everest Boole who wrote in an "open letter to Dr Bose" titled "Indian Thought and Western Science in the Nineteenth Century" written in 1901[37][38]: "Think what must have been the effect of the intense Hinduizing of three such men as Babbage, De Morgan and George Boole on the mathematical atmosphere of 1830-1865"

Logic in China

In China, a contemporary of Confucius, Mozi, "Master Mo", is credited with founding the Mohist school, whose canons dealt with issues relating to valid inference and the conditions of correct conclusions. In particular, one of the schools that grew out of Mohism, the Logicians, are credited by some scholars for their early investigation of formal logic. Due to the harsh rule of Legalism in the subsequent Qin Dynasty, this line of investigation disappeared in China until the introduction of Indian philosophy by Buddhists.

Medieval logic

Logic in Islamic philosophy


A text by Avicenna, founder of Avicennian logic

The works of Al-Farabi, Avicenna, Al-Ghazali, Averroes and other Muslim logicians both criticized and developed Aristotelian logic and were important in communicating the ideas of the ancient world to the medieval West.[39] Al-Farabi (Alfarabi) (873–950) was an Aristotelian logician who discussed the topics of future contingents, the number and relation of the categories, the relation between logic and grammar, and non-Aristotelian forms of inference.[40] Al-Farabi also considered the theories of conditional syllogisms and analogical inference, which were part of the Stoic tradition of logic rather than the Aristotelian.[41]

Ibn Sina (Avicenna) (980–1037) was the founder of Avicennian logic, which replaced Aristotelian logic as the domininant system of logic in the Islamic world,[42] and also had an important influence on Western medieval writers such as Albertus Magnus.[43] Avicenna wrote on the hypothetical syllogism[44] and on the propositional calculus, which were both part of the Stoic logical tradition.[45] He developed an original theory of "temporally modalized" syllogistic[40] and made use of inductive logic, such as the methods of agreement, difference and concomitant variation which are critical to the scientific method.[44] One of Avicenna's ideas had a particularly important influence on Western logicians such as William of Ockham. Avicenna's word for a meaning or notion (*ma'na*), was translated by the scholastic logicians as the Latin *intentio*. In medieval logic and epistemology, this is a sign in the mind that naturally represents a thing.[46] This was crucial to the development of Ockham's conceptualism. A universal term (e.g. "man") does not signify a thing existing in reality, but rather a sign in the mind (*intentio in intellectu*) which represents many things in reality. Ockham cites Avicenna's commentary on *Metaphysics* V in support of this view.[47]

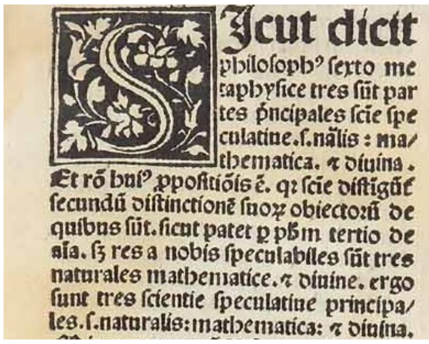
Fakhr al-Din al-Razi (b. 1149) criticised Aristotle's "first figure" and formulated an early system of inductive logic, foreshadowing the system of inductive logic developed by John Stuart Mill (1806–1873).[48] Al-Razi's work was seen by later Islamic scholars as marking a new direction for Islamic logic, towards a Post-Avicennian logic. This was further elaborated by his student Afdaladdîn al-Khûnajî (d. 1249), who developed a form of logic revolving around the subject matter of conceptions and assents. In response to this tradition, Nasir al-Din al-Tusi (1201–1274) began a tradition of Neo-Avicennian logic which remained faithful to Avicenna's work and existed as an alternative to the more dominant Post-Avicennian school over the following centuries.[49]

Systematic refutations of Greek logic were written by the Illuminationist school, founded by Shahab al-Din Suhrawardi (1155–1191), who developed the idea of "decisive necessity", which refers to the reduction of all modalities (necessity, possibility, contingency and impossibility) to the single mode of necessity.[50] Ibn al-Nafis (1213–1288) wrote a book on Avicennian logic, which was a commentary of Avicenna's *Al-Isharat* (*The Signs*) and *Al-Hidayah* (*The Guidance*).[51] Another systematic refutation of Greek logic was written by Ibn Taymiyyah (1263–1328), the *Ar-Radd 'ala al-Mantiqiyyin* (*Refutation of Greek Logicians*), where he argued against the usefulness, though not the validity, of the syllogism[52] and in favour of inductive reasoning.[48] Ibn Taymiyyah also argued against the certainty of syllogistic arguments and in favour of analogy. His argument is that concepts founded on induction are themselves not certain but only probable, and thus a syllogism based on such concepts is no more certain than an argument based on analogy. He further claimed that induction itself is founded on a process of analogy. His model of analogical reasoning was based on that of juridical arguments.[53][54] This model of analogy has been used in the recent work of John F. Sowa.[54]

The *Sharh al-takmil fi'l-mantiq* written by Muhammad ibn Fayd Allah ibn Muhammad Amin al-Sharwani in the 15th century is the last major Arabic work on logic that has been studied.[55] However, "thousands upon thousands of pages" on logic were written between the 14th and 19th centuries, though only a fraction of the texts written during this period have been studied by historians, hence little is known about the original work on Islamic logic produced during this later period.[49]

Logic in medieval Europe

"Medieval logic" (also known as "Scholastic logic") generally means the form of Aristotelian logic developed in medieval Europe throughout the period c 1200–1600.[56] For centuries after Stoic logic had been formulated, it was the dominant system of logic in the classical world. When the study of logic resumed after the Dark Ages, the main source was the work of the Christian philosopher Boethius, who was familiar with some of Aristotle's logic, but almost none of the work of the Stoics.[57] Until the twelfth century the only works of Aristotle available in the West were the *Categories*, *On Interpretation* and Boethius' translation of the Isagoge of Porphyry (a commentary on the Categories). These works were known as the "Old Logic" (*Logica Vetus* or *Ars Vetus*). An important work in this tradition was the *Logica Ingredientibus* of Peter Abelard (1079–1142). His direct influence was small,[58] but his influence through pupils such as John of Salisbury was great, and his method of applying rigorous logical analysis to theology shaped the way that theological criticism developed in the period that followed.[59]


Sicut dicit philosophus sexto metaphysice tres sunt partes principales scie speculative. s. nālis: mathematica. et divina. Et rō huius ppositiōis ē. qz scie distīgūt secundū distinctionē suorū obiectorū de quibus sūt. sicut patet p phm tertio de aīa. sz res a nobis speculabiles sūt tres naturales mathematice. et divine. ergo sunt tres scientie speculative principales. s. naturalis: mathematica: et divina.

Brito's questions on the *Old Logic*

By the early thirteenth century the remaining works of Aristotle's *Organon* (including the *Prior Analytics*, *Posterior Analytics* and the *Sophistical Refutations*) had been recovered in the West.[60] Logical work until then was mostly paraphrasis or commentary on the work of Aristotle.[61] The period from the middle of the thirteenth to the middle of the fourteenth century was one of significant developments in logic, particularly in three areas which were original, with little foundation in the Aristotelian tradition that came before. These were:[62]

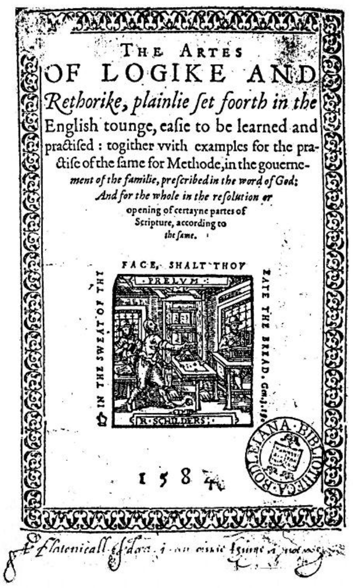
- The theory of supposition. Supposition theory deals with the way that predicates (e.g. 'man') range over a domain of individuals (e.g. all men).[63] In the proposition 'every man is an animal', does the term 'man' range over or 'supposit for' men existing in the present? Or does the range include past and future men? Can a term supposit for non-existing individuals? Some medievalists have argued that this idea was a precursor of modern first order logic.[64] "The theory of supposition with the associated theories of *copulatio* (sign-capacity of adjectival terms), *ampliatio* (widening of referential domain), and *distributio* constitute one of the most original achievements of Western medieval logic".[65]
- The theory of syncategoremata. Syncategoremata are terms which are necessary for logic, but which, unlike *categorematic* terms, do not signify on their own behalf, but 'co-signify' with other words. Examples of syncategoremata are 'and', 'not', 'every', 'if', and so on.
- The theory of consequences. A consequence is a hypothetical, conditional proposition: two propositions joined by the terms 'if ... then'. For example 'if a man runs, then God exists' (*Si homo currit, Deus est*).[66] A fully developed theory of consequences is given in Book III of William of Ockham's work Summa Logicae. There, Ockham distinguishes between 'material' and 'formal' consequences, which are roughly equivalent to the modern material implication and logical implication respectively. Similar accounts are given by Jean Buridan and Albert of Saxony.

The last great works in this tradition are the *Logic* of John Poinsot (1589–1644, known as John of St Thomas), the *Metaphysical Disputations* of Francisco Suarez (1548–1617), and the *Logica Demonstrativa* of Giovanni Girolamo Saccheri (1667–1733).

Traditional logic

The textbook tradition

Traditional logic generally means the textbook tradition that begins with Antoine Arnauld and Pierre Nicole's *Logic, or the Art of Thinking*, better known as the *Port-Royal Logic*.[67] Published in 1662, it was the most influential work on logic in England until the nineteenth century.[68] The book presents a loosely Cartesian doctrine (that the proposition is a combining of ideas rather than terms, for example) within a framework that is broadly derived from Aristotelian and medieval term logic. Between 1664 and 1700 there were eight editions, and the book had considerable influence after that.[68] The account of propositions that Locke gives in the *Essay* is essentially that of Port-Royal: "Verbal propositions, which are words, [are] the signs of our ideas, put together or separated in affirmative or negative sentences. So that proposition consists in the putting together or separating these signs, according as the things which they stand for agree or disagree." (Locke, *An Essay Concerning Human Understanding*, IV. 5. 6)



Dudley Fenner's *Art of Logic* (1584)

Another influential work was the *Novum Organum* by Francis Bacon, published in 1620. The title translates as "new instrument". This is a reference to Aristotle's work *Organon*. In this work, Bacon rejected the syllogistic method of Aristotle in favour of an alternative procedure "which by slow and faithful toil gathers information from things and brings it into understanding".[69] This method is known as inductive reasoning. The inductive method starts from empirical observation and proceeds to lower axioms or propositions. From the lower axioms more general ones can be derived (by induction). In finding the cause of a *phenomenal* *nature* such as heat, one must list all of the situations where heat is found. Then another list should be drawn up, listing situations that are similar to those of the first list except for the lack of heat. A third table lists situations where heat can vary. The *form nature*, or cause, of heat must be that which is common to all instances in the first table, is lacking from all instances of the second table and varies by degree in instances of the third table.

Other works in the textbook tradition include Isaac Watts' *Logick: Or, the Right Use of Reason* (1725), Richard Whately's *Logic* (1826), and John Stuart Mill's *A System of Logic* (1843). Although the latter was one of the last great works in the tradition, Mill's view that the foundations of logic lay in introspection[70] influenced the view that logic is best understood as a branch of psychology, an approach to the subject which dominated the next fifty years of its development, especially in Germany.[71]

Logic in Hegel's philosophy

Georg Wilhelm Friedrich Hegel

G.W.F. Hegel indicated the importance of logic to his philosophical system when he condensed his extensive *Science of Logic* into a shorter work published in 1817 as the first volume of his *Encyclopaedia of the Philosophical Sciences.* The "Shorter" or "Encyclopaedia" *Logic*, as it is often known, lays out a series of transitions which leads from the most empty and abstract of categories: Hegel begins with "Pure Being" and "Pure Nothing"—to the "Absolute"—the category which contains and resolves all the categories which preceded it. Despite the title, Hegel's *Logic* is not really a contribution to the science of valid inference. Rather than deriving conclusions about concepts through valid inference from premises, Hegel seeks to show that thinking about one concept compels thinking about another concept (one cannot, he argues, possess the concept of "Quality" without the concept of "Quantity"); and the compulsion here is not a matter of individual psychology, but arises almost organically from the content of the concepts themselves. His purpose is to show the rational structure of the "Absolute"—indeed of rationality itself. The method by which thought is driven from one concept to its contrary, and then to further concepts, is known as the Hegelian dialectic.

Although Hegel's *Logic* has had little impact on mainstream logical studies, its influence can be seen in Carl von Prantl's *Geschichte der Logik in Abendland* (1855–1867),[72] and in the work of the British Idealists—for example in F.H. Bradley's *Principles of Logic* (1883)—and in the economic, political and philosophical studies of Karl Marx and the various schools of Marxism.

Logic and psychology

Between the work of Mill and Frege stretched half a century during which logic was widely treated as a descriptive science, an empirical study of the structure of reasoning, and thus essentially as a branch of psychology.[73] The German psychologist Wilhelm Wundt, for example, discussed deriving "the logical from the psychological laws of thought", emphasizing that "psychological thinking is always the more comprehensive form of thinking."[74] This view was widespread among German philosophers of the period: Theodor Lipps described logic as "a specific discipline of psychology";[75] Christoph von Sigwart understood logical necessity as grounded in the individual's compulsion to think in a certain way;[76] and Benno Erdmann argued that "logical laws only hold within the limits of our thinking"[77] Such was the dominant view of logic in the years following Mill's work.[78] This psychological approach to logic was rejected by Gottlob Frege. It was also subjected to an extended and destructive critique by Edmund Husserl in the first volume of his *Logical Investigations* (1900), an assault which has been described as "overwhelming".[79] Husserl argued forcefully that grounding logic in psychological observations implied that all logical truths remained unproven, and that skepticism and relativism were unavoidable consequences.

Such criticisms did not immediately extirpate so-called "psychologism". For example, the American philosopher Josiah Royce, while acknowledging the force of Husserl's critique, remained "unable to doubt" that progress in psychology would be accompanied by progress in logic, and vice versa.[80]

Rise of modern logic

The period between the fourteenth century and the beginning of the nineteenth century had been largely one of decline and neglect, and is generally regarded as barren by historians of logic.[1] The revival of logic occurred in the mid-nineteenth century, at the beginning of a revolutionary period where the subject developed into a rigorous and formalistic discipline whose exemplar was the exact method of proof used in mathematics. The development of the modern so-called "symbolic" or "mathematical" logic during this period is the most significant in the 2,000-year history of logic, and is arguably one of the most important and remarkable events in human intellectual history.[2]

A number of features distinguish modern logic from the old Aristotelian or traditional logic, the most important of which are as follows:[81] Modern logic is fundamentally a *calculus* whose rules of operation are determined only by the *shape* and not by the *meaning* of the symbols it employs, as in mathematics. Many logicians were impressed by the "success" of mathematics, in that there had been no prolonged dispute about any truly mathematical result. C.S. Peirce noted[82] that even though a mistake in the evaluation of a definite integral by Laplace led to an error concerning the moon's orbit that persisted for nearly 50 years, the mistake, once spotted, was corrected without any serious dispute. Peirce contrasted this with the disputation and uncertainty surrounding traditional logic, and especially reasoning in metaphysics. He argued that a truly "exact" logic would depend upon mathematical, i.e., "diagrammatic" or "iconic" thought. "Those who follow such methods will ... escape all error except such as will be speedily corrected after it is once suspected". Modern logic is also "constructive" rather than "abstractive"; i.e., rather than abstracting and formalising theorems derived from ordinary language (or from psychological intuitions about validity), it constructs theorems by formal methods, then looks for an interpretation in ordinary language. It is entirely symbolic, meaning that even the logical constants (which the medieval logicians called "syncategoremata") and the categoric terms are expressed in symbols.

Periods of modern logic

The development of modern logic falls into roughly five periods:[83]

- The **embryonic period** from Leibniz to 1847, when the notion of a logical calculus was discussed and developed, particularly by Leibniz, but no schools were formed, and isolated periodic attempts were abandoned or went unnoticed.
- The **algebraic period** from Boole's Analysis to Schröder's *Vorlesungen*. In this period there were more practitioners, and a greater continuity of development.
- The **logicist period** from the Begriffsschrift of Frege to the *Principia Mathematica* of Russell and Whitehead. This was dominated by the "logicist school", whose aim was to incorporate the logic of all mathematical and scientific discourse in a single unified system, and which, taking as a fundamental principle that all mathematical truths are logical, did not accept any non-logical terminology. The major logicists were Frege, Russell, and the early Wittgenstein.[84] It culminates with the *Principia*, an important work which includes a thorough examination and attempted solution of the antinomies which had been an obstacle to earlier progress.
- The **metamathematical period** from 1910 to the 1930s, which saw the development of metalogic, in the finitist system of Hilbert, and the non-finitist system of Löwenheim and Skolem, the combination of logic and metalogic in the work of Gödel and Tarski. Gödel's incompleteness theorem of 1931 was one of the greatest achievements in the history of logic. Later in the 1930s Gödel developed the notion of set-theoretic constructibility.
- The **period after World War II**, when mathematical logic branched into four inter-related but separate areas of research: model theory, proof theory, computability theory, and set theory, and its ideas and methods began to

influence philosophy.

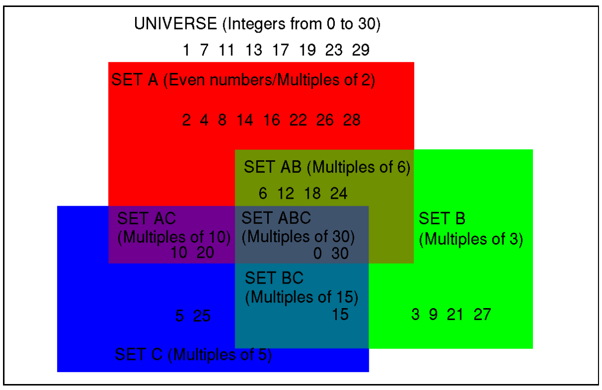
Embryonic period

Life of Raymond Lull. 14th-century manuscript.

The idea that inference could be represented by a purely mechanical process is found as early as Raymond Llull, who proposed a (somewhat eccentric) method of drawing conclusions by a system of concentric rings. The work of logicians such as the Oxford Calculators[85] led to a method of using letters instead of writing out logical calculations (*calculationes*) in words, a method used, for instance, in the *Logica magna* of Paul of Venice. Three hundred years after Llull, the English philosopher and logician Thomas Hobbes suggested that all logic and reasoning could be reduced to the mathematical operations of addition and subtraction.[86] The same idea is found in the work of Leibniz, who had read both Llull and Hobbes, and who argued that logic can be represented through a combinatorial process or calculus. But, like Llull and Hobbes, he failed to develop a detailed or comprehensive system, and his work on this topic was not published until long after his death. Leibniz says that ordinary languages are subject to "countless ambiguities" and are unsuited for a calculus, whose task is to expose mistakes in inference arising from the forms and structures of words;[87] hence, he proposed to identify an alphabet of human thought comprising fundamental concepts which could be composed to express complex ideas,[88] and create a *calculus ratiocinator* which would make reasoning "as tangible as those of the Mathematicians, so that we can find our error at a glance, and when there are disputes among persons, we can simply say: Let us calculate."[89]

Gergonne (1816) said that reasoning does not have to be about objects about which we have perfectly clear ideas, since algebraic operations can be carried out without our having any idea of the meaning of the symbols involved.[90] Bolzano anticipated a fundamental idea of modern proof theory when he defined logical consequence or "deducibility" in terms of variables: a set of propositions *n*, *o*, *p* ... are *deducible* from propositions *a*, *b*, *c* ... in respect of the variables *i*, *j*, ... if any substitution for *i*, *j* that have the effect of making *a*, *b*, *c* ... true, simultaneously make the propositions *n*, *o*, *p* ... also.[91] This is now known as semantic validity.

Algebraic period



Boolean multiples

Modern logic begins with the so-called "algebraic school", originating with Boole and including Peirce, Jevons, Schröder and Venn.[92] Their objective was to develop a calculus to formalise reasoning in the area of classes, propositions and probabilities. The school begins with Boole's seminal work *Mathematical Analysis of Logic* which appeared in 1847, although De Morgan (1847) is its immediate precursor.[93] The fundamental idea of Boole's system is that algebraic formulae can be used to express logical relations. This idea occurred to Boole in his teenage years, working as an usher in a private school in Lincoln, Lincolnshire.[94] For example, let x and y stand for classes let the symbol = signify that the classes have the same members, xy stand for the class containing all and only the members of x and y and so on. Boole calls these *elective symbols*, i.e. symbols which select certain objects for consideration.[95] An expression in which elective symbols are used is called an *elective function*, and an equation of which the members are elective functions, is an *elective equation*.[96] The theory of elective functions and their "development" is essentially the modern idea of truth-functions and their expression in disjunctive normal form.[95]

Boole's system admits of two interpretations, in class logic, and propositional logic. Boole distinguished between "primary propositions" which are the subject of syllogistic theory, and "secondary propositions", which are the subject of propositional logic, and showed how under different "interpretations" the same algebraic system could represent both. An example of a primary proposition is "All inhabitants are either Europeans or Asiatics." An example of a secondary proposition is "Either all inhabitants are Europeans or they are all Asiatics."[97] These are easily distinguished in modern propositional calculus, where it is also possible to show that the first follows from the second, but it is a significant disadvantage that there is no way of representing this in the Boolean system.[98]

In his *Symbolic Logic* (1881), John Venn used diagrams of overlapping areas to express Boolean relations between classes or truth-conditions of propositions. In 1869 Jevons realised that Boole's methods could be mechanised, and constructed a "logical machine" which he showed to the Royal Society the following year.[95] In 1885 Allan Marquand proposed an electrical version of the machine that is still extant (picture at the Firestone Library [99]).

The defects in Boole's system (such as the use of the letter *v* for existential propositions) were all remedied by his followers. Jevons published *Pure Logic, or the Logic of Quality apart from Quantity* in 1864, where he suggested a symbol to signify exclusive or, which allowed Boole's system to be greatly simplified.[100] This was usefully exploited by Schröder when he set out theorems in parallel columns in his *Vorlesungen* (1890–1905). Peirce (1880) showed how all the Boolean elective functions could be expressed by the use of a single primitive binary operation, "neither ... nor ..." and equally well "not both ... and ...",[101] however, like many of Peirce's innovations, this remained unknown or unnoticed until Sheffer rediscovered it in 1913.[102] Boole's early work also lacks the idea of the logical sum which originates in Peirce (1867), Schröder (1877) and Jevons (1890),[103] and the concept of inclusion, first suggested by Gergonne (1816) and clearly articulated by Peirce (1870).

The success of Boole's algebraic system suggested that all logic must be capable of algebraic representation, and there were attempts to express a logic of relations in such form, of which the most ambitious was Schröder's monumental *Vorlesungen über die Algebra der Logik* ("Lectures on the Algebra of Logic", vol iii 1895), although the original idea was again anticipated by Peirce.[104]

Logicist period

Frege's "Concept Script"

After Boole, the next great advances were made by the German mathematician Gottlob Frege. Frege's objective was the program of Logicism, i.e. demonstrating that arithmetic is identical with logic.[105] Frege went much further than any of his predecessors in his rigorous and formal approach to logic, and his calculus or Begriffsschrift is important.[105] Frege also tried to show that the concept of number can be defined by purely logical means, so that (if he was right) logic includes arithmetic and all branches of mathematics that are reducible to arithmetic. He was not the first writer to suggest this. In his pioneering work *Die Grundlagen der Arithmetik* (The Foundations of Arithmetic), sections 15–17, he acknowledges the efforts of Leibniz, J.S. Mill as well as Jevons, citing the latter's claim that "algebra is a highly developed logic, and number but logical discrimination."[106]

Frege's first work, the *Begriffsschrift* ("concept script") is a rigorously axiomatised system of propositional logic, relying on just two connectives (negational and conditional), two rules of inference (*modus ponens* and substitution), and six axioms. Frege referred to the "completeness" of this system, but was unable to prove this.[107] The most significant innovation, however, was his explanation of the quantifier in terms of mathematical functions. Traditional logic regards the sentence "Caesar is a man" as of fundamentally the same form as "all men are mortal." Sentences with a proper name subject were regarded as universal in character, interpretable as "every Caesar is a man".[108] Frege argued that the quantifier expression "all men" does not have the same logical or semantic form as "all men", and that the universal proposition "every A is B" is a complex proposition involving two *functions*, namely ' – is A' and ' – is B' such that whatever satisfies the first, also satisfies the second. In modern notation, this would be expressed as

(x) Ax -> Bx

In English, "for all x, if Ax then Bx". Thus only singular propositions are of subject-predicate form, and they are irreducibly singular, i.e. not reducible to a general proposition. Universal and particular propositions, by contrast, are not of simple subject-predicate form at all. If "all mammals" were the logical subject of the sentence "all mammals are land-dwellers", then to negate the whole sentence we would have to negate the predicate to give "all mammals are *not* land-dwellers". But this is not the case.[109] This functional analysis of ordinary-language sentences later had a great impact on philosophy and linguistics.

This means that in Frege's calculus, Boole's "primary" propositions can be represented in a different way from "secondary" propositions. "All inhabitants are either Europeans or Asiatics" is

(x) [I(x) -> (E(x) v A(x))]

whereas "All the inhabitants are Europeans or all the inhabitants are Asiatics" is

(x) (I(x) -> E(x)) v (x) (I(x) -> A(x))

As Frege remarked in a critique of Boole's calculus:

> "The real difference is that I avoid [the Boolean] division into two parts ... and give a homogeneous presentation of the lot. In Boole the two parts run alongside one another, so that one is like the mirror image of the other, but for that very reason stands in no organic relation to it'[110]

As well as providing a unified and comprehensive system of logic, Frege's calculus also resolved the ancient problem of multiple generality. The ambiguity of "every girl kissed a boy" is difficult to express in traditional logic, but Frege's logic captures this through the different scope of the quantifiers. Thus

(x) [girl(x) -> E(y) (boy(y) & kissed(x,y))]

means that to every girl there corresponds some boy (any one will do) who the girl kissed. But

E(x) [boy(x) & (y) (girl(y) -> kissed(y,x))]

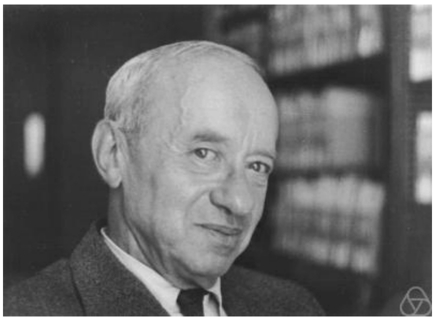
means that there is some particular boy whom every girl kissed. Without this device, the project of logicism would have been doubtful or impossible. Using it, Frege provided a definition of the ancestral relation, of the many-to-one relation, and of mathematical induction.[111]

This period overlaps with the work of the so-called "mathematical school", which included Dedekind, Pasch, Peano, Hilbert, Zermelo, Huntington, Veblen and Heyting. Their objective was the axiomatisation of branches of mathematics like geometry, arithmetic, analysis and set theory.

The logicist project received a near-fatal setback with the discovery of a paradox in 1901 by Bertrand Russell. This proved that the Frege's naive set theory led to a contradiction. Frege's theory is that for any formal criterion, there is a set of all objects that meet the criterion. Russell showed that a set containing exactly the sets that are not members of themselves would contradict its own definition (if it is not a member of itself, it is a member of itself, and if it is a member of itself, it is not).[112] This contradiction is now known as Russell's paradox. One important method of resolving this paradox was proposed by Ernst Zermelo.[113] Zermelo set theory was the first axiomatic set theory. It was developed into the now-canonical Zermelo–Fraenkel set theory (ZF).

The monumental Principia Mathematica, a three-volume work on the foundations of mathematics, written by Russell and Alfred North Whitehead and published 1910–13 also included an attempt to resolve the paradox, by means of an elaborate system of types: a set of elements is of a different type than is each of its elements (set is not the element; one element is not the set) and one cannot speak of the "set of all sets". The *Principia* was an attempt to derive all mathematical truths from a well-defined set of axioms and inference rules in symbolic logic.

Metamathematical period


Alfred Tarski

The names of Gödel and Tarski dominate the 1930s,[114] a crucial period in the development of metamathematics – the study of mathematics using mathematical methods to produce metatheories, or mathematical theories about other mathematical theories. Early investigations into metamathematics had been driven by Hilbert's program. which sought to resolve the ongoing crisis in the foundations of mathematics by grounding all of mathematics to a finite set of axioms, proving its consistency by "finitistic" means and providing a procedure which would decide the truth or falsity of any mathematical statement. Work on metamathematics culminated in the work of Gödel, who in 1929 showed that a given first-order sentence is deducible if and only if it is logically valid – i.e. it is true in every structure for its language. This is known as Gödel's completeness theorem. A year later, he proved two important theorems, which showed Hibert's program to be unattainable in its original form. The first is that no consistent system of axioms whose theorems can be listed by an effective procedure such as an algorithm or computer program is capable of proving all facts about the natural numbers. For any such system, there will always be statements about the natural numbers that are true, but that are unprovable within the system. The second is that if such a system is also capable of proving certain basic facts about the natural numbers, then the system cannot prove the consistency of the system itself. These two results are known as Gödel's incompleteness theorems, or simply *Gödel's Theorem*. Later in the decade, Gödel developed the concept of set-theoretic constructibility, as part of his proof that the axiom of choice and the continuum hypothesis are consistent with Zermelo–Fraenkel set theory.

In proof theory, Gerhard Gentzen developed natural deduction and the sequent calculus. The former attempts to model logical reasoning as it 'naturally' occurs in practice and is most easily applied to intuitionistic logic, while the latter was devised to clarify the derivation of logical proofs in any formal system. Since Gentzen's work, natural deduction and sequent calculi have been widely applied in the fields of proof theory, mathematical logic and computer science. Gentzen also proved normalization and cut-elimination theorems for intuitionistic and classical logic which could be used to reduce logical proofs to a normal form.[115][116]

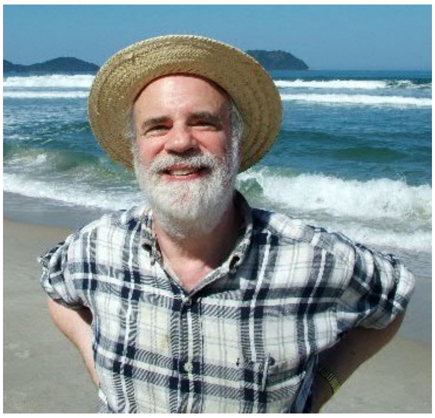
Alfred Tarski, a pupil of Łukasiewicz, is best known for his definition of truth and logical consequence, and the semantic concept of logical satisfaction. In 1933, he published (in Polish) *The concept of truth in formalized languages*, in which he proposed his semantic theory of truth: a sentence such as "snow is white" is true if and only if snow is white. Tarski's theory separated the metalanguage, which makes the statement about truth, from the object language, which contains the sentence whose truth is being asserted, and gave a correspondence (the T-schema) between phrases in the object language and elements of an interpretation. Tarski's approach to the difficult idea of explaining truth has been enduringly influential in logic and philosophy, especially in the development of model theory.[117] Tarski also produced important work on the methodology of deductive systems, and on fundamental principles such as completeness, decidability, consistency and definability. According to Anita Feferman, Tarski "changed the face of logic in the twentieth century".[118]

Alonzo Church and Alan Turing proposed formal models of computability, giving independent negative solutions to Hilbert's *Entscheidungsproblem* in 1936 and 1937, respectively. The *Entscheidungsproblem* asked for a procedure that, given any formal mathematical statement, would algorithmically determine whether the statement is true. Church and Turing proved there is no such procedure; Turing's paper introduced the halting problem as a key example of a mathematical problem without an algorithmic solution.

Church's system for computation developed into the modern λ-calculus, while the Turing machine became a standard model for a general-purpose computing device. It was soon shown that many other proposed models of computation were equivalent in power to those proposed by Church and Turing. These results led to the Church–Turing thesis that any deterministic algorithm that can be carried out by a human can be carried out by a Turing machine. Church proved additional undecidability results, showing that both Peano arithmetic and first-order logic are undecidable. Later work by Emil Post and Stephen Cole Kleene in the 1940s extended the scope of computability theory and introduced the concept of degrees of unsolvability.

The results of the first few decades of the twentieth century also had an impact upon analytic philosophy and philosophical logic, particularly from the 1950s onwards, in subjects such as modal logic, temporal logic, deontic logic, and relevance logic.

Logic after WWII



Saul Kripke

After World War II, mathematical logic branched into four inter-related but separate areas of research: model theory, proof theory, computability theory, and set theory.[119]

In set theory, the method of forcing revolutionized the field by providing a robust method for constructing models and obtaining independence results. Paul Cohen introduced this method in 1962 to prove the independence of the continuum hypothesis and the axiom of choice from Zermelo–Fraenkel set theory.[120] His technique, which was simplified and extended soon after its introduction, has since been applied to many other problems in all areas of mathematical logic.

Computability theory had its roots in the work of Turing, Church, Kleene, and Post in the 1930s and 40s. It developed into a study of abstract computability, which became known as recursion theory.[121] The priority method, discovered independently by Albert Muchnik and Richard Friedberg in the 1950s, led to major advances in the understanding of the degrees of unsolvability and related structures. Research into higher-order computability theory demonstrated its connections to set theory. The fields of constructive analysis and computable analysis were developed to study the effective content of classical mathematical theorems; these in turn inspired the program of reverse mathematics. A separate branch of computability theory, computational complexity theory, was also characterized in logical terms as a result of investigations into descriptive complexity.

Abraham Robinson

Model theory applies the methods of mathematical logic to study models of particular mathematical theories. Alfred Tarski published much pioneering work in the field, which is named after a series of papers he published under the title *Contributions to the theory of models*. In the 1960s, Abraham Robinson used model-theoretic techniques to develop calculus and analysis based on infinitesimals, a problem that first had been proposed by Leibniz.

In proof theory, the relationship between classical mathematics and intuitionistic mathematics was clarified via tools such as the realizability method invented by Georg Kreisel and Gödel's *Dialectica* interpretation. This work inspired the contemporary area of proof mining. The Curry-Howard correspondence emerged as a deep analogy between logic and computation, including a correspondence between systems of natural deduction and typed lambda calculi used in computer science. As a result, research into this class of formal systems began to address both logical and computational aspects; this area of research came to be known as modern type theory. Advances were also made in ordinal analysis and the study of independence results in arithmetic such as the Paris–Harrington theorem.

This was also a period, particularly in the 1950s and afterwards, when the ideas of mathematical logic begin to influence philosophical thinking. For example, tense logic is a formalised system for representing, and reasoning about, propositions qualified in terms of time. The philosopher Arthur Prior played a significant role in its development in the 1960s. Modal logics extend the scope of formal logic to include the elements of modality (for example, possibility and necessity). The ideas of Saul Kripke, particularly about possible worlds, and the formal system now called Kripke semantics have had a profound impact on analytic philosophy.[122] His best known and most influential work is *Naming and Necessity* (1980).[123] Deontic logics are closely related to modal logics: they attempt to capture the logical features of obligation, permission and related concepts. Ernst Mally, a pupil of Alexius Meinong, was the first to propose a formal deontic system in his *Grundgesetze des Sollens*, based on the syntax of Whitehead's and Russell's propositional calculus. Another logical system founded after World War II was fuzzy logic by Iranian mathematician Lotfi Asker Zadeh in 1965.

Notes

[1] Oxford Companion p. 498; Bochenski, Part I Introduction, *passim*

[2] Oxford Companion p. 500

[3] Kneale, p. 2

[4] Kneale p. 3

[5] H. F. J. Horstmanshoff, Marten Stol, Cornelis Tilburg (2004), *Magic and Rationality in Ancient Near Eastern and Graeco-Roman Medicine*, p. 99, Brill Publishers, ISBN 90-04-13666-5.

[6] D. Brown (2000), *Mesopotamian Planetary Astronomy-Astrology* , Styx Publications, ISBN 90-5693-036-2.

[7] Heath, *Mathematics in Aristotle*, cited in Kneale, p. 5

[8] Kneale p. 15

[9] Kneale, p. 16

[10] Kneale p. 17

[11] "forming an opinion is talking, and opinion is speech that is held not with someone else or aloud but in silence with oneself" *Theaetetus* 189E–190A

[12] Kneale p. 20. For example, the proof given in the *Meno* that the square on the diagonal is double the area of the original square presumably involves the forms of the square and the triangle, and the necessary relation between them

[13] Kneale p. 21

[14] Zalta, Edward N. " Aristotle's Logic (http://plato.stanford.edu/entries/aristotle-logic/#Def)". Stanford University, 18 March 2000. Retrieved 13 March 2010.

[15] See e.g. Aristotle's logic (http://plato.stanford.edu/entries/aristotle-logic/), Stanford Encyclopedia of Philosophy

[16] Bochenski p. 63

[17] "Throughout later antiquity two great schools of logic were distinguished, the Peripatetic which was derived from Aristotle, and the Stoic which was developed by Chrysippus from the teachings of the Megarians" – Kneale p. 113

[18] *Oxford Companion*, article "Chrysippus", p. 134

[19] (http://plato.stanford.edu/entries/logic-ancient/) Stanford Encyclopedia of Philosophy: Susanne Bobzien, *Ancient Logic*

[20] K. Huelser, Die Fragmente zur Dialektik der Stoiker, 4 vols, Stuttgart 1986-7

[21] Kneale 117–158

[22] *Metaphysics* Eta 3, 1046b 29

[23] Boethius, *Commentary on the Perihermenias*, Meiser p. 234

[24] Epictetus, *Dissertationes* ed. Schenkel ii. 19. I.

[25] Alexander p. 177

[26] Sextus, *Adv. Math.* Bk viii, Section 113

[27] See e.g. Lukasiewicz p. 21

[28] Sextus Bk viii., Sections 11, 12

[29] Bochenski p. 446

[30] S. C. Vidyabhusana (1971). *A History of Indian Logic: Ancient, Mediaeval, and Modern Schools*.

[31] R. P. Kangle (1986). *The Kautiliya Arthashastra* (1.2.11). Motilal Banarsidass.

[32] Bochenski p. 417 and *passim*

[33] Bochenski pp. 431–7

[34] Bochenski p. 438

[35] Bochenksi p. 441

[36] Kisor Kumar Chakrabarti (June 1976). "Some Comparisons Between Frege's Logic and Navya-Nyaya Logic". *Philosophy and Phenomenological Research* (International Phenomenological Society) **36** (4): 554–563. doi:10.2307/2106873. JSTOR 2106873. "This paper consists of three parts. The first part deals with Frege's distinction between sense and reference of proper names and a similar distinction in Navya-Nyaya logic. In the second part we have compared Frege's definition of number to the Navya-Nyaya definition of number. In the third part we have shown how the study of the so-called 'restrictive conditions for universals' in Navya-Nyaya logic anticipated some of the developments of modern set theory."
[37] Boole, Mary Everest "Collected Works" eds E M Cobham and E S Dummer London, Daniel 1931. Letter also published in the Ceylon National Review in 1909, and published as a separate pamphlet "The Psychologic Aspect of Imperialism" in 1911.
[38] Jonardon Ganeri (2001). *Indian logic: a reader*. Routledge. p. vii. ISBN 0-7007-1306-9
[39] See e.g. Routledge Encyclopedia of Philosophy Online Version 2.0 (http://www.rep.routledge.com/article/H057), article 'Islamic philosophy'
[40] History of logic: Arabic logic (http://www.britannica.com/ebc/article-65928), *Encyclopædia Britannica*.
[41] Feldman, Seymour (1964-11-26). "Rescher on Arabic Logic". *The Journal of Philosophy* (Journal of Philosophy, Inc.) **61** (22): 724–734. doi:10.2307/2023632. ISSN 0022-362X. JSTOR 2023632. [726]. Long, A. A.; D. N. Sedley (1987). *The Hellenistic Philosophers. Vol 1: Translations of the principal sources with philosophical commentary*. Cambridge: Cambridge University Press. ISBN 0-521-27556-3.
[42] Dag Nikolaus Hasse (September 19, 2008). "Influence of Arabic and Islamic Philosophy on the Latin West" (http://plato.stanford.edu/entries/arabic-islamic-influence/). Stanford Encyclopedia of Philosophy. . Retrieved 2009-10-13.
[43] Richard F. Washell (1973), "Logic, Language, and Albert the Great", *Journal of the History of Ideas* **34** (3), pp. 445–450 [445].
[44] Goodman, Lenn Evan (2003), *Islamic Humanism*, p. 155, Oxford University Press, ISBN 0-19-513580-6.
[45] Goodman, Lenn Evan (1992); *Avicenna*, p. 188, Routledge, ISBN 0-415-01929-X.
[46] Kneale p. 229
[47] Kneale: p. 266; Ockham: Summa Logicae i. 14; Avicenna: *Avicennae Opera* Venice 1508 f87rb
[48] Muhammad Iqbal, *The Reconstruction of Religious Thought in Islam*, "The Spirit of Muslim Culture" (cf. (http://www.allamaiqbal.com/works/prose/english/reconstruction) and (http://www.witness-pioneer.org/vil/Books/MI_RRTI/chapter_05.htm))
[49] Tony Street (July 23, 2008). "Arabic and Islamic Philosophy of Language and Logic" (http://plato.stanford.edu/entries/arabic-islamic-language). Stanford Encyclopedia of Philosophy. . Retrieved 2008-12-05.
[50] Dr. Lotfollah Nabavi, Sohrevardi's Theory of Decisive Necessity and kripke's QSS System (http://public.ut.ac.ir/html/fac/lit/articles.html), *Journal of Faculty of Literature and Human Sciences*.
[51] Dr. Abu Shadi Al-Roubi (1982), "Ibn Al-Nafis as a philosopher", *Symposium on Ibn al-Nafis*, Second International Conference on Islamic Medicine: Islamic Medical Organization, Kuwait (cf. Ibn al-Nafis As a Philosopher (http://www.islamset.com/isc/nafis/drroubi.html), *Encyclopedia of Islamic World*).
[52] See pp. 253–254 of Street, Tony (2005). "Logic". In Peter Adamson and Richard C. Taylor (edd.). *The Cambridge Companion to Arabic Philosophy*. Cambridge University Press. pp. 247–265. ISBN 978-0-521-52069-0.
[53] Ruth Mas (1998). "Qiyas: A Study in Islamic Logic" (http://www.colorado.edu/ReligiousStudies/faculty/mas/LOGIC.pdf). *Folia Orientalia* **34**: 113–128. ISSN 0015-5675. .
[54] John F. Sowa; Arun K. Majumdar (2003). "Analogical reasoning" (http://www.jfsowa.com/pubs/analog.htm). *Conceptual Structures for Knowledge Creation and Communication, Proceedings of ICCS 2003*. Berlin: Springer-Verlag. ., pp. 16-36
[55] Nicholas Rescher and Arnold vander Nat, "The Arabic Theory of Temporal Modal Syllogistic", in George Fadlo Hourani (1975), *Essays on Islamic Philosophy and Science*, pp. 189–221, State University of New York Press, ISBN 0-87395-224-3.
[56] Boehner p. xiv
[57] Kneale p. 198
[58] Stephen Dumont, article "Peter Abelard" in Gracia and Noone p. 492
[59] Kneale, pp. 202–3
[60] See e.g. Kneale p. 225
[61] Boehner p. 1
[62] Boehner pp. 19–76
[63] Boehner p. 29
[64] Boehner p. 30
[65] Ebbesen 1981
[66] Boehner pp. 54–5
[67] *Oxford Companion* p. 504, article "Traditional logic"
[68] Buroker xxiii
[69] Farrington, 1964, 89
[70] N. Abbagnano, "Psychologism" in P. Edwards (ed) *The Encyclopaedia of Philosophy*, MacMillan, 1967
[71] Of the German literature in this period, Robert Adamson wrote "*Logics* swarm as bees in springtime..."; Robert Adamson, *A Short History of Logic*, Wm. Blackwood & Sons, 1911, page 242
[72] Carl von Prantl (1855-1867), *Geschichte von Logik in Abendland*, Leipsig: S. Hirzl, anastatically reprinted in 1997, Hildesheim: Georg Olds.
[73] See e.g. Psychologism (http://plato.stanford.edu/entries/psychologism), Stanford Encyclopedia of Philosophy

[74] Wilhelm Wundt, *Logik* (1880–1883); quoted in Edmund Husserl, *Logical Investigations,* translated J.N. Findlay, Routledge, 2008, Volume 1, pp. 115–116.
[75] Theodor Lipps, *Grundzüge der Logik* (1893); quoted in Edmund Husserl, *Logical Investigations,* translated J.N. Findlay, Routledge, 2008, Volume 1, p. 40
[76] Christoph von Sigwart, *Logik* (1873–78); quoted in Edmund Husserl, *Logical Investigations,* translated J.N. Findlay, Routledge, 2008, Volume 1, p. 51
[77] Benno Erdmann, *Logik* (1892); quoted in Edmund Husserl, *Logical Investigations,* translated J.N. Findlay, Routledge, 2008, Volume 1, p. 96
[78] Dermot Moran, "Introduction"; Edmund Husserl, *Logical Investigations,* translated J.N. Findlay, Routledge, 2008, Volume 1, p. xxi
[79] Michael Dummett, "Preface"; Edmund Husserl, *Logical Investigations,* translated J.N. Findlay, Routledge, 2008, Volume 1, p. xvii
[80] Josiah Royce, "Recent Logical Enquiries and their Psychological Bearings" (1902) in John J. McDermott (ed) *The Basic Writings of Josiah Royce* Volume 2, Fordham University Press, 2005, p. 661
[81] Bochenski, p. 266
[82] Peirce 1896
[83] See Bochenski p. 269
[84] *Oxford Companion* p. 499
[85] Edith Sylla (1999), "Oxford Calculators", in *The Cambridge Dictionary of Philosophy*, Cambridge, Cambridgeshire: Cambridge.
[86] El. philos. sect. I de corp 1.1.2.
[87] Bochenski p. 274
[88] Rutherford, Donald, 1995, "Philosophy and language" in Jolley, N., ed., *The Cambridge Companion to Leibniz*. Cambridge Univ. Press.
[89] Wiener, Philip, 1951. *Leibniz: Selections*. Scribner.
[90] *Essai de dialectique rationelle*, 211n, quoted in Bochenski p. 277.
[91] *Wissenschaftslehre II* 198ff, quoted in Bochenski 280; see *Oxford 'Companion* p. 498.
[92] See e.g. Bochenski p. 296 and *passim*
[93] Before publishing, he wrote to De Morgan, who was just finishing his work *Formal Logic*. De Morgan suggested they should publish first, and thus the two books appeared at the same time, possibly even reaching the bookshops on the same day. cf. Kneale p. 404
[94] Kneale p. 404
[95] Kneale p. 407
[96] Boole (1847) p. 16
[97] Boole 1847 pp. 58–9
[98] Beaney p. 11
[99] http://finelib.princeton.edu/instruction/wri172_demonstration.php
[100] Kneale p. 422
[101] Peirce, "A Boolean Algebra with One Constant", 1880 MS, *Collected Papers* v. 4, paragraphs 12–20, reprinted *Writings* v. 4, pp. 218-21. Google Preview (http://books.google.com/books?id=E7ZUnx3FqrcC&q=378+Winter).
[102] *Trans. Amer. Math. Soc., xiv (1913)*, pp. 481–8. This is now known as the Sheffer stroke
[103] Bochenski 296
[104] See CP III
[105] Kneale p. 435
[106] Jevons, *The Principles of Science*, London 1879, p. 156, quoted in *Grundlagen* 15
[107] Beaney p. 10 – the completeness of Frege's system was eventually proved by Jan Łukasiewicz in 1934
[108] See for example the argument by the medieval logician William of Ockham that singular propositions are universal, in Summa Logicae III. 8 (??)
[109] "On concept and object" p. 198; Geach p. 48
[110] BLC p. 14, quoted in Beaney p. 12
[111] See e.g. The Internet Encyclopedia of Philosophy (http://www.utm.edu/research/iep/f/frege.htm), article "Frege"
[112] See e.g. Potter 2004
[113] Zermelo 1908
[114] Feferman 1999 p. 1
[115] Girard, Jean-Yves; Paul Taylor, Yves Lafont (1990) [1989]. *Proofs and Types* (http://www.paultaylor.eu/stable/Proofs+Types.html). Cambridge University Press (Cambridge Tracts in Theoretical Computer Science, 7). ISBN 0-521-37181-3. .
[116] Alex Sakharov, " Cut Elimination Theorem (http://mathworld.wolfram.com/CutEliminationTheorem.html)" from MathWorld.
[117] Feferman and Feferman 2004, p. 122, discussing "The Impact of Tarski's Theory of Truth".
[118] Feferman 1999, p. 1
[119] See e.g. Barwise, *Handbook of Mathematical Logic*
[120] The Independence of the Continuum Hypothesis, II (http://links.jstor.org/sici?sici=0027-8424(19640115)51:1<105:TIOTCH>2.0.CO;2-U) Paul J. Cohen Proceedings of the National Academy of Sciences of the United States of America, Vol. 51, No. 1. (Jan. 15, 1964), pp. 105-110.
[121] Many of the foundational papers are collected in *The Undecidable* (1965) edited by Martin Davis

[122] Jerry Fodor, " Water's water everywhere (http://www.lrb.co.uk/v26/n20/jerry-fodor/waters-water-everywhere)", *London Review of Books*, 21 October 2004

[123] See *Philosophical Analysis in the Twentieth Century: Volume 2: The Age of Meaning*, Scott Soames: "*Naming and Necessity* is among the most important works ever, ranking with the classical work of Frege in the late nineteenth century, and of Russell, Tarski and Wittgenstein in the first half of the twentieth century". Cited in Byrne, Alex and Hall, Ned. 2004. 'Necessary Truths'. *Boston Review* October/November 2004

References

- Alexander of Aphrodisias, *In Aristotelis An. Pr. Lib. I Commentarium*, ed. Wallies, C.I.A.G.
- Avicenna, *Avicennae Opera* Venice 1508.
- Barwise, Jon, ed. (1982), *Handbook of Mathematical Logic*, Studies in Logic and the Foundations of Mathematics, North Holland, ISBN 978-0-444-86388-1 .
- Beaney, Michael, *The Frege Reader*, London: Blackwell 1997.
- Bochenski, I.M., *A History of Formal Logic*, Notre Dame press, 1961.
- Philotheus Boehner, *Medieval Logic*, Manchester 1950.
- Boethius *Commentary on the Perihermenias*, Secunda Editio, ed. Meiser.
- Bolzano, Bernard *Wissenschaftslehre*, 4 Bde Neudr., 2. verb, A. hrsg. W. Schultz, Leipzig I-II 1929, III 1930, IV 1931 (trans. as *Theory of science, attempt at a detailed and in the main novel exposition of logic with constant attention to earlier authors.* (Edited and translated by Rolf George University of California Press, Berkeley and Los Angeles 1972).
- Bolzano, Bernard *Theory of science* (Edited, with an introduction, by Jan Berg. Translated from the German by Burnham Terrell – D. Reidel Publishing Company, Dordrecht and Boston 1973).
- Boole, George (1847) *The Mathematical Analysis of Logic* (Cambridge and London); repr. in *Studies in Logic and Probability*, ed. R. Rhees (London 1952).
- Boole, George (1854) *The Laws of Thought* (London and Cambridge); repr. as *Collected Logical Works*. Vol. 2, (Chicago and London: Open Court, 1940).
- Jill Vance Buroker (transl. and introduction), *Logic or the Art of Thinking*, Cambridge University Press, 1996, ISBN 0-521-48249-6.
- Church, Alonzo, 1936-8. "A bibliography of symbolic logic". *Journal of Symbolic Logic 1*: 121–218; *3*:178–212.
- Ebbesen, S. "Early supposition theory (12th–13th Century)" *Histoire, Épistémologie, Langage* 3/1: 35–48 (1981).
- Epictetus, *Dissertationes* ed. Schenkl.
- Farrington, B., *The Philosophy of Francis Bacon*, Liverpool 1964.
- Feferman, Anita B. (1999). "Alfred Tarski". American National Biography. 21. Oxford University Press. pp. 330–332. ISBN 978-0-19-512800-0.
- Feferman, Anita B.; Feferman, Solomon (2004). *Alfred Tarski: Life and Logic*. Cambridge University Press. ISBN 978-0-521-80240-6. OCLC 54691904.
- Frege, G., *Boole's Logical Calculus and the Concept Script*, 1882, in *Posthumous Writings transl. P.Long and R. White 1969, pp. 9–46.*
- Gabbay, Dov and John Woods, eds, *Handbook of the History of Logic* 2004. 1. Greek, Indian and Arabic logic; 2. Mediaeval and Renaissance logic; 3. The rise of modern logic: from Leibniz to Frege; 4. British logic in the Nineteenth century; 5. Logic from Russell to Church; 6. Sets and extensions in the Twentieth century (not yet published); 7. Logic and the modalities in the Twentieth century; 8. The many-valued and nonmonotonic turn in logic; 9. Logic and computation (not yet published); 10. Inductive logic (not yet published); 11. Logic: A history of its central concepts (not yet published) Elsevier, ISBN 0-444-51611-5.
- Geach, P.T. *Logic Matters*, Blackwell 1972.
- Gergonne, Joseph Diaz, (1816) "*Essai de dialectique rationelle*", in *Annales de mathem, pures et appl.* 7, 1816/7, 189–228.
- Goodman, Lenn Evan (2003). *Islamic Humanism*. Oxford University Press, ISBN 0-19-513580-6.
- Goodman, Lenn Evan (1992). *Avicenna*. Routledge, ISBN 0-415-01929-X.

- Grattan-Guinness, Ivor, 2000. *The Search for Mathematical Roots 1870–1940*. Princeton University Press.
- Gracia, J.G. and Noone, T.B., *A Companion to Philosophy in the Middle Ages*, London 2003.
- Haaparanta, Leila (ed.) 2009. *The Development of Modern Logic* Oxford University Press.
- Heath, T.L., 1949. *Mathematics in Aristotle* Oxford University Press.
- Heath, T.L., 1931, *A Manual of Greek Mathematics*, Oxford (Clarendon Press).
- Honderich, Ted (ed.). The Oxford Companion to Philosophy (New York: Oxford University Press, 1995) ISBN 0-19-866132-0.
- Jevons, *The Principles of Science*, London 1879.
- Kneale, William and Martha, 1962. *The development of logic*. Oxford University Press, ISBN 0-19-824773-7.
- Lukasiewicz, *Aristotle's Syllogistic*, Oxford University Press 1951.
- *Ockham's Theory of Terms*: Part I of the Summa Logicae, translated and introduced by Michael J. Loux (Notre Dame, IN: University of Notre Dame Press 1974). Reprinted: South Bend, IN: St. Augustine's Press, 1998.
- *Ockham's Theory of Propositions*: Part II of the Summa Logicae, translated by Alfred J. Freddoso and Henry Schuurman and introduced by Alfred J. Freddoso (Notre Dame, IN: University of Notre Dame Press, 1980). Reprinted: South Bend, IN: St. Augustine's Press, 1998.
- Peirce, C.S., (1896), "The Regenerated Logic", *The Monist*, vol. VII (http://books.google.com/books?id=pa0LAAAAIAAJ&printsec=titlepage), No. 1, p pp. 19 (http://books.google.com/books?id=pa0LAAAAIAAJ&pg=PA19)-40, The Open Court Publishing Co., Chicago, IL, 1896, for the Hegeler Institute. Reprinted (CP 3.425–455). *Internet Archive The Monist* 7 (http://www.archive.org/details/monistquart07hegeuoft).
- Michael Potter (2004), *Set Theory and its Philosophy*, Oxford Univ. Press.
- Sextus Empiricus, *Against the Grammarians (Adversos Mathematicos I)*. David Blank (trans.) (Oxford: Clarendon Press, 1998). ISBN 0-19-824470-3.
- Zermelo, Ernst (1908). "Untersuchungen über die Grundlagen der Mengenlehre I". *Mathematische Annalen* **65** (2): 261–281. doi:10.1007/BF01449999. English translation in Heijenoort, Jean van (1967). "Investigations in the foundations of set theory". *From Frege to Gödel: A Source Book in Mathematical Logic, 1879–1931*. Source Books in the History of the Sciences. Harvard Univ. Press. pp. 199–215. ISBN 978-0-674-32449-7..

External links

- History of Logic in Relationship to Ontology (http://www.ontology.co/history-of-logic.htm) Annotated bibliography on the history of logic
- Ancient Logic (http://plato.stanford.edu/entries/logic-ancient) entry by Susanne Bobzien in the *Stanford Encyclopedia of Philosophy*
- Peter of Spain (http://plato.stanford.edu/entries/peter-spain) entry by Joke Spruyt in the *Stanford Encyclopedia of Philosophy*
- Paul Spade's "Thoughts Words and Things" (http://pvspade.com/Logic/docs/thoughts1_1a.pdf)
- John of St Thomas (http://www.newadvent.org/cathen/08479b.htm)
- Insights, Images, and Bios of 116 logicians (http://humbox.ac.uk/3682/)

Topics in logic

Aristotelian logic

In philosophy, **term logic**, also known as **traditional logic** or **Aristotelian logic**, is a loose name for the way of doing logic that began with Aristotle and that was dominant until the advent of modern predicate logic in the late nineteenth century. This entry is an introduction to the term logic needed to understand philosophy texts written before predicate logic came to be seen as the only formal logic of interest. Readers lacking a grasp of the basic terminology and ideas of term logic can have difficulty understanding such texts, because their authors typically assumed an acquaintance with term logic.

Aristotle's system

Aristotle's logical work is collected in the six texts that are collectively known as the *Organon*. Two of these texts in particular, namely the *Prior Analytics* and *De Interpretatione* contain the heart of Aristotle's treatment of judgements and formal inference, and it is principally this part of Aristotle's works that is about term logic.

The basics

The fundamental assumption behind the theory is that propositions are composed of two terms – hence the name "two-term theory" or "term logic" – and that the reasoning process is in turn built from propositions:

- **The term** is a part of speech representing something, but which is not true or false in its own right, such as "man" or "mortal".
- **The proposition** consists of two terms, in which one term (the "predicate") is "affirmed" or "denied" of the other (the "subject"), and which is capable of truth or falsity.
- **The syllogism** is an inference in which one proposition (the "conclusion") follows of necessity from two others (the "premises").

A proposition may be universal or particular, and it may be affirmative or negative. Thus there are just four kinds of propositions:

- A-type: Universal and affirmative or ("All men are mortal")
- I-type: Particular and affirmative ("Some men are philosophers")
- E-type: Universal and negative ("No men are immortal")
- O-type: Particular and negative ("Some men are not philosophers").

This was called the *fourfold scheme* of propositions. (See types of syllogism for the origin of the letters A, I, E, and O.) Aristotle summarised the logical relationship between four types of propositions with his square of oppositions. The syllogistic is a formal theory explaining which combinations of true premises yield true conclusions.

The term

A term (Greek *horos*) is the basic component of the proposition. The original meaning of the *horos* (and also of the Latin *terminus*) is "extreme" or "boundary". The two terms lie on the outside of the proposition, joined by the act of affirmation or denial. For Aristotle, a term is simply a "thing", a part of a proposition. For early modern logicians like Arnauld (whose *Port-Royal Logic* was the best-known text of his day), it is a psychological entity like an "idea" or "concept". Mill considers it a word. None of these interpretations are quite satisfactory. In asserting that something is a unicorn, we are not asserting *anything* of *anything*. Nor does "all Greeks are men" say that the ideas of Greeks are ideas of men, or that word "Greeks" is the word "men". A proposition cannot be built from real things or ideas, but it is not just meaningless words either. This is a problem about the meaning of language that is still not entirely resolved. (See the book by Prior below for an excellent discussion of the problem).

The proposition

In term logic, a "proposition" is simply a *form of language*: a particular kind of sentence, in which the subject and predicate are combined, so as to assert something true or false. It is not a thought, or an abstract entity. The word "propositio" is from the Latin, meaning the first premise of a syllogism. Aristotle uses the word premise (*protasis*) as a sentence affirming or denying one thing of another (*Posterior Analytics* 1. 1 24a 16), so a premise is also a form of words. However, in modern philosophical logic, it now means what is asserted as the result of uttering a sentence, and is regarded as something peculiarly mental or intentional. Writers before Frege and Russell, such as Bradley, sometimes spoke of the "judgment" as something distinct from a sentence, but this is not quite the same. As a further confusion the word "sentence" derives from the Latin, meaning an opinion or judgment, and so is equivalent to "proposition". The **quality** of a proposition is whether it is affirmative (the predicate is affirmed of the subject) or negative (the predicate is denied of the subject). Thus "every man is a mortal" is affirmative, since "mortal" is affirmed of "man". "No men are immortals" is negative, since "immortal" is denied of "man". The **quantity** of a proposition is whether it is universal (the predicate is affirmed or denied of "the whole" of the subject) or particular (the predicate is affirmed or denied of only "part of" the subject).

Singular terms

For Aristotle, the distinction between singular and universal is a fundamental metaphysical one, and not merely grammatical. A singular term for Aristotle is that which is of such a nature as to be predicated of only one thing, thus "Callias". (*De Int.* 7). It is not predicable of more than one thing: "*Socrates* is not predicable of more than one subject, and therefore we do not say *every Socrates* as we say *every man*". (*Metaphysics* D 9, 1018 a4). It may feature as a grammatical predicate, as in the sentence "the person coming this way is Callias". But it is still a *logical* subject.

He contrasts it with "universal" (*katholou* - "of a whole"). Universal terms are the basic materials of Aristotle's logic, propositions containing singular terms do not form part of it at all. They are mentioned briefly in the *De Interpretatione*. Afterwards, in the chapters of the *Prior Analytics* where Aristotle methodically sets out his theory of the syllogism, they are entirely ignored.

The reason for this omission is clear. The essential feature of term logic is that, of the four terms in the two premises, one must occur twice. Thus

All Greeks are **men**

All **men** are mortal.

What is subject in one premise, must be predicate in the other, and so it is necessary to eliminate from the logic any terms which cannot function both as subject and predicate. Singular terms do not function this way, so they are omitted from Aristotle's syllogistic.

In later versions of the syllogistic, singular terms were treated as universals. See for example (where it is clearly stated as received opinion) Part 2, chapter 3, of the *Port-Royal Logic*. Thus

All men are mortals

All Socrates are men

All Socrates are mortals

This is clearly awkward, and is a weakness exploited by Frege in his devastating attack on the system (from which, ultimately, it never recovered). See concept and object.

The famous syllogism "Socrates is a man ...", is frequently quoted as though from Aristotle. See for example Kapp, *Greek Foundations of Traditional Logic*, New York 1942, p. 17, Copleston *A History of Philosophy* Vol. I., p. 277, Russell, *A History of Western Philosophy* London 1946 p. 218. In fact it is nowhere in the *Organon*. It is first mentioned by Sextus Empiricus in his *Hyp. Pyrrh.* ii. 164.

Decline of term logic

Term logic began to decline in Europe during the Renaissance, when logicians like Rodolphus Agricola Phrisius (1444–1485) and Ramus began to promote place logics. The logical tradition called Port-Royal Logic, or sometimes "traditional logic", claimed that a proposition was a combination of ideas rather than terms, but otherwise followed many of the conventions of term logic and was influential, especially in England, until the 19th century. Spinoza's "way of geometry" was far more influenced by Euclid's *Elements* than by Aristotelian concepts. Leibniz created a distinctive logical calculus, but nearly all of his work on logic was unpublished and unremarked until Louis Couturat went through the Leibniz *Nachlass* around 1900, and published many Leibniz manuscripts and a pioneering study of Leibniz's logic.

19th century attempts to algebratize logic, such as the work of Boole and Venn, typically yielded systems highly influenced by the term logic tradition. The first predicate logic was that of Frege's landmark *Begriffsschrift*, little read before 1950, in part because of its eccentric notation. Modern predicate logic as we know it began in the 1880s with the writings of Charles Sanders Peirce, who influenced Peano and even more, Ernst Schröder. It reached full fruition in the hands of Bertrand Russell and A. N. Whitehead, whose *Principia Mathematica* (1910–13) made splendid use of a variant of Peano's predicate logic.

Predicate logic was designed as a form of mathematics, and as such is capable of all sorts of mathematical reasoning beyond the powers of term logic. Predicate logic is also capable of many commonsense inferences that elude term logic. Term logic cannot, for example, explain the inference from "every car is a vehicle", to "every owner of a car is an owner of a vehicle." Syllogistic reasoning cannot explain inferences involving multiple generality. Relations and identity must be treated as subject-predicate relations, which make the identity statements of mathematics difficult to handle. Term logic contains no analog of the singular term and singular proposition, both essential features of predicate logic.

With the ascension of predicate logic, term and syllogistic logic gradually fell into disuse except among students of ancient and medieval philosophy. Since the development of predicate logic, introductory texts on logic have ignored or disparaged term logic, except perhaps as a source of examples for beginning students. A notable exception to this generalization are the four editions of Quine's *Methods of Logic*, which discussed term logic (which Quine called "Boolean term schemata") and syllogisms at some length. Quine's writings on logic contain much that is in the spirit of term logic in that they frequently invoke grammatical concepts and examples taken from natural language, even employing bits of scholastic terminology such as "syncategorematic."

Term logic also survived to some extent in traditional Roman Catholic education, especially in seminaries. Medieval Catholic theology, especially the writings of Thomas Aquinas, had a powerfully Aristotelean cast, and thus term logic became a part of Catholic theological reasoning. For example, Joyce (1949), written for use in Catholic seminaries, made no mention of Frege or Bertrand Russell. On Aristotle, term logic, and Roman Catholicism, see Copleston's *A History of Philosophy*.

A revival

Some philosophers have complained that predicate logic:

- Is unnatural in a sense, in that its syntax does not follow the syntax of the sentences that figure in our everyday reasoning. It is, as Quine acknowledged, "Procrustean," employing an artificial language of function and argument, quantifier and bound variable.
- Suffers from theoretical problems, probably the most serious being empty names and identity statements.

Even academic philosophers entirely in the mainstream, such as Gareth Evans, have written as follows:

> "I come to semantic investigations with a preference for *homophonic* theories; theories which try to take serious account of the syntactic and semantic devices which actually exist in the language ...I would prefer [such] a theory ... over a theory which is only able to deal with [sentences of the form "all A's are B's"] by "discovering" hidden logical constants ... The objection would not be that such [Fregean] truth conditions are not correct, but that, in a sense which we would all dearly love to have more exactly explained, the syntactic shape of the sentence is treated as so much misleading surface structure" (Evans 1977)

The writings of Fred Sommers (e.g., Sommers 1970) and his students have modified term logic so that it can address these criticisms of predicate logic and overcome the well-known weaknesses of term logic. The result is the "term functor logic" of Sommers (1982), and Sommers and Englebretsen (2000). This logic has a very Boolean appearance, in that '+' and '-' are the sole operational signs and all statements are equations. It has sufficient expressive power to handle relational terms generally, and to capture the validity of arguments that elude syllogistic reasoning. Term functor logic has similarities to Quine's predicate functor logic, an algebraic formalism Quine devised to do first-order logic without quantifiers.

In a less formal vein, term logic has acquired a following among those advocating a return to educational methods grounded in the medieval Trivium: grammar, logic, and rhetoric. Advocates of the Trivium include the Paideia Proposal by philosopher Mortimer J. Adler, and some homeschoolers. The Trivium views logic not as a form of mathematics, but as part of a classical education in language. Those advocating this line see predicate logic as excessively nominalistic, as primarily concerned with the manipulation of symbols (syntax) and not with the whys and essences of things (ontology and metaphysics).

A variant of term logic, probabilistic term logic, which assigns a probability value and a confidence value to the truth of both terms and propositions, is gaining popularity in artificial intelligence systems. Variants include both Pei Wang's "Non-Axiomatic Reasoning System" (NARS) and Ben Goertzel's "OpenCog" system.

References

- Bocheński, I. M., 1951. *Ancient Formal Logic*. North-Holland.
- Louis Couturat, 1961 (1901). *La Logique de Leibniz*. Hildesheim: Georg Olms Verlagsbuchhandlung.
- Gareth Evans, 1977, "Pronouns, Quantifiers and Relative Clauses," *Canadian Journal of Philosophy*.
- Peter Geach, 1976. *Reason and Argument*. University of California Press.
- Hammond and Scullard, 1992. *The Oxford Classical Dictionary*. Oxford University Press, ISBN 0-19-869117-3.
- Joyce, George Hayward, 1949 (1908). *Principles of Logic* [1], 3rd ed. Longmans. A manual written for use in Catholic seminaries. Authoritative on traditional logic, with many references to medieval and ancient sources. Contains no hint of modern formal logic. The author lived 1864-1943.
- Jan Łukasiewicz, 1951. *Aristotle's Syllogistic, from the Standpoint of Modern Formal Logic*. Oxford Univ. Press.
- John Stuart Mill, 1904. *A System of Logic*, 8th ed. London.
- Parry and Hacker, 1991. *Aristotelian Logic*. State University of New York Press.
- Arthur Prior, 1962. *Formal Logic*, 2nd ed. Oxford Univ. Press. While primarily devoted to modern formal logic, contains much on term and medieval logic.
- --------, 1976. *The Doctrine of Propositions and Terms*. Peter Geach and A. J. P. Kenny, eds. London: Duckworth.

- Willard Quine, 1986. *Philosophy of Logic* 2nd ed. Harvard Univ. Press.
- Rose, Lynn E., 1968. *Aristotle's Syllogistic*. Springfield: Clarence C. Thomas.
- Sommers, Fred, 1970, "The Calculus of Terms," *Mind 79*: 1-39. Reprinted in Englebretsen, G., ed., 1987. *The new syllogistic* New York: Peter Lang. ISBN 0-8204-0448-9
- --------, 1982. *The logic of natural language*. Oxford University Press.
- --------, 1990, "Predication in the Logic of Terms, [2]" *Notre Dame Journal of Formal Logic 31*: 106-26.
- -------- and Englebretsen, George, 2000. *An invitation to formal reasoning. The logic of terms*. Aldershot UK: Ashgate. ISBN 0-7546-1366-6.
- Szabolcsi Lorne, 2008. *Numerical Term Logic*. Lewiston: Edwin Mellen Press.

External links

- Stanford Encyclopedia of Philosophy:
  - Aristotle's Logic [3] -- by Robin Smith.
  - Traditional Square of Opposition [4] -- by Terence Parsons.
- Internet Encyclopedia of Philosophy: "Aristotle. [5]" Discusses how Aristotle's logic was viewed by his many successors.
- Aristotle's term logic online [6] -- This online program provides a platform for experimentation and research on Aristotelian logic.
- Annotated bibliographies of writings by:
  - Fred Sommers. [7]
  - George Englebretsen. [8]
- PlanetMath: Aristotelian Logic.
- Interactive Syllogistic Machine for Term Logic [9] A web based syllogistic machine for exploring fallacies, figures, terms, and modes of syllogisms.

References

[1] http://www.webcitation.org/query?url=http://uk.geocities.com/frege%40btinternet.com/joyce/principlesoflogic.htm&date=2009-10-25+04:16:58

[2] http://projecteuclid.org/Dienst/UI/1.0/Summarize/euclid.ndjfl/1093635336

[3] http://plato.stanford.edu/entries/aristotle-logic

[4] http://plato.stanford.edu/entries/square/

[5] http://www.utm.edu/research/iep/a/aristotl.htm

[6] http://aristotelianlogic.glashoff.net

[7] http://www.ontology.co/sommersf.htm

[8] http://www.ontology.co/biblio/englebretseng.htm

[9] http://thefirstscience.org/syllogistic-machine/

Propositional calculus

In mathematical logic, a **propositional calculus** or **logic** (also called **sentential calculus** or **sentential logic**) is a formal system in which formulas of a formal language may be interpreted as representing propositions. A system of inference rules and axioms allows certain formulas to be derived, called theorems; which may be interpreted as true propositions. The series of formulas which is constructed within such a system is called a derivation and the last formula of the series is a theorem, whose derivation may be interpreted as a proof of the truth of the proposition represented by the theorem.

Truth-functional propositional logic is a propositional logic whose interpretation limits the truth values of its propositions to two, usually *true* and *false*. Truth-functional propositional logic and systems isomorphic to it are considered to be **zeroth-order logic**.

History

Although propositional logic (which is interchangeable with propositional calculus) had been hinted by earlier philosophers, it was developed into a formal logic by Chrysippus[1] and expanded by the Stoics. The logic was focused on propositions. This advancement was different from the traditional syllogistic logic which was focused on terms. However, later in antiquity, the propositional logic developed by the Stoics was no longer understood. As a result, the system was essentially reinvented by Peter Abelard.[2]

Propositional logic was eventually refined using symbolic logic. Gottfried Leibniz has been credited with being the founder of symbolic logic for his work with the calculus ratiocinator. Although his work was the first of its kind, it was unknown to the larger logical community. As a result, many of the advances achieved by Leibniz were reachieved by logicians like George Boole and Augustus De Morgan completely independent of Leibniz.[3]

Just as propositional logic can be seen as an advancement from the earlier syllogistic logic, Gottlob Frege's predicate logic was an advancement from the earlier propositional logic. Predicate logic has been described as combining "the distinctive features of syllogistic logic and propositional logic."[4] As a result, it ushered a new era in the history of logic. However, advances in propositional logic were still made after Frege. These include Natural Deduction, Truth-Trees and Truth-Tables. Natural deduction was invented by Jan Łukasiewicz. Truth-Trees were invented by Evert Willem Beth.[5] The invention of truth-tables, however, is of controversial attribution.

The ideas preceding truth tables have been found in both Frege[6] and Bertrand Russell[7] whereas the actual 'tabular structure' (i.e. being formed as a table) is generally credited to either Ludwig Wittgenstein, Emil Post or both (independently of one another).[6] Besides Frege and Russell, others credited for having preceding ideas of truth-tables include Philo, Boole, Charles Sanders Peirce, and Ernst Schröder. And besides Post and Wittgenstein, others credited with the tabular structure include Łukasiewicz, Schröder, Alfred North Whitehead, William Stanley Jevons, John Venn, and Clarence Irving Lewis.[7] Ultimately, some, like John Shosky, have concluded "It is far from clear that any one person should be given the title of 'inventor' of truth-tables.".[7]

Terminology

In general terms, a calculus is a formal system that consists of a set of syntactic expressions (*well-formed formulæ* or *wffs*), a distinguished subset of these expressions (axioms), plus a set of formal rules that define a specific binary relation, intended to be interpreted as logical equivalence, on the space of expressions.

When the formal system is intended to be a logical system, the expressions are meant to be interpreted as statements, and the rules, known as *inference rules*, are typically intended to be truth-preserving. In this setting, the rules (which may include axioms) can then be used to derive ("infer") formulæ representing true statements from given formulæ representing true statements.

The set of axioms may be empty, a nonempty finite set, a countably infinite set, or be given by axiom schemata. A formal grammar recursively defines the expressions and well-formed formulæ (wffs) of the language. In addition a semantics may be given which defines truth and valuations (or interpretations).

The language of a propositional calculus consists of

1. a set of primitive symbols, variously referred to as *atomic formulae*, *placeholders*, *proposition letters*, or *variables*, and
2. a set of operator symbols, variously interpreted as *logical operators* or *logical connectives*.

A *well-formed formula* (*wff*) is any atomic formula, or any formula that can be built up from atomic formulæ by means of operator symbols according to the rules of the grammar.

Mathematicians sometimes distinguish between propositional constants, propositional variables, and schemata. Propositional constants represent some particular proposition, while propositional variables range over the set of all atomic propositions. Schemata, however, range over all propositions. It is common to represent propositional constants by A , B , and C , propositional variables by P, Q , and R , and schematic letters are often Greek letters, most often φ , ψ , and χ .

Basic concepts

The following outlines a standard propositional calculus. Many different formulations exist which are all more or less equivalent but differ in the details of

1. their language, that is, the particular collection of primitive symbols and operator symbols,
2. the set of axioms, or distinguished formulæ, and
3. the set of inference rules.

We may represent any given proposition with a letter which we call a propositional constant, analogous to representing a number by a letter in mathematics, for instance, $a = 5$. We require that all propositions have exactly one of two truth-values: true or false. To take an example, let P be the proposition that it is raining outside. This will be true if it is raining outside and false otherwise.

- We then define truth-functional operators, beginning with negation. We write $\neg P$ to represent the negation of P , which can be thought of as the denial of P. In the example above, $\neg P$ expresses that it is not raining outside, or by a more standard reading: "It is not the case that it is raining outside." When P is true, $\neg P$ is false; and when P is false, $\neg P$ is true. $\neg\neg P$ always has the same truth-value as P.
- Conjunction is a truth-functional connective which forms a proposition out of two simpler propositions, for example, P and Q . The conjunction of P and Q is written $P \land Q$, and expresses that each are true. We read $P \land Q$ as " P and Q ". For any two propositions, there are four possible assignments of truth values:
  1. P is true and Q is true
  2. P is true and Q is false
  3. P is false and Q is true
  4. P is false and Q is false

  The conjunction of P and Q is true in case 1 and is false otherwise. Where P is the proposition that it is raining outside and Q is the proposition that a cold-front is over Kansas, $P \land Q$ is true when it is raining outside and there is a cold-front over Kansas. If it is not raining outside, then $P \land Q$ is false; and if there is no cold-front over Kansas, then $P \land Q$ is false.
- Disjunction resembles conjunction in that it forms a proposition out of two simpler propositions. We write it $P \lor Q$, and it is read " P or Q ". It expresses that either P or Q is true. Thus, in the cases listed above, the disjunction of P and Q is true in all cases except 4. Using the example above, the disjunction expresses that it is either raining outside or there is a cold front over Kansas. (Note, this use of disjunction is supposed to resemble the use of the English word "or". However, it is most like the English inclusive "or", which can be used to express

the truth of at least one of two propositions. It is not like the English exclusive "or", which expresses the truth of exactly one of two propositions. That is to say, the exclusive "or" is false when both P and Q are true (case 1). An example of the exclusive or is: You may have a bagel or a pastry, but not both. Often in natural language, given the appropriate context, the addendum "but not both" is omitted but implied. In mathematics, however, "or" is always used as inclusive or; if exclusive or is meant it will be specified, possibly by "xor".)

- Material conditional also joins two simpler propositions, and we write $P \rightarrow Q$, which is read "if P then Q ". The proposition to the left of the arrow is called the antecedent and the proposition to the right is called the consequent. (There is no such designation for conjunction or disjunction, since they are commutative operations.) It expresses that Q is true whenever P is true. Thus it is true in every case above except case 2, because this is the only case when P is true but Q is not. Using the example, if P then Q expresses that if it is raining outside then there is a cold-front over Kansas. The material conditional is often confused with physical causation. The material conditional, however, only relates two propositions by their truth-values—which is not the relation of cause and effect. It is contentious in the literature whether the material implication represents logical causation.
- Biconditional joins two simpler propositions, and we write $P \leftrightarrow Q$, which is read " P if and only if Q ". It expresses that P and Q have the same truth-value, thus P if and only if Q is true in cases 1 and 4, and false otherwise.

It is extremely helpful to look at the truth tables for these different operators, as well as the method of analytic tableaux.

Closure under operations

Propositional logic is closed under truth-functional connectives. That is to say, for any proposition φ , $\neg\varphi$ is also a proposition. Likewise, for any propositions φ and ψ , $\varphi \wedge \psi$ is a proposition, and similarly for disjunction, conditional, and biconditional. This implies that, for instance, $P \wedge Q$ is a proposition, and so it can be conjoined with another proposition. In order to represent this, we need to use parentheses to indicate which proposition is conjoined with which. For instance, $P \wedge Q \wedge R$ is not a well-formed formula, because we do not know if we are conjoining $P \wedge Q$ with R or if we are conjoining P with $Q \wedge R$. Thus we must write either $(P \wedge Q) \wedge R$ to represent the former, or $P \wedge (Q \wedge R)$ to represent the latter. By evaluating the truth conditions, we see that both expressions have the same truth conditions (will be true in the same cases), and moreover that any proposition formed by arbitrary conjunctions will have the same truth conditions, regardless of the location of the parentheses. This means that conjunction is associative, however, one should not assume that parentheses never serve a purpose. For instance, the sentence $P \wedge (Q \vee R)$ does not have the same truth conditions as $(P \wedge Q) \vee R$, so they are different sentences distinguished only by the parentheses. One can verify this by the truth-table method referenced above.

Note: For any arbitrary number of propositional constants, we can form a finite number of cases which list their possible truth-values. A simple way to generate this is by truth-tables, in which one writes P, Q , ..., Z for any list of k propositional constants—that is to say, any list of propositional constants with k entries. Below this list, one writes 2^k rows, and below P one fills in the first half of the rows with true (or T) and the second half with false (or F). Below Q one fills in one-quarter of the rows with T, then one-quarter with F, then one-quarter with T and the last quarter with F. The next column alternates between true and false for each eighth of the rows, then sixteenths, and so on, until the last propositional constant varies between T and F for each row. This will give a complete listing of cases or truth-value assignments possible for those propositional constants.

Argument

The propositional calculus then defines an *argument* as a set of propositions. A valid argument is a set of propositions, the last of which follows from—or is implied by—the rest. All other arguments are invalid. The simplest valid argument is modus ponens, one instance of which is the following set of propositions:

$$\begin{array}{rl} 1. & P \to Q \\ 2. & P \\ \hline \therefore & Q \end{array}$$

This is a set of three propositions, each line is a proposition, and the last follows from the rest. The first two lines are called premises, and the last line the conclusion. We say that any proposition C follows from any set of propositions $(P_1, ..., P_n)$, if C must be true whenever every member of the set $(P_1, ..., P_n)$ is true. In the argument above, for any P and Q , whenever $P \to Q$ and P are true, necessarily Q is true. Notice that, when P is true, we cannot consider cases 3 and 4 (from the truth table). When $P \to Q$ is true, we cannot consider case 2. This leaves only case 1, in which Q is also true. Thus Q is implied by the premises.

This generalizes schematically. Thus, where φ and ψ may be any propositions at all,

$$\begin{array}{rl} 1. & \varphi \to \psi \\ 2. & \varphi \\ \hline \therefore & \psi \end{array}$$

Other argument forms are convenient, but not necessary. Given a complete set of axioms (see below for one such set), modus ponens is sufficient to prove all other argument forms in propositional logic, and so we may think of them as derivative. Note, this is not true of the extension of propositional logic to other logics like first-order logic. First-order logic requires at least one additional rule of inference in order to obtain completeness.

The significance of argument in formal logic is that one may obtain new truths from established truths. In the first example above, given the two premises, the truth of Q is not yet known or stated. After the argument is made, Q is deduced. In this way, we define a deduction system as a set of all propositions that may be deduced from another set of propositions. For instance, given the set of propositions $A = \{P \lor Q, \neg Q \land R, (P \lor Q) \to R\}$, we can define a deduction system, Γ, which is the set of all propositions which follow from A . Reiteration is always assumed, so $P \lor Q, \neg Q \land R, (P \lor Q) \to R \in \Gamma$. Also, from the first element of A , last element, as well as modus ponens, R is a consequence, and so $R \in \Gamma$. Because we have not included sufficiently complete axioms, though, nothing else may be deduced. Thus, even though most deduction systems studied in propositional logic are able to deduce $(P \lor Q) \leftrightarrow (\neg P \to Q)$, this one is too weak to prove such a proposition.

Generic description of a propositional calculus

A **propositional calculus** is a formal system $\mathcal{L} = \mathcal{L}\left(\mathrm{A},\ \Omega,\ \mathrm{Z},\ \mathrm{I}\right)$, where:

- The *alpha set* A is a finite set of elements called *proposition symbols* or *propositional variables*. Syntactically speaking, these are the most basic elements of the formal language $\mathcal{L}$, otherwise referred to as *atomic formulæ* or *terminal elements*. In the examples to follow, the elements of A are typically the letters p , q , r , and so on.
- The *omega set* Ω is a finite set of elements called *operator symbols* or *logical connectives*. The set Ω is partitioned into disjoint subsets as follows:

$$\Omega = \Omega_0 \cup \Omega_1 \cup \ldots \cup \Omega_j \cup \ldots \cup \Omega_m.$$

In this partition, Ω_j is the set of operator symbols of *arity* j .

In the more familiar propositional calculi, Ω is typically partitioned as follows:

$$\Omega_1 = \{\neg\},$$
$$\Omega_2 \subseteq \{\land, \lor, \to, \leftrightarrow\}.$$

A frequently adopted convention treats the constant logical values as operators of arity zero, thus:

$$\Omega_0 = \{0, 1\}.$$

Some writers use the tilde (~), or N, instead of $\neg$; and some use the ampersand (&), the prefixed K, or $\cdot$ instead of $\wedge$. Notation varies even more for the set of logical values, with symbols like {false, true}, {F, T}, or $\{\bot, \top\}$ all being seen in various contexts instead of {0, 1}.

- The *zeta set* Z is a finite set of *transformation rules* that are called *inference rules* when they acquire logical applications.
- The *iota set* I is a finite set of *initial points* that are called *axioms* when they receive logical interpretations.

The *language* of $\mathcal{L}$, also known as its set of *formulæ*, *well-formed formulas* or *wffs*, is inductively defined by the following rules:

1. Base: Any element of the alpha set A is a formula of $\mathcal{L}$.
2. If $p_1, p_2, \ldots, p_j$ are formulæ and f is in Ω_j, then $(f(p_1, p_2, \ldots, p_j))$ is a formula.
3. Closed: Nothing else is a formula of $\mathcal{L}$.

Repeated applications of these rules permits the construction of complex formulæ. For example:

1. By rule 1, p is a formula.
2. By rule 2, $\neg p$ is a formula.
3. By rule 1, q is a formula.
4. By rule 2, $(\neg p \vee q)$ is a formula.

Example 1. Simple axiom system

Let $\mathcal{L}_1 = \mathcal{L}(\mathrm{A}, \Omega, \mathrm{Z}, \mathrm{I})$, where A, Ω, Z, I are defined as follows:

- The alpha set A, is a finite set of symbols that is large enough to supply the needs of a given discussion, for example:

$$\mathrm{A} = \{p, q, r, s, t, u\}.$$

- Of the three connectives for conjunction, disjunction, and implication ($\wedge$, $\vee$, and $\rightarrow$), one can be taken as primitive and the other two can be defined in terms of it and negation ($\neg$). Indeed, all of the logical connectives can be defined in terms of a sole sufficient operator. The biconditional ($\leftrightarrow$) can of course be defined in terms of conjunction and implication, with $a \leftrightarrow b$ defined as $(a \to b) \wedge (b \to a)$.

  Adopting negation and implication as the two primitive operations of a propositional calculus is tantamount to having the omega set $\Omega = \Omega_1 \cup \Omega_2$ partition as follows:

$$\Omega_1 = \{\neg\},$$
$$\Omega_2 = \{\rightarrow\}.$$

- An axiom system discovered by Jan Łukasiewicz formulates a propositional calculus in this language as follows. The axioms are all substitution instances of:
  - $(p \to (q \to p))$
  - $((p \to (q \to r)) \to ((p \to q) \to (p \to r)))$
  - $((\neg p \to \neg q) \to (q \to p))$
- The rule of inference is modus ponens (i.e., from p and $(p \to q)$, infer q). Then $a \vee b$ is defined as $\neg a \to b$, and $a \wedge b$ is defined as $\neg(a \to \neg b)$.

Example 2. Natural deduction system

Let $\mathcal{L}_2 = \mathcal{L}(\mathrm{A}, \Omega, \mathrm{Z}, \mathrm{I})$, where A, Ω, Z, I are defined as follows:

- The alpha set A, is a finite set of symbols that is large enough to supply the needs of a given discussion, for example:

  $\mathrm{A} = \{p, q, r, s, t, u\}.$
- The omega set $\Omega = \Omega_1 \cup \Omega_2$ partitions as follows:

  $\Omega_1 = \{\neg\},$

  $\Omega_2 = \{\land, \lor, \to, \leftrightarrow\}.$

In the following example of a propositional calculus, the transformation rules are intended to be interpreted as the inference rules of a so-called *natural deduction system*. The particular system presented here has no initial points, which means that its interpretation for logical applications derives its theorems from an empty axiom set.

- The set of initial points is empty, that is, $\mathrm{I} = \varnothing$.
- The set of transformation rules, Z, is described as follows:

Our propositional calculus has ten inference rules. These rules allow us to derive other true formulae given a set of formulae that are assumed to be true. The first nine simply state that we can infer certain wffs from other wffs. The last rule however uses hypothetical reasoning in the sense that in the premise of the rule we temporarily assume an (unproven) hypothesis to be part of the set of inferred formulae to see if we can infer a certain other formula. Since the first nine rules don't do this they are usually described as *non-hypothetical* rules, and the last one as a *hypothetical* rule.

In describing the transformation rules, we may introduce a metalanguage symbol $\vdash$. It is basically a convenient shorthand for saying "infer that". The format is $\Gamma \vdash \psi$, in which Γ is a (possibly empty) set of formulae called premises, and ψ is a formula called conclusion. The transformation rule $\Gamma \vdash \psi$ means that if every proposition in Γ is a theorem (or has the same truth value as the axioms), then ψ is also a theorem. Note that considering the following rule Conjunction introduction, we will know whenever Γ has more than one formula, we can always safely reduce it into one formula using conjunction. So for short, from that time on we may represent Γ as one formula instead of a set. Another omission for convenience is when Γ is an empty set, in which case Γ may not appear.

Reductio ad absurdum (negation introduction)

From $(p \to q)$ and $\neg q$, infer $\neg p$.

That is, $\{p \to q, \neg q\} \vdash \neg p$.

Double negative elimination

From $\neg\neg p$, infer p.

That is, $\neg\neg p \vdash p$.

Conjunction introduction

From p and q, infer $(p \land q)$.

That is, $\{p, q\} \vdash (p \land q)$.

Conjunction elimination

From $(p \land q)$, infer p.

From $(p \land q)$, infer q.

That is, $(p \land q) \vdash p$ and $(p \land q) \vdash q$.

Disjunction introduction

From p, infer $(p \lor q)$.

From q, infer $(p \vee q)$.

That is, $p \vdash (p \vee q)$ and $q \vdash (p \vee q)$.

Disjunction elimination

From $(p \vee q)$ and $(p \rightarrow r)$ and $(q \rightarrow r)$, infer r.

That is, $\{p \vee q, p \rightarrow r, q \rightarrow r\} \vdash r$.

Biconditional introduction

From $(p \rightarrow q)$ and $(q \rightarrow p)$, infer $(p \leftrightarrow q)$.

That is, $\{p \rightarrow q, q \rightarrow p\} \vdash (p \leftrightarrow q)$.

Biconditional elimination

From $(p \leftrightarrow q)$, infer $(p \rightarrow q)$.

From $(p \leftrightarrow q)$, infer $(q \rightarrow p)$.

That is, $(p \leftrightarrow q) \vdash (p \rightarrow q)$ and $(p \leftrightarrow q) \vdash (q \rightarrow p)$.

Modus ponens (conditional elimination)

From p and $(p \rightarrow q)$, infer q.

That is, $\{p, p \rightarrow q\} \vdash q$.

Conditional proof (conditional introduction)

From [accepting p allows a proof of q], infer $(p \rightarrow q)$.

That is, $(p \vdash q) \vdash (\vdash (p \rightarrow q))$.

Basic and derived argument forms

Basic and Derived Argument Forms		
Name	**Sequent**	**Description**
Modus Ponens	$((p \rightarrow q) \wedge p) \vdash q$	If p then q; p; therefore q
Modus Tollens	$((p \rightarrow q) \wedge \neg q) \vdash \neg p$	If p then q; not q; therefore not p
Hypothetical Syllogism	$((p \rightarrow q) \wedge (q \rightarrow r)) \vdash (p \rightarrow r)$	If p then q; if q then r; therefore, if p then r
Disjunctive Syllogism	$((p \vee q) \wedge \neg p) \vdash q$	Either p or q, or both; not p; therefore, q
Constructive Dilemma	$((p \rightarrow q) \wedge (r \rightarrow s) \wedge (p \vee r)) \vdash (q \vee s)$	If p then q; and if r then s; but p or r; therefore q or s
Destructive Dilemma	$((p \rightarrow q) \wedge (r \rightarrow s) \wedge (\neg q \vee \neg s)) \vdash (\neg p \vee \neg r)$	If p then q; and if r then s; but not q or not s; therefore not p or not r
Bidirectional Dilemma	$((p \rightarrow q) \wedge (r \rightarrow s) \wedge (p \vee \neg s)) \vdash (q \vee \neg r)$	If p then q; and if r then s; but p or not s; therefore q or not r
Simplification	$(p \wedge q) \vdash p$	p and q are true; therefore p is true
Conjunction	$p, q \vdash (p \wedge q)$	p and q are true separately; therefore they are true conjointly
Addition	$p \vdash (p \vee q)$	p is true; therefore the disjunction (p or q) is true
Composition	$((p \rightarrow q) \wedge (p \rightarrow r)) \vdash (p \rightarrow (q \wedge r))$	If p then q; and if p then r; therefore if p is true then q and r are true
De Morgan's Theorem (1)	$\neg(p \wedge q) \vdash (\neg p \vee \neg q)$	The negation of (p and q) is equiv. to (not p or not q)

De Morgan's Theorem (2)	$\neg(p \vee q) \vdash (\neg p \wedge \neg q)$	The negation of (p or q) is equiv. to (not p and not q)
Commutation (1)	$(p \vee q) \vdash (q \vee p)$	(p or q) is equiv. to (q or p)
Commutation (2)	$(p \wedge q) \vdash (q \wedge p)$	(p and q) is equiv. to (q and p)
Commutation (3)	$(p \leftrightarrow q) \vdash (q \leftrightarrow p)$	(p is equiv. to q) is equiv. to (q is equiv. to p)
Association (1)	$(p \vee (q \vee r)) \vdash ((p \vee q) \vee r)$	p or (q or r) is equiv. to (p or q) or r
Association (2)	$(p \wedge (q \wedge r)) \vdash ((p \wedge q) \wedge r)$	p and (q and r) is equiv. to (p and q) and r
Distribution (1)	$(p \wedge (q \vee r)) \vdash ((p \wedge q) \vee (p \wedge r))$	p and (q or r) is equiv. to (p and q) or (p and r)
Distribution (2)	$(p \vee (q \wedge r)) \vdash ((p \vee q) \wedge (p \vee r))$	p or (q and r) is equiv. to (p or q) and (p or r)
Double Negation	$p \vdash \neg\neg p$	p is equivalent to the negation of not p
Transposition	$(p \rightarrow q) \vdash (\neg q \rightarrow \neg p)$	If p then q is equiv. to if not q then not p
Material Implication	$(p \rightarrow q) \vdash (\neg p \vee q)$	If p then q is equiv. to not p or q
Material Equivalence (1)	$(p \leftrightarrow q) \vdash ((p \rightarrow q) \wedge (q \rightarrow p))$	(p is equiv. to q) means (if p is true then q is true) and (if q is true then p is true)
Material Equivalence (2)	$(p \leftrightarrow q) \vdash ((p \wedge q) \vee (\neg p \wedge \neg q))$	(p is equiv. to q) means either (p and q are true) or (both p and q are false)
Material Equivalence (3)	$(p \leftrightarrow q) \vdash ((p \vee \neg q) \wedge (\neg p \vee q))$	(p is equiv. to q) means, both (p or not q is true) and (not p or q is true)
Exportation[8]	$((p \wedge q) \rightarrow r) \vdash (p \rightarrow (q \rightarrow r))$	from (if p and q are true then r is true) we can prove (if q is true then r is true, if p is true)
Importation	$(p \rightarrow (q \rightarrow r)) \vdash ((p \wedge q) \rightarrow r)$	If p then (if q then r) is equivalent to if p and q then r
Tautology (1)	$p \vdash (p \vee p)$	p is true is equiv. to p is true or p is true
Tautology (2)	$p \vdash (p \wedge p)$	p is true is equiv. to p is true and p is true
Tertium non datur (Law of Excluded Middle)	$\vdash (p \vee \neg p)$	p or not p is true
Law of Non-Contradiction	$\vdash \neg(p \wedge \neg p)$	p and not p is false, is a true statement

Proofs in propositional calculus

One of the main uses of a propositional calculus, when interpreted for logical applications, is to determine relations of logical equivalence between propositional formulæ. These relationships are determined by means of the available transformation rules, sequences of which are called *derivations* or *proofs*.

In the discussion to follow, a proof is presented as a sequence of numbered lines, with each line consisting of a single formula followed by a *reason* or *justification* for introducing that formula. Each premise of the argument, that is, an assumption introduced as an hypothesis of the argument, is listed at the beginning of the sequence and is marked as a "premise" in lieu of other justification. The conclusion is listed on the last line. A proof is complete if every line follows from the previous ones by the correct application of a transformation rule. (For a contrasting approach, see proof-trees).

Example of a proof

- To be shown that $A \to A$.
- One possible proof of this (which, though valid, happens to contain more steps than are necessary) may be arranged as follows:

Example of a Proof

Number	Formula	Reason
1	A	premise
2	$A \vee A$	From (1) by disjunction introduction
3	$(A \vee A) \wedge A$	From (1) and (2) by conjunction introduction
4	A	From (3) by conjunction elimination
5	$A \vdash A$	Summary of (1) through (4)
6	$\vdash A \to A$	From (5) by conditional proof

Interpret $A \vdash A$ as "Assuming A , infer A ". Read $\vdash A \to A$ as "Assuming nothing, infer that A implies A ", or "It is a tautology that A implies A ", or "It is always true that A implies A ".

Soundness and completeness of the rules

The crucial properties of this set of rules are that they are *sound* and *complete*. Informally this means that the rules are correct and that no other rules are required. These claims can be made more formal as follows.

We define a *truth assignment* as a function that maps propositional variables to **true** or **false**. Informally such a truth assignment can be understood as the description of a possible state of affairs (or possible world) where certain statements are true and others are not. The semantics of formulae can then be formalized by defining for which "state of affairs" they are considered to be true, which is what is done by the following definition.

We define when such a truth assignment A satisfies a certain wff with the following rules:

- A satisfies the propositional variable P if and only if $A(P) = \text{true}$
- A satisfies $\neg\phi$ if and only if A does not satisfy ϕ
- A satisfies $(\phi \wedge \psi)$ if and only if A satisfies both ϕ and ψ
- A satisfies $(\phi \vee \psi)$ if and only if A satisfies at least one of either ϕ or ψ
- A satisfies $(\phi \to \psi)$ if and only if it is not the case that A satisfies ϕ but not ψ
- A satisfies $(\phi \leftrightarrow \psi)$ if and only if A satisfies both ϕ and ψ or satisfies neither one of them

With this definition we can now formalize what it means for a formula ϕ to be implied by a certain set S of formulae. Informally this is true if in all worlds that are possible given the set of formulae S the formula ϕ also holds. This leads to the following formal definition: We say that a set S of wffs *semantically entails* (or *implies*) a certain wff ϕ if all truth assignments that satisfy all the formulae in S also satisfy ϕ .

Finally we define *syntactical entailment* such that ϕ is syntactically entailed by S if and only if we can derive it with the inference rules that were presented above in a finite number of steps. This allows us to formulate exactly what it means for the set of inference rules to be sound and complete:

Soundness

If the set of wffs S syntactically entails wff ϕ then S semantically entails ϕ

Completeness

If the set of wffs S semantically entails wff ϕ then S syntactically entails ϕ

For the above set of rules this is indeed the case.

Sketch of a soundness proof

(For most logical systems, this is the comparatively "simple" direction of proof)

Notational conventions: Let G be a variable ranging over sets of sentences. Let A , B, and C range over sentences. For " G syntactically entails A " we write " G proves A ". For " G semantically entails A " we write " G implies A ".

We want to show: (A)(G)(if G proves A , then G implies A).

We note that " G proves A " has an inductive definition, and that gives us the immediate resources for demonstrating claims of the form "If G proves A , then ...". So our proof proceeds by induction.

I. Basis. Show: If A is a member of G , then G implies A .

II. Basis. Show: If A is an axiom, then G implies A .

III. Inductive step (induction on n , the length of the proof):

  a. Assume for arbitrary G and A that if G proves A in n or fewer steps, then G implies A .
  b. For each possible application of a rule of inference at step $n+1$, leading to a new theorem B , show that G implies B .

Notice that Basis Step II can be omitted for natural deduction systems because they have no axioms. When used, Step II involves showing that each of the axioms is a (semantic) logical truth.

The Basis step(s) demonstrate(s) that the simplest provable sentences from G are also implied by G , for any G . (The is simple, since the semantic fact that a set implies any of its members, is also trivial.) The Inductive step will systematically cover all the further sentences that might be provable—by considering each case where we might reach a logical conclusion using an inference rule—and shows that if a new sentence is provable, it is also logically implied. (For example, we might have a rule telling us that from " A " we can derive " A or B ". In III.a We assume that if A is provable it is implied. We also know that if A is provable then " A or B " is provable. We have to show that then " A or B " too is implied. We do so by appeal to the semantic definition and the assumption we just made. A is provable from G , we assume. So it is also implied by G . So any semantic valuation making all of G true makes A true. But any valuation making A true makes " A or B " true, by the defined semantics for "or". So any valuation which makes all of G true makes " A or B " true. So " A or B " is implied.) Generally, the Inductive step will consist of a lengthy but simple case-by-case analysis of all the rules of inference, showing that each "preserves" semantic implication.

By the definition of provability, there are no sentences provable other than by being a member of G , an axiom, or following by a rule; so if all of those are semantically implied, the deduction calculus is sound.

Sketch of completeness proof

(This is usually the much harder direction of proof.)

We adopt the same notational conventions as above.

We want to show: If G implies A , then G proves A . We proceed by contraposition: We show instead that if G does **not** prove A then G does **not** imply A .

I. G does not prove A . (Assumption)

II. If G does not prove A , then we can construct an (infinite) "Maximal Set", G^* , which is a superset of G and which also does not prove A .

  1. Place an "ordering" on all the sentences in the language (e.g., shortest first, and equally long ones in extended alphabetical ordering), and number them $E_1, E_2, \ldots$
  2. Define a series G_n of sets ($G_0, G_1, \ldots$) inductively:
     i. $G_0 = G$
     ii. If $G_k \cup \{E_{k+1}\}$ proves A , then $G_{k+1} = G_k$

   iii. If $G_k \cup \{E_{k+1}\}$ does **not** prove A , then $G_{k+1} = G_k \cup \{E_{k+1}\}$
3. Define G^* as the union of all the G_n . (That is, G^* is the set of all the sentences that are in any G_n .)
4. It can be easily shown that
   i. G^* contains (is a superset of) G (by (b.i));
   ii. G^* does not prove A (because if it proves A then some sentence was added to some G_n which caused it to prove ' A ; but this was ruled out by definition); and
   iii. G^* is a "Maximal Set" (with respect to A): If *any* more sentences whatever were added to G^* , it *would* prove A . (Because if it were possible to add any more sentences, they should have been added when they were encountered during the construction of the G_n , again by definition)

III. If G^* is a Maximal Set (wrt A), then it is "truth-like". This means that it contains the sentence " C " only if it does *not* contain the sentence not- C ; If it contains " C " and contains "If C then B " then it also contains " B "; and so forth.
IV. If G^* is truth-like there is a " G^* -Canonical" valuation of the language: one that makes every sentence in G^* true and everything outside G^* false while still obeying the laws of semantic composition in the language.
V. A G^* -canonical valuation will make our original set G all true, and make A false.
VI. If there is a valuation on which G are true and A is false, then G does not (semantically) imply A .

QED

Another outline for a completeness proof

If a formula is a tautology, then there is a truth table for it which shows that each valuation yields the value true for the formula. Consider such a valuation. By mathematical induction on the length of the subformulae, show that the truth or falsity of the subformula follows from the truth or falsity (as appropriate for the valuation) of each propositional variable in the subformula. Then combine the lines of the truth table together two at a time by using "(P is true implies S) implies ((P is false implies S) implies S)". Keep repeating this until all dependencies on propositional variables have been eliminated. The result is that we have proved the given tautology. Since every tautology is provable, the logic is complete.

Interpretation of a truth-functional propositional calculus

An **interpretation of a truth-functional propositional calculus** $\mathcal{P}$ is an assignment to each propositional symbol of $\mathcal{P}$ of one or the other (but not both) of the truth values truth (**T**) and falsity (**F**), and an assignment to the connective symbols of $\mathcal{P}$ of their usual truth-functional meanings. An interpretation of a truth-functional propositional calculus may also be expressed in terms of truth tables.[9]

For n distinct propositional symbols there are 2^n distinct possible interpretations. For any particular symbol a , for example, there are $2^1 = 2$ possible interpretations:

1. a is assigned **T**, or
2. a is assigned **F**.

For the pair a , b there are $2^2 = 4$ possible interpretations:

1. both are assigned **T**,
2. both are assigned **F**,
3. a is assigned **T** and b is assigned **F**, or
4. a is assigned **F** and b is assigned **T**.[9]

Since $\mathcal{P}$ has $\aleph_0$, that is, denumerably many propositional symbols, there are $2^{\aleph_0} = \mathfrak{c}$, and therefore uncountably many distinct possible interpretations of $\mathcal{P}$.[9]

Interpretation of a sentence of truth-functional propositional logic

If ϕ and ψ are formulas of $\mathcal{P}$ and $\mathcal{I}$ is an interpretation of $\mathcal{P}$ then:

- A sentence of propositional logic is *true under an interpretation* $\mathcal{I}$ iff $\mathcal{I}$ assigns the truth value **T** to that sentence. If a sentence is true under an interpretation, then that interpretation is called a *model* of that sentence.
- ϕ is *false under an interpretation* $\mathcal{I}$ iff ϕ is not true under $\mathcal{I}$.[9]
- A sentence of propositional logic is *logically valid* iff it is true under every interpretation

  $\models \phi$ means that ϕ is logically valid
- A sentence ψ of propositional logic is a *semantic consequence* of a sentence ϕ iff there is no interpretation under which ϕ is true and ψ is false.
- A sentence of propositional logic is *consistent* iff it is true under at least one interpretation. It is inconsistent if it is not consistent.

Some consequences of these definitions:

- For any given interpretation a given formula is either true or false.[9]
- No formula is both true and false under the same interpretation.[9]
- ϕ is false for a given interpretation iff $\neg\phi$ is true for that interpretation; and ϕ is true under an interpretation iff $\neg\phi$ is false under that interpretation.[9]
- If ϕ and $(\phi \to \psi)$ are both true under a given interpretation, then ψ is true under that interpretation.[9]
- If $\models_{\mathrm{P}} \phi$ and $\models_{\mathrm{P}} (\phi \to \psi)$, then $\models_{\mathrm{P}} \psi$.[9]
- $\neg\phi$ is true under $\mathcal{I}$ iff ϕ is not true under $\mathcal{I}$.
- $(\phi \to \psi)$ is true under $\mathcal{I}$ iff either ϕ is not true under $\mathcal{I}$ or ψ is true under $\mathcal{I}$.[9]
- A sentence ψ of propositional logic is a semantic consequence of a sentence ϕ iff $(\phi \to \psi)$ is logically valid, that is, $\phi \models_{\mathrm{P}} \psi$ iff $\models_{\mathrm{P}} (\phi \to \psi)$.[9]

Alternative calculus

It is possible to define another version of propositional calculus, which defines most of the syntax of the logical operators by means of axioms, and which uses only one inference rule.

Axioms

Let ϕ, χ and ψ stand for well-formed formulæ. (The wffs themselves would not contain any Greek letters, but only capital Roman letters, connective operators, and parentheses.) Then the axioms are as follows:

Axioms		
Name	**Axiom Schema**	**Description**
THEN-1	$\phi \to (\chi \to \phi)$	Add hypothesis χ, implication introduction
THEN-2	$(\phi \to (\chi \to \psi)) \to ((\phi \to \chi) \to (\phi \to \psi))$	Distribute hypothesis ϕ over implication
AND-1	$\phi \land \chi \to \phi$	Eliminate conjunction
AND-2	$\phi \land \chi \to \chi$	
AND-3	$\phi \to (\chi \to (\phi \land \chi))$	Introduce conjunction
OR-1	$\phi \to \phi \lor \chi$	Introduce disjunction
OR-2	$\chi \to \phi \lor \chi$	
OR-3	$(\phi \to \psi) \to ((\chi \to \psi) \to (\phi \lor \chi \to \psi))$	Eliminate disjunction
NOT-1	$(\phi \to \chi) \to ((\phi \to \neg\chi) \to \neg\phi)$	Introduce negation
NOT-2	$\phi \to (\neg\phi \to \chi)$	Eliminate negation

NOT-3	$\phi \vee \neg\phi$	Excluded middle, classical logic
IFF-1	$(\phi \leftrightarrow \chi) \to (\phi \to \chi)$	Eliminate equivalence
IFF-2	$(\phi \leftrightarrow \chi) \to (\chi \to \phi)$	
IFF-3	$(\phi \to \chi) \to ((\chi \to \phi) \to (\phi \leftrightarrow \chi))$	Introduce equivalence

- Axiom THEN-2 may be considered to be a "distributive property of implication with respect to implication."
- Axioms AND-1 and AND-2 correspond to "conjunction elimination". The relation between AND-1 and AND-2 reflects the commutativity of the conjunction operator.
- Axiom AND-3 corresponds to "conjunction introduction."
- Axioms OR-1 and OR-2 correspond to "disjunction introduction." The relation between OR-1 and OR-2 reflects the commutativity of the disjunction operator.
- Axiom NOT-1 corresponds to "reductio ad absurdum."
- Axiom NOT-2 says that "anything can be deduced from a contradiction."
- Axiom NOT-3 is called "tertium non datur" (Latin: "a third is not given") and reflects the semantic valuation of propositional formulae: a formula can have a truth-value of either true or false. There is no third truth-value, at least not in classical logic. Intuitionistic logicians do not accept the axiom NOT-3.

Inference rule

The inference rule is modus ponens:

$$\phi,\ \phi \to \chi \vdash \chi .$$

Meta-inference rule

Let a demonstration be represented by a sequence, with hypotheses to the left of the turnstile and the conclusion to the right of the turnstile. Then the deduction theorem can be stated as follows:

If the sequence

$$\phi_1,\ \phi_2,\ \ldots,\ \phi_n,\ \chi \vdash \psi$$

has been demonstrated, then it is also possible to demonstrate the sequence

$$\phi_1,\ \phi_2,\ \ldots,\ \phi_n \vdash \chi \to \psi .$$

This deduction theorem (DT) is not itself formulated with propositional calculus: it is not a theorem of propositional calculus, but a theorem about propositional calculus. In this sense, it is a meta-theorem, comparable to theorems about the soundness or completeness of propositional calculus.

On the other hand, DT is so useful for simplifying the syntactical proof process that it can be considered and used as another inference rule, accompanying modus ponens. In this sense, DT corresponds to the natural conditional proof inference rule which is part of the first version of propositional calculus introduced in this article.

The converse of DT is also valid:

If the sequence

$$\phi_1,\ \phi_2,\ \ldots,\ \phi_n \vdash \chi \to \psi$$

has been demonstrated, then it is also possible to demonstrate the sequence

$$\phi_1,\ \phi_2,\ \ldots,\ \phi_n,\ \chi \vdash \psi$$

in fact, the validity of the converse of DT is almost trivial compared to that of DT:

If

$$\phi_1,\ \ldots,\ \phi_n \vdash \chi \to \psi$$

then

1: $\phi_1, \ \dots, \ \phi_n, \ \chi \vdash \chi \to \psi$

2: $\phi_1, \ \dots, \ \phi_n, \ \chi \vdash \chi$

and from (1) and (2) can be deduced

3: $\phi_1, \ \dots, \ \phi_n, \ \chi \vdash \psi$

by means of modus ponens, Q.E.D.

The converse of DT has powerful implications: it can be used to convert an axiom into an inference rule. For example, the axiom AND-1,

$\vdash \phi \land \chi \to \phi$

can be transformed by means of the converse of the deduction theorem into the inference rule

$\phi \land \chi \vdash \phi$

which is conjunction elimination, one of the ten inference rules used in the first version (in this article) of the propositional calculus.

Example of a proof

The following is an example of a (syntactical) demonstration, involving only axioms THEN-1 and THEN-2:

Prove: $A \to A$ (Reflexivity of implication).

Proof:

1. $(A \to ((B \to A) \to A)) \to ((A \to (B \to A)) \to (A \to A))$
   Axiom THEN-2 with $\phi = A$, $\chi = B \to A$, $\psi = A$
2. $A \to ((B \to A) \to A)$
   Axiom THEN-1 with $\phi = A$, $\chi = B \to A$
3. $(A \to (B \to A)) \to (A \to A)$
   From (1) and (2) by modus ponens.
4. $A \to (B \to A)$
   Axiom THEN-1 with $\phi = A$, $\chi = B$
5. $A \to A$
   From (3) and (4) by modus ponens.

Equivalence to equational logics

The preceding alternative calculus is an example of a Hilbert-style deduction system. In the case of propositional systems the axioms are terms built with logical connectives and the only inference rule is modus ponens. Equational logic as standardly used informally in high school algebra is a different kind of calculus from Hilbert systems. Its theorems are equations and its inference rules express the properties of equality, namely that it is a congruence on terms that admits substitution.

Classical propositional calculus as described above is equivalent to Boolean algebra, while intuitionistic propositional calculus is equivalent to Heyting algebra. The equivalence is shown by translation in each direction of the theorems of the respective systems. Theorems ϕ of classical or intuitionistic propositional calculus are translated as equations $\phi = 1$of Boolean or Heyting algebra respectively. Conversely theorems $x = y$ of Boolean or Heyting algebra are translated as theorems $(x \to y) \land (y \to x)$of classical or propositional calculus respectively, for which $x \equiv y$is a standard abbreviation. In the case of Boolean algebra $x = y$can also be translated as $(x \land y) \lor (\neg x \land \neg y)$, but this translation is incorrect intuitionistically.
In both Boolean and Heyting algebra, inequality $x \leq y$can be used in place of equality. The equality $x = y$is expressible as a pair of inequalities $x \leq y$and $y \leq x$. Conversely the inequality $x \leq y$is expressible as the

equality $x \wedge y = x$, or as $x \vee y = y$. The significance of inequality for Hilbert-style systems is that it corresponds to the latter's or entailment symbol $\vdash$. An entailment

$$\phi_1,\ \phi_2,\ \ldots,\ \phi_n \vdash \psi$$

is translated in the inequality version of the algebraic framework as

$$\phi_1\ \wedge\ \phi_2\ \wedge\ \ldots\ \wedge\ \phi_n\ \ \leq\ \ \psi$$

Conversely the algebraic inequality $x \leq y$ is translated as the entailment

$$x\ \vdash\ y.$$

The difference between implication $x \to y$ and inequality or entailment $x \leq y$ or $x\ \vdash\ y$ is that the former is internal to the logic while the latter is external. Internal implication between two terms is another term of the same kind. Entailment as external implication between two terms expresses a metatruth outside the language of the logic, and is considered part of the metalanguage. Even when the logic under study is intuitionistic, entailment is ordinarily understood classically as two-valued: either the left side entails, or is less-or-equal to, the right side, or it is not.

Similar but more complex translations to and from algebraic logics are possible for natural deduction systems as described above and for the sequent calculus. The entailments of the latter can be interpreted as two-valued, but a more insightful interpretation is as a set, the elements of which can be understood as abstract proofs organized as the morphisms of a category. In this interpretation the cut rule of the sequent calculus corresponds to composition in the category. Boolean and Heyting algebras enter this picture as special categories having at most one morphism per homset, i.e., one proof per entailment, corresponding to the idea that existence of proofs is all that matters: any proof will do and there is no point in distinguishing them.

Graphical calculi

It is possible to generalize the definition of a formal language from a set of finite sequences over a finite basis to include many other sets of mathematical structures, so long as they are built up by finitary means from finite materials. What's more, many of these families of formal structures are especially well-suited for use in logic.

For example, there are many families of graphs that are close enough analogues of formal languages that the concept of a calculus is quite easily and naturally extended to them. Indeed, many species of graphs arise as *parse graphs* in the syntactic analysis of the corresponding families of text structures. The exigencies of practical computation on formal languages frequently demand that text strings be converted into pointer structure renditions of parse graphs, simply as a matter of checking whether strings are wffs or not. Once this is done, there are many advantages to be gained from developing the graphical analogue of the calculus on strings. The mapping from strings to parse graphs is called *parsing* and the inverse mapping from parse graphs to strings is achieved by an operation that is called *traversing* the graph.

Other logical calculi

Propositional calculus is about the simplest kind of logical calculus in current use. It can be extended in several ways. (Aristotelian "syllogistic" calculus, which is largely supplanted in modern logic, is in *some* ways simpler – but in other ways more complex – than propositional calculus.) The most immediate way to develop a more complex logical calculus is to introduce rules that are sensitive to more fine-grained details of the sentences being used.

First-order logic (aka first-order predicate logic) results when the "atomic sentences" of propositional logic are broken up into terms, variables, predicates, and quantifiers, all keeping the rules of propositional logic with some new ones introduced. (For example, from "All dogs are mammals" we may infer "If Rover is a dog then Rover is a mammal".) With the tools of first-order logic it is possible to formulate a number of theories, either with explicit axioms or by rules of inference, that can themselves be treated as logical calculi. Arithmetic is the best known of these; others include set theory and mereology. Second-order logic and other higher-order logics are formal extensions of first-order logic. Thus, it makes sense to refer to propositional logic as *"zeroth-order logic"*, when comparing it with these logics.

Modal logic also offers a variety of inferences that cannot be captured in propositional calculus. For example, from "Necessarily p" we may infer that p. From p we may infer "It is possible that p". The translation between modal logics and algebraic logics is as for classical and intuitionistic logics but with the introduction of a unary operator on Boolean or Heyting algebras, different from the Boolean operations, interpreting the possibility modality, and in the case of Heyting algebra a second operator interpreting necessity (for Boolean algebra this is redundant since necessity is the De Morgan dual of possibility). The first operator preserves 0 and disjunction while the second preserves 1 and conjunction.

Many-valued logics are those allowing sentences to have values other than *true* and *false*. (For example, *neither* and *both* are standard "extra values"; "continuum logic" allows each sentence to have any of an infinite number of "degrees of truth" between *true* and *false*.) These logics often require calculational devices quite distinct from propositional calculus. When the values form a Boolean algebra (which may have more than two or even infinitely many values), many-valued logic reduces to classical logic; many-valued logics are therefore only of independent interest when the values form an algebra that is not Boolean.

Solvers

Finding solutions to propositional logic formulas is an NP-complete problem. However, practical methods exist (e.g., DPLL algorithm, 1962; Chaff algorithm, 2001) that are very fast for many useful cases. Recent work has extended the SAT solver algorithms to work with propositions containing arithmetic expressions; these are the SMT solvers.

References

[1] Ancient Logic (Stanford Encyclopedia of Philosophy) (http://plato.stanford.edu/entries/logic-ancient/)
[2] Marenbon, John (2007). *Medieval philosophy: an historical and philosophical introduction*. Routledge. p. 137.
[3] Leibniz's Influence on 19th Century Logic (http://plato.stanford.edu/entries/leibniz-logic-influence/#Con)
[4] Hurley, Patrick (2007). *A Concise Introduction to Logic 10th edition*. Wadsworth Publishing. p. 392.
[5] Beth, Evert W.; "Semantic entailment and formal derivability", series: Mededlingen van de Koninklijke Nederlandse Akademie van Wetenschappen, Afdeling Letterkunde, Nieuwe Reeks, vol. 18, no. 13, Noord-Hollandsche Uitg. Mij., Amsterdam, 1955, pp. 309–42. Reprinted in Jaakko Intikka (ed.) *The Philosophy of Mathematics*, Oxford University Press, 1969
[6] Truth in Frege (http://frege.brown.edu/heck/pdf/unpublished/TruthInFrege.pdf)
[7] Russell's Use of Truth-Tables (http://digitalcommons.mcmaster.ca/cgi/viewcontent.cgi?article=1219&context=russelljournal)
[8] Toida, Shunichi (2 August 2009). "Proof of Implications" (http://www.cs.odu.edu/~toida/nerzic/content/logic/prop_logic/implications/implication_proof.html). *CS381 Discrete Structures/Discrete Mathematics Web Course Material*. Department Of Computer Science, Old Dominion University. . Retrieved 10 March 2010.
[9] Hunter, Geoffrey (1971). *Metalogic: An Introduction to the Metatheory of Standard First-Order Logic*. University of California Pres. ISBN 0-520-02356-0.

Further reading

- Brown, Frank Markham (2003), *Boolean Reasoning: The Logic of Boolean Equations*, 1st edition, Kluwer Academic Publishers, Norwell, MA. 2nd edition, Dover Publications, Mineola, NY.
- Chang, C.C. and Keisler, H.J. (1973), *Model Theory*, North-Holland, Amsterdam, Netherlands.
- Kohavi, Zvi (1978), *Switching and Finite Automata Theory*, 1st edition, McGraw–Hill, 1970. 2nd edition, McGraw–Hill, 1978.
- Korfhage, Robert R. (1974), *Discrete Computational Structures*, Academic Press, New York, NY.
- Lambek, J. and Scott, P.J. (1986), *Introduction to Higher Order Categorical Logic*, Cambridge University Press, Cambridge, UK.
- Mendelson, Elliot (1964), *Introduction to Mathematical Logic*, D. Van Nostrand Company.

Related works

- Hofstadter, Douglas (1979). *Gödel, Escher, Bach: An Eternal Golden Braid*. Basic Books. ISBN 978-0-465-02656-2.

External links

- Klement, Kevin C. (2006), "Propositional Logic", in James Fieser and Bradley Dowden (eds.), *Internet Encyclopedia of Philosophy*, Eprint (http://www.iep.utm.edu/p/prop-log.htm).
- Introduction to Mathematical Logic (http://www.ltn.lv/~podnieks/mlog/ml2.htm) by V. Detlovs and K. Podnieks
- Formal Predicate Calculus (http://www.qedeq.org/current/doc/math/qedeq_formal_logic_v1_en.pdf), contains a systematic formal development along the lines of Alternative calculus
- *forall x: an introduction to formal logic* (http://www.fecundity.com/logic/), by P.D. Magnus, covers formal semantics and proof theory for sentential logic.
- Propositional Logic on PlanetMath (GFDLed)
- Category:Propositional Calculus (http://www.proofwiki.org/wiki/Category:Propositional_Calculus) on ProofWiki (GFDLed)

Predicate logic

In mathematical logic, **predicate logic** is the generic term for symbolic formal systems like first-order logic, second-order logic, many-sorted logic, or infinitary logic. This formal system is distinguished from other systems in that its formulae contain variables which can be quantified. Two common quantifiers are the existential ∃ ("there exists") and universal ∀ ("for all") quantifiers. The variables could be elements in the universe under discussion, or perhaps relations or functions over that universe. For instance, an existential quantifier over a function symbol would be interpreted as modifier "there is a function".

In informal usage, the term "predicate logic" occasionally refers to first-order logic. Some authors consider the **predicate calculus** to be an axiomatized form of predicate logic, and the predicate logic to be derived from an informal, more intuitive development.[1]

Predicate logics also include logics mixing modal operators and quantifiers. See Modal logic, Saul Kripke, Barcan Marcus formulae, A. N. Prior, and Nicholas Rescher.

Syntax

Predicate calculus symbols may represent either variables, constants, functions or predicates.

1. **Constants** name specific objects or properties in the domain of discourse. Thus George, tree, tall and blue are examples of well formed constant symbols. The constants $\top$ (true) and $\bot$ (false) are sometimes included.
2. **Variable symbols** are used to designate general classes or objects or properties in the domain of discourse.
3. **Functions** denote a mapping of one or more elements in a set(called the *domain* of the function) into a unique element of another set(the *range* of the function). Elements of the domain and range are objects in the world of discourse. Every function symbol has an associated *arity*, indicating the number of elements in the domain mapped onto each element of range.

A *function expression* is a function symbol followed by its arguments. The arguments are elements from the domain of the function; the number of arguments is equal to the arity of the function. The arguments are enclosed in parentheses and separated by commas. e.g.:

- f(X,Y)

- father(david)
- price(apple)

are all well-formed function expressions.

Predicate logics may be viewed syntactically as Chomsky grammars. As such, predicate logics (as well as modal logics and mixed modal predicate logics) may be viewed as context-sensitive, or more typically as context-free, grammars. As each one of the four Chomsky-type grammars have equivalent automata, these logics can be viewed as automata just as well.

Footnotes

[1] Among these authors is Stolyar, p. 166. Hamilton considers both to be calculi but divides them into an informal calculus and a formal calculus.

References

- A. G. Hamilton 1978, *Logic for Mathematicians*, Cambridge University Press, Cambridge UK ISBN 0-521-21838-1.
- Abram Aronovic Stolyar 1970, *Introduction to Elementary Mathematical Logic*, Dover Publications, Inc. NY. ISBN 0-486-64561
- George F Luger, *Artificial Intelligence*, Pearson Education, ISBN 978-81-317-2327-2

Modal logic

Modal logic is a type of formal logic primarily developed in the 1960s that extends classical propositional and predicate logic to include operators expressing modality. Modals—words that express modalities—qualify a statement. For example, the statement "John is happy" might be qualified by saying that John is usually happy, in which case the term "usually" is functioning as a modal. The traditional alethic modalities, or modalities of truth, include possibility ("Possibly, p", "It is possible that p"), necessity ("Necessarily, p", "It is necessary that p"), and impossibility ("It is impossible that p").[1] Other modalities that have been formalized in modal logic include temporal modalities, or modalities of time (notably, "It was the case that p", "It has always been that p", "It will be that p", "It will always be that p"),[2][3] deontic modalities (notably, "It is obligatory that p", and "It is permissible that p"), epistemic modalities, or modalities of knowledge ("It is known that p")[4] and doxastic modalities, or modalities of belief ("It is believed that p").[5]

A formal modal logic represents modalities using modal operators. For example, "It might rain today" and "It is possible that rain will fall today" both contain the notion of possibility. In a modal logic this is represented as an operator, Possibly, attached to the sentence "It will rain today".

The basic unary (1-place) modal operators are usually written $\Box$ for Necessarily and $\Diamond$ for Possibly. In a classical modal logic, each can be expressed by the other with negation:

$$\Diamond P \leftrightarrow \neg\Box\neg P;$$
$$\Box P \leftrightarrow \neg\Diamond\neg P.$$

Thus it is *possible* that it will rain today if and only if it is *not necessary* that it will *not* rain today; and it is *necessary* that it will rain today if and only if it is *not possible* that it will *not* rain today. Alternative symbols used for the modal operators are "L" for Necessarily and "M" for Possibly.[6]

Development of modal logic

Although Aristotle's logic is in large parts concerned with the theory of non-modalized categorical syllogisms, he also developed a modal syllogistic.[7] Moreover, there are passages in his work, such as the famous sea-battle argument in *De Interpretatione* § 9, that are now seen as anticipations of modal logic and its connection with potentiality and time. So are the modal systems developed by Diodorus Cronus, Philo of Megara and the Stoic Chrysippus, which each contained precursors of modal axioms T and D as well as inter-defined notions of necessity and possibility.[8] Modal logic as a self-aware subject owes much to the writings of the Scholastics, in particular William of Ockham and John Duns Scotus, who reasoned informally in a modal manner, mainly to analyze statements about essence and accident.

C. I. Lewis founded modern modal logic in his 1910 Harvard thesis and in a series of scholarly articles beginning in 1912. This work culminated in his 1932 book *Symbolic Logic* (with C. H. Langford), which introduced the five systems *S1* through *S5*.

Ruth C. Barcan (later Ruth Barcan Marcus) developed the first axiomatic systems of quantified modal logic — first and second order extensions of Lewis's "S2", "S4", and "S5".

The contemporary era in modal semantics began in 1959, when Saul Kripke (then only a 19-year-old Harvard University undergraduate) introduced the now-standard Kripke semantics for modal logics. These are commonly referred to as "possible worlds" semantics. Kripke and A. N. Prior had previously corresponded at some length. Kripke semantics is basically simple, but proofs are eased using semantic-tableaux or analytic tableaux, as explained by E. W. Beth.

A. N. Prior created modern temporal logic, closely related to modal logic, in 1957 by adding modal operators [F] and [P] meaning "eventually" and "previously". Vaughan Pratt introduced dynamic logic in 1976. In 1977, Amir Pnueli proposed using temporal logic to formalise the behaviour of continually operating concurrent programs. Flavors of temporal logic include propositional dynamic logic (PDL), propositional linear temporal logic (PLTL), linear temporal logic (LTL), computational tree logic (CTL), Hennessy–Milner logic, and *T*.

The mathematical structure of modal logic, namely Boolean algebras augmented with unary operations (often called modal algebras), began to emerge with J. C. C. McKinsey's 1941 proof that *S2* and *S4* are decidable, and reached full flower in the work of Alfred Tarski and his student Bjarni Jonsson (Jonsson and Tarski 1951–52). This work revealed that *S4* and *S5* are models of interior algebra, a proper extension of Boolean algebra originally designed to capture the properties of the interior and closure operators of topology. Texts on modal logic typically do little more than mention its connections with the study of Boolean algebras and topology. For a thorough survey of the history of formal modal logic and of the associated mathematics, see [[Robert Goldblatt [9]] (2006).]

Formalizations

Semantics

The semantics for modal logic are usually given like so:[10] First we define a *frame*, which consists of a non-empty set, *G*, whose members are generally called possible worlds, and a binary relation, *R*, that holds (or not) between the possible worlds of *G*. This binary relation is called the *accessibility relation*. For example, $w\ R\ v$ means that the world v is accessible from world w. That is to say, the state of affairs known as v is a live possibility for w. This gives a pair, <*G*, *R*>.

Next, the *frame* is extended to a *model* by specifying the truth-values of all propositions at each of the worlds in *G*. We do so by defining a relation $\vDash$ between possible worlds and propositional letters. If there is a world w such that $w \vDash P$, then P is true at w. A model is thus an ordered triple, <*G*, *R*, $\vDash$>.

Then we recursively define the truth of a formula in a model:

- $w \vDash \neg P$ if and only if $w \nvDash P$

- $w \models (P \land Q)$ if and only if $w \models P$ and $w \models Q$
- $w \models \Box P$ if and only if for every element v of G, if $w\ R\ v$ then $v \models P$
- $w \models \Diamond P$ if and only if for some element v of G, it holds that $w\ R\ v$ and $v \models P$

According to these semantics, a truth is *necessary* with respect to a possible world *w* if it is true at every world that is accessible to *w*, and *possible* if it is true at some world that is accessible to *w*. Possibility thereby depends upon the accessibility relation *R*, which allows us to express the relative nature of possibility. For example, we might say that given our laws of physics it is not possible for humans to travel faster than the speed of light, but that given other circumstances it could have been possible to do so. Using the accessibility relation we can translate this scenario as follows: At all of the worlds accessible to our own world, it is not the case that humans can travel faster than the speed of light, but at one of these accessible worlds there is *another* world accessible from *those* worlds but not accessible from our own at which humans can travel faster than the speed of light.

It should also be noted that the definition of $\Box$ makes vacuously true certain sentences, since when it speaks of "every world that is accessible to w" it takes for granted the usual mathematical interpretation of the word "every" (see vacuous truth). Hence, if a world w doesn't have any accessible worlds, any sentence beginning with $\Box$ is true.

The different systems of modal logic are distinguished by the properties of their corresponding accessibility relations. There are several systems that have been espoused (often called *frame conditions*). An accessibility relation is:

- **reflexive** iff $w\ R\ w$, for every w in G
- **symmetric** iff $w\ R\ v$ implies $v\ R\ w$, for all w and v in G
- **transitive** iff $w\ R\ v$ and $v\ R\ q$ together imply $w\ R\ q$, for all w, v, q in G.
- **serial** iff, for each w in G there is some v in G such that $w\ R\ v$.
- **euclidean** iff, for every u,v and w, $w\ R\ u$ and $w\ R\ v$ implies $u\ R\ v$ (note that it also implies: $v\ R\ u$)

The logics that stem from these frame conditions are:

- **K** := no conditions
- **D** := serial
- **T** := reflexive
- **S4** := reflexive and transitive
- **S5** := reflexive, symmetric, transitive and Euclidean

S5 models are reflexive transitive and euclidean. The accessibility relation R is an equivalence relation. The relation *R* is reflexive, symmetric and transitive. It is interesting to note how the euclidean property along with reflexivity yields symmetry and transitivity. We can prove that these frames produce the same set of valid sentences as do any frames where all worlds can see all other worlds of *W*(*i.e.*, where *R* is a "total" relation). This gives the corresponding *modal graph* which is total complete (*i.e.*, no more edges (relations) can be added).

For example, in S4:

$w \models \Diamond P$ if and only if for some element v of G, it holds that $v \models P$ and $w\ R\ v$.

However, in S5, we can just say that

$w \models \Diamond P$ if and only if for some element v of G, it holds that $v \models P$.

We can drop the accessibility clause from the latter stipulation because it is trivially true of all S5 frames that $w\ R\ v$.

All of these logical systems can also be defined axiomatically, as is shown in the next section. For example, in S5, the axioms $P \to \Box \Diamond P$, $\Box P \to \Box \Box P$, and $\Box P \to P$ (corresponding to *symmetry*, *transitivity* and *reflexivity*, respectively) hold, whereas at least one of these axioms does not hold in each of the other, weaker logics.

Axiomatic systems

The first formalizations of modal logic were axiomatic. Numerous variations with very different properties have been proposed since C. I. Lewis began working in the area in 1910. Hughes and Cresswell (1996), for example, describe 42 normal and 25 non-normal modal logics. Zeman (1973) describes some systems Hughes and Cresswell omit.

Modern treatments of modal logic begin by augmenting the propositional calculus with two unary operations, one denoting "necessity" and the other "possibility". The notation of C. I. Lewis, much employed since, denotes "necessarily *p*" by a prefixed "box" ($\Box p$) whose scope is established by parentheses. Likewise, a prefixed "diamond" ($\Diamond p$) denotes "possibly *p*". Regardless of notation, each of these operators is definable in terms of the other:

- $\Box p$ (necessarily *p*) is equivalent to $\neg\Diamond\neg p$ ("not possible that not-*p*")
- $\Diamond p$ (possibly *p*) is equivalent to $\neg\Box\neg p$ ("not necessarily not-*p*")

Hence $\Box$ and $\Diamond$ form a dual pair of operators.

In many modal logics, the necessity and possibility operators satisfy the following analogs of de Morgan's laws from Boolean algebra:

"It is **not necessary that** *X*" is logically equivalent to "It is **possible that not** *X*".

"It is **not possible that** *X*" is logically equivalent to "It is **necessary that not** *X*".

Precisely what axioms and rules must be added to the propositional calculus to create a usable system of modal logic is a matter of philosophical opinion, often driven by the theorems one wishes to prove; or, in computer science, it is a matter of what sort of computational or deductive system one wishes to model. Many modal logics, known collectively as normal modal logics, include the following rule and axiom:

- **N**, **Necessitation Rule**: If *p* is a theorem (of any system invoking **N**), then $\Box p$ is likewise a theorem.
- **K**, **Distribution Axiom**: $\Box(p \to q) \to (\Box p \to \Box q)$.

The weakest normal modal logic, named *K* in honor of Saul Kripke, is simply the propositional calculus augmented by $\Box$, the rule **N**, and the axiom **K**. *K* is weak in that it fails to determine whether a proposition can be necessary but only contingently necessary. That is, it is not a theorem of *K* that if $\Box p$ is true then $\Box\Box p$ is true, i.e., that necessary truths are "necessarily necessary". If such perplexities are deemed forced and artificial, this defect of *K* is not a great one. In any case, different answers to such questions yield different systems of modal logic.

Adding axioms to *K* gives rise to other well-known modal systems. One cannot prove in *K* that if "*p* is necessary" then *p* is true. The axiom **T** remedies this defect:

- **T**, **Reflexivity Axiom**: $\Box p \to p$ (If *p* is necessary, then *p* is the case.) **T** holds in most but not all modal logics. Zeman (1973) describes a few exceptions, such as $S1^0$.

Other well-known elementary axioms are:

- **4**: $\Box p \to \Box\Box p$
- **B**: $p \to \Box\Diamond p$
- **D**: $\Box p \to \Diamond p$
- **5**: $\Diamond p \to \Box\Diamond p$

These yield the systems (axioms in bold, systems in italics):

- *K* := **K** + **N**
- *T* := *K* + **T**
- *S4* := *T* + **4**
- *S5* := *S4* + **5**
- *D* := *K* + **D**.

K through *S5* form a nested hierarchy of systems, making up the core of normal modal logic. But specific rules or sets of rules may be appropriate for specific systems. For example, in deontic logic, $\Box p \rightarrow \Diamond p$ (If it ought to be that *p*, then it is permitted that *p*) seems appropriate, but we should probably not include that $p \rightarrow \Box\Diamond p$. In fact, to do so is to commit the naturalistic fallacy (i.e. to state that what is natural is also good, by saying that if *p* is the case, *p* ought to be permitted).

The commonly employed system *S5* simply makes all modal truths necessary. For example, if *p* is possible, then it is "necessary" that *p* is possible. Also, if *p* is necessary, then it is necessary that *p* is necessary. Other systems of modal logic have been formulated, in part because *S5* does not describe every kind of modality of interest.

Alethic logic

Modalities of necessity and possibility are called *alethic* modalities. They are also sometimes called *special* modalities, from the Latin *species*. Modal logic was first developed to deal with these concepts, and only afterward was extended to others. For this reason, or perhaps for their familiarity and simplicity, necessity and possibility are often casually treated as *the* subject matter of modal logic. Moreover it is easier to make sense of relativizing necessity, e.g. to legal, physical, nomological, epistemic, and so on, than it is to make sense of relativizing other notions.

In classical modal logic, a proposition is said to be

- **possible** if and only if it is *not necessarily false* (regardless of whether it is actually true or actually false);
- **necessary** if and only if it is *not possibly false*; and
- **contingent** if and only if it is *not necessarily false* and *not necessarily true* (i.e. possible but not necessarily true).

In classical modal logic, therefore, either the notion of possibility or necessity may be taken to be basic, where these other notions are defined in terms of it in the manner of De Morgan duality. Intuitionistic modal logic treats possibility and necessity as not perfectly symmetric.

For those with difficulty with the concept of something being possible but not true, the meaning of these terms may be made more comprehensible by thinking of multiple "possible worlds" (in the sense of Leibniz) or "alternate universes"; something "necessary" is true in all possible worlds, something "possible" is true in at least one possible world. These "possible world semantics" are formalized with Kripke semantics.

Physical possibility

Something is physically, or nomically, possible if it is permitted by the laws of physics. For example, current theory is thought to allow for there to be an atom with an atomic number of 126,[11] even if there are no such atoms in existence. Similarly, while it is logically possible to accelerate beyond the speed of light,[12] modern science stipulates that it is not physically possible for material particles or information.[13]

Metaphysical possibility

Philosophers ponder the properties that objects have independently of those dictated by scientific laws. For example, it might be metaphysically necessary, as some have thought, that all thinking beings have bodies and can experience the passage of time. Saul Kripke has argued that every person necessarily has the parents they do have: anyone with different parents would not be the same person.[14]

Metaphysical possibility is generally thought to be more restricting than bare logical possibility (i.e., fewer things are metaphysically possible than are logically possible). Its exact relation to physical possibility is a matter of some dispute. Philosophers also disagree over whether metaphysical truths are necessary merely "by definition", or whether they reflect some underlying deep facts about the world, or something else entirely.

Confusion with epistemic modalities

Alethic modalities and epistemic modalities (see below) are often expressed in English using the same words. "It is possible that bigfoot exists" can mean either "Bigfoot *could* exist, whether or not bigfoot does in fact exist" (alethic), or more likely, "For all I know, bigfoot exists" (epistemic).

It has been questioned whether these modalities should be considered distinct from each other. The criticism states that there is no real difference between "the truth in the world" (alethic) and "the truth in an individual's mind" (epistemic).[15] An investigation has not found a single language in which alethic and epistemic modalities are formally distinguished, as by the means of a grammatical mood.[16]

Epistemic logic

Epistemic modalities (from the Greek *episteme*, knowledge), deal with the *certainty* of sentences. The $\Box$ operator is translated as "x knows that…", and the $\Diamond$ operator is translated as "For all x knows, it may be true that…" In ordinary speech both metaphysical and epistemic modalities are often expressed in similar words; the following contrasts may help:

A person, Jones, might reasonably say *both*: (1) "No, it is *not* possible that Bigfoot exists; I am quite certain of that"; *and*, (2) "Sure, Bigfoot possibly *could* exist". What Jones means by (1) is that given all the available information, there is no question remaining as to whether Bigfoot exists. This is an epistemic claim. By (2) he makes the *metaphysical* claim that it is *possible for* Bigfoot to exist, *even though he does not* (which is not equivalent to "it is *possible that* Bigfoot exists – for all I know", which contradicts (1)).

From the other direction, Jones might say, (3) "It is *possible* that Goldbach's conjecture is true; but also *possible* that it is false", and *also* (4) "if it *is* true, then it is necessarily true, and not possibly false". Here Jones means that it is *epistemically possible* that it is true or false, for all he knows (Goldbach's conjecture has not been proven either true or false), but if there *is* a proof (heretofore undiscovered), then it would show that it is not *logically* possible for Goldbach's conjecture to be false—there could be no set of numbers that violated it. Logical possibility is a form of *alethic* possibility; (4) makes a claim about whether it is possible (i.e., logically speaking) that a mathematical truth to have been false, but (3) only makes a claim about whether it is possible, for all Jones knows, (i.e., speaking of certitude) that the mathematical claim is specifically either true or false, and so again Jones does not contradict himself. It is worthwhile to observe that Jones is not necessarily correct: It is possible (epistemically) that Goldbach's conjecture is both true and unprovable.[17]

Epistemic possibilities also bear on the actual world in a way that metaphysical possibilities do not. Metaphysical possibilities bear on ways the world *might have been,* but epistemic possibilities bear on the way the world *may be* (for all we know). Suppose, for example, that I want to know whether or not to take an umbrella before I leave. If you tell me "it is *possible that* it is raining outside" – in the sense of epistemic possibility – then that would weigh on whether or not I take the umbrella. But if you just tell me that "it is *possible for* it to rain outside" – in the sense of *metaphysical possibility* – then I am no better off for this bit of modal enlightenment.

Some features of epistemic modal logic are in debate. For example, if *x* knows that *p*, does *x* know that it knows that *p*? That is to say, should $\Box P \to \Box\Box P$ be an axiom in these systems? While the answer to this question is unclear, there is at least one axiom that is generally included in epistemic modal logic, because it is minimally true of all normal modal logics (see the section on axiomatic systems):

- **K**, *Distribution Axiom*: $\Box(p \to q) \to (\Box p \to \Box q)$.

But this is disconcerting, because with **K**, we can prove that we know all the logical consequences of our beliefs: If *q* is a logical consequence of *p*, then $\Box(p \to q)$. And if so, then we can deduce that $(\Box p \to \Box q)$ using **K**. When we translate this into epistemic terms, this says that if *q* is a logical consequence of *p*, then we know that it is, and if we know *p*, we know *q*. That is to say, we know all the logical consequences of our beliefs. This must be true for all possible Kripkean modal interpretations of epistemic cases where $\Box$ is translated as "knows that". But then, for example, if x knows that prime numbers are divisible only by themselves and the number one, then x knows that 8683317618811886495518194401279999999 is prime (since this number is only divisible by itself and the number one). That is to say, under the modal interpretation of knowledge, anyone who knows the definition of a prime number knows that this number is prime. This shows that epistemic modal logics that are based on normal modal systems provide an idealized account of knowledge, and explain objective, rather than subjective knowledge (if anything).

Temporal logic

Temporal logic is an approach to the semantics of expressions with tense, that is, expressions with qualifications of when. Some expressions, such as '2 + 2 = 4', are true at all times, while tensed expressions such as 'John is happy' are only true sometimes.

In temporal logic, tense constructions are treated in terms of modalities, where a standard method for formalizing talk of time is to use *two* pairs of operators, one for the past and one for the future (P will just mean 'it is presently the case that P'). For example:

F*P* : It will sometimes be the case that *P*

G*P* : It will always be the case that *P*

P*P* : It was sometime the case that *P*

H*P* : It has always been the case that *P*

There are then at least three modal logics that we can develop. For example, we can stipulate that,

$\Diamond P$ = *P* is the case at some time *t*

$\Box P$ = *P* is the case at every time *t*

Or we can trade these operators to deal only with the future (or past). For example,

$\Diamond_1 P$ = **F***P*

$\Box_1 P$ = **G***P*

or,

$\Diamond_2 P$ = *P* and/or **F***P*

$\Box_2 P$ = *P* and **G***P*

The operators **F** and **G** may seem initially foreign, but they create normal modal systems. Note that **F***P* is the same as ¬**G**¬*P*. We can combine the above operators to form complex statements. For example, **P***P* → $\Box$**P***P* says (effectively), *Everything that is past and true is necessary*.

It seems reasonable to say that possibly it will rain tomorrow, and possibly it won't; on the other hand, since we can't change the past, if it is true that it rained yesterday, it probably isn't true that it may not have rained yesterday. It seems the past is "fixed", or necessary, in a way the future is not. This is sometimes referred to as accidental necessity. But if the past is "fixed", and everything that is in the future will eventually be in the past, then it seems plausible to say that future events are necessary too.

Similarly, the problem of future contingents considers the semantics of assertions about the future: is either of the propositions 'There will be a sea battle tomorrow', or 'There will not be a sea battle tomorrow' now true? Considering this thesis led Aristotle to reject the principle of bivalence for assertions concerning the future.

Additional binary operators are also relevant to temporal logics, *q.v.* Linear Temporal Logic.

Versions of temporal logic can be used in computer science to model computer operations and prove theorems about them. In one version, $\Diamond P$ means "at a future time in the computation it is possible that the computer state will be such that P is true"; $\Box P$ means "at all future times in the computation P will be true". In another version, $\Diamond P$ means "at the immediate next state of the computation, P might be true"; $\Box P$ means "at the immediate next state of the computation, P will be true". These differ in the choice of Accessibility relation. (P always means "P is true at the current computer state".) These two examples involve nondeterministic or not-fully-understood computations; there are many other modal logics specialized to different types of program analysis. Each one naturally leads to slightly different axioms.

A variation, closely related to Temporal or Chronological or Tense logics, are Modal logics based upon "topology", "place", or "spatial position".[18][19] One might also take note that in the Russian language, verbs have an aspect, based commonly upon time, but position also.

Deontic logic

Likewise talk of morality, or of obligation and norms generally, seems to have a modal structure. The difference between "You must do this" and "You may do this" looks a lot like the difference between "This is necessary" and "This is possible". Such logics are called *deontic*, from the Greek for "duty".

Deontic logics commonly lack the axiom **T** semantically corresponding to the reflexivity of the accessibility relation in Kripke semantics: in symbols, $\Box\phi \to \phi$. Interpreting $\Box$ as "it is obligatory that", **T** informally says that every obligation is true. For example, if it is obligatory not to kill others (i.e. killing is morally forbidden), then **T** implies that people actually do not kill others. The consequent is obviously false.

Instead, using Kripke semantics, we say that though our own world does not realize all obligations, the worlds accessible to it do (i.e., **T** holds at these worlds). These worlds are called idealized worlds. *P* is obligatory with respect to our own world if at all idealized worlds accessible to our world, *P* holds. Though this was one of the first interpretations of the formal semantics, it has recently come under criticism.[20]

One other principle that is often (at least traditionally) accepted as a deontic principle is *D*, $\Box\phi \to \Diamond\phi$, which corresponds to the seriality (or extendability or unboundedness) of the accessibility relation. It is an embodiment of the Kantian idea that "ought implies can". (Clearly the "can" can be interpreted in various senses, e.g. in a moral or alethic sense.)

Intuitive problems with deontic logic

When we try and formalize ethics with standard modal logic, we run into some problems. Suppose that we have a proposition *K*: you have killed the victim, and another, *Q*: you have killed the victim quickly. Now suppose we want to express the thought that "if you have killed the victim, it ought to be the case that you have killed him quickly". There are two likely candidates,

(1) $(K \to \Box Q)$

(2) $\Box(K \to Q)$

But (1) says that if you have killed the victim, then it ought to be the case that you have killed him quickly. This surely isn't right, because you ought not to have killed him at all. And (2) doesn't work either. If the right representation of "if you have killed the victim then you ought to have killed him quickly" is (2), then the right representation of (3) "if you have killed the victim then you ought to have killed him slowly" is $\Box(K \to \neg Q)$. Now suppose (as seems reasonable) that you should not have killed the victim, or $\Box\neg K$. But then we can deduce $\Box(K \to \neg Q)$, which would express sentence (3). So if you should not have killed the victim, then if you did kill him, you should have killed him slowly. But that can't be right, and is not right when we use natural language. Telling someone they should not kill the victim certainly does not imply that they should kill the victim slowly if they do kill him.[21]

Doxastic logic

Doxastic logic concerns the logic of belief (of some set of agents). The term doxastic is derived from the ancient Greek *doxa* which means "belief". Typically, a doxastic logic uses $\Box$, often written "B", to mean "It is believed that", or when relativized to a particular agent s, "It is believed by s that".

Other modal logics

Significantly, modal logics can be developed to accommodate most of these idioms; it is the fact of their common logical structure (the use of "intensional" sentential operators) that make them all varieties of the same thing.

The ontology of possibility

Further information: Accessibility relation and Possible worlds

In the most common interpretation of modal logic, one considers "logically possible worlds". If a statement is true in all possible worlds, then it is a necessary truth. If a statement happens to be true in our world, but is not true in all possible worlds, then it is a contingent truth. A statement that is true in some possible world (not necessarily our own) is called a possible truth.

Under this "possible worlds idiom," to maintain that Bigfoot's existence is possible but not actual, one says, "There is some possible world in which Bigfoot exists; but in the actual world, Bigfoot does not exist". However, it is unclear what this claim commits us to. Are we really alleging the existence of possible worlds, every bit as real as our actual world, just not actual? Saul Kripke believes that 'possible world' is something of a misnomer – that the term 'possible world' is just a useful way of visualizing the concept of possibility.[22] For him, the sentences "you could have rolled a 4 instead of a 6" and "there is a possible world where you rolled a 4, but you rolled a 6 in the actual world" are not significantly different statements, and neither commit us to the existence of a possible world.[23] David Lewis, on the other hand, made himself notorious by biting the bullet, asserting that all merely possible worlds are as real as our own, and that what distinguishes our world as *actual* is simply that it is indeed our world – *this* world.[24] That position is a major tenet of "modal realism". Some philosophers decline to endorse any version of modal realism, considering it ontologically extravagant, and prefer to seek various ways to paraphrase away these ontological commitments. Robert Adams holds that 'possible worlds' are better thought of as 'world-stories', or consistent sets of propositions. Thus, it is possible that you rolled a 4 if such a state of affairs can be described coherently.[25]

Computer scientists will generally pick a highly specific interpretation of the modal operators specialized to the particular sort of computation being analysed. In place of "all worlds", you may have "all possible next states of the computer", or "all possible future states of the computer".

Applications

- Modality has also been treated from the viewpoint of "counter-factuals" in literature (see Victorian Studies).[26][27][28]
- Modality modifies propositions and modalities provide closure (i.e.: propositions with modalities are still propositions). Thus, as propositions constitute a part of language, they may be understood as subject to linguistic analysis such as that of Noam Chomsky. Modalities might then be viewed as being context-free, context-sensitive, or even fully phrase-structured (Chomsky type-0) languages. This broadens the view of modalities which are usually viewed as context-free. A discussion of this may be found under the Philosophy of language.
- Aristotle classified and discussed rhetoric as being based upon the enthymeme, thus closely related to logic. However, it is clear that if logic is extended by modal logics, multi-valued logics, etc., then rhetoric must also be extended by modern developments.

Further applications

Modal logics have begun to be used in areas of the humanities such as literature, poetry, art and history.[29][30][31]

Controversies

Modal logic has been rejected by many philosophers. (Historically, philosophers starting with Aristotle seem to have had priority of interest in modal logic over mathematicians.) The primary reason for this rejection is that modality provides a different logical framework with which to analyze society. Nicholas Rescher has pointed out how radically conservative were the views of Bertrand Russell in his unreasoned rejection of both Modal Logic and the ideas of the philosopher Alexius Meinong. Indeed, Arthur Norman Prior warned his protégé Ruth Barcan to prepare well in the debates concerning Quantified Modal Logic with Willard Van Orman Quine, due to the biases against Modal Logic.[32]

Notes

[1] "Formal Logic", by A. N. Prior, Oxford Univ. Press, 1962, p. 185

[2] "Temporal Logic", by Rescher and Urquhart, Springer-Verlag, 1971, p. 52

[3] "Past, Present and Future", by A. N. Prior, Oxford Univ. Press, 1967

[4] "Knowledge and Belief", by Jaakko Hinntikka, Cornell Univ. Press, 1962

[5] "Topics in Philosophical Logic", by N. Rescher, Humanities Press, 1968, p. 41

[6] So in the standard work *A New Introduction to Modal Logic*, by G. E. Hughes and M. J. Cresswell, Routledge, 1996, *passim*.

[7] (http://plato.stanford.edu/entries/logic-ancient/#ModLog) Stanford Encyclopedia of Philosophy: Susanne Bobzien, *Ancient Logic*

[8] (http://plato.stanford.edu/entries/logic-ancient/#DioCroPhiLog) Stanford Encyclopedia of Philosophy: Susanne Bobzien, *Ancient Logic*

[9] http://www.mcs.vuw.ac.nz/~rob/papers/modalhist.pdf

[10] Fitting and Mendelsohn. *First-Order Modal Logic*. Kluwer Academic Publishers, 1998. Section 1.6

[11] "Superheavy Element 114 Confirmed: A Stepping Stone to the Island of Stability" (http://phys.org/news173028810.html). phys.org. .

[12] Feinberg, G. (1967). "Possibility of Faster-Than-Light Particles". *Physical Review* **159** (5): 1089–1105. Bibcode 1967PhRv..159.1089F. doi:10.1103/PhysRev.159.1089. See also Feinberg's later paper: Phys. Rev. D 17, 1651 (1978)

[13] Einstein, Albert (1905-06-30). "Zur Elektrodynamik bewegter Körper". *Annalen der Physik* **17** (10): 891–921. Bibcode 1905AnP...322..891E. doi:10.1002/andp.19053221004.

[14] Saul Kripke. *Naming and Necessity*. Harvard University Press, 1980. pg 113

[15] Eschenroeder, Erin; Sarah Mills; Thao Nguyen (2006-09-30). William Frawley. ed. *The Expression of Modality* (http://books.google.co.uk/books?id=72URszHq2SEC&pg=PT18). The Expression of Cognitive Categories. Mouton de Gruyter. pp. 8–9. ISBN 3-11-018436-2. . Retrieved 2010-01-03.

[16] Nuyts, Jan (November 2000). *Epimestic Modality, Language, and Conceptualization: A Cognitive-pragmatic Perspective*. Human Cognitive Processing. John Benjamins Publishing Co. p. 28. ISBN 90-272-2357-2.

[17] See Goldbach's conjecture > Origins

[18] Nicholas Rescher, James Garson, "Topological Logic" in The Journal of Symbolic Logic, 33(4):537-548, December, 1968

[19] Georg Henrik von Wright, "A Modal Logic of Place", in E. Sosa (Editor), pp. 65-73, "The Philosophy of Nicholas Rescher: Discussion and Replies", D. Reidel, Dordrecht, Holland, 1979

[20] See, e.g., Sven Hansson, "Ideal Worlds—Wishful Thinking in Deontic Logic", Studia Logica, Vol. 82 (3), pp. 329–336, 2006.

[21] Ted Sider's *Logic for Philosophy*, unknown page.

[22] Kripke, Saul. *Naming and Necessity*. (1980; Harvard UP), pp. 43–5.

[23] Kripke, Saul. *Naming and Necessity*. (1980; Harvard UP), pp. 15–6.

[24] David Lewis, *On the Plurality of Worlds* (1986; Blackwell)

[25] Adams, Robert M. *Theories of Actuality* (http://www.jstor.org/stable/2214751). Noûs, Vol. 8, No. 3 (Sep., 1974), particularly pp. 225–31.

[26] "Possible Worlds of Fiction and History", by Dolezel, Lubomír, New Literary History, 1998, 29(4): 785-809

[27] "Lives Unled in Realist Fiction", by Miller, Andrew H., Representations 98, Spring 2007, The Regents of the University of California, ISSN 1553-855X, pp. 118-134.

[28] "Not Forthcoming", by Miller, Andrew H., Dickens Universe, 2009, Univ. of California, Santa Cruz

[29] See http://www.estherlederberg.com/EImages/Extracurricular/Dickens%20Universe/Counter%20Factuals.html

[30] Andrew H. Miller, "Lives Unled in Realist Fiction", Representations 98, Spring 2007, The Regents of the University of California, ISSN 1553-855X, pp. 118-134

[31] See also http://www.estherlederberg.com/EImages/Extracurricular/Dickens%20Universe/Page%2017%20CounterFactuals.html

[32] "Modalities: Philosophical Essays", by Ruth Barcan Marcus, Oxford Univ. Press, 1993, Chapter 14

References

- *This article includes material from the* Free On-line Dictionary of Computing, *used with permission under the* GFDL.
- Blackburn, Patrick; de Rijke, Maarten; and Venema, Yde (2001) *Modal Logic*. Cambridge University Press. ISBN 0-521-80200-8
- Barcan-Marcus, Ruth JSL 11 (1946) and JSL 112 (1947) and "Modalities", OUP, 1993, 1995
- Beth, Evert W., 1955. "Semantic entailment and formal derivability", Mededlingen van de Koninklijke Nederlandse Akademie van Wetenschappen, Afdeling Letterkunde, N.R. Vol 18, no 13, 1955, pp 309–42. Reprinted in Jaakko Intikka (ed.) The Philosophy of Mathematics, Oxford University Press, 1969 (Semantic Tableaux proof methods)
- Beth, Evert W., "Formal Methods: An Introduction to Symbolic Logic and to the Study of Effective Operations in Arithmetic and Logic", D. Reidel, 1962 (Semantic Tableaux proof methods)
- Blackburn, P.; van Benthem, J.; and Wolter, Frank; Eds. (2006) *Handbook of Modal Logic* (http://www.csc.liv.ac.uk/~frank/MLHandbook/). North Holland.
- Chagrov, Aleksandr; and Zakharyaschev, Michael (1997) *Modal Logic*. Oxford University Press. ISBN 0-19-853779-4
- Chellas, B. F. (1980) *Modal Logic: An Introduction*. Cambridge University Press. ISBN 0-521-22476-4
- Cresswell, M. J. (2001) "Modal Logic" in Goble, Lou; Ed., *The Blackwell Guide to Philosophical Logic*. Basil Blackwell: 136–58. ISBN 0-631-20693-0
- Fitting, Melvin; and Mendelsohn, R. L. (1998) *First Order Modal Logic*. Kluwer. ISBN 0-7923-5335-8
- James Garson (2006) *Modal Logic for Philosophers*. Cambridge University Press. ISBN 0-521-68229-0. A thorough introduction to modal logic, with coverage of various derivation systems and a distinctive approach to the use of diagrams in aiding comprehension.
- Girle, Rod (2000) *Modal Logics and Philosophy*. Acumen (UK). ISBN 0-7735-2139-9. Proof by refutation trees. A good introduction to the varied interpretations of modal logic.
- Goldblatt, Robert (http://www.mcs.vuw.ac.nz/~rob/) (1992) "Logics of Time and Computation", 2nd ed., CSLI Lecture Notes No. 7. University of Chicago Press.
- —— (1993) *Mathematics of Modality*, CSLI Lecture Notes No. 43. University of Chicago Press.
- —— (2006) " Mathematical Modal Logic: a View of its Evolution (http://www.mcs.vuw.ac.nz/~rob/papers/modalhist.pdf)", in Gabbay, D. M.; and Woods, John; Eds., *Handbook of the History of Logic, Vol. 6*. Elsevier BV.
- Goré, Rajeev (1999) "Tableau Methods for Modal and Temporal Logics" in D'Agostino, M.; Gabbay, Dov; Haehnle, R.; and Posegga, J.; Eds., *Handbook of Tableau Methods*. Kluwer: 297–396.
- Hughes, G. E., and Cresswell, M. J. (1996) *A New Introduction to Modal Logic*. Routledge. ISBN 0-415-12599-5
- Jónsson, B. and Tarski, A., 1951–52, "Boolean Algebra with Operators I and II", *American Journal of Mathematics 73*: 891-939 and *74*: 129–62.
- Kracht, Marcus (1999) *Tools and Techniques in Modal Logic*, Studies in Logic and the Foundations of Mathematics No. 142. North Holland.
- Lemmon, E. J. (with Scott, D.) (1977) *An Introduction to Modal Logic*, American Philosophical Quarterly Monograph Series, no. 11 (Krister Segerberg, series ed.). Basil Blackwell.
- Lewis, C. I. (with Langford, C. H.) (1932). *Symbolic Logic*. Dover reprint, 1959.
- Prior, A. N. (1957) *Time and Modality*. Oxford University Press.
- Snyder, D. Paul "Modal Logic and its applications", Van Nostrand Reinhold Company, 1971 (proof tree methods)
- Zeman, J. J. (1973) *Modal Logic.* (http://www.clas.ufl.edu/users/jzeman/modallogic/) Reidel. Employs Polish notation.

Further reading

- D.M. Gabbay, A. Kurucz, F. Wolter and M. Zakharyaschev, *Many-Dimensional Modal Logics: Theory and Applications*, Elsevier, Studies in Logic and the Foundations of Mathematics, volume 148, 2003, ISBN 0-444-50826-0. Covers many varieties of modal logics, e.g. temporal, epistemic, dynamic, description, spatial from a unified perspective with emphasis on computer science aspects, e.g. decidability and complexity.

External links

- Stanford Encyclopedia of Philosophy:
  - " Modal logic (http://plato.stanford.edu/entries/logic-modal)" – by James Garson.
  - " Provability Logic (http://plato.stanford.edu/entries/logic-provability/)" – by Rineke Verbrugge.
- Edward N. Zalta, 1995, " Basic Concepts in Modal Logic. (http://mally.stanford.edu/notes.pdf)"
- John McCarthy, 1996, " Modal Logic. (http://www-formal.stanford.edu/jmc/mcchay69/node22.html)"
- Molle (http://molle.sourceforge.net/) a Java prover for experimenting with modal logics
- Suber, Peter, 2002, " Bibliography of Modal Logic. (http://www.earlham.edu/~peters/courses/logsys/nonstbib.htm#modal)"
- Marcus, Ruth Barcan, "Modalities" OUP 1993, 1995
- List of Logic Systems (http://www.cc.utah.edu/~nahaj/logic/structures/systems/index.html) List of many modal logics with sources, by John Halleck.
- Advances in Modal Logic. (http://aiml.net/) Biannual international conference and book series in modal logic.
- S4prover (http://teachinglogic.imag.fr/TableauxS4) A tableaux prover for S4 logic

Informal logic

Informal logic, intuitively, refers to the principles of logic and logical thought outside of a formal setting. However, perhaps because of the **informal** in the title, the precise definition of **informal logic** is a matter of some dispute.[1] Ralph H. Johnson and J. Anthony Blair define informal logic as "a branch of logic whose task is to develop non-formal standards, criteria, procedures for the analysis, interpretation, evaluation, criticism and construction of argumentation."[2] This definition reflects what had been implicit in their practice and what others[3][4][5] were doing in their informal logic texts.

Informal logic is associated with (informal) fallacies, critical thinking, the Thinking Skills Movement[6] and the interdisciplinary inquiry known as argumentation theory. Frans H. van Eemeren writes that the label "informal logic" covers a "collection of normative approaches to the study of reasoning in ordinary language that remain closer to the practice of argumentation than formal logic."[7]

History

Informal logic as a distinguished enterprise under this name emerged roughly in the late 1970s as a sub-field of philosophy. The naming of the field was preceded by the appearance of a number of textbooks that rejected the symbolic approach to logic on pedagogical grounds as inappropriate and unhelpful for introductory textbooks on logic for a general audience, for example Howard Kahane's *Logic and Contemporary Rhetoric*, subtitled "The Use of Reason in Everyday Life", first published in 1971. Kahane's textbook was described on the notice of his death in the *Proceedings And Addresses of the American Philosophical Association* (2002) as "a text in informal logic, [that] was intended to enable students to cope with the misleading rhetoric one frequently finds in the media and in political discourse. It was organized around a discussion of fallacies, and was meant to be a practical instrument for dealing with the problems of everyday life. [It has] ... gone through many editions; [it is] ... still in print; and the thousands upon thousands of students who have taken courses in which his text [was] ... used can thank Howard for contributing to their ability to dissect arguments and avoid the deceptions of deceitful rhetoric. He tried to put into practice the ideal of discourse that aims at truth rather than merely at persuasion. (Hausman et al. 2002)"[8][9] Other textbooks from the era taking this approach were Michael Scriven's *Reasoning* (Edgepress, 1976) and *Logical Self-Defense* by Ralph H. Johnson and J. Anthony Blair, first published in 1977.[8] Earlier precursors in this tradition can be considered Monroe Beardsley's *Practical Logic* (1950) and Stephen Toulmin's *The Uses of Argument* (1958).[10]

The field perhaps became recognized under its current name with the *First International Symposium on Informal Logic* held in 1978. Although initially motivated by a new pedagogical approach to undergraduate logic textbooks, the scope of the field was basically defined by a list of 13 problems and issues which Blair and Johnson included as an appendix to their keynote address at this symposium:[8][11]

- the theory of logical criticism
- the theory of argument
- the theory of fallacy
- the fallacy approach vs. the critical thinking approach
- the viability of the inductive/deductive dichotomy
- the ethics of argumentation and logical criticism
- the problem of assumptions and missing premises
- the problem of context
- methods of extracting arguments from context
- methods of displaying arguments
- the problem of pedagogy
- the nature, division and scope of informal logic
- the relationship of informal logic to other inquiries

David Hitchcock argues that the naming of the field was unfortunate, and that *philosophy of argument* would have been more appropriate. He argues that more undergraduate students in North America study informal logic than any other branch of philosophy, but that as of 2003 informal logic (or philosophy of argument) was not recognized as separate sub-field by the World Congress of Philosophy.[8] Frans H. van Eemeren wrote that "informal logic" is mainly an approach to argumentation advanced by a group of US and Canadian philosophers and largely based on the previous works of Stephen Toulmin and to a lesser extent those of Chaïm Perelman.[7]

Alongside the symposia, since 1983 the journal *Informal Logic* has been the publication of record of the field, with Blair and Johnson as initial editors, with the editorial board now including two other colleagues from the University of Windsor—Christopher Tindale and Hans V. Hansen.[12] Other journals that regularly publish articles on informal logic include *Argumentation* (founded in 1986), *Philosophy and Rhetoric*, *Argumentation and Advocacy* (the journal of the American Forensic Association), and *Inquiry: Critical Thinking Across the Disciplines* (founded in 1988).[13]

Proposed definitions

Johnson and Blair (2000) proposed the following definition: "Informal logic designates that branch of logic whose task is to develop non-formal^2 standards, criteria, procedures for the analysis, interpretation, evaluation, critique and construction of argumentation in everyday discourse." Their meaning of non-formal^2 is taken from Barth and Krabbe (1982), which is explained below.

To understand the definition above, one must understand "informal" which takes its meaning in contrast to its counterpart "formal." (This point was not made for a very long time, hence the nature of informal logic remained opaque, even to those involved in it, for a period of time.) Here it is helpful to have recourse[14] to Barth and Krabbe (1982:14f) where they distinguish three senses of the term "form." By "form^1," Barth and Krabbe mean the sense of the term which derives from the Platonic idea of form—the ultimate metaphysical unit. Barth and Krabbe claim that most traditional logic is formal in this sense. That is, syllogistic logic is a logic of terms where the terms could naturally be understood as place-holders for Platonic (or Aristotelian) forms. In this first sense of "form," almost all logic is informal (not-formal). Understanding informal logic this way would be much too broad to be useful.

By "form^2," Barth and Krabbe mean the form of sentences and statements as these are understood in modern systems of logic. Here validity is the focus: if the premises are true, the conclusion must then also be true. Now validity has to do with the logical form of the statement that makes up the argument. In this sense of "formal," most modern and contemporary logic is "formal." That is, such logics canonize the notion of logical form, and the notion of validity plays the central normative role. In this second sense of form, informal logic is not-formal, because it abandons the notion of logical form as the key to understanding the structure of arguments, and likewise retires validity as normative for the purposes of the evaluation of argument. It seems to many that validity is too stringent a requirement, that there are good arguments in which the conclusion is supported by the premises even though it does not follow necessarily from them (as validity requires). An argument in which the conclusion is thought to be "beyond reasonable doubt, given the premises" is sufficient in law to cause a person to be sentenced to death, even though it does not meet the standard of logical validity. This type of argument, based on accumulation of evidence rather than pure deduction, is called a conductive argument.

By "form^3," Barth and Krabbe mean to refer to "procedures which are somehow regulated or regimented, which take place according to some set of rules." Barth and Krabbe say that "we do not defend formality^3 of all kinds and under all circumstances." Rather "we defend the thesis that verbal dialectics must have a certain form (i.e., must proceed according to certain rules) in order that one can speak of the discussion as being won or lost" (19). In this third sense of "form", informal logic can be formal, for there is nothing in the informal logic enterprise that stands opposed to the idea that argumentative discourse should be subject to norms, i.e., subject to rules, criteria, standards or procedures. Informal logic does present standards for the evaluation of argument, procedures for detecting missing premises etc.

Johnson and Blair (2000) noticed a limitation of their own definition, particularly with respect to "everyday discourse", which could indicate that it does not seek to understand specialized, domain-specific arguments made in natural languages. Consequently, they have argued that the crucial divide is between arguments made in formal languages and those made in natural languages.

Fisher and Scriven (1997) proposed a more encompassing definition, seeing informal logic as "the discipline which studies the practice of critical thinking and provides its intellectual spine". By "critical thinking" they understand "skilled and active interpretation and evaluation of observations and communications, information and argumentation."[15]

Criticisms

Some hold the view that informal logic is not a branch or subdiscipline of logic, or even the view that there cannot be such a thing as informal logic.[16][17][18] Massey criticizes informal logic on the grounds that it has no theory underpinning it. Informal logic, he says, requires detailed classification schemes to organize it, which in other disciplines is provided by the underlying theory. He maintains that there is no method of establishing the invalidity of an argument aside from the formal method, and that the study of fallacies may be of more interest to other disciplines, like psychology, than to philosophy and logic.[16]

Relation to critical thinking

Since the 1980s, informal logic has been partnered and even equated,[19] in the minds of many, with critical thinking. The precise definition of "critical thinking" is a subject of much dispute.[20] Critical thinking, as defined by Johnson, is the evaluation of an intellectual product (an argument, an explanation, a theory) in terms of its strengths and weaknesses.[20] While critical thinking will include evaluation of arguments and hence require skills of argumentation including informal logic, critical thinking requires additional abilities not supplied by informal logic, such as the ability to obtain and assess information and to clarify meaning. Also, many believe that critical thinking requires certain dispositions.[21] Understood in this way, "critical thinking" is a broad term for the attitudes and skills that are involved in analyzing and evaluating arguments. The critical thinking movement promotes critical thinking as an educational ideal. The movement emerged with great force in the 80s in North America as part of an ongoing critique of education as regards the thinking skills not being taught.

Relation to argumentation theory

The social, communicative practice of argumentation can and should be distinguished from implication (or entailment)—a relationship between propositions; and from inference—a mental activity typically thought of as the drawing of a conclusion from premises. Informal logic may thus be said to be a logic of argumentation, as distinguished from implication and inference.[22]

Argumentation theory (or the theory of argumentation) has come to be the term that designates the theoretical study of argumentation. This study is interdisciplinary in the sense that no one discipline will be able to provide a complete account. A full appreciation of argumentation requires insights from logic (both formal and informal), rhetoric, communication theory, linguistics, psychology, and, increasingly, computer science. Since the 1970s, there has been significant agreement that there are three basic approaches to argumentation theory: the logical, the rhetorical and the dialectical. According to Wenzel,[23] the logical approach deals with the product, the dialectical with the process, and the rhetorical with the procedure. Thus, informal logic is one contributor to this inquiry, being most especially concerned with the norms of argument.

Footnotes

[1] See Johnson 1999 for a survey of definitions.
[2] Johnson, Ralph H., and Blair, J. Anthony (1987), "The Current State of Informal Logic", *Informal Logic*, 9(2–3), 147–151. Johnson & Blair added "... in everyday discourse" but in (2000), modified their definition, and broadened the focus now to include the sorts of argument that occurs not just in everyday discourse but also disciplined inquiry—what Weinstein (1990) calls "stylized discourse."
[3] Scriven, 1976
[4] Munson, 1976
[5] Fogelin, 1978
[6] Resnick, 1989
[7] Frans H. van Eemeren (2009). "The Study of Argumentation" (http://books.google.com/books?id=RYRf2JACLGkC&pg=PA117). In Andrea A. Lunsford, Kirt H. Wilson, Rosa A. Eberly. *The SAGE handbook of rhetorical studies*. SAGE. p. 117. ISBN 978-1-4129-0950-1. .
[8] David Hitchcock, Informal logic 25 years later (http://www.humanities.mcmaster.ca/~hitchckd/25.pdf) in *Informal Logic at 25: Proceedings of the Windsor Conference (OSSA 2003)*
[9] JSTOR 3218569
[10] Fisher (2004) p. vii
[11] J. Anthony Blair and Ralph H. Johnson (eds.), Informal Logic: The First International Symposium, 3-28. Pt. Reyes, CA: Edgepress
[12] http://ojs.uwindsor.ca/ojs/leddy/index.php/informal_logic/about/editorialTeam
[13] Johnson and Blair (2000), p. 100
[14] As Johnson (1999) does.
[15] Johnson and Blair (2000), p. 95
[16] Massey, 1981
[17] Woods, 1980
[18] Woods, 2000
[19] Johnson (2000) takes the conflation to be part of the Network Problem and holds that settling the issue will require a theory of reasoning.

[20] Johnson, 1992
[21] Ennis, 1987
[22] Johnson, 1999
[23] Wenzel (1990)

References

- Barth, E. M., & Krabbe, E. C. W. (Eds.). (1982). From axiom to dialogue: A philosophical study of logics and argumentation. Berlin: Walter De Gruyter.
- Blair, J. A & Johnson, R.H. (1980). The recent development of informal logic. In J. Anthony Blair and Ralph H. Johnson (Eds.). Informal logic: The first international symposium, (pp. 3–28). Inverness, CA: Edgepress.
- Ennis, R.H. (1987). A taxonomy of critical thinking dispositions and abilities. In J.B. Baron and R.J. Sternberg (Eds.), Teaching critical thinking skills: Theory and practice, (pp. 9–26). New York: Freeman.
- Eemeren, F. H. van, & Grootendorst, R. (1992). Argumentation, communication and fallacies. Hillsdale, NJ: Lawrence Erlbaum Associates.
- Fisher, A. and Scriven, M. (1997). Critical thinking: It's definition and assessment. Point Reyes, CA: Edgepress
- Fisher, Alec (2004). *The logic of real arguments* (2nd ed.). Cambridge University Press. ISBN 978-0-521-65481-4.
- Govier, T. (1987). Problems in argument analysis and evaluation. Dordrecht: Foris.
- Govier, T. (1999). The Philosophy of Argument. Newport News, VA: Vale Press.
- Groarke, L. (2006). Informal Logic. Stanford Encyclopedia of Philosophy, from http://plato.stanford.edu/entries/logic-informal/
- Hitchcock, David (2007). "Informal logic and the concept of argument". In Jacquette, Dale. *Philosophy of logic*. Elsevier. ISBN 978-0-444-51541-4. preprint (http://www.humanities.mcmaster.ca/~hitchckd/informal.pdf)
- Johnson, R. H. (1992). The problem of defining critical thinking. In S. P. Norris (Ed.), The generalizability of critical thinking (pp. 38–53). New York: Teachers College Press. (Reprinted in Johnson (1996).)
- Johnson, R. H. (1996). The rise of informal logic. Newport News, VA: Vale Press
- Johnson, R. H. (1999). The relation between formal and informal logic. *Argumentation*, 13(3) 265-74.
- Johnson, R. H. (2000). Manifest rationality: A pragmatic theory of argument. Mahwah, NJ: Lawrence Erlbaum Associates.
- Johnson, R. H. & Blair, J. A. (1987). The current state of informal logic. *Informal Logic* 9, 147-51.
- Johnson, R. H. & Blair, J. A. (1996). Informal logic and critical thinking. In F. van Eemeren, R. Grootendorst, & F. Snoeck Henkemans (Eds.), Fundamentals of argumentation theory (pp. 383–86). Mahwah, NJ: Lawrence Erlbaum Associates
- Johnson, R. H. & Blair, J. A. (2002). Informal logic and the reconfiguration of logic. In D. Gabbay, R. H. Johnson, H.-J. Ohlbach and J. Woods (Eds.). Handbook of the logic of argument and inference: The turn towards the practical (pp. 339–396). Elsivier: North Holland.
- MacFarlane, J. (2005). Logical Constants. Stanford Encyclopedia of Philosophy.
- Massey, G. (1981). The fallacy behind fallacies. *Midwest Studies of Philosophy*, 6, 489-500.
- Munson, R. (1976). The way of words: an informal logic. Boston: Houghton Mifflin.
- Resnick, L. (1987). Education and learning to think. Washington, DC: National Academy Press..
- Walton, D. N. (1990). What is reasoning? What is an argument? *The Journal of Philosophy*, 87, 399-419.
- Weinstein, M. (1990) Towards a research agenda for informal logic and critical thinking. *Informal Logic*, 12, 121-143.
- Wenzel, J. 1990 Three perspectives on argumentation. In R Trapp and J Scheutz, (Eds.), Perspectives on argumentation: Essays in honour of Wayne Brockreide, 9-26 Waveland Press: Prospect Heights, IL
- Woods, J. (1980). What is informal logic? In J.A. Blair & R. H. Johnson (Eds.), Informal Logic: The First International Symposium (pp. 57–68). Point Reyes, CA: Edgepress.

Special journal issue

The open access issue 20(2) (http://www.phaenex.uwindsor.ca/ojs/leddy/index.php/informal_logic/issue/view/277) of *Informal Logic* from year 2000 groups a number of papers addressing foundational issues, based on the Panel on Informal Logic that was held at the 1998 World Congress of Philosophy, including:

- Hitchcock, D. (2000) The significance of informal logic for philosophy. *Informal Logic* 20(2), 129-138.
- Johnson, R. H. & Blair, J. A. (2000). Informal logic: An overview. *Informal Logic* 20(2): 93-99.
- Woods, J. (2000). How Philosophical is Informal Logic? *Informal Logic* 20(2): 139-167. 2000

Textbooks

- Kahane, H. (1971). Logic and contemporary rhetoric:The use of reasoning in everyday life. Belmont: Wadsworth. Still in print as Nancy Cavender; Howard Kahane (2009). *Logic and Contemporary Rhetoric: The Use of Reason in Everyday Life* (11th ed.). Cengage Learning. ISBN 978-0-495-80411-6.
- Scriven, M. (1976). Reasoning. New York. McGraw Hill.
- Johnson, R. H. & Blair, J. A. (1977). Logical self-defense. Toronto: McGraw-Hill Ryerson. US Edition. (2006). New York: Idebate Press.
- Fogelin, R.J. (1978). Understanding arguments: An introduction to informal logic. New York: Harcourt, Brace, Jovanovich. Still in print as Sinnott-Armstrong, Walter; Fogelin, Robert (2010), *Understanding Arguments: An Introduction to Informal Logic* (8th ed.), Belmont, California: Wadsworth Cengage Learning, ISBN 978-0-495-60395-5
- Stephen N. Thomas (1997). *Practical reasoning in natural language* (4th ed.). Prentice Hall. ISBN 978-0-13-678269-8.
- Irving M. Copi; Keith Burgess-Jackson (1996). *Informal logic* (3rd ed.). Prentice Hall. ISBN 978-0-13-229048-7.
- Woods, John, Andrew Irvine and Douglas Walton, 2004. Argument: Critical Thinking, Logic and the Fallacies. Toronto: Prentice Hall
- Groarke, Leo and Christopher Tindale, 2004. Good Reasoning Matters! (3rd edition). Toronto: Oxford University Press
- Douglas N. Walton (2008). *Informal logic: a pragmatic approach* (2nd ed.). Cambridge University Press. ISBN 978-0-521-71380-1.
- Trudy Govier (2009). *A Practical Study of Argument* (7th ed.). Cengage Learning. ISBN 978-0-495-60340-5.

External links

- Informal Logic (http://plato.stanford.edu/entries/logic-informal) entry by Leo Groarke in the *Stanford Encyclopedia of Philosophy*

Mathematical logic

Mathematical logic (also known as **symbolic logic**) is a subfield of mathematics with close connections to the foundations of mathematics, theoretical computer science and philosophical logic.[1] The field includes both the mathematical study of logic and the applications of formal logic to other areas of mathematics. The unifying themes in mathematical logic include the study of the expressive power of formal systems and the deductive power of formal proof systems.

Mathematical logic is often divided into the fields of set theory, model theory, recursion theory, and proof theory. These areas share basic results on logic, particularly first-order logic, and definability. In computer science (particularly in the ACM Classification) mathematical logic encompasses additional topics not detailed in this article; see logic in computer science for those.

Since its inception, mathematical logic has both contributed to, and has been motivated by, the study of foundations of mathematics. This study began in the late 19th century with the development of axiomatic frameworks for geometry, arithmetic, and analysis. In the early 20th century it was shaped by David Hilbert's program to prove the consistency of foundational theories. Results of Kurt Gödel, Gerhard Gentzen, and others provided partial resolution to the program, and clarified the issues involved in proving consistency. Work in set theory showed that almost all ordinary mathematics can be formalized in terms of sets, although there are some theorems that cannot be proven in common axiom systems for set theory. Contemporary work in the foundations of mathematics often focuses on establishing which parts of mathematics can be formalized in particular formal systems (as in reverse mathematics) rather than trying to find theories in which all of mathematics can be developed.

Subfields and scope

The *Handbook of Mathematical Logic* makes a rough division of contemporary mathematical logic into four areas:

1. set theory
2. model theory
3. recursion theory, and
4. proof theory and constructive mathematics (considered as parts of a single area).

Each area has a distinct focus, although many techniques and results are shared between multiple areas. The border lines between these fields, and the lines between mathematical logic and other fields of mathematics, are not always sharp. Gödel's incompleteness theorem marks not only a milestone in recursion theory and proof theory, but has also led to Löb's theorem in modal logic. The method of forcing is employed in set theory, model theory, and recursion theory, as well as in the study of intuitionistic mathematics.

The mathematical field of category theory uses many formal axiomatic methods, and includes the study of categorical logic, but category theory is not ordinarily considered a subfield of mathematical logic. Because of its applicability in diverse fields of mathematics, mathematicians including Saunders Mac Lane have proposed category theory as a foundational system for mathematics, independent of set theory. These foundations use toposes, which resemble generalized models of set theory that may employ classical or nonclassical logic.

History

Mathematical logic emerged in the mid-19th century as a subfield of mathematics independent of the traditional study of logic (Ferreirós 2001, p. 443). Before this emergence, logic was studied with rhetoric, through the syllogism, and with philosophy. The first half of the 20th century saw an explosion of fundamental results, accompanied by vigorous debate over the foundations of mathematics.

Early history

Further information: History of logic

Theories of logic were developed in many cultures in history, including China, India, Greece and the Islamic world. In 18th century Europe, attempts to treat the operations of formal logic in a symbolic or algebraic way had been made by philosophical mathematicians including Leibniz and Lambert, but their labors remained isolated and little known.

19th century

In the middle of the nineteenth century, George Boole and then Augustus De Morgan presented systematic mathematical treatments of logic. Their work, building on work by algebraists such as George Peacock, extended the traditional Aristotelian doctrine of logic into a sufficient framework for the study of foundations of mathematics (Katz 1998, p. 686).

Charles Sanders Peirce built upon the work of Boole to develop a logical system for relations and quantifiers, which he published in several papers from 1870 to 1885. Gottlob Frege presented an independent development of logic with quantifiers in his *Begriffsschrift*, published in 1879, a work generally considered as marking a turning point in the history of logic. Frege's work remained obscure, however, until Bertrand Russell began to promote it near the turn of the century. The two-dimensional notation Frege developed was never widely adopted and is unused in contemporary texts.

From 1890 to 1905, Ernst Schröder published *Vorlesungen über die Algebra der Logik* in three volumes. This work summarized and extended the work of Boole, De Morgan, and Peirce, and was a comprehensive reference to symbolic logic as it was understood at the end of the 19th century.

Foundational theories

Concerns that mathematics had not been built on a proper foundation led to the development of axiomatic systems for fundamental areas of mathematics such as arithmetic, analysis, and geometry.

In logic, the term *arithmetic* refers to the theory of the natural numbers. Giuseppe Peano (1888) published a set of axioms for arithmetic that came to bear his name (Peano axioms), using a variation of the logical system of Boole and Schröder but adding quantifiers. Peano was unaware of Frege's work at the time. Around the same time Richard Dedekind showed that the natural numbers are uniquely characterized by their induction properties. Dedekind (1888) proposed a different characterization, which lacked the formal logical character of Peano's axioms. Dedekind's work, however, proved theorems inaccessible in Peano's system, including the uniqueness of the set of natural numbers (up to isomorphism) and the recursive definitions of addition and multiplication from the successor function and mathematical induction.

In the mid-19th century, flaws in Euclid's axioms for geometry became known (Katz 1998, p. 774). In addition to the independence of the parallel postulate, established by Nikolai Lobachevsky in 1826 (Lobachevsky 1840), mathematicians discovered that certain theorems taken for granted by Euclid were not in fact provable from his axioms. Among these is the theorem that a line contains at least two points, or that circles of the same radius whose centers are separated by that radius must intersect. Hilbert (1899) developed a complete set of axioms for geometry, building on previous work by Pasch (1882). The success in axiomatizing geometry motivated Hilbert to seek complete axiomatizations of other areas of mathematics, such as the natural numbers and the real line. This would prove to be a major area of research in the first half of the 20th century.

The 19th century saw great advances in the theory of real analysis, including theories of convergence of functions and Fourier series. Mathematicians such as Karl Weierstrass began to construct functions that stretched intuition, such as nowhere-differentiable continuous functions. Previous conceptions of a function as a rule for computation, or a smooth graph, were no longer adequate. Weierstrass began to advocate the arithmetization of analysis, which sought to axiomatize analysis using properties of the natural numbers. The modern (ε, δ)-definition of limit and continuous functions was already developed by Bolzano in 1817 (Felscher 2000), but remained relatively unknown. Cauchy in 1821 defined continuity in terms of infinitesimals (see Cours d'Analyse, page 34). In 1858, Dedekind proposed a definition of the real numbers in terms of Dedekind cuts of rational numbers (Dedekind 1872), a definition still employed in contemporary texts.

Georg Cantor developed the fundamental concepts of infinite set theory. His early results developed the theory of cardinality and proved that the reals and the natural numbers have different cardinalities (Cantor 1874). Over the next twenty years, Cantor developed a theory of transfinite numbers in a series of publications. In 1891, he published a new proof of the uncountability of the real numbers that introduced the diagonal argument, and used this method to prove Cantor's theorem that no set can have the same cardinality as its powerset. Cantor believed that every set could be well-ordered, but was unable to produce a proof for this result, leaving it as an open problem in 1895 (Katz 1998, p. 807).

20th century

In the early decades of the 20th century, the main areas of study were set theory and formal logic. The discovery of paradoxes in informal set theory caused some to wonder whether mathematics itself is inconsistent, and to look for proofs of consistency.

In 1900, Hilbert posed a famous list of 23 problems for the next century. The first two of these were to resolve the continuum hypothesis and prove the consistency of elementary arithmetic, respectively; the tenth was to produce a method that could decide whether a multivariate polynomial equation over the integers has a solution. Subsequent work to resolve these problems shaped the direction of mathematical logic, as did the effort to resolve Hilbert's *Entscheidungsproblem*, posed in 1928. This problem asked for a procedure that would decide, given a formalized mathematical statement, whether the statement is true or false.

Set theory and paradoxes

Ernst Zermelo (1904) gave a proof that every set could be well-ordered, a result Georg Cantor had been unable to obtain. To achieve the proof, Zermelo introduced the axiom of choice, which drew heated debate and research among mathematicians and the pioneers of set theory. The immediate criticism of the method led Zermelo to publish a second exposition of his result, directly addressing criticisms of his proof (Zermelo 1908a). This paper led to the general acceptance of the axiom of choice in the mathematics community.

Skepticism about the axiom of choice was reinforced by recently discovered paradoxes in naive set theory. Cesare Burali-Forti (1897) was the first to state a paradox: the Burali-Forti paradox shows that the collection of all ordinal numbers cannot form a set. Very soon thereafter, Bertrand Russell discovered Russell's paradox in 1901, and Jules Richard (1905) discovered Richard's paradox.

Zermelo (1908b) provided the first set of axioms for set theory. These axioms, together with the additional axiom of replacement proposed by Abraham Fraenkel, are now called Zermelo–Fraenkel set theory (ZF). Zermelo's axioms incorporated the principle of limitation of size to avoid Russell's paradox.

In 1910, the first volume of *Principia Mathematica* by Russell and Alfred North Whitehead was published. This seminal work developed the theory of functions and cardinality in a completely formal framework of type theory, which Russell and Whitehead developed in an effort to avoid the paradoxes. *Principia Mathematica* is considered one of the most influential works of the 20th century, although the framework of type theory did not prove popular as a foundational theory for mathematics (Ferreirós 2001, p. 445).

Fraenkel (1922) proved that the axiom of choice cannot be proved from the remaining axioms of Zermelo's set theory with urelements. Later work by Paul Cohen (1966) showed that the addition of urelements is not needed, and the axiom of choice is unprovable in ZF. Cohen's proof developed the method of forcing, which is now an important tool for establishing independence results in set theory.

Symbolic logic

Leopold Löwenheim (1915) and Thoralf Skolem (1920) obtained the Löwenheim–Skolem theorem, which says that first-order logic cannot control the cardinalities of infinite structures. Skolem realized that this theorem would apply to first-order formalizations of set theory, and that it implies any such formalization has a countable model. This counterintuitive fact became known as Skolem's paradox.

In his doctoral thesis, Kurt Gödel (1929) proved the completeness theorem, which establishes a correspondence between syntax and semantics in first-order logic. Gödel used the completeness theorem to prove the compactness theorem, demonstrating the finitary nature of first-order logical consequence. These results helped establish first-order logic as the dominant logic used by mathematicians.

In 1931, Gödel published *On Formally Undecidable Propositions of Principia Mathematica and Related Systems*, which proved the incompleteness (in a different meaning of the word) of all sufficiently strong, effective first-order theories. This result, known as Gödel's incompleteness theorem, establishes severe limitations on axiomatic foundations for mathematics, striking a strong blow to Hilbert's program. It showed the impossibility of providing a consistency proof of arithmetic within any formal theory of arithmetic. Hilbert, however, did not acknowledge the importance of the incompleteness theorem for some time.

Gödel's theorem shows that a consistency proof of any sufficiently strong, effective axiom system cannot be obtained in the system itself, if the system is consistent, nor in any weaker system. This leaves open the possibility of consistency proofs that cannot be formalized within the system they consider. Gentzen (1936) proved the consistency of arithmetic using a finitistic system together with a principle of transfinite induction. Gentzen's result introduced the ideas of cut elimination and proof-theoretic ordinals, which became key tools in proof theory. Gödel (1958) gave a different consistency proof, which reduces the consistency of classical arithmetic to that of intutitionistic arithmetic in higher types.

Beginnings of the other branches

Alfred Tarski developed the basics of model theory.

Beginning in 1935, a group of prominent mathematicians collaborated under the pseudonym Nicolas Bourbaki to publish a series of encyclopedic mathematics texts. These texts, written in an austere and axiomatic style, emphasized rigorous presentation and set-theoretic foundations. Terminology coined by these texts, such as the words *bijection*, *injection*, and *surjection*, and the set-theoretic foundations the texts employed, were widely adopted throughout mathematics.

The study of computability came to be known as recursion theory, because early formalizations by Gödel and Kleene relied on recursive definitions of functions.[2] When these definitions were shown equivalent to Turing's formalization involving Turing machines, it became clear that a new concept – the computable function – had been discovered, and that this definition was robust enough to admit numerous independent characterizations. In his work on the incompleteness theorems in 1931, Gödel lacked a rigorous concept of an effective formal system; he immediately realized that the new definitions of computability could be used for this purpose, allowing him to state the incompleteness theorems in generality that could only be implied in the original paper.

Numerous results in recursion theory were obtained in the 1940s by Stephen Cole Kleene and Emil Leon Post. Kleene (1943) introduced the concepts of relative computability, foreshadowed by Turing (1939), and the arithmetical hierarchy. Kleene later generalized recursion theory to higher-order functionals. Kleene and Kreisel studied formal versions of intuitionistic mathematics, particularly in the context of proof theory.

Formal logical systems

At its core, mathematical logic deals with mathematical concepts expressed using formal logical systems. These systems, though they differ in many details, share the common property of considering only expressions in a fixed formal language, or signature. The systems of propositional logic and first-order logic are the most widely studied today, because of their applicability to foundations of mathematics and because of their desirable proof-theoretic properties.[3] Stronger classical logics such as second-order logic or infinitary logic are also studied, along with nonclassical logics such as intuitionistic logic.

First-order logic

First-order logic is a particular formal system of logic. Its syntax involves only finite expressions as well-formed formulas, while its semantics are characterized by the limitation of all quantifiers to a fixed domain of discourse.

Early results about formal logic established limitations of first-order logic. The Löwenheim–Skolem theorem (1919) showed that if a set of sentences in a countable first-order language has an infinite model then it has at least one model of each infinite cardinality. This shows that it is impossible for a set of first-order axioms to characterize the natural numbers, the real numbers, or any other infinite structure up to isomorphism. As the goal of early foundational studies was to produce axiomatic theories for all parts of mathematics, this limitation was particularly stark.

Gödel's completeness theorem (Gödel 1929) established the equivalence between semantic and syntactic definitions of logical consequence in first-order logic. It shows that if a particular sentence is true in every model that satisfies a particular set of axioms, then there must be a finite deduction of the sentence from the axioms. The compactness theorem first appeared as a lemma in Gödel's proof of the completeness theorem, and it took many years before logicians grasped its significance and began to apply it routinely. It says that a set of sentences has a model if and only if every finite subset has a model, or in other words that an inconsistent set of formulas must have a finite inconsistent subset. The completeness and compactness theorems allow for sophisticated analysis of logical consequence in first-order logic and the development of model theory, and they are a key reason for the prominence of first-order logic in mathematics.

Gödel's incompleteness theorems (Gödel 1931) establish additional limits on first-order axiomatizations. The **first incompleteness theorem** states that for any sufficiently strong, effectively given logical system there exists a statement which is true but not provable within that system. Here a logical system is effectively given if it is possible to decide, given any formula in the language of the system, whether the formula is an axiom. A logical system is sufficiently strong if it can express the Peano axioms. When applied to first-order logic, the first incompleteness theorem implies that any sufficiently strong, consistent, effective first-order theory has models that are not elementarily equivalent, a stronger limitation than the one established by the Löwenheim–Skolem theorem. The **second incompleteness theorem** states that no sufficiently strong, consistent, effective axiom system for arithmetic can prove its own consistency, which has been interpreted to show that Hilbert's program cannot be completed.

Other classical logics

Many logics besides first-order logic are studied. These include infinitary logics, which allow for formulas to provide an infinite amount of information, and higher-order logics, which include a portion of set theory directly in their semantics.

The most well studied infinitary logic is $L_{\omega_1,\omega}$. In this logic, quantifiers may only be nested to finite depths, as in first-order logic, but formulas may have finite or countably infinite conjunctions and disjunctions within them. Thus, for example, it is possible to say that an object is a whole number using a formula of $L_{\omega_1,\omega}$ such as

$$(x = 0) \vee (x = 1) \vee (x = 2) \vee \cdots .$$

Higher-order logics allow for quantification not only of elements of the domain of discourse, but subsets of the domain of discourse, sets of such subsets, and other objects of higher type. The semantics are defined so that, rather than having a separate domain for each higher-type quantifier to range over, the quantifiers instead range over all objects of the appropriate type. The logics studied before the development of first-order logic, for example Frege's logic, had similar set-theoretic aspects. Although higher-order logics are more expressive, allowing complete axiomatizations of structures such as the natural numbers, they do not satisfy analogues of the completeness and compactness theorems from first-order logic, and are thus less amenable to proof-theoretic analysis.

Another type of logics are fixed-point logics that allow inductive definitions, like one writes for primitive recursive functions.

One can formally define an extension of first-order logic — a notion which encompasses all logics in this section because they behave like first-order logic in certain fundamental ways, but does not encompass all logics in general, e.g. it does not encompass intuitionistic, modal or fuzzy logic. Lindström's theorem implies that the only extension of first-order logic satisfying both the compactness theorem and the Downward Löwenheim–Skolem theorem is first-order logic.

Nonclassical and modal logic

Modal logics include additional modal operators, such as an operator which states that a particular formula is not only true, but necessarily true. Although modal logic is not often used to axiomatize mathematics, it has been used to study the properties of first-order provability (Solovay 1976) and set-theoretic forcing (Hamkins and Löwe 2007).

Intuitionistic logic was developed by Heyting to study Brouwer's program of intuitionism, in which Brouwer himself avoided formalization. Intuitionistic logic specifically does not include the law of the excluded middle, which states that each sentence is either true or its negation is true. Kleene's work with the proof theory of intuitionistic logic showed that constructive information can be recovered from intuitionistic proofs. For example, any provably total function in intuitionistic arithmetic is computable; this is not true in classical theories of arithmetic such as Peano arithmetic.

Algebraic logic

Algebraic logic uses the methods of abstract algebra to study the semantics of formal logics. A fundamental example is the use of Boolean algebras to represent truth values in classical propositional logic, and the use of Heyting algebras to represent truth values in intuitionistic propositional logic. Stronger logics, such as first-order logic and higher-order logic, are studied using more complicated algebraic structures such as cylindric algebras.

Set theory

Set theory is the study of sets, which are abstract collections of objects. Many of the basic notions, such as ordinal and cardinal numbers, were developed informally by Cantor before formal axiomatizations of set theory were developed. The first such axiomatization, due to Zermelo (1908b), was extended slightly to become Zermelo–Fraenkel set theory (ZF), which is now the most widely used foundational theory for mathematics.

Other formalizations of set theory have been proposed, including von Neumann–Bernays–Gödel set theory (NBG), Morse–Kelley set theory (MK), and New Foundations (NF). Of these, ZF, NBG, and MK are similar in describing a cumulative hierarchy of sets. New Foundations takes a different approach; it allows objects such as the set of all sets at the cost of restrictions on its set-existence axioms. The system of Kripke–Platek set theory is closely related to generalized recursion theory.

Two famous statements in set theory are the axiom of choice and the continuum hypothesis. The axiom of choice, first stated by Zermelo (1904), was proved independent of ZF by Fraenkel (1922), but has come to be widely accepted by mathematicians. It states that given a collection of nonempty sets there is a single set C that contains exactly one element from each set in the collection. The set C is said to "choose" one element from each set in the collection. While the ability to make such a choice is considered obvious by some, since each set in the collection is nonempty, the lack of a general, concrete rule by which the choice can be made renders the axiom nonconstructive. Stefan Banach and Alfred Tarski (1924) showed that the axiom of choice can be used to decompose a solid ball into a finite number of pieces which can then be rearranged, with no scaling, to make two solid balls of the original size. This theorem, known as the Banach–Tarski paradox, is one of many counterintuitive results of the axiom of choice.

The continuum hypothesis, first proposed as a conjecture by Cantor, was listed by David Hilbert as one of his 23 problems in 1900. Gödel showed that the continuum hypothesis cannot be disproven from the axioms of Zermelo–Fraenkel set theory (with or without the axiom of choice), by developing the constructible universe of set theory in which the continuum hypothesis must hold. In 1963, Paul Cohen showed that the continuum hypothesis cannot be proven from the axioms of Zermelo–Fraenkel set theory (Cohen 1966). This independence result did not completely settle Hilbert's question, however, as it is possible that new axioms for set theory could resolve the hypothesis. Recent work along these lines has been conducted by W. Hugh Woodin, although its importance is not yet clear (Woodin 2001).

Contemporary research in set theory includes the study of large cardinals and determinacy. Large cardinals are cardinal numbers with particular properties so strong that the existence of such cardinals cannot be proved in ZFC. The existence of the smallest large cardinal typically studied, an inaccessible cardinal, already implies the consistency of ZFC. Despite the fact that large cardinals have extremely high cardinality, their existence has many ramifications for the structure of the real line. *Determinacy* refers to the possible existence of winning strategies for certain two-player games (the games are said to be *determined*). The existence of these strategies implies structural properties of the real line and other Polish spaces.

Model theory

Model theory studies the models of various formal theories. Here a theory is a set of formulas in a particular formal logic and signature, while a model is a structure that gives a concrete interpretation of the theory. Model theory is closely related to universal algebra and algebraic geometry, although the methods of model theory focus more on logical considerations than those fields.

The set of all models of a particular theory is called an elementary class; classical model theory seeks to determine the properties of models in a particular elementary class, or determine whether certain classes of structures form elementary classes.

The method of quantifier elimination can be used to show that definable sets in particular theories cannot be too complicated. Tarski (1948) established quantifier elimination for real-closed fields, a result which also shows the theory of the field of real numbers is decidable. (He also noted that his methods were equally applicable to algebraically closed fields of arbitrary characteristic.) A modern subfield developing from this is concerned with o-minimal structures.

Morley's categoricity theorem, proved by Michael D. Morley (1965), states that if a first-order theory in a countable language is categorical in some uncountable cardinality, i.e. all models of this cardinality are isomorphic, then it is categorical in all uncountable cardinalities.

A trivial consequence of the continuum hypothesis is that a complete theory with less than continuum many nonisomorphic countable models can have only countably many. Vaught's conjecture, named after Robert Lawson Vaught, says that this is true even independently of the continuum hypothesis. Many special cases of this conjecture have been established.

Recursion theory

Recursion theory, also called **computability theory**, studies the properties of computable functions and the Turing degrees, which divide the uncomputable functions into sets which have the same level of uncomputability. Recursion theory also includes the study of generalized computability and definability. Recursion theory grew from the work of Alonzo Church and Alan Turing in the 1930s, which was greatly extended by Kleene and Post in the 1940s.

Classical recursion theory focuses on the computability of functions from the natural numbers to the natural numbers. The fundamental results establish a robust, canonical class of computable functions with numerous independent, equivalent characterizations using Turing machines, λ calculus, and other systems. More advanced results concern the structure of the Turing degrees and the lattice of recursively enumerable sets.

Generalized recursion theory extends the ideas of recursion theory to computations that are no longer necessarily finite. It includes the study of computability in higher types as well as areas such as hyperarithmetical theory and α-recursion theory.

Contemporary research in recursion theory includes the study of applications such as algorithmic randomness, computable model theory, and reverse mathematics, as well as new results in pure recursion theory.

Algorithmically unsolvable problems

An important subfield of recursion theory studies algorithmic unsolvability; a decision problem or function problem is **algorithmically unsolvable** if there is no possible computable algorithm which returns the correct answer for all legal inputs to the problem. The first results about unsolvability, obtained independently by Church and Turing in 1936, showed that the Entscheidungsproblem is algorithmically unsolvable. Turing proved this by establishing the unsolvability of the halting problem, a result with far-ranging implications in both recursion theory and computer science.

There are many known examples of undecidable problems from ordinary mathematics. The word problem for groups was proved algorithmically unsolvable by Pyotr Novikov in 1955 and independently by W. Boone in 1959. The busy beaver problem, developed by Tibor Radó in 1962, is another well-known example.

Hilbert's tenth problem asked for an algorithm to determine whether a multivariate polynomial equation with integer coefficients has a solution in the integers. Partial progress was made by Julia Robinson, Martin Davis and Hilary Putnam. The algorithmic unsolvability of the problem was proved by Yuri Matiyasevich in 1970 (Davis 1973).

Proof theory and constructive mathematics

Proof theory is the study of formal proofs in various logical deduction systems. These proofs are represented as formal mathematical objects, facilitating their analysis by mathematical techniques. Several deduction systems are commonly considered, including Hilbert-style deduction systems, systems of natural deduction, and the sequent calculus developed by Gentzen.

The study of **constructive mathematics**, in the context of mathematical logic, includes the study of systems in non-classical logic such as intuitionistic logic, as well as the study of predicative systems. An early proponent of predicativism was Hermann Weyl, who showed it is possible to develop a large part of real analysis using only predicative methods (Weyl 1918).

Because proofs are entirely finitary, whereas truth in a structure is not, it is common for work in constructive mathematics to emphasize provability. The relationship between provability in classical (or nonconstructive) systems and provability in intuitionistic (or constructive, respectively) systems is of particular interest. Results such as the Gödel–Gentzen negative translation show that it is possible to embed (or *translate*) classical logic into intuitionistic logic, allowing some properties about intuitionistic proofs to be transferred back to classical proofs.

Recent developments in proof theory include the study of proof mining by Ulrich Kohlenbach and the study of proof-theoretic ordinals by Michael Rathjen.

Connections with computer science

The study of computability theory in computer science is closely related to the study of computability in mathematical logic. There is a difference of emphasis, however. Computer scientists often focus on concrete programming languages and feasible computability, while researchers in mathematical logic often focus on computability as a theoretical concept and on noncomputability.

The theory of semantics of programming languages is related to model theory, as is program verification (in particular, model checking). The Curry–Howard isomorphism between proofs and programs relates to proof theory, especially intuitionistic logic. Formal calculi such as the lambda calculus and combinatory logic are now studied as idealized programming languages.

Computer science also contributes to mathematics by developing techniques for the automatic checking or even finding of proofs, such as automated theorem proving and logic programming.

Descriptive complexity theory relates logics to computational complexity. The first significant result in this area, Fagin's theorem (1974) established that NP is precisely the set of languages expressible by sentences of existential second-order logic.

Foundations of mathematics

In the 19th century, mathematicians became aware of logical gaps and inconsistencies in their field. It was shown that Euclid's axioms for geometry, which had been taught for centuries as an example of the axiomatic method, were incomplete. The use of infinitesimals, and the very definition of function, came into question in analysis, as pathological examples such as Weierstrass' nowhere-differentiable continuous function were discovered.

Cantor's study of arbitrary infinite sets also drew criticism. Leopold Kronecker famously stated "God made the integers; all else is the work of man," endorsing a return to the study of finite, concrete objects in mathematics. Although Kronecker's argument was carried forward by constructivists in the 20th century, the mathematical community as a whole rejected them. David Hilbert argued in favor of the study of the infinite, saying "No one shall expel us from the Paradise that Cantor has created."

Mathematicians began to search for axiom systems that could be used to formalize large parts of mathematics. In addition to removing ambiguity from previously-naive terms such as function, it was hoped that this axiomatization would allow for consistency proofs. In the 19th century, the main method of proving the consistency of a set of axioms was to provide a model for it. Thus, for example, non-Euclidean geometry can be proved consistent by defining *point* to mean a point on a fixed sphere and *line* to mean a great circle on the sphere. The resulting structure, a model of elliptic geometry, satisfies the axioms of plane geometry except the parallel postulate.

With the development of formal logic, Hilbert asked whether it would be possible to prove that an axiom system is consistent by analyzing the structure of possible proofs in the system, and showing through this analysis that it is impossible to prove a contradiction. This idea led to the study of proof theory. Moreover, Hilbert proposed that the analysis should be entirely concrete, using the term *finitary* to refer to the methods he would allow but not precisely defining them. This project, known as Hilbert's program, was seriously affected by Gödel's incompleteness theorems, which show that the consistency of formal theories of arithmetic cannot be established using methods formalizable in those theories. Gentzen showed that it is possible to produce a proof of the consistency of arithmetic in a finitary system augmented with axioms of transfinite induction, and the techniques he developed to so do were seminal in proof theory.

A second thread in the history of foundations of mathematics involves nonclassical logics and constructive mathematics. The study of constructive mathematics includes many different programs with various definitions of *constructive*. At the most accommodating end, proofs in ZF set theory that do not use the axiom of choice are called constructive by many mathematicians. More limited versions of constructivism limit themselves to natural numbers, number-theoretic functions, and sets of natural numbers (which can be used to represent real numbers, facilitating the study of mathematical analysis). A common idea is that a concrete means of computing the values of the function must be known before the function itself can be said to exist.

In the early 20th century, Luitzen Egbertus Jan Brouwer founded intuitionism as a philosophy of mathematics. This philosophy, poorly understood at first, stated that in order for a mathematical statement to be true to a mathematician, that person must be able to *intuit* the statement, to not only believe its truth but understand the reason for its truth. A consequence of this definition of truth was the rejection of the law of the excluded middle, for there are statements that, according to Brouwer, could not be claimed to be true while their negations also could not be claimed true. Brouwer's philosophy was influential, and the cause of bitter disputes among prominent mathematicians. Later, Kleene and Kreisel would study formalized versions of intuitionistic logic (Brouwer rejected formalization, and presented his work in unformalized natural language). With the advent of the BHK interpretation and Kripke models, intuitionism became easier to reconcile with classical mathematics.

Notes

[1] Undergraduate texts include Boolos, Burgess, and Jeffrey (2002), Enderton (2001), and Mendelson (1997). A classic graduate text by Shoenfield (2001) first appeared in 1967.

[2] A detailed study of this terminology is given by Soare (1996).

[3] Ferreirós (2001) surveys the rise of first-order logic over other formal logics in the early 20th century.

References

Undergraduate texts

- Walicki, Michał (2011), *Introduction to Mathematical Logic*, Singapore: World Scientific Publishing, ISBN 978-981-4343-87-9 (pb.).
- ; Burgess, John; (2002), *Computability and Logic* (4th ed.), Cambridge: Cambridge University Press, ISBN 978-0-521-00758-0 (pb.).
- Enderton, Herbert (2001), *A mathematical introduction to logic* (2nd ed.), Boston, MA: Academic Press, ISBN 978-0-12-238452-3.
- Hamilton, A.G. (1988), *Logic for Mathematicians* (2nd ed.), Cambridge: Cambridge University Press, ISBN 978-0-521-36865-0.
- Ebbinghaus, H.-D.; Flum, J.; Thomas, W. (1994), *Mathematical Logic* (http://www.springer.com/mathematics/book/978-0-387-94258-2) (2nd ed.), New York: Springer, ISBN 0-387-94258-0.
- Katz, Robert (1964), *Axiomatic Analysis*, Boston, MA: D. C. Heath and Company.
- Mendelson, Elliott (1997), *Introduction to Mathematical Logic* (4th ed.), London: Chapman & Hall, ISBN 978-0-412-80830-2.
- Rautenberg, Wolfgang (2010), *A Concise Introduction to Mathematical Logic* (http://www.springerlink.com/content/978-1-4419-1220-6/) (3rd ed.), New York: Springer Science+Business Media, doi:10.1007/978-1-4419-1221-3, ISBN 978-1-4419-1220-6.
- Schwichtenberg, Helmut (2003–2004), *Mathematical Logic* (http://www.mathematik.uni-muenchen.de/~schwicht/lectures/logic/ws03/ml.pdf), Munich, Germany: Mathematisches Institut der Universität München.
- Shawn Hedman, *A first course in logic: an introduction to model theory, proof theory, computability, and complexity*, Oxford University Press, 2004, ISBN 0-19-852981-3. Covers logics in close relation with

computability theory and complexity theory

Graduate texts

- Andrews, Peter B. (2002), *An Introduction to Mathematical Logic and Type Theory: To Truth Through Proof* (2nd ed.), Boston: Kluwer Academic Publishers, ISBN 978-1-4020-0763-7.
- Barwise, Jon, ed. (1989), *Handbook of Mathematical Logic*, Studies in Logic and the Foundations of Mathematics, North Holland, ISBN 978-0-444-86388-1.
- Hodges, Wilfrid (1997), *A shorter model theory*, Cambridge: Cambridge University Press, ISBN 978-0-521-58713-6.
- (2003), *Set Theory: Millennium Edition*, Springer Monographs in Mathematics, Berlin, New York: Springer-Verlag, ISBN 978-3-540-44085-7.
- Shoenfield, Joseph R. (2001) [1967], *Mathematical Logic* (2nd ed.), A K Peters, ISBN 978-1-56881-135-2.
- ; Schwichtenberg, Helmut (2000), *Basic Proof Theory*, Cambridge Tracts in Theoretical Computer Science (2nd ed.), Cambridge: Cambridge University Press, ISBN 978-0-521-77911-1.

Research papers, monographs, texts, and surveys

- Cohen, P. J. (1966), *Set Theory and the Continuum Hypothesis*, Menlo Park, CA: W. A. Benjamin.
- Davis, Martin (1973), "Hilbert's tenth problem is unsolvable", *The American Mathematical Monthly* (The American Mathematical Monthly, Vol. 80, No. 3) **80** (3): 233–269, doi:10.2307/2318447, JSTOR 2318447, reprinted as an appendix in Martin Davis, Computability and Unsolvability, Dover reprint 1982. JStor (http://links.jstor.org/sici?sici=0002-9890(197303)80:3<233:HTPIU>2.0.CO;2-E)
- Felscher, Walter (2000), "Bolzano, Cauchy, Epsilon, Delta", *The American Mathematical Monthly* (The American Mathematical Monthly, Vol. 107, No. 9) **107** (9): 844–862, doi:10.2307/2695743, JSTOR 2695743. JSTOR (http://links.jstor.org/sici?sici=0002-9890(200011)107:9<844:BCED>2.0.CO;2-L)
- Ferreirós, José (2001), "The Road to Modern Logic-An Interpretation", *Bulletin of Symbolic Logic* (The Bulletin of Symbolic Logic, Vol. 7, No. 4) **7** (4): 441–484, doi:10.2307/2687794, JSTOR 2687794. JStor (http://links.jstor.org/sici?sici=1079-8986(200112)7:4<441:TRTMLI>2.0.CO;2-O)
- Hamkins, Joel David; Löwe, Benedikt, "The modal logic of forcing", *Transactions of the American Mathematical Society*, to appear. Electronic posting by the journal (http://www.ams.org/tran/0000-000-00/S0002-9947-07-04297-3/home.html)
- Katz, Victor J. (1998), *A History of Mathematics*, Addison–Wesley, ISBN 0-321-01618-1.
- Morley, Michael (1965), "Categoricity in Power", *Transactions of the American Mathematical Society* (Transactions of the American Mathematical Society, Vol. 114, No. 2) **114** (2): 514–538, doi:10.2307/1994188, JSTOR 1994188.
- Soare, Robert I. (1996), "Computability and recursion", *Bulletin of Symbolic Logic* (The Bulletin of Symbolic Logic, Vol. 2, No. 3) **2** (3): 284–321, doi:10.2307/420992, JSTOR 420992.
- Solovay, Robert M. (1976), "Provability Interpretations of Modal Logic", *Israel Journal of Mathematics* **25** (3–4): 287–304, doi:10.1007/BF02757006.
- Woodin, W. Hugh (2001), "The Continuum Hypothesis, Part I", *Notices of the American Mathematical Society* **48** (6). PDF (http://www.ams.org/notices/200106/fea-woodin.pdf)

Classical papers, texts, and collections

- Burali-Forti, Cesare (1897), *A question on transfinite numbers*, reprinted in van Heijenoort 1976, pp. 104–111.
- Dedekind, Richard (1872), *Stetigkeit und irrationale Zahlen*. English translation of title: "Consistency and irrational numbers".
- Dedekind, Richard (1888), *Was sind und was sollen die Zahlen?* Two English translations:
  - 1963 (1901). *Essays on the Theory of Numbers*. Beman, W. W., ed. and trans. Dover.
  - 1996. In *From Kant to Hilbert: A Source Book in the Foundations of Mathematics*, 2 vols, Ewald, William B., ed., Oxford University Press: 787–832.
- Fraenkel, Abraham A. (1922), "Der Begriff 'definit' und die Unabhängigkeit des Auswahlsaxioms", *Sitzungsberichte der Preussischen Akademie der Wissenschaften, Physikalisch-mathematische Klasse*, pp. 253–257 (German), reprinted in English translation as "The notion of 'definite' and the independence of the axiom of choice", van Heijenoort 1976, pp. 284–289.
- Frege Gottlob (1879), *Begriffsschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens*. Halle a. S.: Louis Nebert. Translation: *Concept Script, a formal language of pure thought modelled upon that of arithmetic*, by S. Bauer-Mengelberg in Jean Van Heijenoort, ed., 1967. *From Frege to Gödel: A Source Book in Mathematical Logic, 1879–1931*. Harvard University Press.
- Frege Gottlob (1884), *Die Grundlagen der Arithmetik: eine logisch-mathematische Untersuchung über den Begriff der Zahl*. Breslau: W. Koebner. Translation: J. L. Austin, 1974. *The Foundations of Arithmetic: A logico-mathematical enquiry into the concept of number*, 2nd ed. Blackwell.
- Gentzen, Gerhard (1936), "Die Widerspruchsfreiheit der reinen Zahlentheorie", *Mathematische Annalen* **112**: 132–213, doi:10.1007/BF01565428, reprinted in English translation in Gentzen's *Collected works*, M. E. Szabo, ed., North-Holland, Amsterdam, 1969.
- Gödel, Kurt (1929), *Über die Vollständigkeit des Logikkalküls*, doctoral dissertation, University Of Vienna. English translation of title: "Completeness of the logical calculus".
- Gödel, Kurt (1930), "Die Vollständigkeit der Axiome des logischen Funktionen-kalküls", *Monatshefte für Mathematik und Physik* **37**: 349–360, doi:10.1007/BF01696781. English translation of title: "The completeness of the axioms of the calculus of logical functions".
- Gödel, Kurt (1931), "Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I", *Monatshefte für Mathematik und Physik* **38** (1): 173–198, doi:10.1007/BF01700692, see On Formally Undecidable Propositions of Principia Mathematica and Related Systems for details on English translations.
- Gödel, Kurt (1958), "Über eine bisher noch nicht benützte Erweiterung des finiten Standpunktes", *Dialectica. International Journal of Philosophy* **12** (3–4): 280–287, doi:10.1111/j.1746-8361.1958.tb01464.x, reprinted in English translation in Gödel's *Collected Works*, vol II, Soloman Feferman et al., eds. Oxford University Press, 1990.
- van Heijenoort, Jean, ed. (1967, 1976 3rd printing with corrections), *From Frege to Gödel: A Source Book in Mathematical Logic, 1879–1931* (3rd ed.), Cambridge, Mass: Harvard University Press, ISBN 0-674-32449-8, (pbk.)
- Hilbert, David (1899), *Grundlagen der Geometrie*, Leipzig: Teubner, English 1902 edition (*The Foundations of Geometry*) republished 1980, Open Court, Chicago.
- David, Hilbert (1929), "Probleme der Grundlegung der Mathematik", *Mathematische Annalen* **102**: 1–9, doi:10.1007/BF01782335. Lecture given at the International Congress of Mathematicians, 3 September 1928. Published in English translation as "The Grounding of Elementary Number Theory", in Mancosu 1998, pp. 266–273.
- (1943), "Recursive Predicates and Quantifiers", *American Mathematical Society Transactions* (Transactions of the American Mathematical Society, Vol. 53, No. 1) **54** (1): 41–73, doi:10.2307/1990131, JSTOR 1990131.

- Lobachevsky, Nikolai (1840), *Geometrishe Untersuchungen zur Theorie der Parellellinien* (German). Reprinted in English translation as "Geometric Investigations on the Theory of Parallel Lines" in *Non-Euclidean Geometry*, Robert Bonola (ed.), Dover, 1955. ISBN 0-486-60027-0
- (1915), "Über Möglichkeiten im Relativkalkül", *Mathematische Annalen* **76** (4): 447–470, doi:10.1007/BF01458217, ISSN 0025-5831 (German). Translated as "On possibilities in the calculus of relatives" in Jean van Heijenoort, 1967. *A Source Book in Mathematical Logic, 1879–1931*. Harvard Univ. Press: 228–251.
- Mancosu, Paolo, ed. (1998), *From Brouwer to Hilbert. The Debate on the Foundations of Mathematics in the 1920s*, Oxford: Oxford University Press.
- Pasch, Moritz (1882), *Vorlesungen über neuere Geometrie*.
- Peano, Giuseppe (1888), *Arithmetices principia, nova methodo exposita* (Italian), excerpt reprinted in English stranslation as "The principles of arithmetic, presented by a new method", van Heijenoort 1976, pp. 83 97.
- Richard, Jules (1905), "Les principes des mathématiques et le problème des ensembles", *Revue générale des sciences pures et appliquées* **16**: 541 (French), reprinted in English translation as "The principles of mathematics and the problems of sets", van Heijenoort 1976, pp. 142–144.
- Skolem, Thoralf (1920), "Logisch-kombinatorische Untersuchungen über die Erfüllbarkeit oder Beweisbarkeit mathematischer Sätze nebst einem Theoreme über dichte Mengen", *Videnskapsselskapet Skrifter, I. Matematisk-naturvidenskabelig Klasse* **6**: 1–36.
- Tarski, Alfred (1948), *A decision method for elementary algebra and geometry*, Santa Monica, California: RAND Corporation
- Turing, Alan M. (1939), "Systems of Logic Based on Ordinals", *Proceedings of the London Mathematical Society* **45** (2): 161–228, doi:10.1112/plms/s2-45.1.161
- Zermelo, Ernst (1904), "Beweis, daß jede Menge wohlgeordnet werden kann", *Mathematische Annalen* **59** (4): 514–516, doi:10.1007/BF01445300 (German), reprinted in English translation as "Proof that every set can be well-ordered", van Heijenoort 1976, pp. 139–141.
- Zermelo, Ernst (1908a), "Neuer Beweis für die Möglichkeit einer Wohlordnung", *Mathematische Annalen* **65**: 107–128, doi:10.1007/BF01450054, ISSN 0025-5831 (German), reprinted in English translation as "A new proof of the possibility of a well-ordering", van Heijenoort 1976, pp. 183–198.
- Zermelo, Ernst (1908b), "Untersuchungen über die Grundlagen der Mengenlehre", *Mathematische Annalen* **65** (2): 261–281, doi:10.1007/BF01449999.

External links

- Logic and set theory around the world (http://settheory.net/world)
- Polyvalued logic (http://home.swipnet.se/~w-33552/logic/home/index.htm)
- *forall x: an introduction to formal logic* (http://www.fecundity.com/logic/), by P.D. Magnus, is a free textbook.
- *A Problem Course in Mathematical Logic* (http://euclid.trentu.ca/math/sb/pcml/), by Stefan Bilaniuk, is another free textbook.
- Detlovs, Vilnis, and Podnieks, Karlis (University of Latvia) *Introduction to Mathematical Logic.* (http://www.ltn.lv/~podnieks/mlog/ml.htm) A hyper-textbook.
- Stanford Encyclopedia of Philosophy: Classical Logic (http://plato.stanford.edu/entries/logic-classical/) – by Stewart Shapiro.
- Stanford Encyclopedia of Philosophy: First-order Model Theory (http://plato.stanford.edu/entries/modeltheory-fo/) – by Wilfrid Hodges.
- The London Philosophy Study Guide (http://www.ucl.ac.uk/philosophy/LPSG/) offers many suggestions on what to read, depending on the student's familiarity with the subject:
  - Mathematical Logic (http://www.ucl.ac.uk/philosophy/LPSG/MathLogic.htm)
  - Set Theory & Further Logic (http://www.ucl.ac.uk/philosophy/LPSG/SetTheory.htm)

- Philosophy of Mathematics (http://www.ucl.ac.uk/philosophy/LPSG/PhilMath.htm)

Algebraic logic

In mathematical logic, **algebraic logic** is the reasoning obtained by manipulating equations with free variables.

What is now usually called *classical algebraic logic* focuses on the identification and algebraic description of models appropriate for the study of various logics (in the form of classes of algebras that constitute the algebraic semantics for these deductive systems) and connected problems like representation and duality. Well known results like the representation theorem for Boolean algebras and Stone duality fall under the umbrella of classical algebraic logic.[1]

Works in the more recent abstract algebraic logic (AAL) focus on the process of algebraization itself, like classifying various forms of algebraizability using the Leibniz operator.[1]

Algebras as models of logics

Algebraic logic treats algebraic structures, often bounded lattices, as models (interpretations) of certain logics, making logic a branch of the order theory.

In algebraic logic:

- Variables are tacitly universally quantified over some universe of discourse. There are no existentially quantified variables or open formulas;
- Terms are built up from variables using primitive and defined operations. There are no connectives;
- Formulas, built from terms in the usual way, can be equated if they are logically equivalent. To express a tautology, equate a formula with a truth value;
- The rules of proof are the substitution of equals for equals, and uniform replacement. Modus ponens remains valid, but is seldom employed.

In the table below, the column on the left contains one or more logical or mathematical systems, and the algebraic structure which are its models are shown on the right in the same row. Some of these structures are either Boolean algebras or proper extensions thereof. Modal and other nonclassical logics are typically modeled by what are called "Boolean algebras with operators."

Algebraic formalisms going beyond first-order logic in at least some respects include:

- Combinatory logic, having the expressive power of set theory;
- Relation algebra, arguably the paradigmatic algebraic logic, can express Peano arithmetic and most axiomatic set theories, including the canonical ZFC.

Logical system	**Its models**
Classical sentential logic	Lindenbaum-Tarski algebra Two-element Boolean algebra
Intuitionistic propositional logic	Heyting algebra
Łukasiewicz logic	MV-algebra
Modal logic K	Modal algebra
Lewis's S4	Interior algebra
Lewis's S5; Monadic predicate logic	Monadic Boolean algebra
First-order logic	complete Boolean algebra Cylindric algebra Polyadic algebra Predicate functor logic

Set theory	Combinatory logic Relation algebra

Algebraic logic is mainly based on square roots.

History

Algebraic logic is, perhaps, the oldest approach to formal logic, arguably beginning with a number of memoranda Leibniz wrote in the 1680s, some of which were published in the 19th century and translated into English by Clarence Lewis in 1918. But nearly all of Leibniz's known work on algebraic logic was published only in 1903 after Louis Couturat discovered it in Leibniz's Nachlass. Parkinson (1966) and Loemker (1969) translated selections from Couturat's volume into English.

Brady (2000) discusses the rich historical connections between algebraic logic and model theory. The founders of model theory, Ernst Schröder and Leopold Loewenheim, were logicians in the algebraic tradition. Alfred Tarski, the founder of set theoretic model theory as a major branch of contemporary mathematical logic, also:

- Co-discovered Lindenbaum-Tarski algebra;
- Invented cylindric algebra;
- Wrote the 1941 paper that revived relation algebra, which can be viewed as the starting point of abstract algebraic logic.

Modern mathematical logic began in 1847, with two pamphlets whose respective authors were Augustus DeMorgan and George Boole. They, and later C.S. Peirce, Hugh MacColl, Frege, Peano, Bertrand Russell, and A. N. Whitehead all shared Leibniz's dream of combining symbolic logic, mathematics, and philosophy. Relation algebra is arguably the culmination of Leibniz's approach to logic. With the exception of some writings by Leopold Loewenheim and Thoralf Skolem, algebraic logic went into eclipse soon after the 1910-13 publication of *Principia Mathematica*, not to be revived until Tarski's 1940 re-exposition of relation algebra.

Leibniz had no influence on the rise of algebraic logic because his logical writings were little studied before the Parkinson and Loemker translations. Our present understanding of Leibniz as a logician stems mainly from the work of Wolfgang Lenzen, summarized in Lenzen (2004). [2] To see how present-day work in logic and metaphysics can draw inspiration from, and shed light on, Leibniz's thought, see Zalta (2000). [3]

References

[1] JSTOR 3094793
[2] http://www.philosophie.uni-osnabrueck.de/Publikationen%20Lenzen/Lenzen%20Leibniz%20Logic.pdf
[3] http://mally.stanford.edu/Papers/leibniz.pdf

Further reading

- J. Michael Dunn; Gary M. Hardegree (2001). *Algebraic methods in philosophical logic*. Oxford University Press. ISBN 978-0-19-853192-0. Good introduction for readers with prior exposure to non-classical logics but without much background in order theory and/or universal algebra; the book covers these prerequisites at length. This book however has been criticized for poor and sometimes incorrect presentation of AAL results. (http://www.jstor.org/stable/3094793)
- Hajnal Andréka, István Németi and Ildikó Sain (2001). "Algebraic logic". In Dov M. Gabbay, Franz Guenthner. *Handbook of philosophical logic, vol 2* (2nd ed.). Springer. ISBN 978-0-7923-7126-7. draft (http://www.math-inst.hu/pub/algebraic-logic/handbook.pdf)
- Willard Quine, 1976, "Algebraic Logic and Predicate Functors" in *The Ways of Paradox*. Harvard Univ. Press: 283-307.

Historical perspective

- Burris, Stanley, 2009. The Algebra of Logic Tradition (http://plato.stanford.edu/entries/algebra-logic-tradition/). Stanford Encyclopedia of Philosophy.
- Brady, Geraldine, 2000. *From Peirce to Skolem: A neglected chapter in the history of logic*. North-Holland/Elsevier Science BV: catalog page (http://www.elsevier.com/wps/find/bookdescription.cws_home/621535/description), Amsterdam, Netherlands, 625 pages.
- Lenzen, Wolfgang, 2004, " Leibniz's Logic (http://www.philosophie.uni-osnabrueck.de/Publikationen Lenzen/Lenzen Leibniz Logic.pdf)" in Gabbay, D., and Woods, J., eds., *Handbook of the History of Logic, Vol. 3: The Rise of Modern Logic from Leibniz to Frege*. North-Holland: 1-84.
- Roger Maddux, 1991, "The Origin of Relation Algebras in the Development and Axiomatization of the Calculus of Relations," *Studia Logica 50*: 421-55.
- Parkinson, G.H.R., 1966. *Leibniz: Logical Papers.* Oxford Uni. Press.
- Ivor Grattan-Guinness, 2000. *The Search for Mathematical Roots*. Princeton Univ. Press.
- Loemker, Leroy (1969 (1956)), *Leibniz: Philosophical Papers and Letters*, Reidel.
- Zalta, E. N., 2000, " A (Leibnizian) Theory of Concepts (http://mally.stanford.edu/leibniz.pdf)," *Philosophiegeschichte und logische Analyse / Logical Analysis and History of Philosophy 3*: 137-183.

External links

- Stanford Encyclopedia of Philosophy: " Propositional Consequence Relations and Algebraic Logic (http://plato.stanford.edu/entries/consequence-algebraic/)" -- by Ramon Jansana. (mainly about abstract algebraic logic)

Multi-valued logic

In logic, a **many-valued logic** (also **multi-** or **multiple-valued logic**) is a propositional calculus in which there are more than two truth values. Traditionally, in Aristotle's logical calculus, there were only two possible values (i.e., "true" and "false") for any proposition. An obvious extension to classical two-valued logic is an *n*-valued logic for *n* greater than 2. Those most popular in the literature are three-valued (e.g., Łukasiewicz's and Kleene's, which accept the values "true", "false", and "unknown"), the finite-valued with more than three values, and the infinite-valued, such as fuzzy logic and probability logic.

History

The first known classical logician who didn't fully accept the law of excluded middle was Aristotle (who, ironically, is also generally considered to be the first classical logician and the "father of logic"[1]). Aristotle admitted that his laws did not all apply to future events (*De Interpretatione*, *ch. IX*), but he didn't create a system of multi-valued logic to explain this isolated remark. Until the coming of the 20th century, later logicians followed Aristotelian logic, which includes or assumes the law of the excluded middle.

The 20th century brought back the idea of multi-valued logic. The Polish logician and philosopher, Jan Łukasiewicz, began to create systems of many-valued logic in 1920, using a third value, "possible", to deal with Aristotle's paradox of the sea battle. Meanwhile, the American mathematician, Emil L. Post (1921), also introduced the formulation of additional truth degrees with $n \geq 2$, where n are the truth values. Later, Jan Łukasiewicz and Alfred Tarski together formulated a logic on n truth values where $n \geq 2$. In 1932 Hans Reichenbach formulated a logic of many truth values where $n \to$infinity. Kurt Gödel in 1932 showed that intuitionistic logic is not a finitely-many valued logic, and defined a system of Gödel logics intermediate between classical and intuitionistic logic; such logics are known as intermediate logics.

Examples

Kleene (K_3) and Priest logic (P_3)

Kleene's "(strong) logic of indeterminacy" K_3 and Priest's "logic of paradox" add a third "undefined" or "indeterminate" truth value *I*. The truth functions for negation (¬), conjunction (∧), disjunction (∨), implication ($\to_K$), and biconditional ($\leftrightarrow_K$) are given by:[2]

¬	
T	F
I	I
F	T

▯	T	I	F
T	T	I	F
I	I	I	F
F	F	F	F

▯	T	I	F
T	T	T	T
I	T	I	I
F	T	I	F

$\to_K$	T	I	F
T	T	I	F
I	T	I	I
F	T	T	T

$\leftrightarrow_K$	T	I	F
T	T	I	F
I	I	I	I
F	F	I	T

The difference between the two logics lies in how tautologies are defined. In K_3 only T is a *designated truth value*, while in P_3 both T and I are. In Kleene's logic I can be interpreted as being "underdetermined", being neither true not false, while in Priest's logic I can be interpreted as being "overdetermined", being both true and false. K_3 does not have any tautologies, while P_3 has the same tautologies as classical two-valued logic.

K_3 has additional connectives for conjunction ($\wedge_+$), disjunction ($\vee_+$) and implication ($\to_+$):[3]

▯$_+$	T	I	F
T	T	I	F
I	I	I	I
F	F	I	F

$\to_+$	T	I	F
T	T	I	F
I	I	I	I
F	T	I[4]	T

Belnap logic (B_4)

Belnap's logic B_4 combines K_3 and P_3. The overdetermined truth value is here denoted as *B* and the underdetermined truth value as *N*.

$f_\neg$	
T	F
B	B
N	N
F	T

$f_{▯}$	T	B	N	F
T	T	B	N	F
B	B	B	F	F
N	N	F	N	F
F	F	F	F	F

$f_{▯}$	T	B	N	F
T	T	T	T	T
B	T	B	T	B
N	T	T	N	N
F	T	B	N	F

Semantics

Relation to classical logic

Logics are usually systems intended to codify rules for preserving some semantic property of propositions across transformations. In classical logic, this property is "truth." In a valid argument, the truth of the derived proposition is guaranteed if the premises are jointly true, because the application of valid steps preserves the property. However, that property doesn't have to be that of "truth"; instead, it can be some other concept.

Multi-valued logics are intended to preserve the property of designationhood (or being designated). Since there are more than two truth values, rules of inference may be intended to preserve more than just whichever corresponds (in the relevant sense) to truth. For example, in a three-valued logic, sometimes the two greatest truth-values (when they are represented as e.g. positive integers) are designated and the rules of inference preserve these values. Precisely, a valid argument will be such that the value of the premises taken jointly will always be less than or equal to the conclusion.

For example, the preserved property could be *justification*, the foundational concept of intuitionistic logic. Thus, a proposition is not true or false; instead, it is justified or flawed. A key difference between justification and truth, in this case, is that the law of excluded middle doesn't hold: a proposition that is not flawed is not necessarily justified; instead, it's only not proven that it's flawed. The key difference is the determinacy of the preserved property: One may prove that P is justified, that P is flawed, or be unable to prove either. A valid argument preserves justification across transformations, so a proposition derived from justified propositions is still justified. However, there are proofs in classical logic that depend upon the law of excluded middle; since that law is not usable under this scheme, there are propositions that cannot be proven that way.

Relation to fuzzy logic

Multi-valued logic is closely related to fuzzy set theory and fuzzy logic. The notion of fuzzy subset was introduced by Lotfi Zadeh as a formalization of vagueness; i.e., the phenomenon that a predicate may apply to an object not absolutely, but to a certain degree, and that there may be borderline cases. Indeed, as in multi-valued logic, fuzzy logic admits truth values different from "true" and "false". As an example, usually the set of possible truth values is the whole interval [0,1]. Nevertheless, the main difference between fuzzy logic and multi-valued logic is in the aims. In fact, in spite of its philosophical interest (it can be used to deal with the Sorites paradox), fuzzy logic is devoted mainly to the applications. More precisely, there are two approaches to fuzzy logic. The first one is very closely linked with multi-valued logic tradition (Hajek school). So a set of designed values is fixed and this enables us to define an entailment relation. The deduction apparatus is defined by a suitable set of logical axioms and suitable inference rules. Another approach (Goguen, Pavelka and others) is devoted to defining a deduction apparatus in which *approximate reasonings* are admitted. Such an apparatus is defined by a suitable fuzzy subset of logical axioms and by a suitable set of fuzzy inference rules. In the first case the logical consequence operator gives the set of logical consequence of a given set of axioms. In the latter the logical consequence operator gives the fuzzy subset of logical consequence of a given fuzzy subset of hypotheses.

Applications

Applications of many-valued logic can be roughly classified into two groups[5]. The first group uses many-valued logic domain to solve binary problems more efficiently. For example, a well-known approach to represent a multiple-output Boolean function is to treat its output part as a single many-valued variable and convert it to a single-output characteristic function. Other applications of many-valued logic include design of Programmable Logic Arrays (PLAs) with input decoders, optimization of finite state machines, testing, and verification.

The second group targets the design of electronic circuits which employ more than two discrete levels of signals, such as many-valued memories, arithmetic circuits, Field Programmable Gate Arrays (FPGA) etc. Many-valued circuits have a number of theoretical advantages over standard binary circuits. For example, the interconnect on and off chip can be reduced if signals in the circuit assume four or more levels rather than only two. In memory design, storing two instead of one bit of information per memory cell doubles the density of the memory in the same die size. Applications using arithmetic circuits often benefit from using alternatives to binary number systems. For example, residue and redundant number systems can reduce or eliminate the ripple-through carries which are involved in normal binary addition or subtraction, resulting in high-speed arithmetic operations. These number systems have a natural implementation using many-valued circuits. However, the practicality of these potential advantages heavily depends on the availability of circuit realizations, which must be compatible or competitive with present-day standard technologies.

Research venues

An IEEE International Symposium on Multiple-Valued Logic (ISMVL) has been held annually since 1970. It mostly caters to applications in digital design and verification.[6] There is also a *Journal of Multiple-Valued Logic and Soft Computing*.[7]

Notes

[1] Hurley, Patrick. *A Concise Introduction to Logic*, 9th edition. (2006).
[2] (Gottwald 2005, p. 19)
[3] (Gottwald 2005, p. 19)
[4] Following the definition in (Gottwald 2005). (Gottwald 2009) states it as T.
[5] Dubrova, Elena (2002). Multiple-Valued Logic Synthesis and Optimization (http://dl.acm.org/citation.cfm?id=566849), in Hassoun S. and Sasao T., editors, *Logic Synthesis and Verification*, Kluwer Academic Publishers, pp. 89-114
[6] http://www.informatik.uni-trier.de/~ley/db/conf/ismvl/index.html
[7] http://www.oldcitypublishing.com/MVLSC/MVLSC.html

References

Further reading

General

- Béziau J.-Y. (1997), What is many-valued logic ? *Proceedings of the 27th International Symposium on Multiple-Valued Logic*, IEEE Computer Society, Los Alamitos, pp. 117–121.
- Malinowski, Gregorz, (2001), *Many-Valued Logics,* in Goble, Lou, ed., *The Blackwell Guide to Philosophical Logic*. Blackwell.
- Merrie Bergmann (2008). *An introduction to many-valued and fuzzy logic: semantics, algebras, and derivation systems*. Cambridge University Press. ISBN 978-0-521-88128-9.
- Cignoli, R. L. O., D'Ottaviano, I, M. L., Mundici, D., (2000). *Algebraic Foundations of Many-valued Reasoning*. Kluwer.
- Grzegorz Malinowski (1993). *Many-valued logics*. Clarendon Press. ISBN 978-0-19-853787-8.
- S. Gottwald, *A Treatise on Many-Valued Logics.* Studies in Logic and Computation, vol. 9, Research Studies Press: Baldock, Hertfordshire, England, 2001.
- Gottwald, Siegfried (2005). *Many-Valued Logics* (http://www.uni-leipzig.de/~logik/gottwald/SGforDJ.pdf).
- D. Michael Miller; Mitchell A. Thornton (2008). *Multiple valued logic: concepts and representations*. Synthesis lectures on digital circuits and systems. **12**. Morgan & Claypool Publishers. ISBN 978-1-59829-190-2.
- Hájek P., (1998), *Metamathematics of fuzzy logic*. Kluwer. (Fuzzy logic understood as many-valued logic sui generis.)

Specific

- Alexandre Zinoviev, *Philosophical Problems of Many-Valued Logic*, D. Reidel Publishing Company, 169p., 1963.
- Prior A. 1957, *Time and Modality. Oxford University Press*, based on his 1956 John Locke lectures
- Goguen J.A. 1968/69, *The logic of inexact concepts*, Synthese, 19, 325–373.
- Chang C.C. and Keisler H. J. 1966. *Continuous Model Theory*, Princeton, Princeton University Press.
- Gerla G. 2001, *Fuzzy logic: Mathematical Tools for Approximate Reasoning*, Kluwer Academic Publishers, Dordrecht.
- Pavelka J. 1979, *On fuzzy logic I: Many-valued rules of inference*, Zeitschr. f. math. Logik und Grundlagen d. Math., 25, 45–52.
- George Metcalfe; Nicola Olivetti; Dov M. Gabbay (2008). *Proof Theory for Fuzzy Logics*. Springer. ISBN 978-1-4020-9408-8. Covers proof theory of many-valued logics as well, in the tradition of Hájek.
- Reiner Hähnle (1993). *Automated deduction in multiple-valued logics*. Clarendon Press. ISBN 978-0-19-853989-6.
- Francisco Azevedo (2003). *Constraint solving over multi-valued logics: application to digital circuits*. IOS Press. ISBN 978-1-58603-304-0.
- Leonard Bolc; Piotr Borowik (2003). *Many-valued Logics 2: Automated reasoning and practical applications*. Springer. ISBN 978-3-540-64507-8.

External links

- Gottwald, Siegfried (2009). "Many-Valued Logic" (http://plato.stanford.edu/entries/logic-manyvalued/). *Stanford Encyclopedia of Philosophy*.
- Stanford Encyclopedia of Philosophy: " Truth Values (http://plato.stanford.edu/entries/truth-values/)" -- by Yaroslav Shramko and Heinrich Wansing.
- IEEE Computer Society's Technical Committee on Multiple-Valued Logic (http://www.lcs.info.hiroshima-cu.ac.jp/~s_naga/MVL/)
- Resources for Many-Valued Logic (http://www.cse.chalmers.se/~reiner/mvl-web/) by Reiner Hähnle, Chalmers University
- Many-valued Logics W3 Server (http://web.archive.org/web/20050211094618/http://www.upmf-grenoble.fr/mvl/) (archived)

Fuzzy logic

Fuzzy logic is a form of many-valued logic or probabilistic logic; it deals with reasoning that is approximate rather than fixed and exact. In contrast with traditional logic theory, where binary sets have two-valued logic, true or false, fuzzy logic variables may have a truth value that ranges in degree between 0 and 1. Fuzzy logic has been extended to handle the concept of partial truth, where the truth value may range between completely true and completely false.[1] Furthermore, when linguistic variables are used, these degrees may be managed by specific functions.

Fuzzy logic began with the 1965 proposal of fuzzy set theory by Lotfi Zadeh.[2][3] Fuzzy logic has been applied to many fields, from control theory to artificial intelligence.

Overview

Fuzzy logic allows for approximate values and inferences as well as incomplete or ambiguous data (fuzzy data) as opposed to only relying on crisp data (binary yes/no choices).

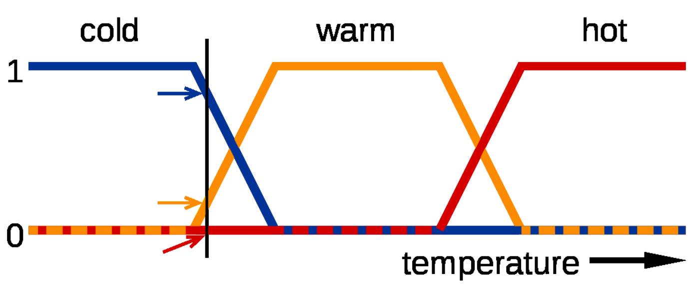
Degrees of truth

Fuzzy logic and probabilistic logic are mathematically similar – both have truth values ranging between 0 and 1 – but conceptually distinct, due to different interpretations—see interpretations of probability theory. Fuzzy logic corresponds to "degrees of truth", while probabilistic logic corresponds to "probability, likelihood"; as these differ, fuzzy logic and probabilistic logic yield different models of the same real-world situations.

Both degrees of truth and probabilities range between 0 and 1 and hence may seem similar at first. For example, let a 100 ml glass contain 30 ml of water. Then we may consider two concepts: Empty and Full. The meaning of each of them can be represented by a certain fuzzy set. Then one might define the glass as being 0.7 empty and 0.3 full. Note that the concept of emptiness would be subjective and thus would depend on the observer or designer. Another designer might equally well design a set membership function where the glass would be considered full for all values down to 50 ml. It is essential to realize that fuzzy logic uses truth degrees as a mathematical model of the vagueness phenomenon while probability is a mathematical model of ignorance.

Applying truth values

A basic application might characterize subranges of a continuous variable. For instance, a temperature measurement for anti-lock brakes might have several separate membership functions defining particular temperature ranges needed to control the brakes properly. Each function maps the same temperature value to a truth value in the 0 to 1 range. These truth values can then be used to determine how the brakes should be controlled.



Fuzzy logic temperature

In this image, the meanings of the expressions *cold*, *warm*, and *hot* are represented by functions mapping a temperature scale. A point on that scale has three "truth values"—one for each of the three functions. The vertical line in the image represents a particular temperature that the three arrows (truth values) gauge. Since the red arrow points to zero, this temperature may be interpreted as "not hot". The orange arrow (pointing at 0.2) may describe it as "slightly warm" and the blue arrow (pointing at 0.8) "fairly cold".

Linguistic variables

While variables in mathematics usually take numerical values, in fuzzy logic applications, the non-numeric *linguistic variables* are often used to facilitate the expression of rules and facts.[4]

A linguistic variable such as *age* may have a value such as *young* or its antonym *old*. However, the great utility of linguistic variables is that they can be modified via linguistic hedges applied to primary terms. The linguistic hedges can be associated with certain functions.

Example

Fuzzy set theory defines fuzzy operators on fuzzy sets. The problem in applying this is that the appropriate fuzzy operator may not be known. For this reason, fuzzy logic usually uses IF-THEN rules, or constructs that are equivalent, such as fuzzy associative matrices.

Rules are usually expressed in the form:
IF *variable* IS *property* THEN *action*

For example, a simple temperature regulator that uses a fan might look like this:

```
IF temperature IS very cold THEN stop fan
IF temperature IS cold THEN turn down fan
IF temperature IS normal THEN maintain level
IF temperature IS hot THEN speed up fan
```

There is no "ELSE" – all of the rules are evaluated, because the temperature might be "cold" and "normal" at the same time to different degrees.

The AND, OR, and NOT operators of boolean logic exist in fuzzy logic, usually defined as the minimum, maximum, and complement; when they are defined this way, they are called the *Zadeh operators*. So for the fuzzy variables x and y:

```
NOT x = (1 - truth(x))
x AND y = minimum(truth(x), truth(y))
x OR y = maximum(truth(x), truth(y))
```

There are also other operators, more linguistic in nature, called *hedges* that can be applied. These are generally adverbs such as "very", or "somewhat", which modify the meaning of a set using a mathematical formula.

Logical analysis

In mathematical logic, there are several formal systems of "fuzzy logic"; most of them belong among so-called t-norm fuzzy logics.

Propositional fuzzy logics

The most important propositional fuzzy logics are:

- Monoidal t-norm-based propositional fuzzy logic MTL is an axiomatization of logic where conjunction is defined by a left continuous t-norm, and implication is defined as the residuum of the t-norm. Its models correspond to MTL-algebras that are prelinear commutative bounded integral residuated lattices.
- Basic propositional fuzzy logic BL is an extension of MTL logic where conjunction is defined by a continuous t-norm, and implication is also defined as the residuum of the t-norm. Its models correspond to BL-algebras.
- Łukasiewicz fuzzy logic is the extension of basic fuzzy logic BL where standard conjunction is the Łukasiewicz t-norm. It has the axioms of basic fuzzy logic plus an axiom of double negation, and its models correspond to MV-algebras.
- Gödel fuzzy logic is the extension of basic fuzzy logic BL where conjunction is Gödel t-norm. It has the axioms of BL plus an axiom of idempotence of conjunction, and its models are called G-algebras.
- Product fuzzy logic is the extension of basic fuzzy logic BL where conjunction is product t-norm. It has the axioms of BL plus another axiom for cancellativity of conjunction, and its models are called product algebras.
- Fuzzy logic with evaluated syntax (sometimes also called Pavelka's logic), denoted by EVŁ, is a further generalization of mathematical fuzzy logic. While the above kinds of fuzzy logic have traditional syntax and many-valued semantics, in EVŁ is evaluated also syntax. This means that each formula has an evaluation. Axiomatization of EVŁ stems from Łukasziewicz fuzzy logic. A generalization of classical Gödel completeness theorem is provable in EVŁ.

Predicate fuzzy logics

These extend the above-mentioned fuzzy logics by adding universal and existential quantifiers in a manner similar to the way that predicate logic is created from propositional logic. The semantics of the universal (resp. existential) quantifier in t-norm fuzzy logics is the infimum (resp. supremum) of the truth degrees of the instances of the quantified subformula.

Decidability issues for fuzzy logic

The notions of a "decidable subset" and "recursively enumerable subset" are basic ones for classical mathematics and classical logic. Then, the question of a suitable extension of these concepts to fuzzy set theory arises. A first proposal in such a direction was made by E.S. Santos by the notions of *fuzzy Turing machine*, *Markov normal fuzzy algorithm* and *fuzzy program* (see Santos 1970). Successively, L. Biacino and G. Gerla argued that the proposed definitions are rather questionable and therefore they proposed the following ones. Denote by $Ü$ the set of rational numbers in [0,1]. Then a fuzzy subset $s : S \longrightarrow [0,1]$ of a set S is *recursively enumerable* if a recursive map $h : S{\times}N \longrightarrow Ü$ exists such that, for every x in S, the function $h(x,n)$ is increasing with respect to n and $s(x) = \lim h(x,n)$. We say that s is *decidable* if both s and its complement $-s$ are recursively enumerable. An extension of such a theory to the general case of the L-subsets is possible (see Gerla 2006). The proposed definitions are well related with fuzzy logic. Indeed, the following theorem holds true (provided that the deduction apparatus of the considered fuzzy logic satisfies some obvious effectiveness property).

Theorem. Any axiomatizable fuzzy theory is recursively enumerable. In particular, the fuzzy set of logically true formulas is recursively enumerable in spite of the fact that the crisp set of valid formulas is not recursively enumerable, in general. Moreover, any axiomatizable and complete theory is decidable.

It is an open question to give supports for a *Church thesis* for fuzzy mathematics the proposed notion of recursive enumerability for fuzzy subsets is the adequate one. To this aim, an extension of the notions of fuzzy grammar and fuzzy Turing machine should be necessary (see for example Wiedermann's paper). Another open question is to start from this notion to find an extension of Gödel's theorems to fuzzy logic.

Fuzzy databases

Once fuzzy relations are defined, it is possible to develop fuzzy relational databases. The first fuzzy relational database, FRDB, appeared in Maria Zemankova's dissertation. Later, some other models arose like the Buckles-Petry model, the Prade-Testemale Model, the Umano-Fukami model or the GEFRED model by J.M. Medina, M.A. Vila et al. In the context of fuzzy databases, some fuzzy querying languages have been defined, highlighting the SQLf by P. Bosc et al. and the FSQL by J. Galindo et al. These languages define some structures in order to include fuzzy aspects in the SQL statements, like fuzzy conditions, fuzzy comparators, fuzzy constants, fuzzy constraints, fuzzy thresholds, linguistic labels and so on.

Comparison to probability

Fuzzy logic and probability are different ways of expressing uncertainty. While both fuzzy logic and probability theory can be used to represent subjective belief, fuzzy set theory uses the concept of fuzzy set membership (i.e., *how much* a variable is in a set), and probability theory uses the concept of subjective probability (i.e., *how probable* do I think that a variable is in a set). While this distinction is mostly philosophical, the fuzzy-logic-derived possibility measure is inherently different from the probability measure, hence they are not *directly* equivalent. However, many statisticians are persuaded by the work of Bruno de Finetti that only one kind of mathematical uncertainty is needed and thus fuzzy logic is unnecessary. On the other hand, Bart Kosko argues that probability is a subtheory of fuzzy logic, as probability only handles one kind of uncertainty. He also claims to have proven a derivation of Bayes' theorem from the concept of fuzzy subsethood. Lotfi Zadeh argues that fuzzy logic is different in character from probability, and is not a replacement for it. He fuzzified probability to fuzzy probability and also generalized it to what is called possibility theory. (cf.[5]) More generally, fuzzy logic is one of many different proposed extensions to classical logic, known as probabilistic logics, intended to deal with issues of uncertainty in classical logic, the inapplicability of probability theory in many domains, and the paradoxes of Dempster-Shafer theory.

Notes

[1] Novák, V., Perfilieva, I. and Močkoř, J. (1999) *Mathematical principles of fuzzy logic* Dodrecht: Kluwer Academic. ISBN 0-7923-8595-0
[2] "Fuzzy Logic" (http://plato.stanford.edu/entries/logic-fuzzy/). *Stanford Encyclopedia of Philosophy*. Stanford University. 2006-07-23. . Retrieved 2008-09-29.
[3] Zadeh, L.A. (1965). "Fuzzy sets", *Information and Control* 8 (3): 338–353.
[4] Zadeh, L. A. et al. 1996 *Fuzzy Sets, Fuzzy Logic, Fuzzy Systems*, World Scientific Press, ISBN 981-02-2421-4
[5] Novák, V. "Are fuzzy sets a reasonable tool for modeling vague phenomena?", *Fuzzy Sets and Systems* **156** (2005) 341—348.

Bibliography

- Von Altrock, Constantin (1995). *Fuzzy logic and NeuroFuzzy applications explained*. Upper Saddle River, NJ: Prentice Hall PTR. ISBN 0-13-368465-2.
- Arabacioglu, B. C. (2010). "Using fuzzy inference system for architectural space analysis" (http://www.sciencedirect.com/science/article/pii/S1568494609002014). *Applied Soft Computing* **10** (3): 926–937. doi:10.1016/j.asoc.2009.10.011.
- Biacino, L.; Gerla, G. (2002). "Fuzzy logic, continuity and effectiveness". *Archive for Mathematical Logic* **41** (7): 643–667. doi:10.1007/s001530100128. ISSN 0933-5846.

- Cox, Earl (1994). *The fuzzy systems handbook: a practitioner's guide to building, using, maintaining fuzzy systems*. Boston: AP Professional. ISBN 0-12-194270-8.
- Gerla, Giangiacomo (2006). "Effectiveness and Multivalued Logics". *Journal of Symbolic Logic* **71** (1): 137–162. doi:10.2178/jsl/1140641166. ISSN 0022-4812.
- Hájek, Petr (1998). *Metamathematics of fuzzy logic*. Dordrecht: Kluwer. ISBN 0-7923-5238-6.
- Hájek, Petr (1995). "Fuzzy logic and arithmetical hierarchy". *Fuzzy Sets and Systems* **3** (8): 359–363. doi:10.1016/0165-0114(94)00299-M. ISSN 0165-0114.
- Halpern, Joseph Y. (2003). *Reasoning about uncertainty*. Cambridge, Mass: MIT Press. ISBN 0-262-08320-5.
- Höppner, Frank; Klawonn, F.; Kruse, R.; Runkler, T. (1999). *Fuzzy cluster analysis: methods for classification, data analysis and image recognition*. New York: John Wiley. ISBN 0-471-98864-2.
- Ibrahim, Ahmad M. (1997). *Introduction to Applied Fuzzy Electronics*. Englewood Cliffs, N.J: Prentice Hall. ISBN 0-13-206400-6.
- Klir, George J.; Folger, Tina A. (1988). *Fuzzy sets, uncertainty, and information*. Englewood Cliffs, N.J: Prentice Hall. ISBN 0-13-345984-5.
- Klir, George J.; St Clair, Ute H.; Yuan, Bo (1997). *Fuzzy set theory: foundations and applications*. Englewood Cliffs, NJ: Prentice Hall. ISBN 0-13-341058-7.
- Klir, George J.; Yuan, Bo (1995). *Fuzzy sets and fuzzy logic: theory and applications*. Upper Saddle River, NJ: Prentice Hall PTR. ISBN 0-13-101171-5.
- Kosko, Bart (1993). *Fuzzy thinking: the new science of fuzzy logic*. New York: Hyperion. ISBN 0-7868-8021-X.
- Kosko, Bart; Isaka, Satoru (July 1993). "Fuzzy Logic". *Scientific American* **269** (1): 76–81. doi:10.1038/scientificamerican0793-76.
- Montagna, F. (2001). "Three complexity problems in quantified fuzzy logic". *Studia Logica* **68** (1): 143–152. doi:10.1023/A:1011958407631. ISSN 0039-3215.
- Mundici, Daniele; Cignoli, Roberto; D'Ottaviano, Itala M. L. (1999). *Algebraic foundations of many-valued reasoning*. Dodrecht: Kluwer Academic. ISBN 0-7923-6009-5.
- Novák, Vilém (1989). *Fuzzy Sets and Their Applications*. Bristol: Adam Hilger. ISBN 0-85274-583-4.
- Novák, Vilém (2005). "On fuzzy type theory". *Fuzzy Sets and Systems* **149** (2): 235–273. doi:10.1016/j.fss.2004.03.027.
- Novák, Vilém; Perfilieva, Irina; Močkoř, Jiří (1999). *Mathematical principles of fuzzy logic*. Dordrecht: Kluwer Academic. ISBN 0-7923-8595-0.
- Onses, Richard (1996). *Second Order Experton: A new Tool for Changing Paradigms in Country Risk Calculation*. ISBN 84-7719-558-7.
- Onses, Richard (1994). *Détermination de l´incertitude inhérente aux investissements en Amérique Latine sur la base de la théorie des sous ensembles flous*. Barcelona. ISBN 84-475-0881-1.
- Passino, Kevin M.; Yurkovich, Stephen (1998). *Fuzzy control*. Boston: Addison-Wesley. ISBN 0-201-18074-X.
- Pedrycz, Witold; Gomide, Fernando (2007). *Fuzzy systems engineering: Toward Human-Centerd Computing*. Hoboken: Wiley-Interscience. ISBN 978-0-471-78857-7.
- Pu, Pao Ming; Liu, Ying Ming (1980). "Fuzzy topology. I. Neighborhood structure of a fuzzy point and Moore-Smith convergence". *Journal of Mathematical Analysis and Applications* **76** (2): 571–599. doi:10.1016/0022-247X(80)90048-7. ISSN 0022-247X
- Santos, Eugene S. (1970). "Fuzzy Algorithms". *Information and Control* **17** (4): 326–339. doi:10.1016/S0019-9958(70)80032-8.
- Scarpellini, Bruno (1962). "Die Nichaxiomatisierbarkeit des unendlichwertigen Prädikatenkalküls von Łukasiewicz". *Journal of Symbolic Logic* (Association for Symbolic Logic) **27** (2): 159–170. doi:10.2307/2964111. ISSN 0022-4812. JSTOR 2964111.
- Steeb, Willi-Hans (2008). *The Nonlinear Workbook: Chaos, Fractals, Cellular Automata, Neural Networks, Genetic Algorithms, Gene Expression Programming, Support Vector Machine, Wavelets, Hidden Markov Models,*

Fuzzy Logic with C++, Java and SymbolicC++ Programs: 4edition. World Scientific. ISBN 981-281-852-9.
- Wiedermann, J. (2004). "Characterizing the super-Turing computing power and efficiency of classical fuzzy Turing machines". *Theor. Comput. Sci.* **317** (1-3): 61–69. doi:10.1016/j.tcs.2003.12.004.
- Yager, Ronald R.; Filev, Dimitar P. (1994). *Essentials of fuzzy modeling and control*. New York: Wiley. ISBN 0-471-01761-2.
- Van Pelt, Miles (2008). *Fuzzy Logic Applied to Daily Life*. Seattle, WA: No No No No Press. ISBN 0-252-16341-9.
- Wilkinson, R.H. (1963). "A method of generating functions of several variables using analog diode logic". *IEEE Transactions on Electronic Computers* **12** (2): 112–129. doi:10.1109/PGEC.1963.263419.
- Zadeh, L.A. (1968). "Fuzzy algorithms". *Information and Control* **12** (2): 94–102. doi:10.1016/S0019-9958(68)90211-8. ISSN 0019-9958.
- Zadeh, L.A. (1965). "Fuzzy sets". *Information and Control* **8** (3): 338–353. doi:10.1016/S0019-9958(65)90241-X. ISSN 0019-9958.
- Zemankova-Leech, M. (1983). *Fuzzy Relational Data Bases*. Ph. D. Dissertation. Florida State University.
- Zimmermann, H. (2001). *Fuzzy set theory and its applications*. Boston: Kluwer Academic Publishers. ISBN 0-7923-7435-5.
- Tsitolovsky, Lev; Sandler, Uziel (2008). *Neural Cell Behavior and Fuzzy Logic*. Springer. ISBN ISBN 978-0-387-09542-4.

External links

Additional articles

- Formal fuzzy logic (http://en.citizendium.org/wiki/Formal_fuzzy_logic) - article at Citizendium
- Fuzzy Logic (http://www.scholarpedia.org/article/Fuzzy_Logic) - article at Scholarpedia
- Modeling With Words (http://www.scholarpedia.org/article/Modeling_with_words) - article at Scholarpedia
- Fuzzy logic (http://plato.stanford.edu/entries/logic-fuzzy/) - article at Stanford Encyclopedia of Philosophy
- Fuzzy Math (http://blog.peltarion.com/2006/10/25/fuzzy-math-part-1-the-theory) - Beginner level introduction to Fuzzy Logic.
- Fuzzy Logic and the Internet of Things: I-o-T (http://www.i-o-t.org/post/WEB_3)

Links pages

- Web page about FSQL (http://www.lcc.uma.es/~ppgg/FSQL/): References and links about FSQL

Applications

- Research conference paper providing a way to establish a fuzzy workload plan while dealing with project planning problem under uncertainty (http://www.enim.fr/mosim2010/articles/279.pdf)
- Research article that describes how industrial foresight could be integrated into capital budgeting with intelligent agents and Fuzzy Logic (http://econpapers.repec.org/paper/amrwpaper/398.htm)
- A doctoral dissertation describing how Fuzzy Logic can be applied in profitability analysis of very large industrial investments (http://econpapers.repec.org/paper/pramprapa/4328.htm)
- A method for asset valuation that uses fuzzy logic and fuzzy numbers for real option valuation (http://payoffmethod.com)

Metatheory

A **metatheory** or **meta-theory** is a theory whose subject matter is some other theory. In other words it is a theory about a theory. Statements made in the metatheory about the theory are called metatheorems.

The following is an example of a meta-theoretical statement:[1]

> Any physical theory is always provisional, in the sense that it is only a hypothesis; you can never prove it. No matter how many times the results of experiments agree with some theory, you can never be sure that the next time the result will not contradict the theory. On the other hand, you can disprove a theory by finding even a single observation that disagrees with the predictions of the theory.

Meta-theory belongs to the philosophical specialty of epistemology and metamathematics, as well as being an object of concern to the area in which the individual theory is conceived. An emerging domain of meta-theories is systemics.

Taxonomy

Examining groups of related theories, a first finding may be to identify classes of theories, thus specifying a taxonomy of theories. A proof engendered by a metatheory is called a *metatheorem*.

History

The concept burst upon the scene of 20th-century philosophy as a result of the work of the German mathematician David Hilbert, who in 1905 published a proposal for proof of the consistency of mathematics, creating the field of metamathematics. His hopes for the success of this proof were dashed by the work of Kurt Gödel who in 1931 proved this to be unattainable by his incompleteness theorems. Nevertheless, his program of unsolved mathematical problems, out of which grew this metamathematical proposal, continued to influence the direction of mathematics for the rest of the 20th century.

The study of metatheory became widespread during the rest of that century by its application in other fields, notably scientific linguistics and its concept of metalanguage.

References

[1] Stephen Hawking in *A Brief History of Time*

External links

- Meta-theoretical Issues (2003), Lyle Flint (http://www.bsu.edu/classes/flint/comm360/metatheo.html)

Metalogic

Metalogic is the study of the metatheory of logic. While *logic* is the study of the manner in which logical systems can be used to decide the correctness of arguments, metalogic studies the properties of the logical systems themselves.[1] According to Geoffrey Hunter, while logic concerns itself with the "truths of logic," metalogic concerns itself with the theory of "sentences used to express truths of logic."[2]

The basic objects of study in metalogic are formal languages, formal systems, and their interpretations. The study of interpretation of formal systems is the branch of mathematical logic known as model theory, while the study of deductive apparatus is the branch known as proof theory.

Overview

Formal language

A *formal language* is an organized set of symbols the essential feature of which is that it can be precisely defined in terms of just the shapes and locations of those symbols. Such a language can be defined, then, without any reference to any meanings of any of its expressions; it can exist before any interpretation is assigned to it—that is, before it has any meaning. First order logic is expressed in some formal language. A formal grammar determines which symbols and sets of symbols are formulas in a formal language.

A formal language can be defined formally as a set *A* of strings (finite sequences) on a fixed alphabet α. Some authors, including Carnap, define the language as the ordered pair <α, *A*>.[3] Carnap also requires that each element of α must occur in at least one string in *A*.

Formation rules

Formation rules (also called *formal grammar*) are a precise description of the well-formed formulas of a formal language. It is synonymous with the set of strings over the alphabet of the formal language which constitute well formed formulas. However, it does not describe their semantics (i.e. what they mean).

Formal systems

A *formal system* (also called a *logical calculus*, or a *logical system*) consists of a formal language together with a deductive apparatus (also called a *deductive system*). The deductive apparatus may consist of a set of transformation rules (also called *inference rules*) or a set of axioms, or have both. A formal system is used to derive one expression from one or more other expressions.

A *formal system* can be formally defined as an ordered triple <α, $\mathcal{I}$, $\mathcal{D}$ d>, where $\mathcal{D}$ d is the relation of direct derivability. This relation is understood in a comprehensive sense such that the primitive sentences of the formal system are taken as directly derivable from the empty set of sentences. Direct derivability is a relation between a sentence and a finite, possibly empty set of sentences. Axioms are laid down in such a way that every first place member of $\mathcal{D}$ d is a member of $\mathcal{I}$ and every second place member is a finite subset of $\mathcal{I}$.

It is also possible to define a *formal system* using only the relation $\mathcal{D}$ d. In this way we can omit $\mathcal{I}$, and α in the definitions of *interpreted formal language*, and *interpreted formal system*. However, this method can be more difficult to understand and work with.[3]

Formal proofs

A *formal proof* is a sequence of well-formed formulas of a formal language, the last one of which is a theorem of a formal system. The theorem is a syntactic consequence of all the well formed formulae preceding it in the proof. For a well formed formula to qualify as part of a proof, it must be the result of applying a rule of the deductive apparatus of some formal system to the previous well formed formulae in the proof sequence.

Interpretations

An *interpretation* of a formal system is the assignment of meanings, to the symbols, and truth-values to the sentences of the formal system. The study of interpretations is called Formal semantics. *Giving an interpretation* is synonymous with *constructing a model.*

Important distinctions in metalogic

Metalanguage–Object language

In metalogic, formal languages are sometimes called *object languages*. The language used to make statements about an object language is called a *metalanguage*. This distinction is a key difference between logic and metalogic. While logic deals with *proofs in a formal system*, expressed in some formal language, metalogic deals with *proofs about a formal system* which are expressed in a metalanguage about some object language.

Syntax–semantics

In metalogic, 'syntax' has to do with formal languages or formal systems without regard to any interpretation of them, whereas, 'semantics' has to do with interpretations of formal languages. The term 'syntactic' has a slightly wider scope than 'proof-theoretic', since it may be applied to properties of formal languages without any deductive systems, as well as to formal systems. 'Semantic' is synonymous with 'model-theoretic'.

Use–mention

In metalogic, the words 'use' and 'mention', in both their noun and verb forms, take on a technical sense in order to identify an important distinction.[2] The *use–mention distinction* (sometimes referred to as the *words-as-words distinction*) is the distinction between *using* a word (or phrase) and *mentioning* it. Usually it is indicated that an expression is being mentioned rather than used by enclosing it in quotation marks, printing it in italics, or setting the expression by itself on a line. The enclosing in quotes of an expression gives us the name of an expression, for example:

> 'Metalogic' is the name of this article.
>
> This article is about metalogic.

Type–token

The *type-token distinction* is a distinction in metalogic, that separates an abstract concept from the objects which are particular instances of the concept. For example, the particular bicycle in your garage is a token of the type of thing known as "The bicycle." Whereas, the bicycle in your garage is in a particular place at a particular time, that is not true of "the bicycle" as used in the sentence: "*The bicycle* has become more popular recently." This distinction is used to clarify the meaning of symbols of formal languages.

History

Metalogical questions have been asked since the time of Aristotle. However, it was only with the rise of formal languages in the late 19th and early 20th century that investigations into the foundations of logic began to flourish. In 1904, David Hilbert observed that in investigating the foundations of mathematics that logical notions are presupposed, and therefore a simultaneous account of metalogical and metamathematical principles was required. Today, metalogic and metamathematics are largely synonymous with each other, and both have been substantially subsumed by mathematical logic in academia.

Results in metalogic

Results in metalogic consist of such things as formal proofs demonstrating the consistency, completeness, and decidability of particular formal systems.

Major results in metalogic include:

- Proof of the uncountability of the set of all subsets of the set of natural numbers (Cantor's theorem 1891)
- Löwenheim-Skolem theorem (Leopold Löwenheim 1915 and Thoralf Skolem 1919)
- Proof of the consistency of truth-functional propositional logic (Emil Post 1920)
- Proof of the semantic completeness of truth-functional propositional logic (Paul Bernays 1918),[4] (Emil Post 1920)[2]
- Proof of the syntactic completeness of truth-functional propositional logic (Emil Post 1920)[2]
- Proof of the decidability of truth-functional propositional logic (Emil Post 1920)[2]
- Proof of the consistency of first order monadic predicate logic (Leopold Löwenheim 1915)
- Proof of the semantic completeness of first order monadic predicate logic (Leopold Löwenheim 1915)
- Proof of the decidability of first order monadic predicate logic (Leopold Löwenheim 1915)
- Proof of the consistency of first order predicate logic (David Hilbert and Wilhelm Ackermann 1928)
- Proof of the semantic completeness of first order predicate logic (Gödel's completeness theorem 1930)
- Proof of the undecidability of first order predicate logic (Church's theorem 1936)
- Gödel's first incompleteness theorem 1931
- Gödel's second incompleteness theorem 1931
- Tarski's undefinability theorem (Gödel and Tarski in the 1930s)

References

[1] Harry J. Gensler, Introduction to Logic, Routledge, 2001, p. 253.
[2] Hunter, Geoffrey, *Metalogic: An Introduction to the Metatheory of Standard First-Order Logic*, University of California Press, 1971
[3] Rudolf Carnap (1958) *Introduction to Symbolic Logic and its Applications*, p. 102.
[4] Hao Wang, Reflections on Kurt Gödel

Philosophical logic

Philosophical logic is a term introduced by Bertrand Russell to represent his idea that the workings of natural language and thought can only be adequately represented by an artificial language;[1] essentially it was his formalization program for the natural language.[2] Today the term is used with several different meanings.[3]

One modern meaning, espoused mainly by philosophers, is that philosophical logic is the study of the more specifically philosophical aspects of logic in contrast with symbolic logic; for example Sybil Wolfram lists the study of the concepts of argument, meaning, and truth.[4] Colin McGinn includes identity, existence, predication, necessity, and truth as the main topics of his book, which he writes was aimed "to bring philosophy back into philosophical logic".[5] John Woods writes that philosophical logic investigates properties such as truth, meaning and reference in natural languages. As contrasting example he argues that Frege's *Begriffsschrift* is an example of mathematical logic, while Frege's discussion of sense and reference belongs to the philosophical logic realm. Woods also points out that there's substantial overlap between philosophy of language and philosophical logic.[6] Susan Haack argued that there is no distinction between philosophical logic seen this way and philosophy of logic.[7][8] A. C. Grayling disagrees however, writing that when "one does philosophy of logic, one is philosophizing about logic; but when one does philosophical logic one is philosophizing." He concedes however that the distinction is not too sharp.[8] In general there is no agreement whether these two fields coincide or not.[9]

Another meaning assigned to philosophical logic today is that it addresses mainly extensions and alternatives to classical logic, the so called non-classical logics. In this sense, philosophical logic is a technical subject. Texts such as John P. Burgess' *Philosophical Logic*,[3] the *Blackwell Companion to Philosophical Logic*,[10] or the multi-volume *Handbook of Philosophical Logic*[11](edited by Dov M. Gabbay and Franz Guenthner) address this latter meaning of the term, with classical logic included as a core component however. According to Burgess, philosophical logic in this sense, has its center of gravity in theoretical computer science, because many non-classical logics find applications there.[3] The Springer *Journal of Philosophical Logic* largely addresses this conception of philosophical logic.

Yet another contemporary meaning proposed by Dale Jacquette is that philosophical logic is philosophy in which any recognized methods of logic are used to solve or advance the discussion of philosophical problems.[9]

References

[1] Richard Mark Sainsbury (2001). *Logical forms: an introduction to philosophical logic* (http://books.google.com/books?id=xAFKgdsB_akC&pg=PA52). Wiley-Blackwell. p. 52. ISBN 978-0-631-21679-7. .

[2] Ted Honderich (2005). *The Oxford companion to philosophy*. Oxford University Press. p. 205. ISBN 978-0-19-926479-7.

[3] John P. Burgess (2009). *Philosophical logic* (http://books.google.com/books?id=k32w3_wjBoYC&pg=PR7). Princeton University Press. pp. vii-viii. ISBN 978-0-691-13789-6. .

[4] Sybil Wolfram (1989). *Philosophical logic: an introduction* (http://books.google.com/books?id=w5sOAAAAQAAJ&pg=PA1). Routledge. p. 1. ISBN 978-0-415-02318-4. .

[5] McGinn, Colin. 2000. *Logical Properties: Identity, Existence, Predication, Necessity, Truth*. Oxford: Oxford University Press. ISBN 0-19-926263-2, preface

[6] John Woods (2007). "Fictions and Their Logic" (http://books.google.com/books?id=1xEVkzuX5e0C&pg=PA1062). In Dale Jacquette. *Philosophy of logic*. Elsevier. p. 1062. ISBN 978-0-444-51541-4. .

[7] Susan Haack (1978). *Philosophy of logics* (http://books.google.com/books?id=0GsZ8SBQrUcC&pg=PA2). Cambridge University Press. p. 2. ISBN 978-0-521-29329-7. .

[8] Grayling, A. C., 1997. *An Introduction to Philosophical Logic.* (http://books.google.ca/books?id=xGHMFdt7w30C&pg=PA3) 3rd ed. Oxford: Blackwell. ISBN 0-631-19982-9, p. 3; Grayling cites S. Haack, *Philosphy of Logics*, p. 2 for her position

[9] Dale Jacquette (2007). *Philosophy of logic* (http://books.google.com/books?id=1xEVkzuX5e0C&pg=PA1). Elsevier. p. 1. ISBN 978-0-444-51541-4. .

[10] Goble, Lou, ed., 2001. *The Blackwell Guide to Philosophical Logic.* (http://books.google.ca/books?id=aaO2f60YAwIC&dq=(The+Blackwell+Guide+to)+Philosophical+Logic&pg=PP1&ots=y5wP1HQkbh&source=bn&sig=gwaoap-A-Xxzx8daLihFbp2kdCk&hl=en&sa=X&oi=book_result&resnum=5&ct=result) Oxford: Blackwell. ISBN 0-631-20693-0.

[11] http://www.springer.com/series/6024

External links

- Journal of Philosophical Logic, Springer Science+Business Media
- Study Guide to Philosophical Logic and the Philosophy of Logic (http://www.ontology.co/pathways-logic.htm) Annotated selection of books on the subject

Logic in computer science

Logic in computer science describes topics where logic is applied to computer science and artificial intelligence. These include:

1. Investigations into logic that are guided by applications in computer science. For example:
   - Rewriting systems were motivated by solving equations algorithmically;
   - Many developments in type theory were motivated by applications in programming language theory;
   - Abstract interpretation was developed to allow proofs of certain program properties;
   - Logics of knowledge and beliefs (of human and artificial agents);
   - Spatial logics, used for reasoning about interaction between concurrent and distributed processes.
   - Logics for spatial reasoning, e.g. about moving in Euclidean space (which should not be confused with spatial logics used for concurrent systems);
   - some developments in categorical logic;
   - Program logics, such as Hoare logic, Hennessy-Milner logic, and dynamic logic are used to reason about program correctness
   - Process calculi were developed to reason about correctness of concurrent systems.
   - Descriptive complexity theory relates logics to computational complexity
2. Applications of logic in computer science, such as Formal methods:
   - Boolean logic, is used for the design of computer circuits;
   - Specification languages are extended logics for reasoning about whether programs behave correctly, such as the Z specification language. Cf. program logics, below, which can be considered to be minimalist specification logics, and cf. also, automated theorem provers, below; specification languages are often integrated into theorem provers.
   - The notion of institution has been developed as an abstract formalization of the notion of logical system, with the goal of handling the "population explosion" of logics used in formal methods.
   - Predicate logic and logical frameworks are used for proving programs correct, and logics such as temporal logic and #Fundamental concepts in computer science that are naturally expressible in formal logic. For example:
   - Formal semantics of programming languages;
   - Logic programming;
   - Definition of formal languages;
3. Aspects of the theory of computation that cast light on fundamental questions of formal logic. For example: Curry-Howard correspondence and Game semantics;
4. Tools for logicians considered as computer science. For example: Automated theorem proving and Model checking;

The study of basic mathematical logic such as propositional logic and predicate logic (normally in conjunction with set theory) is considered an important theoretical underpinning to any undergraduate computer science course. Higher-order logic is usually considered an advanced topic, but is important in theorem proving tools like HOL.

Books

- *Mathematical Logic for Computer Science* by Mordechai Ben-Ari. Springer-Verlag, 2nd edition, 2003. ISBN 1-85233-319-7.
- *Logic in Computer Science: Modelling and Reasoning about Systems* [1] by Michael Huth, Mark Ryan. Cambridge University Press, 2nd edition, 2004. ISBN 0-521-54310-X.
- *Logic for Mathematics and Computer Science* by Stanley N. Burris. Prentice Hall, 1997. ISBN 0-13-285974-2.

External links

- Article on *Logic and Artificial Intelligence* [2] at the Stanford Encyclopedia of Philosophy.
- IEEE Symposium on Logic in Computer Science [3] (LICS)
- Draft book on Logic in Computer Science by Andrei Voronkov [4]

References

[1] http://www.cs.bham.ac.uk/research/lics/
[2] http://plato.stanford.edu/entries/logic-ai/
[3] http://www.informatik.hu-berlin.de/lics/
[4] http://www.voronkov.com/lics.cgi

Controversies in logic

Principle of bivalence

In logic, the semantic **principle (or law) of bivalence** states that every declarative sentence expressing a proposition (of a theory under inspection) has exactly one truth value, either true or false.[1][2] A logic satisfying this principle is called a **two-valued logic**[3] or **bivalent logic**.[2][4]

In formal logic, the principle of bivalence becomes a property that a semantics may or may not possess. It is not the same as the law of excluded middle, however, and a semantics may satisfy that law without being bivalent.[2]

The principle of bivalence is studied in philosophical logic to address the question of which natural-language statements have a well-defined truth value. Sentences which predict events in the future, and sentences which seem open to interpretation, are particularly difficult for philosophers who hold that the principle of bivalence applies to all declarative natural-language statements.[2] Many-valued logics formalize ideas that a realistic characterization of the notion of consequence requires the admissibility of premises which, owing to vagueness, temporal or quantum indeterminacy, or reference-failure, cannot be considered classically bivalent. Reference failures can also be addressed by free logics.[5]

Relationship with the law of the excluded middle

The principle of bivalence is related to the law of excluded middle though the latter is a syntactic expression of the language of a logic of the form "P ∨ ¬P". The difference between the principle and the law is important because there are logics which validate the law but which do not validate the principle.[2] For example, the three-valued Logic of Paradox (LP) validates the law of excluded middle, but not the law of non-contradiction, ¬(P ∧ ¬P), and its intended semantics is not bivalent.[6] In classical two-valued logic both the law of excluded middle and the law of non-contradiction hold.[1]

Many modern logic programming systems replace the law of the excluded middle with the concept of negation as failure. The programmer may wish to add the law of the excluded middle by explicitly asserting it as true; however, it is not assumed *a priori*.

Classical logic

The intended semantics of classical logic is bivalent, but this is not true of every semantics for classical logic. In Boolean-valued semantics (for classical propositional logic), the truth values are the elements of an arbitrary Boolean algebra, "true" corresponds to the maximal element of the algebra, and "false" corresponds to the minimal element. Intermediate elements of the algebra correspond to truth values other than "true" and "false". The principle of bivalence holds only when the Boolean algebra is taken to be the two-element algebra, which has no intermediate elements.

Assign Boolean semantics to classical predicate calculus requires that the model be a complete Boolean algebra because the universal quantifier maps to the infimum operation, and the existential quantifier maps to the supremum;[7] this is called a Boolean-valued model. All finite Boolean algebras are complete.

Criticisms

Future contingents

A famous example[2] is the *contingent sea battle* case found in Aristotle's work, *De Interpretatione*, chapter 9:

> Imagine P refers to the statement "There will be a sea battle tomorrow."

The principle of bivalence here asserts:

> Either it is true that there will be a sea battle tomorrow, or it is not true that there will be a sea battle tomorrow.

Aristotle hesitated to embrace bivalence for such future contingents; Chrysippus, the Stoic logician, did embrace bivalence for this and all other propositions. The controversy continues to be of central importance in both metaphysics and the philosophy of logic.

One of the early motivations for the study of many-valued logics has been precisely this issue. In the early 20th century, the Polish formal logician Jan Łukasiewicz proposed three truth-values: the true, the false and the *as-yet-undetermined*. This approach was later developed by Arend Heyting and L. E. J. Brouwer;[2] see Łukasiewicz logic.

Issues such as this have also been addressed in various temporal logics, where one can assert that "*Eventually*, either there will be a sea battle tomorrow, or there won't be." (Which is true if "tomorrow" eventually occurs.)

Vagueness

Such puzzles as the Sorites paradox and the related continuum fallacy have raised doubt as to the applicability of classical logic and the principle of bivalence to concepts that may be vague in their application. Fuzzy logic and some other multi-valued logics have been proposed as alternatives that handle vague concepts better. Truth (and falsity) in fuzzy logic, for example, comes in varying degrees. Consider the following statement in the circumstance of sorting apples on a moving belt:

> This apple is red.[8]

Upon observation, the apple is an undetermined color between yellow and red, or it is motled both colors. Thus the color falls into neither category " red " nor " yellow ", but these are the only categories available to us as we sort the apples. We might say it is "50% red". This could be rephrased: it is 50% true that the apple is red. Therefore, P is 50% true, and 50% false. Now consider:

> This apple is red and it is not-red.

In other words, P and not-P. This violates the law of noncontradiction and, by extension, bivalence. However, this is only a partial rejection of these laws because P is only partially true. If P were 100% true, not-P would be 100% false, and there is no contradiction because P and not-P no longer holds.

However, the law of the excluded middle is retained, because P and not-P implies P or not-P, since "or" is inclusive. The only two cases where P and not-P is false (when P is 100% true or false) are the same cases considered by two-valued logic, and the same rules apply.

Example of a 3-valued logic applied to vague (undetermined) cases: Kleene 1952[9] (§64, pp. 332–340) offers a 3-valued logic for the cases when algorithms involving partial recursive functions may not return values, but rather end up with circumstances "u" = undecided. He lets "t" = "true", "f" = "false", "u" = "undecided" and redesigns all the propositional connectives. He observes that:

> "We were justified intuitionistically in using the classical 2-valued logic, when we were using the connectives in building primitive and general recursive predicates, since there is a decision procedure for each general recursive predicate; i.e. the law of the excluded middle is proved intuitionistically to apply to general recursive predicates.

"Now if Q(x) is a partial recursive predicate, there is a decision procedure for Q(x) on its range of definition, so the law of the excluded middle or excluded "third" (saying that, Q(x) is either t or f) applies intuitionistically on the range of definition. But there may be no algorithm for deciding, given x, whether Q(x) is defined or not . . .. Hence it is only classically and not intuitionistically that we have a law of the excluded fourth (saying that, for each x, Q(x) is either t, f, or u).

"The third "truth value" u is thus not on par with the other two t and f in our theory. Consideration of its status will show that we are limited to a special kind of truth table".

The following are his "strong tables"[10]:

~Q			QVR	R	t	f	u	Q&R	R	t	f	u	Q→R	R	t	f	u	Q=R	R	t	f	u
Q	**t**	f	**Q**	**t**	t	t	t	**Q**	**t**	t	f	u	**Q**	**t**	t	f	u	**Q**	**t**	t	f	u
	f	T		**f**	t	f	u		**f**	f	f	f		**f**	t	t	t		**f**	f	T	u
	u	u		**u**	t	u	u		**u**	u	f	u		**u**	t	u	u		**u**	u	u	u

For example, if a determination cannot be made as to whether an apple is red or not-red, then the truth value of the assertion Q: " This apple is red " is " u ". Likewise, the truth value of the assertion R " This apple is not-red " is " u ". Thus the AND of these into the assertion Q AND R, i.e. " This apple is red AND this apple is not-red " will, per the tables, yield " u ". And, the assertion Q OR R, i.e. " This apple is red OR this apple is not-red " will likewise yield " u ".

References

[1] Lou Goble (2001). *The Blackwell guide to philosophical logic* (http://books.google.com/books?id=aaO2f60YAwIC&pg=PA309). Wiley-Blackwell. p. 309. ISBN 978-0-631-20693-4. .

[2] Paul Tomassi (1999). *Logic* (http://books.google.com/books?id=TUVQr6InyNYC&pg=PA124). Routledge. p. 124. ISBN 978-0-415-16696-6. .

[3] Lou Goble (2001). *The Blackwell guide to philosophical logic* (http://books.google.com/books?id=aaO2f60YAwIC&pg=PA4). Wiley-Blackwell. p. 4. ISBN 978-0-631-20693-4. .

[4] Mark Hürlimann (2009). *Dealing with Real-World Complexity: Limits, Enhancements and New Approaches for Policy Makers* (http://books.google.com/books?id=_BThojvh-XUC&pg=PA42). Gabler Verlag. pp. 42. ISBN 978-3-8349-1493-4. .

[5] Dov M. Gabbay; John Woods (2007). *The Many Valued and Nonmonotonic Turn in Logic* (http://books.google.com/books?id=3TNj1ZkP3qEC&pg=PA7). The handbook of the history of logic. **8**. Elsevier. p. vii. ISBN 978-0-444-51623-7. . Retrieved 4 April 2011.

[6] Graham Priest (2008). *An introduction to non-classical logic: from if to is* (http://books.google.com/books?id=rMXVbmAw3YwC&pg=PA124). Cambridge University Press. pp. 124–125. ISBN 978-0-521-85433-7. .

[7] Morten Heine Sørensen; Paweł Urzyczyn (2006). *Lectures on the Curry-Howard isomorphism* (http://books.google.com/books?id=_mtnm-9KtbEC&pg=PA206). Elsevier. pp. 206–207. ISBN 978-0-444-52077-7. .

[8] Note the use of the (extremely) definite article: " This " as opposed to a more-vague " The ". " The " would have to be accompanied with a pointing-gesture to make it definitive. Ff *Principia Mathematica* (2nd edition), p. 91. Russell & Whitehead observe that this " this " indicates "something given in sensation" and as such it shall be considered "elementary".

[9] Stephen C. Kleene 1952 *Introduction to Metamathematics*, 6th Reprint 1971, North-Holland Publishing Company, Amsterdam NY, ISBN 0-7294-2130-9.

[10] "Strong tables" is Kleene's choice of words. Note that even though " u " may appear for the value of Q or R, " t " or " f " may, in those occasions, appear as a value in " Q V R ", " Q & R " and " Q → R ". "Weak tables" on the other hand, are "regular", meaning they have " u " appear in all cases when the value " u " is applied to either Q or R or both. Kleene notes that these tables are *not* the same as the original values of the tables of Łukasiewicz 1920. (Kleene gives these differences on page 335). He also concludes that " u " can mean any or all of the following: "undefined", "unknown (or value immaterial)", "value disregarded for the moment", i.e. it is a third category that does not (ultimately) exclude " t " and " f " (page 335).

Further reading

- Devidi, D.; Solomon, G. (1999). "On Confusions About Bivalence and Excluded Middle" (in French). *Dialogue* **38** (04): 785-799. doi:10.1017/S0012217300006715..
- Betti Arianna (2002) The Incomplete Story of Łukasiewicz and Bivalence (http://segr-did2.fmag.unict.it/~polphil/Polphil/Lukas/LukasBival.pdf) in T. Childers (ed.) *The Logica 2002 Yearbook*, Prague: The Czech Academy of Sciences—Filosofia, pp. 21–26
- Jean-Yves Béziau (2003) " Bivalence, excluded middle and non contradiction (http://www.unine.ch/unilog/jyb/new principle of bivalence.pdf)", in *The Logica Yearbook 2003*, L.Behounek (ed), Academy of Sciences, Prague, pp. 73–84.
- Font, J. M. (2009). "Taking Degrees of Truth Seriously". *Studia Logica* **91** (3): 383–406. doi:10.1007/s11225-009-9180-7.

External links

- Truth Values (http://plato.stanford.edu/entries/truth-values) entry by Yaroslav Shramko, Heinrich Wansing in the *Stanford Encyclopedia of Philosophy*

Paradoxes of material implication

The **paradoxes of material implication** are a group of formulas which are truths of classical logic, but which are intuitively problematic. One of these paradoxes is the **paradox of entailment**.

The root of the paradoxes lies in a mismatch between the interpretation of the validity of logical implication in natural language, and its formal interpretation in classical logic, dating back to George Boole's algebraic logic. In classical logic, implication describes conditional if-then statements using a truth-functional interpretation, i.e. "p implies q" is **defined** to be "it is not the case that p is true and q false". Also, "p implies q" is equivalent to "p is false or q is true". For example, "if it is raining, then I will bring an umbrella", is equivalent to "it is not raining, or I will bring an umbrella, or both". This truth-functional interpretation of implication is called material implication or material conditional.

The paradoxes are logical statements which are true but whose truth is intuitively surprising to people who are not familiar with them. If the terms 'p', 'q' and 'r' stand for arbitrary propositions then the main paradoxes are given formally as follows:

1. $(\neg p \wedge p) \to q$, p and its negation imply q. This is the *paradox of entailment*.
2. $p \to (q \to p)$, if p is true then it is implied by every q.
3. $\neg p \to (p \to q)$, if p is false then it implies every q. This is referred to as 'explosion'.
4. $p \to (q \vee \neg q)$, either q or its negation is true, so their disjunction is implied by every p.
5. $(p \to q) \vee (q \to r)$, if p, q and r are three arbitrary propositions, then either p implies q or q implies r. This is because if q is true then p implies it, and if it is false then q implies any other statement. Since r can be p, it follows that given two arbitrary propositions, one must imply the other, even if they are mutually contradictory. For instance, "Nadia is in Barcelona implies Nadia is in Madrid or Nadia is in Madrid implies Nadia is in Barcelona." This truism sounds like nonsense in ordinary discourse.
6. $\neg(p \to q) \to (p \wedge \neg q)$, if p does not imply q then p is true and q is false. NB if p were false then it would imply q, so p is true. If q were also true then p would imply q, hence q is false. This paradox is particularly surprising because it tells us that if one proposition does not imply another then the first is true and the second false.

The paradoxes of material implication arise because of the truth-functional definition of material implication, which is said to be true merely because the antecedent is false or the consequent is true. By this criterion, "If the moon is made of green cheese, then the world is coming to an end," is true merely because the moon isn't made of green cheese. By extension, any contradiction implies anything whatsoever, since a contradiction is never true. (All paraconsistent logics must, by definition, reject (1) as false.) Also, any tautology is implied by anything whatsoever, since a tautology is always true.

To sum up, although it is deceptively similar to what we mean by "logically follows" in ordinary usage, material implication does not capture the meaning of "if... then".

Paradox of entailment

As the most well known of the paradoxes, and most formally simple, the paradox of entailment makes the best introduction.

In natural language, an instance of the paradox of entailment arises:

> *It is raining*

And

> *It is not raining*

Therefore

> *George Washington is made of rakes.*

This arises from the principle of explosion, a law of classical logic stating that inconsistent premises always make an argument valid; that is, inconsistent premises imply any conclusion at all. This seems paradoxical, as it suggests that the above is a valid argument.

Understanding the paradox of entailment

Validity is defined in classical logic as follows:

> *An argument (consisting of premises and a conclusion) is valid if and only if there is no possible situation in which all the premises are true and the conclusion is false.*

For example a valid argument might run:

> *If it is raining, water exists* (1st premise)
>
> *It is raining* (2nd premise)
>
> *Water exists* (Conclusion)

In this example there is no possible situation in which the premises are true while the conclusion is false. Since there is no counterexample, the argument is valid.

But one could construct an argument in which the premises are inconsistent. This would satisfy the test for a valid argument since there would be *no possible situation in which all the premises are true* and therefore *no possible situation in which all the premises are true and the conclusion is false*.

For example an argument with inconsistent premises might run:

> *Matter has mass* (1st premise; true)
>
> *Matter does not have mass* (2nd premise; false)
>
> *All numbers are equal to 12* (Conclusion)

As there is no possible situation where both premises could be true, then there is certainly no possible situation in which the premises could be true while the conclusion was false. So the argument is valid whatever the conclusion is; inconsistent premises imply all conclusions.

Explaining the paradox

The strangeness of the paradox of entailment comes from the fact that the definition of validity in classical logic does not always agree with the use of the term in ordinary language. In everyday use *validity* suggests that the premises are consistent. In classical logic, the additional notion of *soundness* is introduced. A sound argument is a valid argument with all true premises. Hence a valid argument with an inconsistent set of premises can never be sound. A suggested improvement to the notion of logical validity to eliminate this paradox is relevant logic.

Simplification

The classical paradox formulas are closely tied to the formula,

- $(p \land q) \to p$

the principle of Simplification, which can be derived from the paradox formulas rather easily (e.g. from (1) by Importation). In addition, there are serious problems with trying to use material implication as representing the English "if ... then ...". For example, the following are valid inferences:

1. $(p \to q) \land (r \to s) \ \vdash \ (p \to s) \lor (r \to q)$
2. $(p \land q) \to r \ \vdash \ (p \to r) \lor (q \to r)$

but mapping these back to English sentences using "if" gives paradoxes. The first might be read "If John is in London then he is in England, and if he is in Paris then he is in France. Therefore, it is either true that if John is in London then he is in France, or that if he is in Paris then he is in England." Either John is in London or John is not in London. If John is in London, then John is in England. Thus the proposition "if John is in Paris, then John is in England" holds because we have prior knowledge that the conclusion is true. If John is not in London, then the proposition "if John is in London, then John is in France" is true because we have prior knowledge that the premise is false.

The second can be read "If both switch A and switch B are closed, then the light is on. Therefore, it is either true that if switch A is closed, the light is on, or if switch B is closed, the light is on." If the two switches are in series, then the premise is true but the conclusion is false. Thus, using classical logic and taking material implication to mean if-then is an unsafe method of reasoning which can give erroneous results.

References

- Bennett, J. *A Philosophical Guide to Conditionals*. Oxford: Clarendon Press. 2003.
- *Conditionals*, ed. Frank Jackson. Oxford: Oxford University Press. 1991.
- Etchemendy, J. *The Concept of Logical Consequence*. Cambridge: Harvard University Press. 1990.
- Sanford, D. *If P, Then Q: Conditionals and the Foundations of Reasoning*. New York: Routledge. 1989.
- Priest, G. *An Introduction to Non-Classical Logic*, Cambridge University Press. 2001.

Paraconsistent logic

A **paraconsistent logic** is a logical system that attempts to deal with contradictions in a discriminating way. Alternatively, paraconsistent logic is the subfield of logic that is concerned with studying and developing paraconsistent (or "inconsistency-tolerant") systems of logic.

Inconsistency-tolerant logics have been discussed since at least 1910 (and arguably much earlier, for example in the writings of Aristotle); however, the term *paraconsistent* ("beside the consistent") was not coined until 1976, by the Peruvian philosopher Francisco Miró Quesada.[1]

Definition

In classical logic (as well as intuitionistic logic and most other logics), contradictions entail everything. This curious feature, known as the principle of explosion or *ex contradictione sequitur quodlibet* (Latin, "from a contradiction, anything follows")[2] can be expressed formally as

$P \wedge \neg P$	Premise
P	conjunctive elimination
$P \vee A$	weakening
$\neg P$	conjunctive elimination
A	disjunctive syllogism

Which means: if *P* and its negation ¬*P* are both assumed to be true, then *P* is assumed to be true, from which it follows that at least one of the claims *P* and some other (arbitrary) claim *A* is true. However, if we know that either *P* or *A* is true, and also that *P* is not true (that ¬*P* is true) we can conclude that *A*, which could be anything, is true. Thus if a theory contains a single inconsistency, it is trivial—that is, it has every sentence as a theorem. The characteristic or defining feature of a paraconsistent logic is that it rejects the principle of explosion. As a result, paraconsistent logics, unlike classical and other logics, can be used to formalize inconsistent but non-trivial theories.

Paraconsistent logics are propositionally weaker than classical logic

Paraconsistent logics are propositionally *weaker* than classical logic; that is, they deem *fewer* propositional inferences valid. The point is that a paraconsistent logic can never be a propositional extension of classical logic, that is, propositionally validate everything that classical logic does. In that sense, then, paraconsistent logic is more conservative or cautious than classical logic. It is due to such conservativeness that paraconsistent languages can be more *expressive* than their classical counterparts including the hierarchy of metalanguages due to Tarski et al. According to Solomon Feferman [1984]: *"…natural language abounds with directly or indirectly self-referential yet apparently harmless expressions—all of which are excluded from the Tarskian framework."* This expressive limitation can be overcome in paraconsistent logic.

Motivation

The primary motivation for paraconsistent logic is the conviction that it ought to be possible to reason with inconsistent information in a controlled and discriminating way. The principle of explosion precludes this, and so must be abandoned. In non-paraconsistent logics, there is only one inconsistent theory: the trivial theory that has every sentence as a theorem. Paraconsistent logic makes it possible to distinguish between inconsistent theories and to reason with them. Sometimes it is possible to revise a theory to make it consistent. In other cases (e.g., large software systems) it is currently impossible to attain consistency.

Some philosophers take a more radical approach, holding that some contradictions are *true*, and thus a theory's being inconsistent is not always an indication that it is incorrect. This view, known as dialetheism, is motivated by several considerations, most notably an inclination to take certain paradoxes such as the Liar and Russell's paradox at face value. Not all advocates of paraconsistent logic are dialetheists. On the other hand, being a dialetheist rationally commits one to some form of paraconsistent logic, on pain of otherwise having to accept everything as true (i.e. trivialism).

The Philosophical Debate on Consistency

In classical logic Aristotle's three laws, namely, the excluded middle (p or $\neg p$), non-contradiction $\neg$ ($p \wedge \neg p$) and identity (p iff p), are regarded as the same, due to the inter-definition of the connectives. Moreover, traditionally contradictoriness (the presence of contradictions in a theory or in a body of knowledge) and triviality (the fact that such a theory entails all possible consequences) are assumed inseparable, granted that negation is available. These views may be philosophically challenged, precisely on the grounds that they fail to distinguish between contradictoriness and other forms of inconsistency.

On the other hand, it is possible to derive triviality from the 'conflict' between consistency and contradictions, once these notions have been properly distinguished. The very notions of consistency and inconsistency may be furthermore internalized at the object language level.

Tradeoff

Paraconsistency does not come for free: it involves a tradeoff. In particular, abandoning the principle of explosion requires one to abandon at least one of the following three very intuitive principles:[3]

Disjunction introduction	$A \vdash A \vee B$
Disjunctive syllogism	$A \vee B, \neg A \vdash B$
Transitivity or "cut"	$\Gamma \vdash A; A \vdash B \Rightarrow \Gamma \vdash B$

Though each of these principles has been challenged, the most popular approach among logicians is to reject disjunctive syllogism. If one is a dialetheist, it makes perfect sense that disjunctive syllogism should fail. The idea behind this syllogism is that, if ¬ *A*, then *A* is excluded, so the only way *A* ∨ *B* could be true would be if *B* were true. However, if *A* and ¬ *A* can both be true at the same time, then this reasoning fails.

Another approach is to reject disjunction introduction but keep disjunctive syllogism and transitivity. The disjunction (A ∨ B) is defined as ¬ (¬A ∧ ¬B). In this approach all of the rules of natural deduction hold, except for proof by contradiction and disjunction introduction; moreover, $A \vdash B$ does not mean necessarily that $\vdash A \Rightarrow B$, which is also a difference from natural deduction.[4] Also, the following usual Boolean properties hold: excluded middle and (for conjunction and disjunction) associativity, commutativity, distributivity, De Morgan's laws, and idempotence. Furthermore, by defining the implication (A → B) as ¬ (A ∧ ¬B), there is a Two-Way Deduction Theorem allowing implications to be easily proved. Carl Hewitt favours this approach, claiming that having the usual Boolean properties, Natural Deduction, and Deduction Theorem are huge advantages in software engineering.[4][5]

Yet another approach is to do both simultaneously. In many systems of relevant logic, as well as linear logic, there are two separate disjunctive connectives. One allows disjunction introduction, and one allows disjunctive syllogism. Of course, this has the disadvantages entailed by separate disjunctive connectives including confusion between them and complexity in relating them.

The three principles below, when taken together, also entail explosion, so at least one must be abandoned:

Reductio ad absurdum	$A \to (B \wedge \neg B) \vdash \neg A$
Rule of weakening	$A \vdash B \to A$
Double negation elimination	$\neg\neg A \vdash A$

Both reductio ad absurdum and the rule of weakening have been challenged in this respect, but without much success. Double negation elimination is challenged, but for unrelated reasons. By removing it alone, while upholding the other two one may still be able to prove all negative propositions from a contradiction.

A simple paraconsistent logic

One well-known system of paraconsistent logic is the simple system known as LP ("Logic of Paradox"), first proposed by the Argentinian logician F. G. Asenjo in 1966 and later popularized by Priest and others.[6]

One way of presenting the semantics for LP is to replace the usual functional valuation with a relational one.[7] The binary relation V relates a formula to a truth value: $V(A,1)$ means that A is true, and $V(A,0)$ means that A is false. A formula must be assigned *at least* one truth value, but there is no requirement that it be assigned *at most* one truth value. The semantic clauses for negation and disjunction are given as follows:

- $V(\neg A, 1) \Leftrightarrow V(A, 0)$
- $V(\neg A, 0) \Leftrightarrow V(A, 1)$
- $V(A \vee B, 1) \Leftrightarrow V(A, 1) \ or \ V(B, 1)$
- $V(A \vee B, 0) \Leftrightarrow V(A, 0) \ and \ V(B, 0)$

(The other logical connectives are defined in terms of negation and disjunction as usual.) Or to put the same point less symbolically:

- *not A* is true if and only if *A* is false
- *not A* is false if and only if *A* is true
- *A or B* is true if and only if *A* is true or *B* is true
- *A or B* is false if and only if *A* is false and *B* is false

(Semantic) logical consequence is then defined as truth-preservation:

$\Gamma \vDash A$ if and only if A is true whenever every element of Γ is true.

Now consider a valuation V such that $V(A,1)$ and $V(A,0)$ but it is not the case that $V(B,1)$. It is easy to check that this valuation constitutes a counterexample to both explosion and disjunctive syllogism. However, it is also a counterexample to modus ponens for the material conditional of LP. For this reason, proponents of LP usually advocate expanding the system to include a stronger conditional connective that is not definable in terms of negation and disjunction.[8]

As one can verify, LP preserves most other inference patterns that one would expect to be valid, such as De Morgan's laws and the usual introduction and elimination rules for negation, conjunction, and disjunction. Surprisingly, the logical truths (or tautologies) of LP are precisely those of classical propositional logic.[9] (LP and classical logic differ only in the *inferences* they deem valid.) Relaxing the requirement that every formula be either true or false yields the weaker paraconsistent logic commonly known as FDE ("First-Degree Entailment"). Unlike LP, FDE contains no logical truths.

It must be emphasized that LP is but one of *many* paraconsistent logics that have been proposed.[10] It is presented here merely as an illustration of how a paraconsistent logic can work.

Relation to other logics

One important type of paraconsistent logic is relevance logic. A logic is *relevant* iff it satisfies the following condition:

if $A \to B$ is a theorem, then A and B share a non-logical constant.

It follows that a relevance logic cannot have $(p \wedge \neg p) \to q$ as a theorem, and thus (on reasonable assumptions) cannot validate the inference from $\{p, \neg p\}$ to q.

Paraconsistent logic has significant overlap with many-valued logic; however, not all paraconsistent logics are many-valued (and, of course, not all many-valued logics are paraconsistent). Dialetheic logics, which are also many-valued, are paraconsistent, but the converse does not hold.

Intuitionistic logic allows $A \vee \neg A$ not to be equivalent to true, while paraconsistent logic allows $A \wedge \neg A$ not to be equivalent to false. Thus it seems natural to regard paraconsistent logic as the "dual" of intuitionistic logic. However, intuitionistic logic is a specific logical system whereas paraconsistent logic encompasses a large class of systems. Accordingly, the dual notion to paraconsistency is called paracompleteness, and the "dual" of intuitionistic logic (a specific paracomplete logic) is a specific paraconsistent system called *anti-intuitionistic* or *dual-intuitionistic logic* (sometimes referred to as *Brazilian logic*, for historical reasons).[11] The duality between the two systems is best seen within a sequent calculus framework. While in intuitionistic logic the sequent

$$\vdash A \vee \neg A$$

is not derivable, in dual-intuitionistic logic

$$A \wedge \neg A \vdash$$

is not derivable. Similarly, in intuitionistic logic the sequent

$$\neg\neg A \vdash A$$

is not derivable, while in dual-intuitionistic logic

$$A \vdash \neg\neg A$$

is not derivable. Dual-intuitionistic logic contains a connective # known as *pseudo-difference* which is the dual of intuitionistic implication. Very loosely, $A \# B$ can be read as "A but not B". However, # is not truth-functional as one might expect a 'but not' operator to be; similarly, the intuitionistic implication operator cannot be treated like "$\neg (A \wedge \neg B)$". Dual-intuitionistic logic also features a basic connective $\top$ which is the dual of intuitionistic $\bot$: negation may be defined as $\neg A = (\top \# A)$

A full account of the duality between paraconsistent and intuitionistic logic, including an explanation on why dual-intuitionistic and paraconsistent logics do not coincide, can be found in Brunner and Carnielli (2005).

Applications

Paraconsistent logic has been applied as a means of managing inconsistency in numerous domains, including:[12]

- Semantics. Paraconsistent logic has been proposed as means of providing a simple and intuitive formal account of truth that does not fall prey to paradoxes such as the Liar. However, such systems must also avoid Curry's paradox, which is much more difficult as it does not essentially involve negation.
- Set theory and the foundations of mathematics (see paraconsistent mathematics). Some believe that paraconsistent logic has significant ramifications with respect to the significance of Russell's paradox and Gödel's incompleteness theorems.
- Epistemology and belief revision. Paraconsistent logic has been proposed as a means of reasoning with and revising inconsistent theories and belief systems.
- Knowledge management and artificial intelligence. Some computer scientists have utilized paraconsistent logic as a means of coping gracefully with inconsistent information.[13]

- Deontic logic and metaethics. Paraconsistent logic has been proposed as a means of dealing with ethical and other normative conflicts.
- Software engineering. Paraconsistent logic has been proposed as a means for dealing with the pervasive inconsistencies among the documentation, use cases, and code of large software systems.[4][5]
- Electronics design routinely uses a four valued logic, with "hi-impedance (z)" and "don't care (x)" playing similar roles to "don't know" and "both true and false" respectively, in addition to True and False. This logic was developed independently of Philosophical logics.

Criticism

Some philosophers have argued against dialetheism on the grounds that the counterintuitiveness of giving up any of the three principles above outweighs any counterintuitiveness that the principle of explosion might have.

Others, such as David Lewis, have objected to paraconsistent logic on the ground that it is simply impossible for a statement and its negation to be jointly true.[14] A related objection is that "negation" in paraconsistent logic is not really *negation*; it is merely a subcontrary-forming operator.[15]

Alternatives

Approaches exist that allow for resolution of inconsistent beliefs without violating any of the intuitive logical principles. Most such systems use multi-valued logic with Bayesian inference and the Dempster-Shafer theory, allowing that no non-tautological belief is completely (100%) irrefutable because it must be based upon incomplete, abstracted, interpreted, likely unconfirmed, potentially uninformed, and possibly incorrect knowledge (of course, this very assumption, if non-tautological, entails its own refutability, if by "refutable" we mean "not completely [100%] irrefutable"). These systems effectively give up several logical principles in practice without rejecting them in theory.

Notable figures

Notable figures in the history and/or modern development of paraconsistent logic include:

- Alan Ross Anderson (USA, 1925–1973). One of the founders of relevance logic, a kind of paraconsistent logic.
- F. G. Asenjo (Argentina)
- Diderik Batens (Belgium)
- Nuel Belnap (USA, b. 1930). Worked with Anderson on relevance logic.
- Jean-Yves Béziau (France/Switzerland, b. 1965). Has written extensively on the general structural features and philosophical foundations of paraconsistent logics.
- Ross Brady (Australia)
- Bryson Brown (Canada)
- Walter Carnielli (Brazil). The developer of the *possible-translations semantics*, a new semantics which makes paraconsistent logics applicable and philosophically understood.
- Newton da Costa (Brazil, b. 1929). One of the first to develop formal systems of paraconsistent logic.
- Itala M. L. D'Ottaviano (Brazil)
- J. Michael Dunn (USA). An important figure in relevance logic.
- Stanisław Jaśkowski (Poland). One of the first to develop formal systems of paraconsistent logic.
- R. E. Jennings (Canada)
- David Kellogg Lewis (USA, 1941–2001). Articulate critic of paraconsistent logic.
- Jan Łukasiewicz (Poland, 1878–1956)
- Robert K. Meyer (USA/Australia)
- Chris Mortensen (Australia). Has written extensively on paraconsistent mathematics.

- Lorenzo Peña (Spain, b. 1944). Has developed an original line of paraconsistent logic, gradualistic logic (also known as *transitive logic*, TL), akin to Fuzzy Logic.
- Val Plumwood [formerly Routley] (Australia, b. 1939). Frequent collaborator with Sylvan.
- Graham Priest (Australia). Perhaps the most prominent advocate of paraconsistent logic in the world today.
- Francisco Miró Quesada (Peru). Coined the term *paraconsistent logic*.
- Peter Schotch (Canada)
- B. H. Slater (Australia). Another articulate critic of paraconsistent logic.
- Richard Sylvan [formerly Routley] (New Zealand/Australia, 1935–1996). Important figure in relevance logic and a frequent collaborator with Plumwood and Priest.
- Nicolai A. Vasiliev (Russia, 1880–1940). First to construct logic tolerant to contradiction (1910).

Notes

[1] Priest (2002), p. 288 and §3.3.
[2] Carnielli, W. and Marcos, J. (2001) "Ex contradictione non sequitur quodlibet" (http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.107.70) *Proc. 2nd Conf. on Reasoning and Logic* (Bucharest, July 2000)
[3] See the article on the principle of explosion for more on this.
[4] Hewitt (2008b)
[5] Hewitt (2008a)
[6] Priest (2002), p. 306.
[7] LP is also commonly presented as a many-valued logic with three truth values (*true*, *false*, and *both*).
[8] See, for example, Priest (2002), §5.
[9] See Priest (2002), p. 310.
[10] Surveys of various approaches to paraconsistent logic can be found in Bremer (2005) and Priest (2002), and a large family of paraconsistent logics is developed in detail in Carnielli, Congilio and Marcos (2007).
[11] See Aoyama (2004).
[12] Most of these are discussed in Bremer (2005) and Priest (2002).
[13] See, for example, the articles in Bertossi et al. (2004).
[14] See Lewis (1982).
[15] See Slater (1995), Béziau (2000).

Resources

- Jean-Yves Béziau, Walter Carnielli and Dov Gabbay, eds. (2007). *Handbook of Paraconsistency*. London: King's College. ASIN 1904987737. ISBN 978-1-904987-73-4.
- Aoyama, Hiroshi (2004). "LK, LJ, Dual Intuitionistic Logic, and Quantum Logic". *Notre Dame Journal of Formal Logic* **45** (4): 193–213. doi:10.1305/ndjfl/1099238445.
- Bertossi, Leopoldo, eds. (2004). *Inconsistency Tolerance*. Berlin: Springer. ISBN 3-540-24260-0.
- Brunner, Andreas and Carnielli, Walter (2005). "Anti-intuitionism and paraconsistency". *Journal of Applied Logic* **3** (1): 161–184. doi:10.1016/j.jal.2004.07.016.
- Béziau, Jean-Yves (2000). "What is Paraconsistent Logic?". In In D. Batens et al. (eds.). *Frontiers of Paraconsistent Logic*. Baldock: Research Studies Press. pp. 95–111. ISBN 0-86380-253-2.
- Bremer, Manuel (2005). *An Introduction to Paraconsistent Logics*. Frankfurt: Peter Lang. ISBN 3-631-53413-2.
- Brown, Bryson (2002). "On Paraconsistency". In In Dale Jacquette (ed.). *A Companion to Philosophical Logic*. Malden, Massachusetts: Blackwell Publishers. pp. 628–650. ISBN 0-631-21671-5.
- Carnielli, Walter; Coniglio, Marcelo E. and Marcos, J, (2007). "Logics of Formal Inconsistency". In In D. Gabbay and F. Guenthner (eds.). *Handbook of Philosophical Logic, Volume 14* (2nd ed.). The Netherlands: Kluwer Academic Publishers. pp. 1–93. ISBN 1-4020-6323-7.
- Feferman, Solomon (1984). *Toward Useful Type-Free Theories, I*. "Journal of Symbolic Logic". *The Journal of Symbolic Logic* **49** (1): 75–111. JSTOR 2274093.
- Hewitt, Carl (2008a). "Large-scale Organizational Computing requires Unstratified Reflection and Strong Paraconsistency". In Jaime Sichman, Pablo Noriega, Julian Padget and Sascha Ossowski (ed.). *Coordination,*

Organizations, Institutions, and Norms in Agent Systems III. Lecture Notes in Computer Science. **4780**. Springer-Verlag. doi:10.1007/978-3-540-79003-7.
- Hewitt, Carl (2008b). "Common sense for concurrency and inconsistency tolerance using Direct Logic^{TM} and the Actor model". arXiv:0812.4852 [cs.LO].
- Lewis, David (1998) [1982]. "Logic for Equivocators". *Papers in Philosophical Logic*. Cambridge: Cambridge University Press. pp. 97–110. ISBN 0-521-58788-3.
- Peña, Lorenzo (1996). "Graham Priest's 'Dialetheism': Is it altogether true?" (http://lp.jurid.net/articles/logica/dialethe.htm). *Sorites* **7**: 28–56. hdl:10261/9714. Retrieved 2009-05-03.
- Priest, Graham (2002). "Paraconsistent Logic.". In In D. Gabbay and F. Guenthner (eds.). *Handbook of Philosophical Logic, Volume 6* (2nd ed.). The Netherlands: Kluwer Academic Publishers. pp. 287–393. ISBN 1-4020-0583-0.
- Priest, Graham and Tanaka, Koji (1996, 2009). "Paraconsistent Logic" (http://plato.stanford.edu/entries/logic-paraconsistent/). *Stanford Encyclopedia of Philosophy*. Retrieved June 17, 2010. (First published Tue Sep 24, 1996; substantive revision Fri Mar 20, 2009)
- Slater, B. H. (1995). "Paraconsistent Logics?". *Journal of Philosophical Logic* **24** (4): 451–454. doi:10.1007/BF01048355.
- Woods, John (2003). *Paradox and Paraconsistency: Conflict Resolution in the Abstract Sciences*. Cambridge: Cambridge University Press. ISBN 0-521-00934-0.

External links

- Stanford Encyclopedia of Philosophy “Paraconsistent Logic” (http://plato.stanford.edu/entries/logic-paraconsistent/)
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Is logic empirical?

"**Is logic empirical?**" is the title of two articles (by Hilary Putnam and Michael Dummett)[1][2] that discuss the idea that the algebraic properties of logic may, or should, be empirically determined; in particular, they deal with the question of whether empirical facts about quantum phenomena may provide grounds for revising classical logic as a consistent logical rendering of reality. The replacement derives from the work of Garrett Birkhoff and John von Neumann on quantum logic. In their work, they showed that the outcomes of quantum measurements can be represented as binary propositions and that these quantum mechanical propositions can be combined in much the same way as propositions in classical logic. However, the algebraic properties of this structure are somewhat different from those of classical propositional logic in that the principle of distributivity fails.

The idea that the principles of logic might be susceptible to revision on empirical grounds has many roots, including the work of W.V. Quine and the foundational studies of Hans Reichenbach.[3]

W.V. Quine

What is the epistemological status of the laws of logic? What sort of arguments are appropriate for criticising purported principles of logic? In his seminal paper "Two Dogmas of Empiricism," the logician and philosopher W.V. Quine argued that all beliefs are in principle subject to revision in the face of empirical data, including the so-called analytic propositions. Thus the laws of logic, being paradigmatic cases of analytic propositions, are not immune to revision.

To justify this claim he cited the so-called *paradoxes of quantum mechanics*. Birkhoff and von Neumann proposed to resolve those paradoxes by abandoning the principle of distributivity, thus substituting their quantum logic for classical logic.

Quine did not at first seriously pursue this argument, providing no sustained argument for the claim in that paper. In *Philosophy of Logic* (the chapter titled "Deviant Logics"), Quine rejects the idea that classical logic should be revised in response to the paradoxes, being concerned with "a serious loss of simplicity", and "the handicap of having to think within a deviant logic". Quine, though, stood by his claim that logic is in principle not immune to revision.

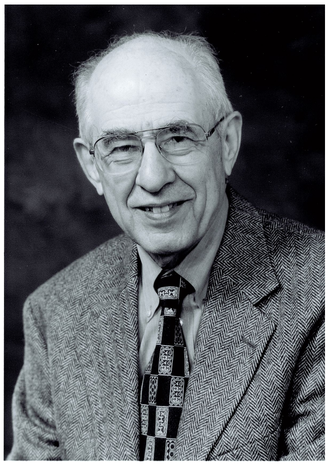
Hans Reichenbach

Reichenbach considered one of the anomalies associated with quantum mechanics, the problem of complementary properties. A pair of properties of a system is said to be *complementary* if each one of them can be assigned a truth value in some experimental setup, but there is no setup which assigns a truth value to both properties. The classic example of complementarity is illustrated by the double-slit experiment in which a photon can be made to exhibit particle-like properties or wave-like properties, depending on the experimental setup used to detect its presence. Another example of complementary properties is that of having a precisely observed position or momentum.

Reichenbach approached the problem within the philosophical program of the logical positivists, wherein the choice of an appropriate language was not a matter of the truth or falsity of a given language – in this case, the language used to describe quantum mechanics – but a matter of "technical advantages of language systems". His solution to the problem was a logic of properties with a three-valued semantics; each property could have one of three possible truth-values: true, false, or indeterminate. The formal properties of such a logical system can be given by a set of fairly simple rules, certainly far simpler than the "projection algebra" that Birkhoff and von Neumann had introduced a few years earlier. However, because of this simplicity, the intended semantics of Reichenbach's three-valued logic is unsuited to provide a foundation for quantum mechanics that can account for observables.

First article: Hilary Putnam

In his paper "Is logic empirical?" Hilary Putnam, whose PhD studies were supervised by Reichenbach, pursued Quine's idea systematically. In the first place, he made an analogy between laws of logic and laws of geometry: once Euclid's postulates were believed to be truths about the physical space in which we live, but modern physical theories are based around non-Euclidean geometries, with a different and fundamentally incompatible notion of straight line.


Hilary Putnam

In particular, he claimed that what physicists have learned about quantum mechanics provides a compelling case for abandoning certain familiar principles of classical logic for this reason: realism about the physical world, which Putnam generally maintains, demands that we square up to the anomalies associated with quantum phenomena. Putnam understands realism about physical objects to entail the existence of the properties of momentum and position for quanta. Since the uncertainty principle says that either of them can be determined, but both cannot be determined at the same time, he faces a paradox. He sees the only possible resolution of the paradox as lying in the embrace of quantum logic, which he believes is not inconsistent.

Quantum logic

The formal laws of a physical theory are justified by a process of repeated controlled observations. This from a physicist's point of view is the meaning of the empirical nature of these laws.

The idea of a propositional logic with rules radically different from Boolean logic in itself was not new. Indeed a sort of analogy had been established in the mid-nineteen thirties by Garrett Birkhoff and John von Neumann between a non-classical propositional logic and some aspects of the measurement process in quantum mechanics. Putnam and the physicist David Finkelstein proposed that there was more to this correspondence than a loose analogy: that in fact there was a logical system whose semantics was given by a lattice of projection operators on a Hilbert space. This, actually, was the correct logic for reasoning about the microscopic world.

In this view, classical logic was merely a limiting case of this new logic. If this were the case, then our "preconceived" Boolean logic would have to be rejected by empirical evidence in the same way Euclidean geometry (taken as the correct geometry of physical space) was rejected on the basis of (the facts supporting the theory of) general relativity. This argument is in favour of the view that the rules of logic are empirical.

That logic came to be known as quantum logic. There are, however, few philosophers today who regard this logic as a replacement for classical logic; Putnam himself may no longer hold that view. Quantum logic is still used as a foundational formalism for quantum mechanics: but in a way in which primitive events are not interpreted as atomic sentences but rather in operational terms as possible outcomes of observations. As such, quantum logic provides a unified and consistent mathematical theory of physical observables and quantum measurement.

Second article: Michael Dummett


Michael Dummett

In an article also titled "Is logic empirical?," Michael Dummett argues that Putnam's desire for realism mandates distributivity: the principle of distributivity is essential for the realist's understanding of how propositions are true of the world, in just the same way as he argues the principle of bivalence is. To grasp why: consider why truth tables work for classical logic: firstly, it must be the case that the variable parts of the proposition are either true or false: if they could be other values, or fail to have truth values at all, then the truth table analysis of logical connectives would not exhaust the possible ways these could be applied; for example intutionistic logic respects the classical truth tables, but not the laws of classical logic, because intuitionistic logic allows propositions to be other than true or false. Second, to be able to apply truth tables to describe a connective depends upon distributivity: a truth table is a disjunction of conjunctive possibilities, and the validity of the exercise depends upon the truth of the whole being a consequence of the bivalence of the propositions, which is true only if the principle of distributivity applies.

Hence Putnam cannot embrace realism without embracing classical logic, and hence his argument to endorse quantum logic because of realism about quanta is a hopeless case.

Dummett's argument is all the more interesting because he is not a proponent of classical logic. His argument for the connection between realism and classical logic is part of a wider argument to suggest that, just as the existence of particular class of entities may be a matter of dispute, so a disputation about the objective existence of such entities is also a matter of dispute. Consequently intuitionistic logic is privileged over classical logic, when it comes to disputation concerning phenomena whose objective existence is a matter of controversy.

Thus the question, "Is logic empirical?," for Dummett, leads naturally into the dispute over realism and anti-realism, one of the deepest issues in modern metaphysics.

Notes

[1] Putnam, H. "Is Logic Empirical?" Boston Studies in the Philosophy of Science, vol. 5, eds. Robert S. Cohen and Marx W. Wartofsky (Dordrecht: D. Reidel, 1968), pp. 216-241. Repr. as "The Logic of Quantum Mechanics" in Mathematics, Matter and Method (1975), pp. 174-197.

[2] Dummett, M. (1976), "Is logic empirical?", in H. D. Lewis (ed.), Contemporary British Philosophy, 4th series (London: Allen and Unwin), pp. 45–68. Reprinted in M. Dummett, *Truth and other Enigmas* (London: Duckworth,1978), pp. 269–289

[3] Reichenbach, H., The Philosophical Foundations of Quantum Mechanics, University of California Press, 1944. Reprinted by Dover 1998,

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