Due Wednesday, May 27 at the beginning of class.

1. **[8 points]** Consider finding the quadratic polynomial, $P_2(x)$, that interpolates the data points (x_i, y_i) given by $(1, 1)$, $(2, 2)$ and $(3, 6)$.

   (a) Suppose we represent $P_2(x)$ with the standard polynomial basis, $P_2(x) = a_0 + a_1x + a_2x^2$. Write down the 3×3 system of linear equations for the coefficients a_i. Solve the system (you can use Matlab) and give the corresponding expression for $P_2(x)$.

   (b) Express $P_2(x)$ in terms of the Lagrange polynomial basis, $P_2(x) = \sum_{i=0}^{2} y_i L_i(x)$. Expand the polynomial and show that it is equal to the polynomial found in part (a). (You must show at least a couple lines of work here to prove that you did it!)

   (c) Compute the table of Newton's divided differences and express $P_2(x)$ in Newton's divided difference form. Again, expand the polynomial and show that it is the same as the polynomial found in part (a).

2. **[9 points]** In this question we consider interpolating the function $f(x) = \sin\left(\frac{\pi}{2}x\right)$ on the interval $x \in [0, 4]$, using five interpolating nodes $x_0, x_1, \ldots x_4$.

   Note: Download the programs `lagrange_interp.m` and `cheb_points.m` for this problem.

   (a) Give an upper bound on the error in the interpolating polynomial, $f(x^*) - P_4(x^*)$, at any point $x^* \in [0, 4]$. The expression will depend on x^* and the location of the nodes $x_0, x_1, \ldots x_4$.

   (b) Suppose we use five equally spaced interpolating nodes at $x = \{0, 1, 2, 3, 4\}$. Use your formula from part (a) to get an upper bound on the error in the polynomial approximation at $x^* = \frac{1}{2}$. Then, use `lagrange_interp.m` to evaluate the polynomial at that point, and verify that the error is less than the upper bound.

   Hint: The `prod` command in Matlab may be helpful for computing $\prod(x^* - x_i)$.

   (c) Use `cheb_points.m` to find the five Chebyshev interpolation points on the interval $[0, 4]$, and repeat part (b) using these interpolation nodes. Is the error larger or smaller than with equally spaced points?

   (d) **Bonus** (Up to two extra points): Provide a nicely labelled graph of $f(x)$ and the two polynomial interpolants you computed in parts (b) and (c), over the interval $[0, 4]$. Are there any regions of the interval where the interpolant that uses equally spaced points is better?

Question 3 is on the next page.

3. **[8 points]** The *Hermite interpolating polynomial* is an extension of the Lagrange interpolant. Given a set of data points (x_i, y_i), $i = 0 \dots n$, and the values of the derivative at these points, (x_i, y_i'), the Hermite polynomial is the unique polynomial of degree $2n+1$ such that

$$H_{2n+1}(x_i) = y_i, \quad \text{and } H'_{2n+1}(x_i) = y_i'$$

at all points x_i. In other words, $H_{2n+1}(x)$ interpolates a function at the points x_i, and has the same first derivative values at those points as well.

Hermite interpolating polynomials can be constructed using divided differences. We will only consider the case $n = 1$. In this case, the table of divided differences looks like the following:

$$\begin{array}{ccccc}
x_0 & y_0 = a_0 & & & \\
x_0 & y_0 & y_0' = a_1 & & \\
x_1 & y_1 & f[x_0, x_1] & \frac{1}{x_1 - x_0}\left(f[x_0, x_1] - y_0'\right) = a_2 & \\
x_1 & y_1 & y_1' & \frac{1}{x_1 - x_0}\left(y_1' - f[x_0, x_1]\right) & a_3
\end{array}$$

where a_3 is the divided difference of the two elements in the previous column (i.e. you subtract them and divide by $x_1 - x_0$ again.) Essentially, we construct a table where every pair (x_i, y_i) is duplicated in the first two columns, and in the third column we use y_i' in place of any undefined quantities of the form $\dfrac{y_i - y_i}{x_i - x_i}$. From that point on, divided differences are computed as normal.

The polynomial is then given by

$$H_3(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^2(x - x_1)$$

We will consider interpolating the function $f(x) = \sin\left(\frac{\pi}{2}x\right)$ at the points $x_0 = 0$, $x_1 = 1$.

(a) Use divided differences to compute (by hand) the following two polynomial approximations to $f(x)$ using the points $x_0 = 0$ and $x_1 = 1$:

  i. The Lagrange interpolant of degree 1, $P_1(x)$.

  ii. The Hermite interpolant of degree 3, $H_3(x)$.

(b) Plot $g(x)$, $P_1(x)$ and $H_3(x)$ on the interval $[-0.2, 1.2]$ and on the same set of axes, with all three curves clearly labelled. Use `x = -0.2:0.01:1.2` as the set of x-values. Comment on the quality of the approximation provided by $P_1(x)$ and $H_3(x)$. **Please include a copy of the plot with your assignment**.

(c) We saw a theorem in class stating that the Lagrange interpolant is unique in some sense. Explain why the existence of the Hermite interpolant does not contradict this theorem.

Note: The assignment will be graded out of 25, plus 3 points for presentation, for a total of 28 points. To ensure that you get 3 points for presentation:

- Write your name, student ID and the assignment number on the front page.
- Staple the assignment in the upper left corner.
- Present your work neatly (typed or handwritten are both OK as long as your work is easy to follow). There should be no torn pages or sections of work that are scribbled out.