1 Vector

Scalars

The physical quantity which has only magnitude but no direction is called scalar quantity or scalars. Length, Mass, time, distance covered, speed, temperature, work, etc. are few examples of scalar quantity. The scalar is specified by mere number and unit, where number represents its magnitude. A scalar may be positive or negative. A scalar can be represented by a single letter.

Vectors

A vector is an object that has both a **magnitude** and a **direction**.

Geometrically, a vector is represented by a directed line segment, whose length is the magnitude of the vector and with an arrow indicating the direction. The direction of the vector is from its tail to its head. If the tail is at point A and the head is at point B, the vector from A to B is written as $\overrightarrow{AB}$.

The length (magnitude) of a $\overrightarrow{AB}$ is written as $|\overrightarrow{AB}|$. The length is always a non-negative real number.

Ex: Displacement (an airplane has flown 150 km to the north east) and velocity (a car moving with velocity 54km/h to the south). Force, acceleration, momentum, electric and field intensities etc..

Types of Vectors:

Zero Vector, Unit Vector, Co-initial Vectors, Coinitial Vectors, Collinear Vectors, Equal Vectors, Negative of a Vector

Zero Vector: A vector whose initial and terminal points coincide, is called a zero vector (or null vector) . Zero vector cannot be assigned a definite direction as it has zero magnitude. Or, alternatively otherwise, it may be regarded as having any direction. The vectors represent the zero vector,

Unit Vector: A vector whose magnitude is unity (i.e., 1 unit) is called a unit vector. Unit vector is denoted by $\hat{a}$. Any vector $\vec{a}$ can be expressed in terms of its unit vector a in the following way $\vec{a} = a\hat{a}$ where $a = |\vec{a}|$. Here $\hat{a}$ is in the same direction as $\vec{a}$. $\hat{a}$ is read as 'a cap'.

Therefore, $$\hat{a} = \frac{\vec{a}}{|\vec{a}|}$$

Coinitial Vectors: Two or more vectors having the same initial point are called coinitial vectors.

Collinear Vectors: Two or more vectors are said to be collinear if they are parallel to the same line, irrespective of their magnitudes and directions.

Equal Vectors: Two vectors and are said to be equal, if they have the same magnitude and direction regardless of the positions of their initial points .

Negative of a Vector: A vector whose magnitude is the same as that of a given vector (say $\overrightarrow{AB}$), but direction is opposite to that of it, is called negative of the given vector. For example, $\overrightarrow{BA}$ is a negative vector of the vector $\overrightarrow{AB}$, and written as $\dot{BA} = -A\dot{B}$

Position vector: Consider a point P in space, having coordinates (x, y) with respect to the origin O (0, 0). Then, the vector having O and P as its initial and terminal points, respectively, is called the position vector of the point P with respect to O. The magnitude of $\overrightarrow{OP}$ is given by $|\overrightarrow{OP}| = \sqrt{x^2 + y^2}$.

Addition of vectors

Let $\vec{a}$ *and* $\vec{b}$ be two vectors then to add them, they are positioned so that the initial point of one coincides with the terminal point of the other. This is known as the **"Triangle law of vector addition"**

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$$

Properties of vector addition:

1. For any two vectors $\vec{a}$ *and* $\vec{b}$,

$\vec{a} + \vec{b} = \vec{b} + \vec{a}$ (Commutative property)

2. For any three vectors $\vec{a}$, $\vec{b}$ *and* $\vec{c}$,

$$(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c}) \quad \text{(Associative property)}$$

Parallelogram law of addition

If we have two vectors $\vec{a}$ *and* $\vec{b}$ represented by two adjacent sides of a parallelogram in magnitude and direction, then their sum $\vec{a}+\vec{b}$ is represented in magnitude and direction by the diagonal of the parallelogram through their common point.

Multiplication of a vector by a scalar

Let $\vec{a}$ be a given vector and λ a scalar. Then the product of $\vec{a}$ by the scalar λ, denoted as $\lambda\vec{a}$ is called multiplication of a vector $\vec{a}$ by a scalar λ.

$$|\lambda\vec{a}|=|\lambda||\vec{a}|$$

1.1

Product of vectors:

Vectors can be multiplied in two different ways: the scalar and vector product. A scalar product of two vectors results in a scalar quantity and a vector product in a vector quantity.

Scalar (or dot) Product of two vectors:

The dot product (also called the inner product or scalar product) of two vectors is denoted by $\vec{a}.\vec{b}$ and is defined as:

$$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta,$$

where $|\vec{a}|$ *and* $|\vec{b}|$ represents the magnitude of $\vec{a}$ *and* $\vec{b}$ and θ is the angle between $\vec{a}$ *and* $\vec{b}$

The dot or scalar product of vectors $\vec{a} = a_1 i + a_2 j$ *and* $\vec{b} = b_1 i + b_2 j$ can be written as

$$\vec{a}.\vec{b} = a_1.b_1 + a_2.b_2$$

Note:

1. Let $\vec{a}$ *and* $\vec{b}$ be two non zero vectors, then $\vec{a}.\vec{b}=0$ if and only if $\vec{a}$ and $\vec{b}$ are perpendicular to each other

i.e., $\vec{a}.\vec{b}=0 \Leftrightarrow \vec{a}\perp\vec{b}$

2. If $\theta=0$, then $\vec{a}.\vec{b}=|\vec{a}||\vec{b}|$ $\quad [cos\theta=cos0^0=1]$

3. If $\theta=\pi$, *then* $\vec{a}.\vec{b}=-|\vec{a}||\vec{b}|$ $\quad [\cos\pi=-1]$

4. For mutually perpendicular units vectors $\hat{i},\hat{j}$ *and* $\hat{k}$, we have

$$\hat{i}.\hat{i}=\hat{j}.\hat{j}=\hat{k}.\hat{k}=1$$

$$\hat{i}.\hat{j}=\hat{j}.\hat{k}=\hat{k}.\hat{i}=0$$

5. The angle between two nonzero vectors $\vec{a}$ *and* $\vec{b}$ is given by

$$cos\theta=\frac{\vec{a}.\vec{b}}{|\vec{a}|\,|\vec{b}|}\Rightarrow\theta=\cos^{-1}\left(\frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}\right)$$

6. The scalar product is commutative i.e.,

$$\vec{a}.\vec{b}=\vec{b}.\vec{a}$$

Vector (or cross) Product of two vectors:

The vector product of two nonzero vectors $\vec{a}$ *and* $\vec{b}$ is denoted by $\vec{a} \times \vec{b}$ and is defined as

$$\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}|\sin\theta.\hat{n}$$

where θ is the angle between $\vec{a}$ *and* $\vec{b}$ and $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ *and* $\vec{b}$ such that $\vec{a}, \vec{b}$ *and* $\hat{n}$ form a right handed system.

The cross product of two vectors $\vec{a} = (a_1, a_2, a_3)$ *and* $\vec{b} = (b_1, b_2, b_3)$ is given by

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \hat{i}(a_2 b_3 - a_3 b_2) - \hat{j}(a_1 b_3 - b_1 a_3) + \hat{k}(a_1 b_2 - b_1 a_2)$$

Note:

1. Let $\vec{a}$ *and* $\vec{b}$ be two nonzero vectors. Then $\vec{a} \times \vec{b} = 0$ if and only if $\vec{a}$ *and* $\vec{b}$ are parallel (or collinear) to each other

i.e., $\vec{a} \times \vec{b} = 0 \Leftrightarrow \vec{a} \parallel \vec{b}$

2. If $\theta = 0$ then $\vec{a} \times \vec{b} = 0$ $[\sin 0 = 0]$

3. If $\theta = \frac{\pi}{2}$ then $\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}|$ $\left[\sin\frac{\pi}{2} = 1\right]$

4. For mutually perpendicular unit vectors $\hat{i}, \hat{j}$ *and* $\hat{k}$ we have

$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$$

$$\hat{i} \times \hat{j} = \hat{k}, \hat{j} \times \hat{k} = \hat{i}, \hat{k} \times \hat{i} = \hat{j}$$

5. The angle between two nonzero vectors $\vec{a}$ *and* $\vec{b}$ is given by

$$\sin\theta = \frac{\vec{a} \times \vec{b}}{|\vec{a}|\,|\vec{b}|} \Rightarrow \theta = \sin^{-1}\left(\frac{\vec{a} \times \vec{b}}{|\vec{a}|\,|\vec{b}|}\right)$$

6. The vector product is not communicative

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$

1.2

Area of a triangle given two of its sides $\vec{a}$ and $\vec{b}$

Let $\overline{AB} = \vec{a}, \overline{AC} = \vec{b}$ and $\angle BAC = \theta$

Area of $\triangle ABC = \frac{1}{2} \times AB \times CD$

$$= \frac{1}{2} \times AB.AC\sin\theta$$

$$= \frac{1}{2}|AB|.|AC|\sin\theta$$

$$= \frac{1}{2}|\overline{AB}|.|\overline{AC}|\sin\theta = = \frac{1}{2}|\vec{a}|.|\vec{b}|\sin\theta$$

Area of $\triangle ABC = = \frac{1}{2}|\vec{a} \times \vec{b}|$ where $\vec{a}$ and $\vec{b}$ are two sides.

Area of Parallelogram

Area of parallelogram $= AB \times DE$

$$= |\vec{a}||\vec{b}|.\sin\theta = |\vec{a} \times \vec{b}|$$

Thus, $|\vec{a} \times \vec{b}|$ represents area of parallelogram with $\vec{a}$ and $\vec{b}$ as adjacent sides

Projection of one vector on another

$\overline{FG}$ and $\overline{CD}$ are two vectors. Let θ be the angle between them. Draw CA and $DB \perp FG$. Then AB is called projection of $\overline{CD}$ on $\overline{LM}$

Take $\overline{FG} = \vec{a}$ $\overline{CD} = \vec{b}$

Projection of $\overline{CD}$ on $\overline{FG}$

$$= AB = CE = CD\cos\theta$$

$$= |\overline{CD}|.\cos\theta = \frac{|\vec{b}||\vec{a}|.\cos\theta}{|\vec{a}|}$$ [Divide and Multiply by $|\vec{a}|$]

i,e, Projection of $\vec{b}$ on $\vec{a}$ $= \dfrac{\vec{a}.\vec{b}}{|\vec{a}|} = \vec{b}$

Note that the projection of $\vec{a}$ on $\vec{b}$ $= \dfrac{\vec{a}.\vec{b}}{|\vec{b}|}$

Worked Problems

1. If $\vec{a} = -\hat{i} + \hat{j} + \widehat{4k}, \vec{b} = 2\hat{i} - \hat{j} - 3\hat{k}$, find (i) $\vec{a} \times \vec{b}$ (ii) $4\vec{a} \times 3\vec{b}$ and (iii) $\left|\vec{a} \times \vec{b}\right|$

Sol: (i) $\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ -1 & 1 & 4 \\ 2 & -1 & -3 \end{vmatrix} = [\hat{i}[1(-3) - (-4)] - \hat{j}[(-1)(-3) - (2)(4)] + \hat{k}[(-1)(-1) - 1.2]]$

$$= \hat{i}[-3+4] - \hat{j}[3-8] + \hat{k}[1-2] = = \hat{i} + 5\hat{j} - \hat{k}$$

(ii) $4\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ -4 & 4 & 16 \\ 2 & -1 & -3 \end{vmatrix} = [\hat{i}[4(-3) - (-16)] - \hat{j}[(-4)(-3) - (2)(16)] + \hat{k}[(-4)(-1) - 4.2]]$

$$= \hat{i}[-12+16] - \hat{j}[12-36] + \hat{k}[4-8] = \hat{i}[4] - \hat{j}[24] + \hat{k}[-4]$$

(iii) $\left|\vec{a} \times \vec{b}\right| = \sqrt{1^2 + 5^2 + (-1)^2} = \sqrt{1+25+1} = \sqrt{27} = 3\sqrt{3}$

2.. Find the area of the triangle, whose two sides are $\vec{a} = -6\hat{i} + 3\hat{j} + \hat{k}$, $\vec{b} = \hat{i} - \hat{j} - 2\hat{k}$.

Sol: $\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ -6 & 3 & 1 \\ 1 & -1 & -2 \end{vmatrix} = [\hat{i}[3(-2) - (-1)] - \hat{j}[(-6)(-2) - 1] + \hat{k}[(-6)(-1) - 3]]$

$$= \hat{i}(-6 + 1) - \hat{j}(12 - 1) + \hat{k}(6 - 3)$$

$$= \hat{i}(-5) - \hat{j}(11) + \hat{k}(3) = -5\hat{i} - 11\hat{j} + 3\hat{k}$$

$$|\vec{a} \times \vec{b}| = \sqrt{(-5)^2 + (-11)^2 + (3)^2} = \sqrt{25 + 121 + 9} = \sqrt{155}$$

Area of triangle $= \frac{1}{2}|\vec{a} \times \vec{b}| = \frac{\sqrt{155}}{2} sq.\ units$

3. Find the area of parallelogram, whose adjacent sides are $\vec{a} = \hat{i} + 2\hat{j} + \hat{k}$, $\vec{b} = 3\hat{i} - \hat{j} - \hat{k}$

Sol: $\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ 1 & 2 & 1 \\ 3 & -1 & -1 \end{vmatrix} = [\hat{i}[2(-1)-(-1)] - \hat{j}[(1)(-1)-1.3] + \hat{k}[(1)(-1)-3.2]]$

$= \hat{i}[-2+1] - \hat{j}[-1-3] + \hat{k}[-1-6]$

$= \hat{i}[-1] - \hat{j}[-4] + \hat{k}[-7] = = -1\hat{i} + 4\hat{j} + -7\hat{k}$

$|\vec{a} \times \vec{b}| = \sqrt{(-1)^2 + 4^2 + (-7)^2} = \sqrt{1+16+49} = \sqrt{66}$

Area of parallelogram $= |\vec{a} \times \vec{b}| = \sqrt{66}$ sq. units.

4. Find the projection of $\vec{a} = 3\hat{i} + 3\hat{j} + \hat{k}$ on $\vec{b} = \hat{i} - 2\hat{j} - \hat{k}$

Sol: Projection of $\vec{a}$ on $\vec{b} = \frac{\vec{a}.\vec{b}}{|\vec{b}|} = \frac{(3)(1)+(3)(-2)+(1)(-1)}{\sqrt{1^2+(-2)^2+(-1)^2}}$

$$= \frac{3-6-1}{\sqrt{1+4+1}} = \frac{-4}{\sqrt{6}} = \frac{-4}{\sqrt{6}} \times \frac{\sqrt{6}}{\sqrt{6}} = \frac{-4\sqrt{6}}{6} = \frac{-2\sqrt{6}}{3}$$

1 Probability

Introduction

The theory of probability originated while dealing with "games of chance" such as gambling, playing cards, coin tossing etc.

"A gambler's dispute in 1654 led to the creation of a mathematical theory of probability by two famous French mathematicians, Blaise Pascal and Pierre de Fermat. But axiomatic approach to the definition of probability was given by Russian mathematician A. Kolmogorov.
Probability is defined as chance of occurring of certain event when expressed quantitatively i.e., it is the measure of how likely an event is.

Some important terms and concepts

Random experiments: These are the experiments whose outcomes cannot be predicted with a certainty before they actually happen.

Trial: When we repeat a random experiment several times, we call each one of them a trial.

Event: The possible outcomes of a trial are called as events.

Sample space: It is a set of all possible outcomes of a random experiment. It is denoted by S.

Illustration:

1. Random experiment: toss a coin: sample space S={heads, tails} or S={H,T}. Here the outcomes of getting head or tail is an event.

2. Random experiment: roll a die; sample space: S={1,2,3,4,5,6}.

3. Random experiment: observe the number of books sold by an book store in Bangalore in 2015; sample space:

   S={0,1,2,3,⋯}.

4. Random experiment: observe the number of goals in a football match; sample space: S={0,1,2,3,⋯}

5. Random experiment: toss a pair of coins simultaneously S={HH,HT,TH,TT}

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5. Random experiment: toss a pair of coins simultaneously S={HH,HT,TH,TT}

Equally likely events: Events which have the same chance of occurring.

Illustration:

1. When a die is thrown all the six faces {1,2,3,4,5,6} are equally likely to occur.

2. When a coin is tossed both head and tail are equally likely to occur.

Types of events

Simple (elementary) events: which contain only one outcome

Compound events: which contains two or more outcomes

Sure or certain events: which contains all the outcomes of the sample space

Impossible (null) event: which contains not outcomes

Exhaustive Event:

When at least one of the events occurs compulsorily from the list of events, then it is also known as exhaustive events.

1) In the experiment of tossing a coin:

A : the event of getting a HEAD

B : the event of getting a TAIL

The two events "A" and "B" are called exhaustive events. [When we conduct the experiment, at least one of these will occur.]

2) In the experiment of throwing a die:

A : the event of getting 1

B : the event of getting 2

...

...

F : the event of getting 6

The six Events "A", "B", "C", "D", "E", "F" together are called exhaustive events. [One of these events will occur whenever the experiment is conducted.]

Algebra of events

If A and B are two events of sample space S then,

1) $A \cup B$ is an event that either A or B or both occur.

2) $A \cap B$ is an event that A and B both occur simultaneously.

3) $\overline{A}$ is the event that A does not occur (complementary event).

4) $\overline{A} \cap \overline{B}$ is an event of non occurrence of both A and B i.e., non of A and B occurs.

Illustration:

1) In a single throw of a die.

$A =$ {event of getting an even number} $A =$ {event of getting a number greater than 3}

then, $A = \{2, 4, 6\}$ $B = \{4, 5, 6\}$

i) $A \cup B$= {event of getting an even number or a number greater than 3}

$A \cup B = \{2, 4, 5, 6\}$

ii) $A \cap B$ = {event of getting an even number greater than 3}

$A \cap B = \{4, 6\}$

iii) $\overline{A}$= {event of not getting an even number i.e., getting an odd number}

$\overline{A} = \{1, 3, 5\}$

$\overline{B}$ = {event of not getting a number greater than 3 i.e., getting a number less than or equal to 3}

$\overline{B} = \{1, 2, 3\}$

$\overline{A} \cap \overline{B}$= {event of neither getting an even number nor a number greater than 3}

$\overline{A} \cap \overline{B} = \{1, 3\}$

Mutually exclusive events:

Two or more events are said to be mutually exclusive if they cannot occur at the same time.

Illustration:

1) Turning left and right simultaneously are mutually exclusive events.

2) In tossing a fair coin getting head and tail simultaneously are mutually exclusive events.

3) A six sided die is rolled once.

A : rolling an odd number

B : rolling an even number

A and B are mutually exclusive, because they cannot occur at the same time.

Probability of an event:

The probability of an event is defined as the number of outcomes favourable to the given event divided by the total number of equally likely outcomes in the sample space of the experiment.

Probability of an event A, denoted as $P(A)$ and is given by

$$P(A) = \frac{\textit{number of outcomes favourable to } A}{\textit{number of possible outcomes}} = \frac{n(A)}{n(S)}$$

Probability of non occurrence of an event A is given by $P(\overline{A})$.

If the probability of the happening of a certain event is denoted by $p = P(A)$ and that of not happening by $q = P(\overline{A})$, then

$$P(A) + P(\overline{A}) = 1 \text{ or } p + q = 1$$

Here, p and q are non negative and cannot exceed unity i.e., $0 \le p \le 1$, $0 \le q \le 1$

Examples

1. A die is rolled, find the probability that an even number is obtained.

Solution: The sample space S of the experiment is

$$S = \{1,2,3,4,5,6\}$$

Let A be the event of getting even number is

$$A = \{2,4,6\}$$

Required probability

$$P(A) = n(A) / n(S) = 3 / 6 = 1 / 2$$

2. A die is rolled and a coin is tossed, find the probability that the die shows an odd number and the coin shows a head.

Solution:

The sample space S of the experiment is

$$S = \{ (1,H),(2,H),(3,H),(4,H),(5,H),(6,H),(1,T),(2,T),(3,T),(4,T),(5,T),(6,T)\}$$

Let E be the event "the die shows an odd number and the coin shows a head" and is as follows

$$A=\{(1,H),(3,H),(5,H)\}$$

The probability P(E) is given by

$$P(A) = n(A) / n(S) = 3 / 12 = 1 / 4$$

3. a) A die is rolled, find the probability that the number obtained is greater than 4.

b) Two coins are tossed, find the probability that one head only is obtained.
c) Two dice are rolled, find the probability that the sum is equal to 5.
d) A card is drawn at random from a deck of cards. Find the probability of getting the King of heart.

Solution: a) $P(A) = n(A) / n(S) = 2 / 6 = 1 / 3$

b) $P(A) = n(A) / n(S) = 2 / 4 = 1 / 2$

c) $P(A) = n(A) / n(S) = 4 / 36 = 1 / 9$

d) $P(A) = n(A) / n(S) = 1 / 52$

4. The letters P, I, N, and K are written on slips of paper. The four slips of paper are placed in a hat. The slips are then selected one at a time from the hat. What is the probability that the order in which they are chosen spells PINK?

Solution: There is only one way to spell PINK; so, $n(A) = 1$.

There are 4! arrangements of the 4 letters; so $n(S) = 4!$

$$P(A) = \frac{n(A)}{n(S)} = \frac{1}{4!} = \frac{1}{24}$$

The probability that the slips spell PINK is $\frac{1}{24}$.

5. Two dice are thrown simultaneously. What is the probability of getting two numbers whose product is even?

Solution: In a simultaneous throw of two dice, we have $n(S) = 6 \times 6 = 36$.

Then,

Then, A = {(1, 2), (1, 4), (1, 6), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (3, 2), (3, 4), (3, 6), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (5, 2), (5, 4), (5, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)}

$n(A) = 27$

$$P(E) = \frac{n(A)}{n(S)} = \frac{27}{36} = \frac{3}{4}$$

Addition rule of probability: If A and B are any two events,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

where,

$P(A)$ is probability of event A,

$P(B)$ is probability of event B

$P(A \cap B)$ is probability of event A and B

Fundamental theorems on Probability:

Theorem1: In a random experiment, if S is the sample space and E is an event then

(i) $P(E) \geq 0$ (ii) $P(\phi) = 0$ (iii) $P(S) = 1$

Note: From the above results it follows that

(i) Probability of occurrence of an event is always non-negative.

(ii) Probability of occurrence of an impossible event is 0.

(iii) Probability of occurrence of a sure event is 1.

Theorem2: If A and B are mutually exclusive events $(A \cap B \neq 0)$, then

(i) $P(A \cap B) = 0$ and (ii) $P(A \cup B) = P(A) + P(B)$

Note: Let $A_1 + A_2 + A_3 + \dots\dots\dots + A_n$ are mutually exclusive events then,

$$P(A_1 \cup A_2 \cup \dots\dots\dots\dots\dots \cup A_n) = P(A_1) + P(A_2) + \dots\dots\dots\dots + P(A_n)$$

Theorem3: If A is any event and $\overline{A}$ its complementary event, then

$$P(A) = 1 - P(\overline{A})$$

Theorem4: If A and B are any two events,

$$P(A - B) = P(A) - P(A \cap B)$$

Theorem 5: If A_1 and A_2 be two events such that $A_1 \subset A_2$, then $P(A_1) \leq P(A_2)$

Theorem6: If A is any event associated with a random experiment, then $0 \leq P(A) \leq 1$

Conditional probability:

The *conditional probability* of an event *B* is the probability that the event will occur given the knowledge that an event *A* has already occurred. It is denoted by *P(B|A) and is defined as*

$$P(A/B) = \frac{P(A \cap B)}{P(B)} \text{ if } P(B) \neq 0$$

http://www.watchknowlearn.org/Video.aspx?VideoID=32843&CategoryID=4455

Illustration:

1. An urn contains 6 red marbles and 4 black marbles. Two marbles are drawn *with replacement* from the urn. What is the probability that both of the marbles are black?

Solution: Let A = the event that the first marble is black; and let B = the event that the second marble is black. We know the following:

- In the beginning, there are 10 marbles in the urn, 4 of which are black. Therefore, P(A) = 4/10.
- After the first selection, we replace the selected marble; so there are still 10 marbles in the urn, 4 of which are black. Therefore, P(B|A) = 4/10.

Therefore, based on the rule of multiplication:

$P(A \cap B) = P(A)\, P(B|A)$

$P(A \cap B) = (4/10)*(4/10) = 16/100 = 0.16$

2. A card is drawn randomly from a deck of ordinary playing cards. You win Rs.10 if the card is a spade or an ace. What is the probability that you will win the game?
Let S = the event that the card is a spade; and let A = the event that the card is an ace.
Solution:
There are 52 cards in the deck.
There are 13 spades, so $P(S) = 13/52$.
There are 4 aces, so $P(A) = 4/52$.
There is 1 ace that is also a spade, so $P(S \cap A) = 1/52$.
Therefore, based on the rule of addition:
$P(S \cup A) = P(S) + P(A) - P(S \cap A)$
$P(S \cup A) = 13/52 + 4/52 - 1/52 = 16/52 = 4/13$

Independent event:

Two events A and B are said to be independent if the result of the second event is not affected by the result of the first event.

Illustration:

1) What is the probability that the total of two dice will be greater than 9, given that the first die is a 5?

Solution: Let *A* = first die is 5

$A = \{(1,4)(4,1)(2,3)(3,2)\}$

Let *B* = total of two dice is greater than 9

$$P(A) = \frac{1}{6}$$

Possible outcomes for *A* and *B*: (5, 5), (5, 6)

$$P(A\ and\ B) = \frac{2}{36} = \frac{1}{18}$$

$$P(B/A) = \frac{P(A\ and\ B)}{P(A)} = \frac{1}{18} \div \frac{1}{6} = \frac{1}{3}$$

2) A bag contains red and blue marbles. Two marbles are drawn without replacement. The probability of selecting a red marble and then a blue marble is 0.28. The probability of selecting a red marble on the first draw is 0.5. What is the probability of selecting a blue marble on the second draw, given that the first marble drawn was red?

Solution: $P(blue/red) = \dfrac{P(blue\ and\ red)}{P(red)} = \dfrac{0.28}{0.5} = 0.56$

3) Susan took two tests. The probability of her passing both tests is . The probability of her passing the first test is . What is the probability of her passing the second test given that she has passed the first test?

Solution: $P(second/first) = \dfrac{P(first\ and\ second)}{P(first)} = \dfrac{0.6}{0.8} = 0.75$

Allied angles

The angles of the form $\left(n.\frac{\pi}{2} \pm \theta\right)$ $or\ (n.90^\circ \pm \theta)$ are called allied angles

Ex: $90^\circ \pm \theta,\ 180^\circ \pm \theta,\ 270^\circ \pm \theta,\ 360^\circ \pm \theta\ etc.$

Consider a point P on any circle which is moving on it or radius 'r''

1. When P is in I quadrant $(i.e.,\ 0^\circ < \theta < 90^\circ)$

Both $x\ and\ y$ are positive

$$\therefore \sin\theta = \frac{y}{r}, \cos\theta = \frac{x}{r}\ and\ \tan\theta = \frac{y}{x}$$

In I quadrant all ratios are positive.

2. When P is in II quadrant $(i.e.,\ 90^0 < \theta < 180^0)$

x is negative and y is positive

$$\therefore sin\theta = \frac{y}{r}, cos\theta = \frac{-x}{r}\ and\ tan\theta = \frac{y}{-x}$$

In II quadrant only sine is positive.

3. When P is in III quadrant $\left(i.e., 180^\circ < \theta < 270^\circ\right)$

x is negative and y is negative

$$\therefore sin\theta = \frac{-y}{r}, cos\theta = \frac{-x}{r} \ and \ tan\theta = \frac{-y}{-x} = \frac{y}{x}$$

In III quadrant only tan is positive.

4. When P is in IV quadrant $\left(i.e., 270^\circ < \theta < 360^\circ\right)$

x is positive and y is negative

$$\therefore sin\theta = \frac{-y}{r}, cos\theta = \frac{x}{r} \ and \ tan\theta = \frac{-y}{x}$$

In IV quadrant only cosine is positive.

ASTC Rule [All Students Take Coffee]

Figure gives the detailed explanation of ASTC rule and allied angles.

Note:

1. Reciprocals

$\sin\theta \leftrightarrow cosec\theta$

$\cos\theta \leftrightarrow sec\theta$

$\tan\theta \leftrightarrow cot\theta$

2. Co-ratios [Complementary ratios]

$sin\theta \leftrightarrow cos\theta$

$\tan\theta \leftrightarrow cot\theta$

$\text{cosec}\theta \leftrightarrow sec\theta$

3. For the angles $(90^\circ \pm \theta)$, $(270^\circ \pm \theta)$ ratios changes to their co-ratios for an acute angle θ where sign is according to ASTC rule.

4. For the angles $(180^\circ \pm \theta)$, $(3690^\circ \pm \theta)$ ratios remain same for an acute angle θ where sign is according to ASTC rule.

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Ratios of $(90^\circ - \theta)$ *and* θ

$\angle POM = \theta$ then $\angle OPM = 90^\circ - \theta$

$$\sin\angle OPM = \sin(90^\circ - \theta) = \frac{OM}{OP} = \cos\theta$$

$$\therefore \sin(90^\circ - \theta) = \cos\theta$$

similarly, $\cos(90^\circ - \theta) = \sin\theta$ and $\tan(90^\circ - \theta) = \cot\theta$

Ratios of $(90^\circ + \theta)$ *and* θ

$\angle POX = \theta$ then $\angle QOX = 90^\circ + \theta$

triangles PMO amd QNO are congruent

$$\therefore PM = ON \text{ and } OM = QN$$

$$\sin(90^\circ + \theta) = \frac{QN}{r} = \frac{OM}{r} = \cos\theta$$

$$\therefore \sin(90^\circ + \theta) = \cos\theta$$

$$\cos(90^\circ + \theta) = \frac{-ON}{r} = \frac{-PM}{r} = -\sin\theta$$

$$\therefore \cos(90^\circ + \theta) = -\sin\theta$$

$$\tan(90^\circ + \theta) = \frac{QN}{-ON} = \frac{-OM}{PM} = -\cot\theta$$

Note: Any angle can be expressed in the form $(n.90^\circ \pm \theta)$ where n is an integer.

If n is even: Same ratio is taken for acute angle θ where sign is according to ASTC rule.

If n is odd: co-ratio is taken for an acute angle θ where sign is according to ASTC rule.

Examples:

1. $\sin(750^\circ)$

$$= \sin(8 \times 90^\circ + 30^\circ)$$
$$= +\sin 30^\circ$$
$$= \frac{1}{2}$$

OR

$$sin(750^\circ)$$
$$= sin(9 \times 90^\circ - 60^\circ)$$
$$= \cos 60^\circ$$
$$= \frac{1}{2}$$

2. $cot(960^\circ)$

$$= cot(10 \times 90^\circ + 60^\circ)$$
$$= +\, cot\ 60^\circ$$
$$= \frac{1}{\sqrt{3}}$$

OR

$$cot(960^\circ)$$
$$= cot(11 \times 90^\circ - 30^\circ)$$
$$= \tan 30^\circ$$
$$= \frac{1}{\sqrt{3}}$$

Worked Examples:

1. $cosec(-1110°) = -cosec(1110°)$

Sol: $= -cosec(3\times 360° + 30°) = -cosec30° = -2$

2. Find the value of $\sin\left(\frac{25\pi}{6}\right)$

Sol: $= sin\left(\frac{25\times 180°}{6}\right) = \sin 750°$

$= sin(8\times 90° + 30°) = +\sin 30° = \frac{1}{2}$

3. Without using tables / calculator find the value of $\sin^2\left(\frac{2\pi}{3}\right) + \cos^2\left(\frac{\pi}{4}\right)$

Sol: $sin^2\left(\frac{2\pi}{3}\right) + cos^2\left(\frac{\pi}{4}\right) = sin^2\left(\frac{2\times 180°}{3}\right) + \cos^2\left(\frac{180°}{4}\right)$

$= \sin^2(120°) + \cos^2(45°) = [\sin(120°)]^2 + [\cos(45°)]^2$

$= [sin(90° + 30°)]^2 + \left(\frac{1}{2}\right)^2 = (\cos 30°)^2 + \frac{1}{2} = \left(\frac{\sqrt{3}}{2}\right)^2 + \frac{1}{2}$

$= \frac{3}{4} + \frac{1}{2} = \frac{3+2}{4} = \frac{5}{4}$

4. If $\sin A = \frac{4}{5}$ where $90^\circ < A < 180^\circ$ find the value of $\frac{3\sin A - \cos A}{4\operatorname{cosec} A + 3\tan A}$

$$5^2 = PM^2 + OM^2$$
$$5^2 = 4^2 + OM^2$$
$$OM^2 = 25 - 16$$
$$OM^2 = 9$$
$$\therefore OM = 3$$

Sol: $$\frac{3\sin A - \cos A}{4\operatorname{cosec} A + 3\tan A} = \frac{3\left(\frac{4}{5}\right) - \left(\frac{-3}{5}\right)}{4\left(\frac{5}{4}\right) + 3\left(\frac{-4}{3}\right)}$$

$$= \frac{\frac{12}{5} + \frac{3}{5}}{\frac{20}{4} - \frac{12}{3}} = \frac{\frac{12+3}{5}}{\frac{60-48}{12}} = \frac{\frac{15}{5}}{\frac{12}{12}}$$

$$= \frac{15}{5} \times \frac{12}{12} = 3 \times 1 = 3$$

5. Evaluate $\cos(-300°)\sin(-300°)+\sin 420° \cos 390°$

Sol: $cos(-300°)sin(-300°)+sin420°cos390°$

$$= -cos300° \times -sin300° + sin420° cos390°$$

$$= -\cos(360° - 60°) \times -\sin(360° - 60°) + \sin(360° + 60°)\cos(360° + 30°)$$

$$= [-\cos 60° \times \sin 60°] + [\sin 60° \times -\cos 30°]$$

$$= \left(-\frac{1}{2} \times \frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{3}}{2} \times \frac{-\sqrt{3}}{2}\right)$$

$$= \frac{-\sqrt{3}}{4} - \frac{3}{4} = \frac{-\sqrt{3}-3}{4}$$

$$= \frac{-(\sqrt{3}+3)}{4}$$

Compound angles

The sum or difference of 2 or more angles are called compound angles.

Ex: $\theta_1 + \theta_2$, $A + B$, $A - B$ etc.

Theorem:

$\sin(A+B) = \sin A \cos B + \cos A \sin B$

Proof: In the ΔOPS

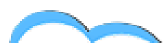
$$\sin(A+B) = \frac{SP}{OS}$$

$$= \frac{ST+TP}{OS} = \frac{ST}{OS} + \frac{TP}{OS}$$

$$= \frac{ST}{SR} \times \frac{SR}{OS} + \frac{RQ}{OR} \times \frac{OR}{OS}$$

$$= \cos A \times \sin B + \sin A \times \cos B$$

$$\therefore \sin(A+B) = \sin A \cos B + \cos A \sin B$$



Similarly $\cos(A+B) = \dfrac{OP}{OS}$

$$= \frac{OQ - PQ}{OS} = \frac{OQ}{OS} - \frac{PQ}{OS}$$

$$= \frac{OQ}{OR} \times \frac{OR}{OS} - \frac{TR}{OS}$$

$$= \frac{OQ}{OR} \times \frac{OR}{OS} - \frac{TR}{SR} \times \frac{SR}{OS}$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

Worked Problems:

1. Find the value of $\sin(15^\circ)$

Sol: $\sin(15^\circ) = \sin(45^\circ - 30^\circ)$

$$= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ$$

$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2}$$

$$= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

In the same we can find the values of
$\cos 15^\circ, \tan 15^\circ, \sin 75^\circ, \cos 75^\circ, \tan 75^\circ, \sin 105^\circ, \cos 105^\circ$ *and* $\tan 105^\circ$

2. Prove that $\sin(A+B)\sin(A-B)=\sin^2 A-\sin^2 B$

Sol: Let L.H.S $=(\sin A\cos B+\cos A\sin B)(\sin A\cos B-\cos A\sin B)$

$$=\sin^2 A\cos^2 B-\cos^2 A\sin^2 B$$

$$=\sin^2 A(1-\sin^2 B)-(1-\sin^2 A)\sin^2 B$$

$$=\sin^2 A-\cancel{\sin^2 A\sin^2 B}-\sin^2 B+\cancel{\sin^2 A\sin^2 B}$$

$$=\sin^2 A-\sin^2 B=R.H.S$$

3. If $A+B+C=\pi$ prove that $\tan A+\tan B+\tan C=\tan A\tan B\tan C$

Sol: $A+B=\pi-C$

Applying tan on both sides

$$\tan(A+B)=\tan(\pi-C)$$

$$\frac{\tan A+\tan B}{1-\tan A\tan B}=-\tan C$$

$$1(\tan A+\tan B)=-\tan C(1-\tan A\tan B)$$

$$\tan A+\tan B=-\tan C+\tan A\tan B\tan C$$

$$\therefore \tan A+\tan B+\tan C=\tan A\tan B\tan C$$

4. Show that $\tan(45^\circ - A) = \dfrac{1-\tan A}{1+\tan A}$

Sol: Let L.H.S $= \tan(45^\circ - A)$

$$= \frac{\tan 45^\circ - \tan A}{1+\tan 45^\circ \tan A} = \frac{1-\tan A}{1+\tan A} = R.H.S$$

5. If $\tan(A-B)=\frac{1}{7}$ and $\tan A = \frac{1}{2}$ show that $A+B=45^\circ$

Sol: Let $\tan(A+B)=\tan(2A-(A-B))$

equals fraction numerator tan 2 A minus tan open parentheses A minus B close parentheses over denominator 1

plus tan 2 A tan open parentheses A minus B close parentheses end fraction

$$\left[\because \tan 2A = \frac{2\tan A}{1-\tan^2 A} = \frac{2\frac{1}{2}}{1-\frac{1}{4}} = \frac{1}{\frac{3}{4}} = \frac{4}{3}\right]$$

$$= \frac{\frac{4}{3}-\frac{1}{7}}{1+\frac{4}{3}\frac{1}{7}} = \frac{\frac{28-3}{21}}{\frac{21+4}{21}} = \frac{\frac{25}{21}}{\frac{25}{21}} = 1$$

$$\tan(A+B)=1$$

$$\therefore A+B=\tan^{-1}1=45^\circ$$

Multiple and submultiple angles

The angles of the type $2A$, $3A$, $4A$, $5A$, $2B$, $3B$, 2θ,.......... etc are known as multiple angles.

Trigonometric ratios in terms of multiple angles

$$\sin 2A = \sin(A + A)$$
$$= \sin A \cos A + \cos A \sin A$$
$$= 2 \sin A \cos A$$
$$\therefore \sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos(A + A)$$
$$= \cos A \cos A - \sin A \sin A$$
$$= \cos^2 A - \sin^2 A$$
$$\therefore \cos 2A = \cos^2 A - \sin^2 A$$

$$
\begin{aligned}
\sin 3A &= \sin(2A + A) \\
&= \sin 2A \cos A + \cos 2A \sin A \\
&= 2\sin A\cos^2 A + (1 - 2\sin^2 A)\sin A \\
&= 2\sin A(1 - \sin^2 A) + \sin A - 2\sin^3 A \\
&= 2\sin A - 2\sin^3 A + \sin A - 2\sin^3 A \\
&= 3\sin A - 4\sin^3 A
\end{aligned}
$$

$$\therefore \sin 3A = 3\sin A - 4\sin^3 A$$

$$
\begin{aligned}
\cos 3A &= \cos(2A + A) \\
&= \cos 2A \cos A - \sin 2A \sin A \\
&= (2\cos^2 A - 1)\cos A - 2\sin^2 A \cos A \\
&= 2\cos^3 A - \cos A - 2\cos A(1 - \cos^2 A) \\
&= 2\cos^3 A - \cos A - 2\cos A + 2\cos^3 A
\end{aligned}
$$

$$\therefore \cos 3A = 4\cos^3 A - 3\cos A$$

$$\tan 3A = \tan(2A + A)$$

$$= \frac{\tan 2A + \tan A}{1 - \tan 2A.\tan A}$$

$$= \frac{\dfrac{2\tan A}{1-\tan^2 A} + \tan A}{1 - \left(\dfrac{2tan^2 A}{1-\tan^2 A}\right).\tan A}$$

$$= \frac{2\tan A + \tan A - \tan^3 A}{1 - \dfrac{2\tan^3 A}{1-\tan^2 A}}$$

$$= \frac{3\tan A - \tan^3 A}{1 - \tan^2 A - 2\tan^2 A}$$

$$\therefore \tan 3A = \frac{3tanA - tan^3 A}{1 - 3tan^2 A}$$

Submultiples

If $2A = \theta$ then $A = \dfrac{\theta}{2}$ and $3A = \theta$ then $A = \dfrac{\theta}{3}$

Substitute these values in all above multiplies formulae then we get

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} \quad \text{or} \quad \cos\theta = 2\cos^2\left(\frac{\theta}{2}\right) - 1 \quad \text{or} \quad \cos\theta = 1 - 2\sin^2\left(\frac{\theta}{2}\right)$$

$$\tan\theta = \frac{2\tan\left(\frac{\theta}{2}\right)}{1 - \tan^2\left(\frac{\theta}{2}\right)}$$

$$sin\theta = 3sin\left(\frac{\theta}{3}\right) - 4\sin^3\left(\frac{\theta}{3}\right)$$

$$\cos\theta = 4\cos^3\left(\frac{\theta}{3}\right) - 3\cos\left(\frac{\theta}{3}\right)$$

$$tan\theta = \frac{3tan\left(\frac{\theta}{3}\right) - \tan^3\left(\frac{\theta}{3}\right)}{1 - 3tan^2\left(\frac{\theta}{3}\right)}$$

Worked problems

1. Prove that $\frac{\sin 3\theta}{\sin\theta} - \frac{\cos 3\theta}{\cos\theta} = 2$

Sol: L.H.S $= \left(\frac{3\sin\theta - 4\sin^3\theta}{\sin\theta}\right) - \left(\frac{4\cos^3\theta - 3\cos\theta}{\cos\theta}\right)$

$= 3 - 4\sin^2\theta - 4\cos^2\theta + 3$

$= 6 - 4(\sin^2\theta + \cos^2\theta)$

$= 6 - 4(1)$

$= 6 - 4 = 2$

2. Prove that $\frac{\sin\theta}{1+\cos\theta} = \tan\left(\frac{\theta}{2}\right)$ and hence find $\tan\left(22\frac{1}{2}^\circ\right)$

Sol: Let L.H.S $= \frac{\sin\theta}{1+\cos\theta}$

$$= \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cancel{1} + 2\cos^2\frac{\theta}{2} \cancel{-1}}$$

$$= \frac{\cancel{2}\sin\frac{\theta}{2}\cancel{\cos\frac{\theta}{2}}}{\cancel{2}\cos^{\cancel{2}}\frac{\theta}{2}}$$

$$= \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \tan\frac{\theta}{2} = \text{R.H.S}$$

To find $\tan\left(22\frac{1}{2}^\circ\right)$

Put $\theta = 45^\circ$

$$\therefore \tan\left(\frac{\theta}{2}\right) = \tan\left(22\frac{1}{2}^{\circ}\right) = \frac{\sin 45^{\circ}}{1+\cos 45^{\circ}}$$

$$= \frac{\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}} = \frac{\frac{1}{\sqrt{2}}}{\frac{\sqrt{2}+1}{\sqrt{2}}}$$

$$\tan\left(22\frac{1}{2}^{\circ}\right) = \frac{1}{\sqrt{2}+1}$$

3. Prove that $\dfrac{1-\cos 2\theta+\sin 2\theta}{1+\cos 2\theta+\sin 2\theta} = \cot(90^{\circ}-\theta)$

Sol: Let L.H.S $= \dfrac{1-\cos 2\theta+\sin 2\theta}{1+\cos 2\theta+\sin 2\theta} = \dfrac{1-(1-2\sin^2\theta)+2\sin\theta\cos\theta}{1+2\cos^2\theta-1+2\sin\theta\cos\theta}$

$$= \frac{2\sin^2\theta+2\sin\theta\cos\theta}{2\cos^2\theta+2\sin\theta\cos\theta}$$

$$= \frac{2\sin\theta(\sin\theta+\cos\theta)}{2\cos\theta(\cos\theta+\sin\theta)} = \frac{\sin\theta}{\cos\theta} = \tan\theta$$

$= \cot(90^{\circ}-\theta)$ = R.H.S

4. Prove that $\sin^6 A + \cos^6 A = 1 - \frac{3}{4}\sin^2(2A)$

Sol: Let $sin^6 A + cos^6 A$

$$= (\sin^2 A)^3 + (\cos^2 A)^3$$

$$= (\sin^2 A + \cos^2 A)^3 - 3\sin^2 A\cos^2 A(\sin^2 A + \cos^2 A)$$

$$= 1^3 - 3(\sin A\cos A)^2(1)$$

$$= 1 - 3\left(\frac{2\sin A\cos A}{2}\right)^2$$

$$= 1 - 3\left(\frac{\sin 2A}{2}\right)^2$$

$$= 1 - \frac{3}{4}\sin^2 2A$$

Transformation formulae [Sum or Difference to product]

$\sin(A+B) = \sin A \cos B + \cos A \sin B$ --------------(1)

$\sin(A-B) = \sin A \cos B - \cos A \sin B$ --------------(2)

$\cos(A+B) = \cos A \cos B - \sin A \sin B$ --------------(3)

$\cos(A-B) = \cos A \cos B + \sin A \sin B$ --------------(4)

Using (1) and (2)

$\sin(A+B) + \sin(A-B) = 2\sin A \cos B$ --------------(5)

Put $A + B = C$ and $A - B = D$

Adding the above we get,

$2A = C + D$

$$A = \frac{C+D}{2}$$

Subtracting the above we get,

$2B = C - D$

$$B = \frac{C-D}{2}$$

Substitute these values in equation (5)

$$\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$$

similarly (1) -(2)

$\sin(A+B) - \sin(A-B) = 2\cos A \sin B$ -------------(6)

$$\therefore \sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$$

(3)+(4)

$\cos(A+B) + \cos(A-B) = 2\cos A\cos B$ ------------(7)

$$\therefore \cos C - \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$$

(3)-(4)

$cos(A+B) - cos(A-B) = -2\sin A\sin B$ --------------(8)

$$\therefore cos C - cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$$

Product to Sum or Difference

From equation (5)

$$\sin A\cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$$

from equation (6)

$$\cos A\sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$$

from equation (7)

$$\cos A\cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$$

from equation (8)

$$\sin A\sin B = -\frac{1}{2}[\cos(A+B) - \cos(A-B)] \; or \; = \frac{1}{2}[\cos(A-B) - \cos(A-B)]$$

Worked problems:

1. Prove that $\frac{\sin 6A + \sin 4A}{\sin 6A - \sin 4A} = \tan 5A.\cot A$

Sol: Let $\frac{\sin 6A + \sin 4A}{\sin 6A - \sin 4A}$

$$= \frac{2\sin\left(\frac{6A+4A}{2}\right)\cos\left(\frac{6A-4A}{2}\right)}{2\cos\left(\frac{6A+4A}{2}\right)\sin\left(\frac{6A-4A}{2}\right)}$$

$$= \frac{\sin 5A.\cos A}{\cos 5A.\sin A}$$

$= \tan 5A\cot A$ = R.H.S

2. Show that $\dfrac{\sin 7x-\sin 5x-\sin 3x+\sin x}{\cos 7x-\cos 5x-\cos 3x+\cos x}=\tan 4x$

Sol: Let $\dfrac{\sin 7x-\sin 5x-\sin 3x+\sin x}{\cos 7x-\cos 5x-\cos 3x+\cos x}$

$$=\frac{\sin 7x+\sin x-(\sin 5x+\sin 3x)}{\cos 7x+\cos x-(\cos 5x+\cos 3x)}$$

$$=\frac{2\sin\left(\frac{7x+x}{2}\right)\cos\left(\frac{7x-x}{2}\right)-2\sin\left(\frac{5x+3x}{2}\right)\cos\left(\frac{5x-3x}{2}\right)}{2\cos\left(\frac{7x+x}{2}\right)\cos\left(\frac{7x-x}{2}\right)-2\cos\left(\frac{5x+3x}{2}\right)\cos\left(\frac{5x-3x}{2}\right)}$$

$$=\frac{2\sin 4x\cos 3x-2\sin 4x\cos x}{2\cos 4x\cos 3x-2\cos 4x\cos x}$$

$$=\frac{(\cos 3x-\cos x)(2\sin 4x)}{(\cos 3x-\cos x)(2\cos 4x)}$$

$=\tan 4x=$ R.H.S

3. Prove that $\cos 20^\circ + \cos 100^\circ + \cos 140^\circ = 0$

Sol: L.H.S = $\cos 20^\circ + \cos 100^\circ + \cos 140^\circ$

$$= 2\cos\left(\frac{20^\circ + 100^\circ}{2}\right)\cos\left(\frac{20^\circ - 100^\circ}{2}\right) + \cos 140^\circ$$

$$= 2\cos 60^\circ \cos(-40^\circ) + \cos(140^\circ)$$

$$= 2.\frac{1}{2}.\cos 40^\circ + \cos 140^\circ$$

$$= \cos 40^\circ + \cos 140^\circ$$

$$= 2\cos\left(\frac{40^\circ + 140^\circ}{2}\right)\cos\left(\frac{40^\circ - 140^\circ}{2}\right)$$

$$= 2\cos\left(\frac{180}{2}\right)\cos\left(\frac{-100}{2}\right)$$

$$= 2\cos 90^\circ.\cos(-50^\circ)$$

$$= 2 \times 0.\cos 50^\circ$$

$= 0$=R.H.S

4. If $A+B+C=180^\circ$, Prove that $\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

Sol: Let L.H.S = $sinA+sinB+sinC$

$$=2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)+2\sin C\cos C$$

$$=2sin\left(\frac{180^\circ-C}{2}\right)cos\left(\frac{A-B}{2}\right)+2sinC\,cosC$$

$$=2\sin(90^\circ-C)\cos\left(\frac{A-B}{2}\right)+2\sin C\cos C$$

$$=2\cos C\left[cos\left(\frac{A-B}{2}\right)+\sin C\right]$$

$$=2\cos C\left[cos\left(\frac{A-B}{2}\right)+\sin C\right]$$

$$=2\cos C\left[cos\left(\frac{A-B}{2}\right)+\sin C\right]$$

$$=2\cos C\left[\cos\left(\frac{A-B}{2}\right)+\sin\left(\frac{180^\circ-(A+B)}{2}\right)\right]$$

$$=2\cos C\left[\cos\left(\frac{A-B}{2}\right)+\cos\left(\frac{A+B}{2}\right)\right]$$

$$=2\cos C\left[2\cos\left(\frac{A}{2}\right)\cos\left(\frac{-B}{2}\right)\right]$$

$$=2\cos C\left[2\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\right]$$

$$=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

5. Prove that $\sin 10^\circ \times sin 50^\circ \times sin 70^\circ = \frac{1}{8}$

Sol: Let L.H.S $= sin 70^\circ \times sin 50^\circ \times sin 10^\circ$

$$= -\frac{1}{2}[\cos(70^\circ + 50^\circ) - \cos(70^\circ - 50^\circ)]\sin 10^\circ$$

$$= -\frac{1}{2}[\cos(120^\circ) - \cos(20^\circ)]\sin 10^\circ$$

$$= -\frac{1}{2}[cos(180^\circ - 60^\circ) - cos(20^\circ)]sin 10^\circ$$

$$= -\frac{1}{2}[-cos(60^\circ) - cos(20^\circ)]sin 10^\circ$$

$$= -\frac{1}{2}[-cos(60^\circ) - cos(20^\circ)]sin 10^\circ$$

$$= -\frac{1}{2}[-cos(60°) - cos(20°)]sin10°"$$

$$= \left[\frac{1}{2}cos60° + \frac{1}{2}cos20°\right]sin10°$$

$$= \left(\frac{1}{2} \times \frac{1}{2} + \frac{1}{2}\cos 20°\right)\sin 10°$$

$$= \frac{1}{4}\sin 10° + \frac{1}{2}\cos 20°\sin 10°$$

$$= \frac{1}{4}\sin 10° + \frac{1}{2}\left[\frac{1}{2}\{\sin(20° + 10°) - \sin(20° - 10°)\}\right]$$

$$= \frac{1}{4}\sin 10° + \frac{1}{4}[\sin 30° - \sin 10°]$$

$$= \frac{1}{4}\sin 10° + \frac{1}{4} \times \frac{1}{2} - \frac{1}{4}\sin 10°$$

$$= \frac{1}{4}\sin 10° + \frac{1}{4} \times \frac{1}{2} - \frac{1}{4}\sin 10°$$

$$= \frac{1}{4}\sin 10° + \frac{1}{4} \times \frac{1}{2} - \frac{1}{4}\sin 10°"$$

$$= \frac{1}{8}$$

1. Complex numbers

Introduction:

The set of Natural numbers N $=\{1,2,3,.............\}$

The set of Integers I or $Z=\{0,\pm 1,\pm 2,.............\}$

The set of rationals Q $=\{p/q \ / \ p,q \in Z \text{ and } q \neq 0\}$

The set of irrationals $Q' = \{\text{square root of rational numbers}\}$

The set of real numbers $R = N^{+}$ or Z or Q or $Q' = R$

$$N^{+} \subset Z \subset Q \subset Q' \subseteq R$$

We have already discussed about real numbers, where all equation of the first degree can be solved over the real number system.

Ex: if $3x+4=0$ then $x = \dfrac{-4}{3}$

similarly equations like $x^2 = 9$ i.e., $x = \pm 3$ also find their solutions over real number system. But we cannot solve $x^2+25=0 \Rightarrow x^2 = -25$ over the set of real number, as there is no real numbers whose square is negative real number.

Euler was the first mathematician to introduce the symbol i (iota) for the square root of -1 with the property $i^2 = -1$. He also called this symbol as the imaginary unit.

Definition $i = \sqrt{-1}$. Therefore, square root of negative number is called **imaginary number**.

i.e., If a<0, then $\sqrt{a} = i\sqrt{|a|}$

Ex: $\sqrt{-2} = \sqrt{-1}\sqrt{2} = i\sqrt{2}$

$\sqrt{3} = \sqrt{-1}\sqrt{3} = i\sqrt{3}$

Note: $i = \sqrt{-1}$, $i^2 = (\sqrt{-1})^2 = -1$, $i^3 = i^2 . i = -1 . i = -i$, $i^4 = i^2 . i^2 = -1 . -1 = 1$

$$i^{2n} = [(i)^2]^n = (-1)^n = \begin{cases} 1 \text{ if } n \text{ is even} \\ -1 \text{ if } n \text{ is odd} \end{cases}$$

$$i^{2n+1} = i^{2n} . i = (-1)^n i = \begin{cases} i \text{ if } n \text{ is even} \\ -i \text{ if } n \text{ is odd} \end{cases}$$

Ex: $i^{10} = (i^2)^5 = (-1)^5 = -1$

$i^{13} = i^{12} . i = (i^2)^6 . i = (-1)^6 . i = 1 . i = i$

Complex numbers

The number of the form $a + ib$, where a and b are real numbers and $i = \sqrt{-1}$ is known as complex number. It is denoted by z. Thus, $z = a + ib$ where, a is real part (R_p) or $Re(z)$ and b is imaginary part (I_p or $Im(z)$).

Ex: If

$z = 5 + i6 \Rightarrow R_p = 5$ and $I_p = 6$

$z_1 = 5 - 2i \Rightarrow R_p = 5,\ I_p = -2$

$z_2 = -6 - 5i \Rightarrow R_p = -6,\ I_p = -5$

Note:

1. If $b = 0$, then $z = a$, such a number is called as a purely real number.

2. If $a = 0$ then $z = ib$, such a complex number is called as a purely imaginary complex number.

Thus z is a real number iff $Im(z) = 0$ and a purely imaginary number iff $Re(z) = 0$

Set of complex numbers denoted by $\mathbb{C} = \{z = a + ib /\ a, b \in R \text{ and } i = \sqrt{-1}\}$

Note:

1. Every real number is a complex number.

$$1 = 1 + i0, \ -3 = -3 + i0, \frac{5}{2} = \frac{5}{2} + i0, \ 0 = 0 + i0$$

2. But every complex number is not a real number.

Argand diagram of complex number ($a + ib$) (Cartesian system)

In plane cartesian co-ordiante plane R_p represent on x-axis is called real axis and I_p represent on y-axis is called imaginary axis.

Ex: $2 + i.3, \ -2 + i, \ -2 - 4i, 4 - 3i, 1 + i, 1 - i \ etc$

A complex number can be represented in a plane called complex plane or it is called Argand plane and these complex numbers represented with diagram (fig) in plane is called Argand diagram.

Equality of complex numbers

If two complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ are said to be equal iff $a_1 = a_2, b_1 = b_2$

Ex: If $z_1 = 3 + 4i \text{ and } z_2 = a + ib \text{ and } z_1 = z_2$ are said to be equal iff $a = 3,\ b = 4$

If $z_1 = x + iy \text{ and } z_2 = -2 + 4i \text{ iff } x = -2,\ y = 4$

If $z_1 = (a + b) + (a - b)i,\quad z_2 = 5 + 7i$ find $a,\ b$

$$\begin{aligned} &a + b = 5 \\ \text{Here, } &a - b = 7 \\ \hline &2a \quad = 12 \\ &\quad a = 6 \end{aligned}$$

If $a = 6$ then $b = -1$ $[a + b = 5 \Rightarrow 6 + b = 5 \Rightarrow b = 5 - 6 \Rightarrow b = -1]$

Conjugate of complex number

If $z = a + ib$ be a complex number then conjugate of a complex number is $a - ib$ denoted by $\overline{z}$. If $z = a + ib$ then conjugate is $\overline{z} = a - ib$

Ex: If $z = 3 + 4i \text{ then } \overline{z} = 3 - 4i \text{ also } z + \overline{z} = 6,\ z - \overline{z} = 8i$

If $z = 4 - 2i \text{ then } \overline{z} = 4 + 2i \text{ also } z + \overline{z} = 8,\ z - \overline{z} = -4i$

i.e., $z + \overline{z} \text{ is only } R_p \text{ and } z - \overline{z} \text{ is only } I_p$

Algebra of complex numbers

Addition and subtraction of two complex numbers is again a complex number.

If $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ then

$$z = z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$$

similarly if $z = z_1 - z_2 = (a_1 + ib_1) - (a_2 + b_2 i) = (a_1 - a_2) + i(b_1 - b_2) \in \mathbb{C}$

Ex: If $z_1 = 3 + 4i$, $z_2 = 5 + 2i$ find $z_1 + z_2, z_1 - z_2, z_2 - z_1$

Solution: $z_1 + z_2 = (3 + 4i) + (5 + 2i)$

$$= (3 + 5) + i(4 + 2) = 8 + 6i$$

$z_1 - z_2 = (3 + 4i) - (5 + 2i)$

$$= 3 + 4i - 5 - 2i = 3 - 5 + 4i - 2i = -2 + i(4 - 2) = -2 + 2i$$

$z_2 - z_1 = 5 + 2i - (3 + 4i) = 5 - 3 + 2i - 4i = 2 - 2i$

Note: Closure law, Associative law, Existence of Identity and existence of inverse w.r.t addition holds good.

Existence of identity $= 0 + 0.i$

Existence of inverse $x + i.y = -x - i.y$

Multiplication of a complex number

If $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ then product of complex number is

$$z_1.z_2 = (a_1 + i.b_1)(a_2 + i.b_2)$$
$$= a_1a_2 + i.a_1b_2 + i.b_1a_1 + i^2b_1b_2$$
$$= a_1a_2 + i.a_1b_2 + ib_1a_1 - b_1b_2 \qquad [by\ i^2 = -1]$$
$$= (a_1a_2 - b_1b_2) + i.(a_1b_2 + b_1a_1)$$

Ex: If $z_1 = 3 + 4i$ and $z_2 = 2 + 5i$ find $z_1.z_2$, $z_2.z_1$, R_p and I_p.

sol: $z_1z_2 = (3+4i)(2+5i) = 6 + 15.i + 8.i + 20.i^2 = 6 + 23.i - 20 = -14 + 23.i$

$z_1z_2 = (2+5i)(3+4i) = 6 + 23.i - 20 = -14 + 23.i$

R_p of $z_1z_2 = -14 \quad I_p$ of $z_1z_2 = 23$

Note: It holds closure law and commutative law.

Illustration:

1. If $z = a + ib$, $\overline{z} = a - ib$ $find\ z.\overline{z}$

sol: $z.\overline{z} = (a + i.b)\,(a - i.b) = a^2 - i.ab + i.ba - i^2b^2$

$= a^2 - i.ab + i.ab + b^2 \quad [since\ ab = ba]$

$= a^2 + b^2$

$z.\overline{z} = a^2 + b^2\ or\ (a^2 + b^2) + i.0$

$R_p\ of\ z.\overline{z} = a^2 + b^2$ and $I_p\ of\ z.\overline{z} = 0$

2. If $z = 1 + i$ find $z\overline{z}$, z^2, $\overline{z}^2$, $\overline{z}z$

Sol: $z.\overline{z} = (1 + i)(1 - i) = 1 - i + i - i^2 = 1 - (-1) = 2$

$z^2 = (1 + i)^2 = 1 + i^2 + 2.i = 1 - 1 + 2i = 2i$

$(\overline{z})^2 = (1 - i)^2 = 1 + i^2 - 2i = 1 - 1 - 2i = -2i$

$z.\overline{z} = (1 - i)(1 + i) = 1 + i - i - i^2 = 1 - (-1) = 1 + 1 = 2$

Modulus of complex number

If $z = a + ib \ then \ |z| = \sqrt{a^2 + b^2}$

Ex: If $z = 3 + 4i \ then \ |z| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$

$z = 2 - 4i \ then \ |z| = \sqrt{2^2 + 4^2} = \sqrt{20} = 2\sqrt{5}$

Properties:

1. $|z| = |\bar{z}|$
2. $z\bar{z} = |z|^2$
3. $|z_1 z_2| = |z_1||z_2|$
4. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \ where \ z_2 \neq 0$

Note: Multiplicative inverse of $z = \frac{\bar{z}}{|z|}$

Properties of conjugate complex number

If $z = a + ib$ $then$ $\overline{z} = a - ib$

1. $z + \overline{z} = a + ib + a - ib = 2a$ or $2Re(z)$

2. $z + -z = (a + ib) - (a - ib) = 2ib$ or $2i.I_p(z)$

3. $z\overline{z} = (a + ib)(a - ib) = a^2 + b^2$ or $(R_p z)^2 + (I_p z)^2$ purely real part

4. $(\overline{\overline{z}}) = z$ I f space z equals a plus i b space t h e n space z with bar on top equals a minus i b space a n d space stack z with bar on top with bar on top equals a plus i b equals z

5. stack z subscript 1. z subscript 2 with bar on top equals stack z subscript 1 with bar on top. stack z subscript 2 with bar on top

If $z_1.z_2 = (3 + 2i)(2 - i) = 6 - 3i + 4i - 2i^2 = 6 + i - 2(-1) = 8 + i$

$\overline{z_1 z_2} = 8 - i$------------(1)

$\overline{z_1}.\overline{z_2} = (3 - 2i)(2 + i) = 6 + 3i - 4i - 2i^2 = 6 - i - 2(-1) = 8 - i$

$\overline{z_1}.\overline{z_2} = 8 - i$ --------(2)

From (1) and (2) we have $\overline{z_1 z_2} = \overline{z_1}.\,\overline{z_2}$

6. $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$

7. $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

8. $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$

Polar form and exponential form of a complex number

If $z = a + ib$ be a complex number represented in argand plane clearly from trigonometry we have

$a = r\cos\theta,\ b = r\sin\theta$

Squaring and adding we have

open vertical bar z close vertical bar equals square root of a squared plus b squared end root equals square root of r squared cos squared theta plus r squared sin squared theta end root equals $\sqrt{r^2(1)} = r$

$$\frac{b}{a} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta$$

$$\theta = \tan^{-1}\left(\frac{b}{a}\right) = arg(z)\ or\ amp(z)$$

This form $z = a + ib = r(\cos\theta + i\sin\theta)$ is called polar form of a complex number.

Or

If $z = a + ib = re^{i\theta}$ is called exponential form of a complex number.

i.e., $re^{i\theta} = r(\cos\theta + i\sin\theta)$

$e^{i\theta} = \cos\theta + i\sin\theta\ if\ r = 1$

The angle θ is called amplitude of z [denoted by $amp(z)$] or argument of z [denoted by $arg(z)$]

The $\theta = amp(z)$ or $arg(z)$ lies $-\pi \le \theta \le \pi$ (which satisfy for $x = r\cos\theta$ and $y = r\sin\theta$) is called principle value and arg(z) or amp(z)

$amp(z) = \theta$ lies in different quadrant depends on a, b value.

(i) If $a > 0$, $b > 0$ (i.e., z is in I quadrant)

then $\theta = arg(z) = \tan^{-1}\dfrac{y}{x}$

(ii) If $a < 0$, $b > 0$ (i.e., z lies in II quadrant)

then $\theta = arg(z) = \pi - tan^{-1}\dfrac{y}{|x|}$

(iii) If $a < 0$, $b < 0$ (i.e., z lies in III quadrant)

then $\theta = argz = -\pi + \tan^{-1}\left(\dfrac{y}{x}\right)$

(iv) If $a > 0$, $b < 0$ (i.e., z lies in IV quadrant)

then $\theta = arg(z) = -tan^{-1}\dfrac{|y|}{x}$

Note: If $z = 0 = 0 + i0$ then $arg(z)$ is not defined

If $z = a + i0$ then $argz = \begin{cases} 0 \ if \ x > 0 \\ \pi \ if \ x < 0 \end{cases}$

If $z = 0 + iy$ then $arg z = \begin{cases} \frac{\pi}{2} \; if \; y > 0 \\ \frac{3\pi}{2} \; if \; y < 0 \end{cases}$

Worked problems

1. Express the following in polar form, also find its magnitude and arg(z) where

(i) $z = 1 - i$ (ii) $z = -1 + \sqrt{3}\,i$ (iii) -3 (iv) $-2i$

(i) Sol: $z = x + iy = (r\cos\theta + i\, r\sin\theta) = 1 - i$

$$1 = r\cos\theta, \quad -1 = r\sin\theta$$

$\therefore |z| = \sqrt{x^2 + y^2} = \sqrt{1^2 + (-1)^2} = \sqrt{1+1} = \sqrt{2}$ is magnitude of z.

$\cos\theta = \frac{1}{r} = \frac{1}{\sqrt{2}} > 0 \; and \; \sin\theta = \frac{1}{r} = \frac{-1}{\sqrt{2}} < 0$ lies in IV quadrant

$$\therefore \theta = -\tan^{-1}\frac{|y|}{x} = -\tan^{-1}\frac{\sqrt{2}}{\sqrt{2}} = -tan^{-1}(1) = -\frac{\pi}{4}$$

$\therefore$ Polar form of $1 - i \; is \; \sqrt{2}(\cos\theta + i\sin\theta)$

$$= \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$$

Worked problems

1. Express the following in polar form, also find its magnitude and arg(z) where

(i) $z = 1 - i$ (ii) $z = -1 + \sqrt{3}\, i$ (iii) -3 (iv) $-2i$

(i) Sol: $z = x + iy = (r\cos\theta + i\, r\sin\theta) = 1 - i$

$$1 = r\cos\theta, \quad -1 = r\sin\theta$$

$\therefore |z| = \sqrt{x^2 + y^2} = \sqrt{1^2 + (-1)^2} = \sqrt{1+1} = \sqrt{2}$ is magnitude of z.

$\cos\theta = \frac{1}{r} = \frac{1}{\sqrt{2}} > 0$ *and* $\sin\theta = \frac{1}{r} = \frac{-1}{\sqrt{2}} < 0$ lies in IV quadrant

$$\therefore \theta = -\tan^{-1}\frac{|y|}{x} = -\tan^{-1}\frac{\sqrt{2}}{\sqrt{2}} = -\tan^{-1}(1) = -\frac{\pi}{4}$$

$\therefore$ Polar form of $1 - i$ *is* $\sqrt{2}(\cos\theta + i\sin\theta)$

$$= \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$$

$$= \sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$$

$$|z| = \sqrt{2}$$

$$arg(z) = \theta = -\frac{\pi}{4}$$

(ii) $z = -1 + \sqrt{3}\,i = (r\cos\theta + i\,r\sin\theta)$

$$-1 = r\cos\theta, \quad \sqrt{3} = r\sin\theta$$

$$\therefore |z| = \sqrt{x^2 + y^2} = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{4} = 2$$

$$\cos\theta = \frac{-1}{r} = \frac{-1}{2} < 0 \text{ and } \sin\theta = \frac{\sqrt{3}}{r} = \frac{\sqrt{3}}{2} > 0$$

$\therefore x < 0$ and $y > 0$ then θ lies in II quadrant

$$\therefore \theta = \arg(z) = \tan^{-1}\frac{y}{|x|} = \tan^{-1}\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}$$

$$= \pi - \tan^{-1}(\sqrt{3}) = \pi - 60^\circ = 120^\circ = 120 \times \frac{\pi}{180} = \frac{2\pi}{3}$$

$\therefore$ polar form of $z = -1 + \sqrt{3}\,i$ is $= 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

$$|z| = 2$$

$$\arg z = amp(z) = \theta = \frac{2\pi}{3}$$

(iii) $z = -3 = -3 + 0\, i = (r\cos\theta + i\, r\sin\theta)$

$$-3 = r\cos\theta, \quad 0 = r\sin\theta$$

$$\therefore |z| = \sqrt{x^2 + y^2} = \sqrt{(-3)^2 + (0)^2} = 3 > 0$$

$$\cos\theta = \frac{-3}{3} = -1 < 0 \text{ and } \sin\theta = 0 \text{ so that } \theta = \pi$$

$\therefore$ polar form of $z = -3 = 3\cos\pi + i\sin\pi$

$$|z| = r = 3 \qquad \arg z = amp(z) = \theta = \pi$$

(iv) $z = -2i = 0 + (-2)\, i = (r\cos\theta + i\, r\sin\theta)$

$$0 = r\cos\theta, \quad -2 = r\sin\theta$$

$$\cos\theta = \frac{0}{r} \text{ and } \sin\theta = \frac{-2}{r}$$

where $\therefore |z| = \sqrt{0^2 + (-2)^2} = \sqrt{4} = 2 = r$

$$|z| = r = 2$$

$$\cos\theta = \frac{0}{2} = 0 \text{ and } \sin\theta = \frac{-2}{2} = -1 < 0$$

$$\therefore \theta = \arg(z) = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{-1}{0}\right) = \tan^{-1}(-\infty) = -\frac{\pi}{2}$$

$$\therefore \arg(z) = \frac{-\pi}{2} = \theta$$

$\therefore$ polar form of $z = 0 + (-2)i = = 2\left(\cos\frac{-\pi}{2} + i\sin\frac{-\pi}{2}\right)$

Introduction:

Calculus is the study of calculation of infinitesimal (very small) values. Calculus mainly consists of two branches *Differential calculus and Integral calculus.*

Two types of quantities are used in calculus.

(i) Constants and (ii) Variables

Constants: The quantities which remain same throughout are called constants

Ex: 2,3,5 -1, π etc

Variables: The quantities which take different values are called variables. There are two kinds of variables i.e., Independent and dependent variables.

Independent variables can be assigned any values, whereas dependent variable depends on independent variable.

Ex: Area of circle (A) $A = \pi r^2$, π is constant, A is dependent and r is independent variable.

Variables are generally by x, y, z, θ, φ etc and constants by $a, b, c, \pi, \alpha, \beta$ etc.

Functions:

When two or more variables are connected by a relation, then each is said to be the function of the other. In other words, a function $f: R \rightarrow R$ is a rule, which associates every real number x with f(x). It is denoted by y=f(x).

Ex: 1. In volume of sphere $V = \frac{4}{3}\pi r^3$ where V is a function of r.

2. In volume of cylinder $V = 4\pi r^2 h$ where V is a function of r and h.

Types of Functions:

1. **Algebraic functions:** an algebraic function is a function that can be defined as the root of a polynomial equation.

Ex: $f(x) = x^2 - 1$, $f(x) = 1/x$ etc.

2. **Transcendental functions** (not algebraic functions)

(a) exponential function

Ex: $y = e^x$, $y = a^x + 2^x$

Logarithmic function

Ex: ,$y = \log x$, $y = \log 5x + \log x$

periodic function

Ex: the sine function is periodic with period 2π, since $\sin(x + 2\pi) = \sin x$

trigonometric functions

Ex: y = sinx, y = tanx+cos2x etc

Odd and Even functions:

A function f(x) is said to be odd function if f(-x) = - f(x).

Ex: 1. $f(x)=x^5+3x$

$\therefore f(-x)=(-x)^5+3(-x)=-x^5-3x=-(x^5+3x)=-f(x)$

$\therefore f(x)$ is odd function.

A function f(x) is said to be odd function if f(-x) = f(x).

Ex: $f(x)= x^2+secx$

$\therefore f(-x)=(-x)^2+sec(-x)=x^2+secx=f(x)$

Indeterminate forms:

There are seven indeterminate forms involving 0, 1, and ∞:

$$\frac{0}{0}, 0-\infty, \frac{\infty}{\infty}, \infty-\infty, 0^0, \infty^0, 1^\infty$$

Limits (formal definition)

Let us consider the function $f(x)=\dfrac{x^2-1}{x-1}$ and try to study its behaviour.

Put $x=1$

$$f(1)=\frac{1^2-1}{1-1}=\frac{0}{0}$$

We don't know the value of $\frac{0}{0}$ (It is indeterminate), to answer this we need another way, so let it be by approaching it to closer and closer value.

x	0.5	0.9	0.99	0.999	0.9999	.....
f(x)	1.5000	1.9000	1.99000	1.99900	1.999900	......

Now we see that as x gets close to 1, then $\frac{x^2-1}{x-1}$ gets closer to 2

Here we can see that when x=1 we don't know the answer (It is indeterminate). But we can see that it is approaching to 2.

Now, this can be defined as

"The limit of $\frac{x^2-1}{x-1}$ as x approaches 1 is 2"

The limit will tell us the value of a function at a certain point

In symbols,

$$\lim_{x \to 1} \frac{x^2-1}{x-1} = 2$$

[i.e., "$f(x)$ gets close to some limit as x gets close to some value"]

Definition: Limit of a function

A function $y = f(x)$ is said to tend to limit $'l'$ as x tends to $'a'$ if the difference between values of $f(x)$ and l becomes smaller, whenever the difference between the values of x and a is smaller.

$$\lim_{x \to a} f(x) = l$$

OR

Let *f* be a function defined on some open interval that contains the number *a*, except possibly at *a* itself. We say *the limit of f(x) as x approaches a is l*, and we write

stack l i m with x rightwards arrow a below f open parentheses x close parentheses equals l

if for every number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$|f(x) - l| < \epsilon$$

whenever

$$0 < |x - a| < \delta$$

Left-Hand Limit

Let *f* be a function defined on some open interval (b,a) [or (a,b)]. We say *the left-hand [or right-hand] limit of f(x) as x approaches a is l*, (or *the limit of f(x) as x approaches a from the left [or right] is l*) and we write

$$\lim_{x \to a^-} f(x) = l$$

if for every number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$|f(x) - l| < \epsilon$$

whenever

$$a - \delta < x < a$$

Right-Hand Limit

Let *f* be a function defined on some open interval (*b*,*a*) [or (*a*,*b*)]. We say *the right-hand limit of f(x) as x approaches a is l*, (or *the limit of f(x) as x approaches a from the right is l*) and we write

$$\lim_{x \to a^+} f(x) = l$$

if for every number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$|f(x) - l| < \epsilon$$

whenever

$$a < x < a + \delta$$

Some theorems on Limit:

1. If k is constant $\lim_{x \to c} k = k$
2. $\lim_{x \to c} x = c$
3. $\lim_{x \to c} k\, f(x) = k \lim_{x \to c} f(x)$ where k is constant.
4. $\lim_{x \to c} \{f(x) \pm g(x)\} = \lim_{x \to c} f(x) \pm \lim_{x \to c} g(x)$ provided limits on R.H.S exist.
5. $\lim_{x \to c} f(x).g(x) = \lim_{x \to c} f(x) . \lim_{x \to c} g(x)$ provided limits on R.H.S exist.
6. $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$ $and \lim_{x \to c} g(x) \neq 0$ provided limits on R.H.S exist.
7. $\lim_{x \to c} \{f(x)\}^n = \left\{\lim_{x \to c} f(x)\right\}^n$ where n is positive integer.
8. If $f(x)$ is polynomial, the $\lim_{x \to c} f(x) = f(c)$, where $f(c)$ is obtained from $f(x)$ by replacing x by c.

Particular cases:

1. $\lim_{x \to c} \log f(x) = \log\left(\lim_{x \to c} f(x)\right)$, provided $\lim_{x \to c} f(x) > 0$
2. $\lim_{x \to c} e^{f(x)} = e^{\lim_{x \to c} f(x)}$

Working rules for evaluating $\lim_{x \to a} f(x)$

Rule 1: Factorisation method

Ex: 1. Evaluate $\lim_{x \to 2} \dfrac{x^2 - 5x + 6}{x^2 - 4}$

Sol: $\lim_{x \to 2} \dfrac{x^2 - 5x + 6}{x^2 - 4} = \lim_{x \to 2} \dfrac{x^2 - 2x - 3x + 6}{(x+2)(x-2)}$

$$= \lim_{x \to 2} \frac{x(x-2) - 3(x-2)}{(x+2)(x-2)}$$

$$= \lim_{x \to 2} \frac{(x-3)\cancel{(x-2)}}{(x+2)\cancel{(x-2)}}$$

$$= \lim_{x \to 2} \frac{(x-3)}{(x+2)}$$

$$= \frac{2-3}{2+2} = \frac{-1}{4}$$

2. Evaluate $\lim_{x \to 1} \frac{x^3 - 1}{x - 1}$

Sol: $\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = \lim_{x \to 1} \frac{(x-1)(x^2+x+1)}{(x-1)}$

$$= \lim_{x \to 1} \frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{(x-1)}}$$

$$= \lim_{x \to 1} (x^2+x+1)$$

$$= 1^2 + 1 + 1 = 3$$

3. Evaluate $\lim\limits_{x \to 1} \dfrac{2x^2 + 9x - 5}{x + 5}$

Sol: $\lim\limits_{x \to -5} \dfrac{2x^2 + 9x - 5}{x + 5} = \lim\limits_{x \to -5} \dfrac{2x^2 + 10x - 1x - 5}{x + 5}$

$$= \lim_{x \to -5} \frac{2x(1x + 5) - (1x + 5)}{x + 5}$$

$$= \lim_{x \to -5} \frac{(2x - 1)(1x + 5)}{(x + 5)}$$

$$= \lim_{x \to -5} \frac{(2x - 1)\cancel{(1x + 5)}}{\cancel{(x + 5)}}$$

$$= \lim_{x \to -5} (2x - 1)$$

$$= 2(-5) - 1 = -11$$

4. Evaluate $\lim_{x \to 0} \frac{(a+x)^2 - a^2}{x}$

Sol: $\lim_{x \to 0} \frac{(a+x)^2 - a^2}{x} = \lim_{x \to 0} \frac{(a+x+a)(a+x-a)}{x}$

$$= \lim_{x \to 0} \frac{(2a+x)(\cancel{a}+x\cancel{-a})}{x}$$

$$= \lim_{x \to 0} \frac{(a+x+a)\cancel{x}}{\cancel{x}}$$

$$= \lim_{x \to 0} (a+x+a)$$

$$= a + 0 + a = 2a$$

Rule 2: Rationalisation method

Ex: 1. Evaluate $\lim_{x \to 0} \frac{\sqrt{2+x}-\sqrt{2}}{x}$

Sol: $\lim_{x \to 0} \frac{\sqrt{2+x}-\sqrt{2}}{x} = \lim_{x \to 0} \frac{(\sqrt{2+x}-\sqrt{2})}{x} \times \frac{(\sqrt{2+x}+\sqrt{2})}{(\sqrt{2+x}+\sqrt{2})}$

$$= \lim_{x \to 0} \frac{(\sqrt{2+x})^2-(\sqrt{2})^2}{x(\sqrt{2+x}+\sqrt{2})}$$

$$= \lim_{x \to 0} \frac{\cancel{2}+x-\cancel{2}}{x(\sqrt{2+x}+\sqrt{2})}$$

$$= \lim_{x \to 0} \frac{\cancel{x}}{\cancel{x}(\sqrt{2+x}+\sqrt{2})}$$

$$= \lim_{x \to 0} \frac{1}{(\sqrt{2+x}+\sqrt{2})}$$

$$= \frac{1}{\sqrt{2+0}+\sqrt{2}} = \frac{1}{2\sqrt{2}}$$

2. Evaluate $\lim\limits_{x \to 2} \dfrac{x^2 - 4}{\sqrt{13x - 2} - \sqrt{x + 2}}$

Sol: $\lim\limits_{x \to 2} \dfrac{x^2 - 4}{\left(\sqrt{3x - 2} - \sqrt{x + 2}\right)} = \lim\limits_{x \to 2} \dfrac{x^2 - 4}{\left(\sqrt{3x - 2} - \sqrt{x + 2}\right)} \times \dfrac{\left(\sqrt{3x - 2} + \sqrt{x + 2}\right)}{\left(\sqrt{3x - 2} + \sqrt{x + 2}\right)}$

$$= \lim_{x \to 2} \frac{(x^2 - 4)\left(\sqrt{3x - 2} + \sqrt{x + 2}\right)}{\left(\sqrt{3x - 2}\right)^2 - \left(\sqrt{x + 2}\right)^2}$$

$$= \lim_{x \to 2} \frac{(x + 2)(x - 2)\left(\sqrt{3x - 2} + \sqrt{x + 2}\right)}{(3x - 2) - (x + 2)}$$

$$= \lim_{x \to 2} \frac{(x + 2)(x - 2)\left(\sqrt{3x - 2} + \sqrt{x + 2}\right)}{3x - 2 - x - 2}$$

$$= \lim_{x \to 2} \frac{(x + 2)(x - 2)\left(\sqrt{3x - 2} + \sqrt{x + 2}\right)}{2x - 4}$$

$$= \lim_{x \to 2} \frac{(x + 2)\cancel{(x - 2)}\left(\sqrt{3x - 2} + \sqrt{x + 2}\right)}{2\cancel{(x - 2)}}$$

$$= \lim_{x \to 2} \frac{(x + 2)\left(\sqrt{3x - 2} + \sqrt{x + 2}\right)}{2}$$

$$= \frac{(2 + 2)\left(\sqrt{3.2 - 2} + \sqrt{2 + 2}\right)}{2}$$

$$= \frac{4(2+2)}{2} = 8$$

3. Evaluate $\lim\limits_{x \to 0} \frac{2x}{\sqrt{a+x} - \sqrt{a-x}}$

Sol: $\lim\limits_{x \to 0} \frac{2x}{\sqrt{a+x} - \sqrt{a-x}} = \lim\limits_{x \to 0} \frac{2x}{\sqrt{a+x} - \sqrt{a-x}} \times \frac{\sqrt{a+x} + \sqrt{a-x}}{\sqrt{a+x} + \sqrt{a-x}}$

$$= \lim_{x \to 0} \frac{2x\left(\sqrt{a+x} + \sqrt{a-x}\right)}{\left(\sqrt{a+x}\right)^2 - \left(\sqrt{a-x}\right)^2}$$

$$= \lim_{x \to 0} \frac{2x\left(\sqrt{a+x} + \sqrt{a-x}\right)}{(a+x) - (a-x)}$$

$$= \lim_{x \to 0} \frac{2x\left(\sqrt{a+x} + \sqrt{a-x}\right)}{\cancel{a} + x \cancel{-a} + x}$$

$$= \lim_{x \to 0} \frac{\cancel{2x}\left(\sqrt{a+x} + \sqrt{a-x}\right)}{\cancel{2x}}$$

$$= \lim_{x \to 0} \left(\sqrt{a+x} + \sqrt{a-x}\right)$$

$$= \sqrt{a+0} + \sqrt{a-0}$$

$$= 2\sqrt{a}$$

4. Evaluate $\lim_{x \to 3} \frac{\sqrt{x+3}-\sqrt{6}}{x^2-9}$

Sol: $\lim_{x \to 3} \frac{\sqrt{x+3}-\sqrt{6}}{x^2-9} = \lim_{x \to 3} \frac{\left(\sqrt{x+3}-\sqrt{6}\right)}{x^2-9} \times \frac{\left(\sqrt{x+3}+\sqrt{6}\right)}{\left(\sqrt{x+3}+\sqrt{6}\right)}$

$$= \lim_{x \to 3} \frac{\left(\sqrt{x+3}\right)^2-\left(\sqrt{6}\right)^2}{(x^2-9)\left(\sqrt{x+3}+\sqrt{6}\right)}$$

$$= \lim_{x \to 3} \frac{(x+3-6)}{(x+3)(x-3)\left(\sqrt{x+3}+\sqrt{6}\right)}$$

$$= \lim_{x \to 3} \frac{\cancel{(x-3)}}{(x+3)\cancel{(x-3)}\left(\sqrt{x+3}+\sqrt{6}\right)}$$

$$= \lim_{x \to 3} \frac{1}{(x+3)\left(\sqrt{x+3}+\sqrt{6}\right)}$$

$$= \frac{1}{(3+3)\left(\sqrt{3+3}+\sqrt{6}\right)}$$

$$= \frac{1}{6\left(2\sqrt{6}\right)} = \frac{1}{12\sqrt{6}}$$

Limit of functions when $n\ (or\ x) \to \infty$

1. Evaluate $\lim\limits_{x \to \infty} \dfrac{(3x-1)(2+x)}{3x^2-2x+7}$

Sol: $\lim\limits_{x \to \infty} \dfrac{(3x-1)(2+x)}{3x^2-2x+7} = \lim\limits_{x \to \infty} \dfrac{3x^2+5x-2}{3x^2-2x+7}$

$$= \lim_{x \to \infty} \frac{\dfrac{3x^2+5x-2}{x^2}}{\dfrac{3x^2-2x+7}{x^2}}$$

$$= \lim_{x \to \infty} \frac{3+\dfrac{5}{x}-\dfrac{2}{x^2}}{3-\dfrac{2}{x}+\dfrac{7}{x^2}}$$

$$= \frac{3+0-0}{3-0+0} = \frac{3}{3} = 1$$

2. Evaluate $\lim\limits_{n \to \infty} \dfrac{(2-3n)(5-2n)}{n^4}$

Sol: $\lim\limits_{n \to \infty} \dfrac{(2-3n)(5-2n)}{n^4} = \lim\limits_{n \to \infty} \dfrac{10-4n-15n+6n^2}{n^4}$

$$= \lim_{n \to \infty} \frac{10-19n+6n^2}{n^4}$$

$$= \lim_{n \to \infty} \frac{10}{n^4} - \frac{19}{n^3} + \frac{6}{n^2}$$

$$= \frac{10}{\infty} - \frac{19}{\infty} + \frac{6}{\infty} = 0$$

3. Evaluate $\lim_{x \to \infty}\left(\frac{x+1}{x+2}\right)\left(\frac{2x+3}{3x+4}\right)$

Sol: $\lim_{x \to \infty}\left(\frac{x+1}{x+2}\right)\left(\frac{2x+3}{3x+4}\right) = \lim_{x \to \infty}\frac{2x^2+3x+2x+3}{3x^2+4x+6x+8}$

$$= \lim_{x \to \infty}\frac{2x^2+5x+3}{3x^2+10x+8}$$

$$= \lim_{x \to \infty}\frac{\frac{2x^2+5x+3}{x^2}}{\frac{3x^2+10x+8}{x^2}}$$

$$= \lim_{x \to \infty}\frac{2+\frac{5}{x}+\frac{3}{x^2}}{3+\frac{10}{x}+\frac{8}{x^2}} = \frac{2}{3}$$

4. Evaluate $\lim\limits_{x \to \infty} \frac{n-1}{\sqrt{4n^2+3}}$

Sol: $\lim\limits_{x \to \infty} \frac{n-1}{\sqrt{4n^2+3}} = \lim\limits_{x \to \infty} \frac{n-1}{\sqrt{n^2\left(4+\frac{3}{n^2}\right)}}$

$$= \lim_{x \to \infty} \frac{\cancel{n}\left(1-\frac{1}{n}\right)}{\cancel{n}\sqrt{\left(4+\frac{3}{n^2}\right)}}$$

$$= \lim_{x \to \infty} \frac{\left(1-\frac{1}{n}\right)}{\sqrt{\left(4+\frac{3}{n^2}\right)}}$$

$$= \frac{1-0}{\sqrt{4+0}} = \frac{1}{2}$$

Algebraic Limits

Statement: $\lim_{x \to a}\left[\frac{x^n - a^n}{x - a}\right] = n.a^{n-1}$, where n is any rational number.

Worked Problems:

1. Evaluate $\lim_{x \to -2}\left[\frac{x^5 + 32}{x + 2}\right]$

Sol: $\lim_{x \to -2}\left[\frac{x^5 + 32}{x + 2}\right] = \lim_{x \to -2}\left[\frac{x^5 - (-32)}{x - (-2)}\right] = \lim_{x \to -2}\left[\frac{x^5 - (-2)^5}{x - (-2)}\right] = 5(-2)^{5-1} = 5(-2)^4 = 80$

2. Evaluate: $\lim_{x \to 3}\left[\frac{\sqrt[5]{x}-\sqrt[5]{3}}{\sqrt[3]{x}-\sqrt[3]{3}}\right]$

Sol: $\lim_{x \to 3}\left[\frac{\sqrt[5]{x}-\sqrt[5]{3}}{\sqrt[3]{x}-\sqrt[3]{3}}\right] = \lim_{x \to 3}\left[\frac{x^{\frac{1}{5}}-3^{\frac{1}{5}}}{x^{\frac{1}{3}}-3^{\frac{1}{3}}}\right]$

$$= \lim_{x \to 3}\left[\frac{\left(\frac{x^{\frac{1}{5}}-3^{\frac{1}{5}}}{x-3}\right)}{\left(\frac{x^{\frac{1}{3}}-3^{\frac{1}{3}}}{x-3}\right)}\right]$$

$$= \frac{\frac{1}{5} \times 3^{\frac{1}{5}-1}}{\frac{1}{3} \times 3^{\frac{1}{3}-1}}$$

$$= \frac{3}{5} \times 3^{\frac{1}{5}-\frac{1}{3}}$$

$$= \frac{3}{5} \times 3^{\frac{-2}{15}} = \frac{3^{1-\frac{2}{15}}}{5} = \frac{3^{\frac{13}{15}}}{5}$$

3. Evaluate $\lim_{x \to 5} \left[\dfrac{\frac{1}{x^3} - \frac{1}{125}}{x-5} \right]$

Sol: $\lim_{x \to 5} \left[\dfrac{\frac{1}{x^3} - \frac{1}{125}}{x-5} \right] = \lim_{x \to 5} \left[\dfrac{x^{-3} - 5^{-3}}{x-5} \right] = -3.5^{3-1} = -3.5^{-4} = \dfrac{-3}{625}$

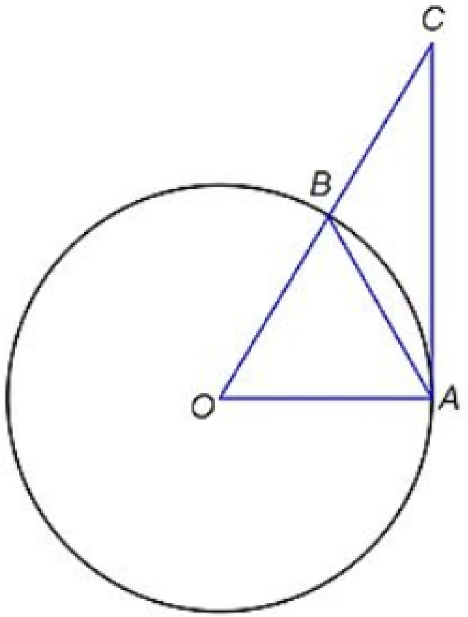
4. Evaluate $\lim_{x \to 1} \dfrac{1 - x^{\frac{-7}{3}}}{1 - x^{\frac{-9}{2}}}$

Sol: $\lim_{x \to 1} \dfrac{1 - x^{\frac{-7}{3}}}{1 - x^{\frac{-9}{2}}} = \lim_{x \to 1} \dfrac{x^{\frac{7}{3}} - 1}{x^{\frac{9}{2}} - 1} = \lim_{x \to 1} \dfrac{\frac{x^{\frac{7}{3}} - 1}{x-1}}{\frac{x^{\frac{9}{2}} - 1}{x-1}} = \dfrac{\frac{7}{3}}{\frac{9}{2}} = \dfrac{14}{27}$

Trigonometric Limit

Prove that $\lim\limits_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$ **where** θ **is measured in radians.**

Proof: Consider a circle of radius r and centre O. Let A and B be points on the circle. $\angle AOB = \theta$ radians. Join OA and OB and AB. At A, a tangent is drawn to circle which cut OB produced at C. Draw $BN \perp OA$



Now from fig

Area of $\triangle OAB$ < Area of sector OAB < Area of $\triangle OAC$ ------------(1)

Area of $\triangle OAB = \frac{1}{2} \times OA \times BN$

$$= \frac{1}{2} \times OA \times OB\sin\theta$$

$$= \frac{1}{2} \times r \times r\sin\theta = \frac{1}{2} \times r^2 sin\theta \text{ -------------(2)}$$

Area of sector $OAB = \frac{1}{2} r^2 \theta$ --------------(3)

Area of $\triangle OAC = \frac{1}{2} \times OA \times AC$

$$= \frac{1}{2} \times OA \times AC$$

$$= \frac{1}{2} \times OA \times OA\tan\theta = = \frac{1}{2} r^2 tan\theta \text{\_\_\_\_(4)}$$

substituting (2), (3), (4) in (1) we get,

$$\frac{1}{2} \times r^2 sin\theta < \frac{1}{2} r^2 \theta < \frac{1}{2} r^2 tan\theta$$

Dividing by $\frac{1}{2}r^2$ we get,

$$\sin\theta < \theta < \tan\theta$$

$$\therefore 1 < \frac{\theta}{\sin\theta} < \frac{\tan\theta}{\sin\theta} \qquad \text{(Dividing by } \sin\theta\text{)}$$

$$i.e., \quad 1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta} \qquad \left[\frac{\tan\theta}{\sin\theta} = \frac{\frac{\sin\theta}{\cos\theta}}{\sin\theta} = \frac{1}{\cos\theta}\right]$$

Taking reciprocal, $1 > \frac{\sin\theta}{\theta} > \cos\theta$

As $\theta \to 0$, $\cos\theta \to \cos 0^\circ = 1$, $\frac{\sin\theta}{\theta}$ lies between 1 and $\cos\theta$, $\frac{\sin\theta}{\theta} \to 1$ as $\theta \to 0$

i.e., applying limit as $\theta \to 0$

$$\lim_{\theta \to 0} 1 > \lim_{\theta \to 0} \frac{\sin\theta}{\theta} > \lim_{\theta \to 0} \cos\theta$$

$$i.e., 1 > \lim_{\theta \to 0} \frac{\sin\theta}{\theta} > 1 \Rightarrow \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$

Note: 1. $\lim_{\theta \to 0} \frac{\tan\theta}{\theta} = \lim_{\theta \to 0} \left[\frac{\frac{\sin\theta}{\cos\theta}}{\theta} \right]$

$$\lim_{\theta \to 0} \left[\frac{1}{\cos\theta} \times \frac{\sin\theta}{\theta} \right] = \frac{1}{\cos\theta} \times 1 = 1$$

$$\therefore \lim_{\theta \to 0} \frac{\tan\theta}{\theta} = 1$$

2. $\therefore \lim_{\theta \to 0} \frac{\cos\theta}{\theta}$ is not defined.

Worked Problem

1. Evaluate $\lim_{\theta \to 0} \frac{\sin m\theta}{\theta}$

Sol: $\lim_{\theta \to 0} \frac{\sin m\theta}{\theta} = \lim_{\theta \to 0} \left[\frac{\sin m\theta}{m\theta} \right] \times m = 1.m = m$

2. Evaluate $\lim\limits_{\theta \to 0} \dfrac{\sin a\theta}{\sin b\theta}$

Sol: $$\lim_{\theta \to 0} \frac{\sin a\theta}{\sin b\theta} = \lim_{\theta \to 0}\left(\frac{\frac{\sin a\theta}{a\theta} \times a}{\frac{\sin b\theta}{b\theta} \times b}\right) = \frac{1 \times a}{1 \times b} = \frac{a}{b}$$

3. Evaluate $\lim\limits_{x \to 0} \dfrac{\sin x^{\circ}}{x}$

Sol: $1^{\circ} = \dfrac{\pi}{180^{\circ}}$ $\qquad \therefore x^{\circ} = \dfrac{\pi}{180}.x$

$$\lim_{x \to 0} \frac{\sin\left(\frac{\pi}{180^{\circ}} x\right)}{x} = \lim_{x \to 0}\left[\frac{\sin\left(\frac{\pi}{180^{\circ}} x\right)}{\frac{\pi}{180^{\circ}} \times x}\right] \times \frac{\pi}{180}$$

$$= 1 \times \frac{\pi}{180} = \frac{\pi}{180}$$

6. Evaluate $\lim\limits_{x \to 0} \dfrac{x^2}{\sin x^2}$

Sol: $$\lim_{x \to 0} \frac{x^2}{\sin x^2} = \lim_{x \to 0} \frac{1}{\frac{\sin x^2}{x^2}} = \frac{1}{\lim\limits_{x \to 0}\left(\frac{\sin x}{x}\right)^2} = \frac{1}{1} = 1$$

8. Evaluate $\lim_{n \to 0} \frac{n\tan n}{1-\cos n}$

Sol: $\lim_{n \to 0} \frac{n\tan n}{1-\cos n} = \lim_{n \to 0} \frac{n\sin n}{\cos n(1-\cos n)}$

$$= \lim_{n \to 0} \frac{n.2\sin\frac{n}{2}\cos\frac{n}{2}}{\cos n\left(2\sin^2\frac{n}{2}\right)} = \lim_{n \to 0} \frac{n.\cancel{2}\,\cancel{\sin\frac{n}{2}}\cos\frac{n}{2}}{\cos n\left(\cancel{2}\sin^{\cancel{2}}\frac{n}{2}\right)}$$

$$= \lim_{n \to 0} \frac{n\cos\frac{n}{2}}{\cos n.\sin\frac{n}{2}} = \lim_{n \to 0} \frac{n}{\cos n.\frac{\sin\frac{n}{2}}{\cos\frac{n}{2}}}$$

$$= \lim_{n \to 0} \frac{1}{\cos n.\left(\frac{\tan\frac{n}{2}}{n}\right)} = \lim_{n \to 0} \frac{1}{\cos n} \lim_{n \to 0} \frac{1}{\left(\frac{\tan\frac{n}{2}}{\frac{n}{2}\times 2}\right)}$$

$$= \frac{1}{\cos 0} \times \frac{1}{\frac{1}{2}} = 1 \times 2 = 2$$

Exponential Limit

1. $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$. The value of e lies between 2 and 3.

2. $\lim_{x \to 0} (1 + x)^{\frac{1}{x}} = e$ changing x to $\frac{1}{x}$

since $x \to \infty \Rightarrow \frac{1}{n} \to 0$

Note: 1. $\lim_{x \to 0} \left(\frac{a^x - 1}{x}\right) = \log_e a$

2. $\lim_{x \to 0} \left(\frac{e^x - 1}{x}\right) = 1$

Worked Problems:

1. Evaluate $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^{-7x}$

Sol: $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^{-7x} = \lim_{x \to \infty} \left\{\left(1 + \frac{1}{x}\right)^x\right\}^{-7} = e^{-7} = \frac{1}{e^7}$

2. Evaluate $\lim_{x \to \infty} \left(1 - \frac{6}{x}\right)^x$

Sol: $\lim_{x \to \infty} \left(1 - \frac{6}{x}\right)^x = \lim_{x \to \infty} \left(1 + \frac{(-6)}{x}\right)^{\frac{x}{-6} \times -6} = \left\{ \lim_{x \to \infty} \left(1 + \frac{(-6)}{x}\right)^{\frac{x}{-6}} \right\}^{-6} = e^{-6} = \frac{1}{e^6}$

3. Evaluate $\lim_{x \to 0} \frac{e^{3+x} - e^3}{x}$

Sol: $\lim_{x \to 0} \frac{e^{3+x} - e^3}{x} = \lim_{x \to 0} \frac{e^3.e^x - e^3}{x} = \lim_{x \to 0} \frac{e^3(e^x - 1)}{x} = e^3 \lim_{x \to 0} \frac{(e^x - 1)}{x} = e^3.1 = e^3$

4. Evaluate $\lim_{\theta \to 0} \frac{a^{m\theta} - b^{n\theta}}{\sin m\theta}$

Sol: $\lim_{\theta \to 0} \frac{a^{m\theta} - b^{n\theta}}{\sin m\theta} = \lim_{\theta \to 0} \frac{\frac{a^{m\theta} - b^{n\theta}}{\theta}}{\frac{\sin m\theta}{\theta}}$

$$= \lim_{\theta \to 0} \frac{\frac{a^{m\theta} - b^{n\theta} - 1 + 1}{\theta}}{\frac{\sin m\theta}{m\theta} \times m}$$

$$= \lim_{\theta \to 0} \frac{\frac{(a^{m\theta} - 1) - (b^{n\theta} - 1)}{\theta}}{\frac{\sin m\theta}{m\theta} \times m}$$

$$= \lim_{\theta \to 0} \frac{\frac{(a^{m\theta} - 1)}{m\theta} \times m - \frac{(b^{n\theta} - 1)}{n\theta} \times n}{\frac{\sin m\theta}{m\theta} \times m}$$

$$= \frac{\log a . m - \log b . n}{1 . m} = \frac{\log a^m - \log b^n}{m} = \frac{1}{m} \log\left(\frac{a^m}{b^n}\right)$$