Due Friday, June 5 at the beginning of class.

1. **[8 points]** *Simpson's three-eighths rule* is the quadrature rule obtained from integrating the Lagrange interpolant of degree 3. Its formula is:

$$\int_a^b f(x)\,dx = \frac{3h}{8}\left[f(a) + 3f(a+h) + 3f(a+2h) + f(a+3h)\right] - \frac{3h^5}{80} f^{(4)}(\xi), \tag{1}$$

where $h = \dfrac{b-a}{3}$ and $\xi \in [a,b]$. The last term on the right is an error term, while the others form the approximation to the integral. As with the trapezoid and Simpson's rule, Simpson's three-eighths rule can be viewed as a weighted average of f on the interval $[a,b]$,

$$\int_a^b f(x)\,dx \approx \sum_{i=0}^{3} w_i f(a+ih), \text{ with } w_i = \int_a^b L_i(x).$$

Although Simpson's rule and Simpsons' three-eighths rule both have the same degree of precision ($n = 3$), the three-eighths rule tends to be somewhat more accurate.

(a) Integrate $L_0(x)$ and $L_1(x)$ on $[a,b]$ and show that they give $w_0 = \frac{3h}{8}$ and $w_1 = \frac{9h}{8}$, respectively. This essentially derives Simpson's three-eighths rule, as $w_2 = \frac{9h}{8}$ and $w_3 = \frac{3h}{8}$ are then obtained from symmetry.

Hint: Making the substitution $u = x - a$ will simplify the integration, just like when we derived Simpson's rule.

(b) Approximate the integral $\displaystyle\int_0^3 \frac{1}{x+1}dx$ using both Simpson's rule and Simpson's three-eighths rule. (Be careful! The value of h is different for the two rules). Compare the results with the true value of the integral, which is easy to find in this case. How many times larger is the error using Simpson's rule?

2. **[9 points]** We will use **composite** Simpson's rule to do an example similar to the one done in class with the composite trapezoid rule.

(a) Modify the file `comp_trap.m` to implement the composite Simpson's rule. **Please include a printout of your code.** It should not be more than a few lines.

Hint: The syntax `x(1:2:end)` in Matlab will give you the 1st, 3rd, 5th, etc. elements of the vector `x`, and `x(2:2:end)` will give you elements 2, 4, 6, etc. Remember that Matlab's indexing convention starts at 1, so the point x_0 in the notation used in class will be `x(1)` in Matlab, and so on.

A suggested test case to see if you have implemented Simpson's rule properly: apply it to the function x^3+x^2+x+1 using the points `x = 1:0.1:2`. Since Simpson's rule is exact for polynomials of degree 3, it should give the exact answer of $\frac{103}{12} \approx 8.58333$

(b) Apply your code from part (a) to the integral

$$\int_0^3 \frac{1}{x+1}dx$$

using a spacing of $h = 0.25$. How close is the result to the true value of the integral?

(c) Derive an upper bound on the error in the approximation,

$$E_n^S(f) = \frac{-(b-a)h^4}{180} f^{(4)}(\xi), \quad \xi \in (a,b).$$

Roughly how many times larger is it than the true error found in part (b)?

(d) Compute the asymptotic error estimate,

$$\widetilde{E_n^S}(f) = \frac{-h^4}{180}\left(f^{(3)}(b) - f^{(3)}(a)\right).$$

How does it compare to the true error?

3. **[8 points]** Consider a quadrature scheme for integrating over $[-1,1]$ using three points,

$$\int_{-1}^{1} f(x)\,dx \approx w_0 f(x_0) + w_1 f(x_1) + w_2 f(x_2)$$

(a) Find the choice of weights w_i and points x_i that give a quadrature scheme with degree of precision equal to **five**. (This is the three-point Gaussian Quadrature rule). You may make the following assumptions:

- The points are distributed symmetrically on the interval. Note that this directly implies that $x_1 = 0$ and that $x_0 = -x_2$.
- $w_0 = w_2$ (again by symmetry).

Using these assumptions, there are only three unknowns for which to solve.

(b) Apply this quadrature rule and Simpson's rule to the following integral:

$$\int_{-1}^{1} \frac{1}{4+x^2}\,dx$$

The true value of this integral is $\arctan\left(\frac{1}{2}\right)$. What is error in the two approximations?

Note: The assignment will be graded out of 25, plus 3 points for presentation, for a total of 28 points. To ensure that you get 3 points for presentation:

- Write your name, student ID and the assignment number on the front page.
- Staple the assignment in the upper left corner.
- Present your work neatly (typed or handwritten are both OK as long as your work is easy to follow). There should be no torn pages or sections of work that are scribbled out.