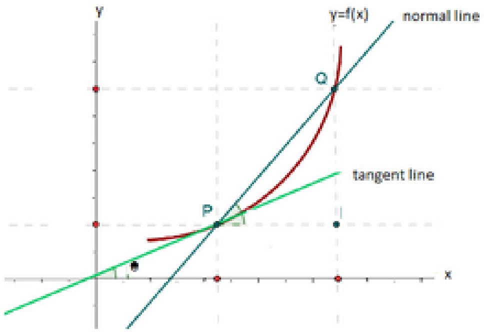
Geometrical meaning of derivative

Consider a curve $y = f(x)$, let $p(x_1, y_1)$ be a point on the curve, draw a tangent to the curve at the point 'p"(as in fig)



let, the tangent line meet the x-axis at A which makes an angle θ to the x-axis. Therefore, slope of the tangent is

$m = \tan\theta$..............................(1)

Similarly it is also defined in derivative, the slope of the tangent to the curve $y = f(x)$ at the point $p(x_1, y_1)$ is defined as $m = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}$

..............................(2)

Note: The line perpendicular to the tangent at the same point $p(x_1, y_1)$ is known as normal line to the curve $y = f(x)$.

Therefore slope of normal is $m = \frac{-1}{(dy/dx)_{(x_1, y_1)}}$.

We know that equation of tangent is defined as $y - y_1 = m(x - x_1)$.

$$y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}(x - x_1)$$

Therefore equation of normal is

$$y - y_1 = \frac{-1}{(dy/dx)_{(x_1, y_1)}}(x - x_1)$$

Note:

1. If the tangent line is parallel to the x- axis then slope is 0. Hence $m = \left(\frac{dy}{dx}\right)_{(x_1,y_1)}$=0.

2. If the tangent line is perpendicular to x-axis then the slope is infinite.

Hence $m = \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = \infty$.

3. If the tangent line is parallel to the line ax+by+c=0.Then slope and the tangent is $\left(\frac{dy}{dx}\right)_{(x_1,y_1)} = \frac{-a}{b}$.

4. If tangent line makes an angle θ with x-axis, then slope of the tangent is

$m = \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = \tan\theta$.

Problems

Find the slope of the tangent to the curve $y = 5x^3 - 4x^2 + 3x - 2$ at the point $(1,2)$.

Soln: Given equation is $y = 5x^3 - 4x^2 + 3x - 2$

Differentiating w.r.t x

$$\frac{dy}{dx} = 15x^2 - 8x + 3.$$

Hence slope of tangent to the curve at the point $(1,2)$ is $m=\left(\frac{dy}{dx}\right)_{(1,2)}$

$$m=15(1)^2-8(1)+3$$
$$m=15-8+3$$
$$m=10$$

∴ slope of tangent =m= 10.

2. Find the equation of the tangent to the curve $2x^2+xy+y^2=8$ at $(-1,3)$

Soln:Given $2x^2+xy+y^2=8$...(1)

Diffg w.r.t x, we get

$$2.2x+x.\frac{dy}{dx}+y.1+2y.\frac{dy}{dx}=0$$

$$4x+x\frac{dy}{dx}+y+2y\frac{dy}{dx}=0$$

$$(x+2y)\frac{dy}{dx}=-4x-y$$

$$\frac{dy}{dx}=\frac{-4x-y}{(x+2y)}$$

∴ slope of the tangent to the curve at (-1,3) is

$$m=\left(\frac{dy}{dx}\right)_{(-1,3)}=\frac{-4(-1)-3}{-1+2(3)}=\frac{4-3}{-1+6}=\frac{1}{5}.$$

$\therefore$ equation of the tangent is $y - y_1 = \left(\dfrac{dy}{dx}\right)_{(x_1,y_1)}(x - x_1)$

$$(y-3) = \frac{1}{5}(x-(-1))$$
$$5(y-3) = 1(x+1)$$
$$5y - 15 = x + 1$$
$$x - 5y + 16 = 0$$

Activity:

1. Find the point on the curve $y = 5x^2 - 20x + 7$ at which the tangent is parallel to the x-axis.
2. Find the equation of the tangent line to the curve $x^2 + 3y - 9 = 0$ which is parallel to the line $4x - y - 5 = 0$.
3. Find the point on the curve $y = 3x^2 - 11x + 2$ at which the tangent make an angle 45° with x-axis.
4. Find the equation of the normal line to the curve $y = 3x^4 - 2x^3 + 5x + 4$ at $(0,4)$

Derivative as rate measure

As we know equations of motions in physics as v=u+at, $s = ut + \frac{1}{2}at^2$, $v^2 = u^2 + 2as$........where S is the distance travelled by a particle (displacement of the particle)

v=velocity of the particle

t=time taken by the particle during the motion

a=acceleration of the particle.

In case of rate measure we define displacement w.r.t 't' is $S = f(t)$ w.r.t 't''time.

velocity is rate of change of displacement i.e.,

$$v = \frac{ds}{dt}$$

acceleration is rate of change of velocity i.e.,

$$a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2s}{dt^2}$$

Note :

1. Rate of change of area w.r.t 't' is $\frac{dA}{dt} cm^2/sec.$

2. Rate of change of volume w.r.t 't' is $\frac{dv}{dt} cm^3/sec.$

3. Rate of change of radius w.r.t 't' is $\frac{dr}{dt} cm/sec$.

4. Area of circle $A = \pi r^2$ sq unit.

5. Area of volume $V = \frac{4}{3}\pi r^3$ sq unit.

6. Circumference of a circle $= 2\pi r\ cm$.

Examples:

1. If $S = t^3 - 6t^2 - 63t + 10$ be the distance travelled by a particle in meter find

a). velocity and acceleration when t=1 sec.

b). at what time the particle stops.

c). find the velocity when acceleration is zero.

Soln: Given $S = t^3 - 6t^2 - 63t + 10$

Diffg w.r.t t

$$v = 3t^2 - 12t - 63$$
$$again\ diffg\ w.r.t\ t$$
$$a = 6t - 12$$

a). velocity when t=1 sec

$$v = 3(1)^2 - 12(1) - 63$$
$$= 3 - 12 - 63$$
$$= -72\ m/s$$

$$a \text{ when } t = 1 \text{ sec}$$
$$a = 6(1) - 12$$
$$= -6\ m/s^2$$

b). w.k.t when the particle stops

when velocity is zero then particle is at rest(stops)

$$3t^2 - 12t - 63 = 0$$
$$3(t^2 - 4t - 21) = 0$$
$$t^2 - 4t - 21 = 0$$
$$(t - 7)(t + 3) = 0$$
$$\therefore t = 7, t = -3$$

c). when acceleration = 0

6t-12=0

6t = 12

Hence t=2 sec

Therefore velocity at t = 2 sec is

$$v = 3t^2 - 12t - 63$$
$$v = 3(2)^2 - 12(2) - 63$$
$$v = -75 m/s$$

3. A volume of a spherical balloon is increasing at the rate of $45cm^3/min$. How fast the surface area of the balloon increases when its radius is 6cm.

Soln: Given

$$\frac{dv}{dt} = 45cm^3/min, \frac{dA}{dt} = ?$$

$$when\ r = 6cm$$

$$Volume\ of\ sphere\ V = \frac{4}{3}\pi r^3$$

$$Diffg\ w.r.t\ 't$$

$$\frac{dv}{dt} = \frac{4}{3}\pi.3r^2\frac{dr}{dt}$$

$$\frac{dv}{dt} = 4\pi r^2\frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{45}{4\pi r^2}$$

Surface area of the sphere is

$$A = 4\pi r^2 sq\ unit$$

$$Diff\ w.r.t\ t$$

$$\frac{dA}{dt} = 4\pi.2r.\frac{dr}{dt}$$

$$= 8\pi r\frac{dr}{dt}$$

2. The radius of the circular plate is increasing at the rate of $\frac{2}{3\pi}$ cm/sec. Find the rate of change of area when its radius is r = 5cm.

Soln: $\frac{dr}{dt} = \frac{2}{3\pi} cm/sec, \frac{dA}{dt} = ?$

$when\ r = 5$

$$A = \pi r^2$$

$$Diffg\ w.r.t\ t$$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

w.k.t area of circle = $\left(\frac{dA}{dt}\right)_{r=5} = 2.\pi.5.\frac{2}{3\pi}$

$$= \frac{20}{3} cm^2/sec$$

$$\therefore area\ of\ circular\ plate\ is\ \frac{20}{3} cm^2/sec\ when\ its\ radius\ is\ 5\ cm.$$

$$\left(\frac{dA}{dt}\right)_{r=6} = 8.\ \pi.6.\ \frac{45}{4\pi 6^2}$$

$$= 15 cm^2/min$$

Maxima and Minima

Increasing and decreasing functions

If the value of the function y = f(x) increases as x increases, then the function f(x) is said to be increasing function of 'x'.

Similarly a function f(x) is called decreasing function, if the value of f(x) decreases as x increases

Nature of a function w.r.t its derivative

Consider a function y = f(x)

1. If $\frac{dy}{dx} > 0\ (+ve)$

$\textit{at the point } x = a, \textit{ then the function } f(x) \textit{ is increasing at } x = a.$

2. If $\frac{dy}{dx} < 0(-ve)$

$\textit{at the point } x = b \textit{ then the function } f(x) \textit{ is decreasing at } x = b.$

3. Suppose if $\frac{dy}{dx} = 0$ at a point say $x = a$, then the function changes its nature from increasing to decreasing or decreasing to increasing

This point is called turning point. At this point the function f(x) obtain either maximum or minimum values.

Examples

Condition for finding maximum and minimum values

Consider a function y = f(x) [is a polynomial equation].

$$\frac{dy}{dx} = f'(x)$$

for maximum or minimum we equate $\frac{dy}{dx} = f'(x) = 0$

After equating to 0, we get the value of x as a,b,c. then find

$$\frac{d^2y}{dx^2} = f''(x)$$

Then we find *Now at* $x = a, f''(x)$ *or* $\frac{d^2y}{dx^2} < 0$

then the function is maximum at $x = a$ *and maximum value is* $f(a)$.

Now at $x = b,\ f''(x)$ *or* $\frac{d^2y}{dx^2} > 0$

then the function is minimum at $x = b$ *and minim*
um value is $f(b)$

Now at $x = c\ ,\ f''(x)$ *or* $\frac{d^2y}{dx^2} = 0$

then the given function is neither maximum nor minimum at $x = c$

Introduction

Definition

The process of finding a function, given its derivative, is called anti-differentiation (or integration). If F'(x) = f(x), we say F(x) is an anti-derivative of f(x).

Examples

F(x) =cos x is an anti-derivative of sin x, and e^x is an anti-derivative of e^x

Note that if F(x) is an anti-derivative of f(x) then F(x) + c, where c is a constant (called the constant of integration) is also an anti-derivative of f(x), as the derivative of a constant function is 0. In fact they are the only anti-derivatives of f(x).

$$\begin{tikzcd} F(x)+c \arrow[r, bend left, "\text{diff}"] & f(x) \arrow[l, bend left, "\text{int}"] \end{tikzcd}$$

We write $\int$f(x) dx = F(x) + c.

if F'(x) = f(x) . We call this the indefinite integral of f(x) .

Thus in order to find the indefinite integral of a function, you need to be familiar with the techniques of differentiation.

List of standard integrals.

Derivatives	Integrals (Anti derivatives)
(i) $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right)=x^n$;	$\int x^n\,dx=\frac{x^{n+1}}{n+1}+C,\ n\neq -1$
Particularly, we note that	
$\frac{d}{dx}(x)=1$;	$\int dx = x+C$
(ii) $\frac{d}{dx}(\sin x)=\cos x$;	$\int \cos x\,dx=\sin x+C$
(iii) $\frac{d}{dx}(-\cos x)=\sin x$;	$\int \sin x\,dx=-\cos x+C$
(iv) $\frac{d}{dx}(\tan x)=\sec^2 x$;	$\int \sec^2 x\,dx=\tan x+C$
(v) $\frac{d}{dx}(-\cot x)=\operatorname{cosec}^2 x$;	$\int \operatorname{cosec}^2 x\,dx=-\cot x+C$
(vi) $\frac{d}{dx}(\sec x)=\sec x\tan x$;	$\int \sec x\tan x\,dx=\sec x+C$
(vii) $\frac{d}{dx}(-\operatorname{cosec} x)=\operatorname{cosec} x\cot x$;	$\int \operatorname{cosec} x\cot x\,dx=-\operatorname{cosec} x+C$
(viii) $\frac{d}{dx}(\sin^{-1} x)=\frac{1}{\sqrt{1-x^2}}$;	$\int \frac{dx}{\sqrt{1-x^2}}=\sin^{-1} x+C$
(ix) $\frac{d}{dx}(-\cos^{-1} x)=\frac{1}{\sqrt{1-x^2}}$;	$\int \frac{dx}{\sqrt{1-x^2}}=-\cos^{-1} x+C$
(x) $\frac{d}{dx}(\tan^{-1} x)=\frac{1}{1+x^2}$;	$\int \frac{dx}{1+x^2}=\tan^{-1} x+C$

(xi) $\frac{d}{dx}\left(-\cot^{-1} x\right)=\frac{1}{1+x^2}$; $\qquad \int\frac{dx}{1+x^2}=-\cot^{-1} x+C$

(xii) $\frac{d}{dx}\left(\sec^{-1} x\right)=\frac{1}{x\sqrt{x^2-1}}$; $\qquad \int\frac{dx}{x\sqrt{x^2-1}}=\sec^{-1} x+C$

(xiii) $\frac{d}{dx}\left(-\operatorname{cosec}^{-1} x\right)=\frac{1}{x\sqrt{x^2-1}}$; $\qquad \int\frac{dx}{x\sqrt{x^2-1}}=-\operatorname{cosec}^{-1} x+C$

(xiv) $\frac{d}{dx}(e^x)=e^x$; $\qquad \int e^x\,dx=e^x+C$

(xv) $\frac{d}{dx}(\log|x|)=\frac{1}{x}$; $\qquad \int\frac{1}{x}\,dx=\log|x|+C$

(xvi) $\frac{d}{dx}\left(\frac{a^x}{\log a}\right)=a^x$; $\qquad \int a^x\,dx=\frac{a^x}{\log a}+C$

$\int x^n=\frac{x^{n+1}}{n+1}+C$	$\int(ax+b)^n\,dx=\frac{1}{a}\frac{(ax+b)^{n+1}}{n+1}+C$
$\int e^x=e^x+C$	$\int e^{ax+b}\,dx=\frac{1}{a}e^{ax+b}+C$
$\int\frac{1}{x}=\ln\lvert x\rvert+C$	$\int\frac{1}{ax+b}\,dx=\frac{1}{a}\ln\lvert ax+b\rvert+C$
$\int\cos x=\sin x+C$	$\int\cos(ax+b)\,dx=\frac{1}{a}\sin(ax+b)+C$
$\int\sin x=-\cos x+C$	$\int\sin(ax+b)\,dx=-\frac{1}{a}\cos(ax+b)+C$
$\int\sec^2 x=\tan x+C$	$\int\sec^2(ax+b)\,dx=\frac{1}{a}\tan(ax+b)+C$
$\int\operatorname{cosec}x\cot x=-\operatorname{cosec}x+C$	$\int\operatorname{cosec}(ax+b)\cot(ax+b)\,dx=-\frac{1}{a}\operatorname{cosec}(ax+b)+C$
$\int\operatorname{cosec}^2 x=-\cot x+C$	$\int\operatorname{cosec}^2(ax+b)\,dx=-\frac{1}{a}\cot(ax+b)+C$

$\int \sec x \tan x = \sec x + C$	$\int \sec(ax+b)\tan(ax+b)dx = \frac{1}{a}\sec(ax+b)+C$

Rules of Integration (only statement)

1. $\int kf(x)dx = k\int f(x)dx$
2. $\int \{f(x) \pm g(x)\}dx = \int f(x)dx \pm \int g(x)dx$

Problems

1. Evaluate $\int 6x\,dx$

Sol: $\int 6x\,dx = 6\int x\,dx = 6.\frac{x^2}{2} + c = 3x^2 + c$

2. Evaluate $\int \frac{-4}{x}dx$

Sol: $\int \frac{-4}{x}dx = -4\int \frac{1}{x}dx = -4\log x + c$

3. Evaluate $\int \sqrt[5]{x^{-2}}\,dx$

Sol: $\int \sqrt[5]{x^{-2}}\,dx = \int x^{-\frac{2}{5}}dx = \frac{x^{-\frac{2}{5}+1}}{-\frac{2}{5}+1} + c = \frac{x^{\frac{3}{5}}}{\frac{3}{5}} + c = \frac{5}{3}x^{\frac{3}{5}} + c$

4. Evaluate $\int sec^2 2x\,dx$

Sol: $\int sec^2 2x\,dx = \frac{\tan 2x}{2} + c$

5. Evaluate $\int(3x^3-3\sin x+7\sqrt{x})dx$

Sol: $\int(3x^3-3\sin x+7\sqrt{x})dx=3\int x^3dx-3\int \sin x dx+7\int\sqrt{x}\,dx$

$$=3\frac{x^4}{4}+3\cos x+7\frac{x^{\frac{3}{2}}}{\frac{3}{2}}+c=3\frac{x^4}{4}+3\cos x+\frac{14}{3}x^{\frac{3}{2}}+c$$

6. Evaluate $\int \sec x(\sec x+\tan x)dx$

Sol: $\int \sec x(\sec x+\tan x)dx=\int(\sec^2x+\sec x\tan x)dx=\int \sec^2xdx+\int \sec x\tan xdx=\tan x+\sec x+c$

7. Evaluate $\int 5t^3-10t^{-6}+4\,dt$

Sol:

$$\int 5t^3-10t^{-6}+4\,dt=5\left(\frac{1}{4}\right)t^4-10\left(\frac{1}{-5}\right)t^{-5}+4t+c$$

$$=\frac{5}{4}t^4+2t^{-5}+4t+c$$

8. Evaluate $\int x^8 + x^{-8}\, dx$

Sol:

$$\int x^8 + x^{-8}\, dx = \frac{1}{9}x^9 - \frac{1}{7}x^{-7} + c$$

9. Evaluate $\int \left(w + \sqrt[3]{w} \right)\left(4 - w^2 \right) dw$

Sol:

$$\int \left(w + \sqrt[3]{w} \right)\left(4 - w^2 \right) dw = \int 4w - w^3 + 4w^{\frac{1}{3}} - w^{\frac{7}{3}} dw$$

$$= 2w^2 - \frac{1}{4}w^4 + 3w^{\frac{4}{3}} - \frac{3}{10}w^{\frac{10}{3}} + c$$

10. Evaluate $\int 3e^x + 5\cos x - 10\sec^2 x\, dx$

Sol: $\int 3e^x + 5\cos x - 10\sec^2 x\, dx = 3e^x + 5\sin x - 10\tan x + c$

Integration by substitution method

1.

Evaluate

$$\int x^2 e^{-4x^3}\,dx$$

Let

$$u = -4x^3.$$

Then

$$du = -4 \cdot 3x^2\,dx = -12x^2\,dx.$$

Solving for $x^2\,dx$, we get

$$x^2\,dx = -\frac{1}{12}\,du$$

Hence,

$$\begin{aligned}\int x^2 e^{-4x^3}\,dx &= \int e^u \cdot -\frac{1}{12}\,du \\ &= -\frac{1}{12}\int e^u\,du \\ &= -\frac{1}{12}e^u + C \\ &= -\frac{1}{12}e^{-4x^3} + C\end{aligned}$$

2.

Evaluate

$$\int x^3 \cos(5x^4)\, dx$$

Let

$$u = 5x^4.$$

Then

$$du = 20x^3\, dx.$$

Solving for $x^3 dx$, we get

$$x^3 dx = \frac{1}{20} du$$

Hence,

$$\begin{aligned} \int x^3 \cos(5x^4)\, dx &= \frac{1}{20} \int \cos(u)\, du \\ &= \frac{1}{20} \sin(u) + C \\ &= \frac{1}{20} \sin(5x^4) + C \end{aligned}$$

3

Evaluate

$$\int \frac{(2-\sqrt{x})^5}{\sqrt{x}}\,dx$$

Let

$$u = 2 - \sqrt{x}.$$

Then

$$du = -\frac{1}{2\sqrt{x}}\,dx.$$

Hence,

$$\begin{aligned}\int \frac{(2-\sqrt{x})^5}{\sqrt{x}}\,dx &= \int -2u^5\,du \\ &= \frac{-2u^6}{6} + C \\ &= \frac{-(2-\sqrt{x})^6}{3} + C\end{aligned}$$

I Integrate the following function w.r.t x

1. $sec^2(5-3x)$

Sol: $I=\int sec^2(5-3x)dx$

Put 5-3x = t so that $-3dx=dt$, $dx=\frac{-dt}{3}$

Therefore $I=\int sec^2(t)\left(\frac{-dt}{3}\right)=-\frac{1}{3}\int sec^2t\ dt=-\frac{1}{3}tant+\ c=-\frac{1}{3}tan(5-3x)+\ c$

2. e^{4x+5}

Sol: $I=\int e^{4x+5}dx=\frac{e^{4x+5}}{4}+c$ using $\int e^{ax+b}dx=\frac{e^{ax+b}}{a}+c$

3. $\frac{7}{3x+2}$

Sol: $I=\int\frac{7}{3x+2}dx=7\int\frac{1}{3x+2}dx=7\frac{log(3x+2)}{3}+c$

II Evaluate the following:

1. $\int cot^2 x dx$

$$I = \int cot^2 x dx$$

Sol: $= \int (cosec^2 x - 1)dx$

$$= \int cosec^2 x dx - \int dx$$

$$= -cot x - x + c$$

2. $\int \sin^3 x dx$

$$I = \int sin^3 x dx$$

$$= \int \frac{3sinx - sin3x}{4} dx$$

Sol: $= \frac{1}{4}\left[\int 3\sin x dx - \int \sin 3x dx\right]$

$$= \frac{1}{4}\left[-3\cos x + \frac{\cos 3x}{3}\right] + c$$

$$I = \frac{-3cosx}{4} + \frac{cos3x}{12} + c$$

3. $\int \sin 3x \sin 5x dx$

$$I = \int \sin 3x \sin 5x dx$$

$$= \frac{1}{2}\int [\cos(3x-5x) - \cos(3x+5x)]dx$$

$$= \frac{1}{2}\int (\cos(-2x) - \cos 8x)dx$$

$$= \frac{1}{2}\int (\cos 2x - \cos 8x)dx$$

$$= \frac{1}{2}\left[\frac{\sin 2x}{2} - \frac{\sin 8x}{8}\right] + c$$

$$I = \frac{\sin 2x}{4} - \frac{\sin 8x}{16} + c$$

Types of Standard Integrals

Standard Integrals of types:

1. $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

2. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c$

1\. 3. $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a} sec^{-1}\left(\frac{x}{a}\right) + c$

4. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right) + c \quad if\ x > a > 0$

5. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a} log\left(\frac{a+x}{a-x}\right) + c \quad if\ a > x > 0$

1. Prove that $\int \frac{dx}{x^2+a^2} = \frac{1}{a}tan^{-1}\left(\frac{x}{a}\right)+c$

Proof: Put $x = \tan\theta$ then, $dx = asec^2\theta d\theta$

$$\int \frac{dx}{x^2+a^2} = \int \frac{a\ sec^2\theta d\theta}{a^2\tan^2\theta + a^2} = \int \frac{a\ sec^2\theta d\theta}{a^2\,(\tan^2\theta+1)}$$

Therefore,

$$= \int \frac{\cancel{a}\ \cancel{sec^2\theta} d\theta}{a^{\cancel{2}}\ \cancel{sec^2\theta}} = \frac{1}{a}\int d\theta$$

$$= \frac{1}{a}\theta + c = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$$

2. Prove that $\int \frac{dx}{\sqrt{a^2-x^2}} = sin^{-1}\left(\frac{x}{a}\right)+c$

Proof: Let $x = a\ \sin\theta\ \ then,\ dx = a\ \cos\theta d\theta$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{a\cos\theta d\theta}{\sqrt{a^2-a^2\sin^2\theta}}$$

Therefore,

$$= \int \frac{a\cos\theta d\theta}{\sqrt{a^2(1-\sin^2\theta)}} = \int \frac{a\cos\theta d\theta}{\sqrt{a^2\cos^2\theta}}$$

$$= \int \frac{\cancel{a\cos\theta} d\theta}{\cancel{a\cos\theta}} = \int d\theta = \theta + c$$

$$= \sin^{-1}\left(\frac{x}{a}\right)+c$$

3 Prove that $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}sec^{-1}\left(\frac{x}{a}\right) + c$

Proof: Let $x = a\ sec\theta\ \ then,\ dx = a\ sec\theta \tan\theta\ d\theta$

Therefore,

$$\int \frac{dx}{x\sqrt{x^2-a^2}} = \int \frac{asec\theta \tan\theta\ d\theta}{asec\theta\sqrt{a^2sec^2\theta - a^2}}$$

$$= \int \frac{\cancel{asec\theta}\ tan\theta\ d\theta}{\cancel{asec\theta}\sqrt{a^2(sec^2\theta - 1)}}$$

$$= \int \frac{tan\theta\ d\theta}{\sqrt{a^2\tan^2\theta}} = \int \frac{\cancel{tan\theta}\ d\theta}{a\ \cancel{tan\theta}} = \frac{1}{a}\int d\theta = \frac{1}{a}\theta + c$$

$$= \frac{1}{a}sec^{-1}\left(\frac{x}{a}\right) + c$$

Problems

1. Evaluate $\int \frac{1}{(x+2)^2+1}dx$

Sol: We have $I = \int \frac{1}{(x+2)^2+1}dx$

Put $x+2=t\ then,\ dx=dt$

Therefore, $I = \int \frac{1}{(t)^2+1}dt = \tan^{-1}t + c = \tan^{-1}(x+2)+c$

2. Evaluate $\int \frac{1}{25t^2+10t-5}dt$

Sol: Given $I = \int \frac{1}{25t^2+10t-5}dt$

$$I = \int \frac{1}{(5t)^2+2(5t)(1)+1-1-5}dt$$

$$= \int \frac{1}{(5t+1)^2-6}dt = \int \frac{1}{(5t+1)^2-(\sqrt{6})^2}dt = \frac{1}{2\sqrt{6}}\log\left(\frac{5t+1-\sqrt{6}}{5t+1+\sqrt{6}}\right)+c$$

3. Evaluate $\int \frac{1}{18-9x^2} dx$

Sol: $I = \int \frac{1}{18-9x^2} dx = \int \frac{1}{18-(3x)^2} dx$

Put $3x = t$ $so\ that\ 3dx = dt \Rightarrow dx = \frac{dt}{3}$

$$I = \frac{1}{3} \int \frac{1}{18-t^2} dt$$

$$= \frac{1}{3} \frac{1}{2\sqrt{18}} \log\left(\frac{\sqrt{18}-t}{\sqrt{18}+t}\right) + c$$

$$= \frac{1}{6\sqrt{18}} log\left(\frac{\sqrt{18}-3x}{\sqrt{18}+3x}\right) + c$$

4. Evaluate $\int \frac{dx}{\sqrt{9-(x+2)^2}}$

Sol:Given $I = \int \frac{dx}{\sqrt{9-(x+2)^2}}$

Put $x+2=t \therefore dx = dt$

$I = \int \frac{dt}{\sqrt{3^2-(t)^2}}$ $\int \frac{dx}{\sqrt{a^2-x^2}} = sin^{-1}\left(\frac{x}{a}\right)+c$

$I = \sin^{-1}\left(\frac{t}{3}\right)+c$

$= sin^{-1}\left(\frac{x+2}{3}\right)+c$

5. Evaluate $\int \frac{x^3}{1-x^8}dx$

Sol: Given $I = \int \frac{x^3}{1-x^8}dx$

$$I = \int \frac{x^3}{1-(x^4)^2}dx$$

Put $x^4 = t$ so that $4x^3 dx = dt \Rightarrow x^3 dx = \frac{dt}{4}$

$$I = \int \frac{\frac{dt}{4}}{1-t^2} = \frac{1}{4}\int \frac{dt}{1-t^2} = \frac{1}{4}\,\frac{1}{2.1}\log\left(\frac{1+t}{1-t}\right)+c$$

$$= \frac{1}{8}\log\left(\frac{1+x^4}{1-x^4}\right)+c$$

Integration By Parts

This method is used for the integration of the product of two functions generally.

Let u and v be two functions of x, then $\frac{d}{dx}(u.v) = u.\frac{dv}{dx} + v.\frac{du}{dx}$

integrating both sides, we have

$$uv = \int u\frac{dv}{dx}.dx + \int v\frac{du}{dx}.dx$$

$$uv = \int u\frac{dv}{dx}.dx + \int v\frac{du}{dx}.dx$$

$$\int u\frac{dv}{dx}.dx = uv - \int v\frac{du}{dx}.dx$$

or

$$\int I.II = I\int II - \int\left(I'\int II\right)$$

In words, integral of the product of two functions =first function x integral of the second -integral of $(diff.coff.of\ first\ function\ x\ integral\ of\ the\ second)$.

problems:

1. $\int \frac{1}{50+2x^2}dx$

2. $\int \frac{1}{x^2-4x+8}dx$

3. $\int \frac{1}{3x^2+13x-10}dx$

4. $\int \frac{1}{1-6x-9x^2}dx$

5. $\int \frac{1}{x^2+8x+20}dx$

6. $\int \frac{e^x}{e^{2x}+6e^x+5}dx$

Problems

1. Evaluate $\int xe^x dx$

Sol: $I = \int xe^x dx = x(e^x) - \int \frac{d}{dx}(x).e^x dx$

$$= xe^x - \int 1.e^x dx = xe^x - e^x + c = e^x(x-1) + c$$

2. Evaluate $\int x\sin 3x\, dx$

sol: $given\ \ I = \int x\sin 3x\, dx$

$$= x\left(-\frac{\cos 3x}{3}\right) - \int .\left(-\frac{\cos 3x}{3}\right)dx$$

$$= -\frac{x\cos 3x}{3} + \frac{1}{3}\int \cos 3x dx$$

$$= -\frac{x\cos 3x}{3} + \frac{1}{3}\left(\frac{\sin 3x}{3}\right) + c$$

$$= -\frac{x\cos 3x}{3} + \frac{\sin 3x}{9} + c$$

3.Evaluate $\int x^2 \cos x \, dx$

sol:I= $\int x^2 \cos x \, dx$

$$= x^2 \int \cos x \, dx - \int \left\{ \frac{d}{dx} x^2 \int \cos x \, dx \right\} dx$$

$$= x^2 \sin x - \int 2x \sin x \, dx$$

$$= x^2 \sin x - 2\left[x \int \sin x \, dx - \int \left\{ \frac{d}{dx}(x) \int \sin x \, dx \right\} dx \right]$$

$$= x^2 \sin x + 2x \cos x - 2\int 1.\cos x \, dx$$

$$= x^2 \sin x + 2x \cos x - 2\sin x + c$$

4.Evalute

$$\int \log x \, dx$$

sol:given $I = \int \log x \, dx$

taking log as first function and 1 as second function and integrating by parts,we have

$$I = \log x \left\{ \int 1 dx \right\} - \left\{ \frac{d}{dx}(\log x) \int 1.dx \right\} dx$$

$$= \log x . x - \int \frac{1}{x} . x dx$$

$$= x \log x - \int 1.dx$$

$$= x \log x - x + c$$

$$I = x(\log x - 1) + c$$

5.Evaluate $\int x^2 \log x \, dx$

sol:I= $\int x^2 \log x \, dx$

$$= \frac{x^3}{3}\log x - \int \frac{x^3}{3}.\frac{1}{x}dx = \frac{x^3}{3}\log x - \frac{1}{3}\int x^2 dx$$

$$= \frac{x^3}{3}\log x - \frac{1}{3}.\frac{x^3}{3} + c$$

$$= \frac{1}{3}x^3\left[\log x - \frac{1}{3}\right] + c$$

6.Evaluate $\int x \cos x \, dx$

sol:given I = $\int x \cos x \, dx$

$$= x\int \cos x \, dx - \int\left[\frac{d}{dx}(x)\int \cos x \, dx\right]dx$$

$$= x \sin x - \int \sin x \, dx$$

$$= x \sin x + \cos x + c$$

7. Evaluate $\int \tan^{-1}x\, dx$

sol: $I = \int tan^{-1}x\, dx$

Let $\tan^{-1}x = t \Rightarrow x = \tan t \Rightarrow dx = sec^2 t\, dt$

$$I = \int \tan^{-1}x dx = \int t\, sec^2 t\, dt$$

$$I = t\tan t - \int 1.\, tan t\, dt = t(\tan t) + \log(\cos t) + c$$ [Integrating by parts]

$$= x\tan^{-1}x + \log\left(\frac{1}{\sqrt{1+x^2}}\right) + c$$ $\left[\tan t = x \Rightarrow \cos t = \frac{1}{\sqrt{1+x^2}}\right]$

$$= x\tan^{-1}x - \frac{1}{2}\log(1+x^2) + c$$

8. Evaluate $\int x\sin^2 x dx$

Sol: Given $I = \int x\sin^2 x dx$

$$I = \int x\left(\frac{1-\cos 2x}{2}\right)dx$$

$$= \frac{1}{2}\int x(1-\cos 2x)dx$$

$$= \frac{1}{2}\int (x - x\cos 2x)dx = \frac{1}{2}\left[\int x\, dx - \int x\cos 2x\, dx\right]$$

$$= \frac{1}{2}\left[\frac{x^2}{2} - \left\{x\left(-\frac{\sin 2x}{2}\right) - \int\left(-\frac{\sin 2x}{2}\right).1dx\right\}\right]$$

$$= \frac{1}{2}\left[\frac{x^2}{2} + \frac{x\sin 2x}{2} - \left(\frac{\cos 2x}{4}\right)\right] + c$$

$$= \frac{x^2}{4} + \frac{x\sin 2x}{4} - \frac{\cos 2x}{8} + c$$

9. Evaluate $\int x\cos^2 x\, dx$

Sol: Given $I = \int x\cos^2 x dx$

$$I = \int x\left(\frac{1+\cos 2x}{2}\right)dx$$

$$= \frac{1}{2}\int x(1+\cos 2x)dx$$

$$= \frac{1}{2}\int (x + x\cos 2x)dx = \frac{1}{2}\left[\int x\, dx + \int x\cos 2x\, dx\right]$$

$$= \frac{1}{2}\left[\frac{x^2}{2} + x\left(-\frac{\sin 2x}{2}\right) - \int\left(-\frac{\sin 2x}{2}\right).1dx\right]$$

$$= \frac{1}{2}\left[\frac{x^2}{2} - \frac{x\sin 2x}{2} + \left(\frac{\cos 2x}{4}\right)\right] + c$$

$$= \frac{x^2}{4} - \frac{x\sin 2x}{4} + \frac{\cos 2x}{8} + c$$

10. Evaluate $\int x \log x dx$

Sol: Given $I = \int x \log x dx$

$$I = \log x . \frac{x^2}{2} - \int \frac{x^2}{2} . \frac{d}{dx}(\log x)\, dx$$

$$= \log x . \frac{x^2}{2} - \int \frac{x^2}{2} . \frac{1}{x} dx$$

$$= \log x . \frac{x^2}{2} - \int \frac{x}{2} dx$$

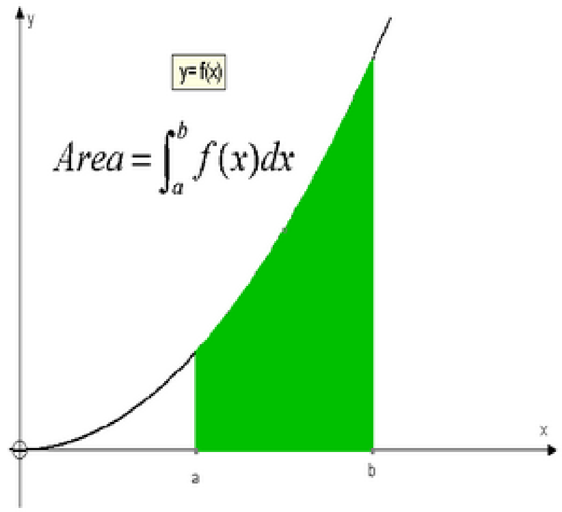
$$= \log x . \frac{x^2}{2} - \frac{x^2}{4} + c$$

Introduction

Definite Integral

If g(x) be an integral of f(x), the g(b)-g(a) is called the definite integral of f(x) between the limits a and b, and is denoted by the symbol

$$\int_a^b f(x)dx$$



The interval [a,b] is called the range of integration. g(b) -g(a) is also denoted by $[g(x)]^b{}_a$ a is called the lower limit of the integral and b is the upper limit of the integral.

Thus, we write

$$\int_a^b f(x)dx = [g(x)]_a{}^b = g(b) - g(a)$$

The result of performing the integral is a number that

represents the area under the curve of *f (x)* between the limits and the *x*-axis if *f(x)* is greater than or equal to zero between the limits.

Problems

1. Evaluate $\int_{25}^{-10} dR$

sol:

$$\int_{25}^{-10} dR = R\Big|_{25}^{-10}$$

$$= -10 - 25$$

$$= -35$$

2. Evaluate $\int_{4}^{9} (2x + 3\sqrt{x})\, dx$

Sol:

$$\int_{4}^{9} (2x + 3\sqrt{x})\, dx = [x^2 + 2x^{3/2}]_4^9$$

$$= [9^2 + 2(9)^{3/2}] - [4^2 + 2(4)^{3/2}]$$

$$= 135 - 32$$

$$= 103$$

3. Evaluate $\int_1^2 y^2 + y^{-2}\, dy$

Sol:

$$\int_1^2 y^2 + y^{-2}\, dy = \left(\frac{1}{3}y^3 - \frac{1}{y}\right)\Big|_1^2$$

$$= \frac{1}{3}(2)^3 - \frac{1}{2} - \left(\frac{1}{3}(1)^3 - \frac{1}{1}\right)$$

$$= \frac{8}{3} - \frac{1}{2} - \frac{1}{3} + 1$$

$$= \frac{17}{6}$$

4. Evaluate $\int_0^{\frac{\pi}{3}} 2\sin\theta - 5\cos\theta\, d\theta$

Sol:

$$\int_0^{\frac{\pi}{3}} 2\sin\theta - 5\cos\theta\, d\theta = \left(-2\cos\theta - 5\sin\theta\right)\Big|_0^{\pi/3}$$

$$= -2\cos\left(\frac{\pi}{3}\right) - 5\sin\left(\frac{\pi}{3}\right) - \left(-2\cos 0 - 5\sin 0\right)$$

$$= -1 - \frac{5\sqrt{3}}{2} + 2$$

$$= 1 - \frac{5\sqrt{3}}{2}$$

5. Evaluate $\int_4^0 \sqrt{t}\,(t-2)\,dt$

Sol:

$$\int_4^0 \sqrt{t}\,(t-2)\,dt = \int_4^0 t^{\frac{3}{2}} - 2t^{\frac{1}{2}}\,dt$$

$$= \left(\frac{2}{5}t^{\frac{5}{2}} - \frac{4}{3}t^{\frac{3}{2}}\right)\Bigg|_4^0$$

$$= 0 - \left(\frac{2}{5}(4)^{\frac{5}{2}} - \frac{4}{3}(4)^{\frac{3}{2}}\right)$$

$$= -\frac{32}{15}$$

$$(4)^{\frac{5}{2}} = \left((4)^{\frac{1}{2}}\right)^5 = (2)^5 = 32$$

$$(4)^{\frac{3}{2}} = \left((4)^{\frac{1}{2}}\right)^3 = (2)^3 = 8$$

6. Evaluate $\int_{\pi/6}^{\pi/4} 5-2\sec z\tan z\,dz$

Sol:

$$\int_{\pi/6}^{\pi/4} 5-2\sec z\tan z\,dz = \left(5z-2\sec z\right)\Big|_{\pi/6}^{\pi/4}$$

$$= 5\left(\frac{\pi}{4}\right)-2\sec\left(\frac{\pi}{4}\right)-\left(5\left(\frac{\pi}{6}\right)-2\sec\left(\frac{\pi}{6}\right)\right)$$

$$= \frac{5\pi}{12}-2\sqrt{2}+\frac{4}{\sqrt{3}}$$

7. Evaluate $\int_{3}^{0} 15w^4-13w^2+w\,dw$

Sol:

$$\int_{3}^{0} 15w^4-13w^2+w\,dw = \left(3w^5-\tfrac{13}{3}w^3+\tfrac{1}{2}w^2\right)\Big|_{3}^{0}$$

$$\int_{3}^{0} 15w^4-13w^2+w\,dw = 0-\tfrac{1233}{2} = \boxed{-\tfrac{1233}{2}}$$

8. Evaluate $\int_{0}^{2} \mathbf{e}^x+\frac{1}{x^2+1}\,dx$

Sol:

$$\int_{0}^{2} \mathbf{e}^x+\frac{1}{x^2+1}\,dx = \left(\mathbf{e}^x+\tan^{-1}(x)\right)\Big|_{0}^{2}$$

$$\int_{0}^{2} \mathbf{e}^x+\frac{1}{x^2+1}\,dx = \left(\mathbf{e}^2+\tan^{-1}(2)\right)-\left(\mathbf{e}^0+\tan^{-1}(0)\right) = \boxed{\mathbf{e}^2+\tan^{-1}(2)-1}$$

9. Evaluate $\int_{-1}^{0} x^3(1-2x^4)^3\,dx$

Sol:

Put

$u = 1 - 2x^4$

Then $du = -8x^3\,dx$

The question has $x^3\,dx$ so we write

$$-\frac{du}{8} = x^3\,dx$$

So we have:

$$\int_{-1}^{0} x^3(1-2x^4)^3\,dx = -\frac{1}{8}\int_{x=-1}^{x=0} u^3\,du$$

$$= -\frac{1}{8} \times \left[\frac{u^4}{4}\right]_{x=-1}^{x=0}$$

$$= -\frac{1}{32}[u^4]_{x=-1}^{x=0}$$

$$= -\frac{1}{32}[(1-2x^4)^4]_{-1}^{0}$$

$$= -\frac{1}{32}[(1-0)^4 - (1-2)^4]$$

$$= -\frac{1}{32}[(1) - (1)]$$

$$= 0$$

10. Find the displacement of an object from t = 2 to t = 3, if the velocity of the object at time t is given by

$$v = \frac{(t^2+1)}{(t^3+3t)^2}$$

Sol: To find the displacement, we need to evaluate:

$$\int_2^3 \frac{(t^2+1)}{(t^3+3t)^2}\,dt$$

Put $u = t^3 + 3t$, then $du = (3t^2 + 3)\,dt = 3(t^2 + 1)\,dt$

So $\frac{du}{3} = (t^2 + 1)\,dt$

So we have:

$$\int_2^3 \frac{(t^2+1)}{(t^3+3t)^2}\,dt = \frac{1}{3}\int_{t=2}^{t=3} \frac{1}{u^2}\,du$$

$$= \frac{1}{3}\int_{t=2}^{t=3} u^{-2}\,du$$

$$= -\frac{1}{3}\left[\frac{1}{u}\right]_{t=2}^{t=3}$$

$$= -\frac{1}{3}\left[\frac{1}{t^3+3t}\right]_2^3$$

$= -\frac{1}{3}$open square brackets fraction numerator 1 over denominator 3 cubed plus 3 open pare

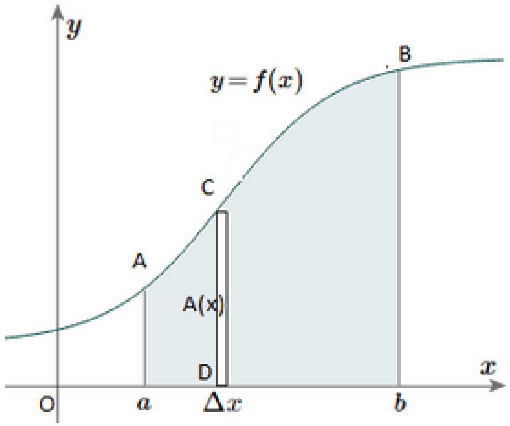
$$= -\frac{1}{3}\left[\frac{1}{36} - \frac{1}{14}\right]$$

$$= 0.014550$$

So the displacement of the object from time $t = 2$ to $t = 3$ is 0.015 units.

Area Function

Area Function



$\int_a^b f(x)dx =$ Area of the region bounded by the curve y=f(x), the x-axis and the ordinates x=a and c=b.

Let C be a point on the curve whose x-coordinate is x, in [a,b]. Then

$\int_a^x f(x)dx =$ Area bounded by the curve y=f(x), x-axis, ordinates x=a and CD, whose abscissa is x.

This area depends upon the value of x. we can say this area is a function of x and denoted by A(x) and its called area function.

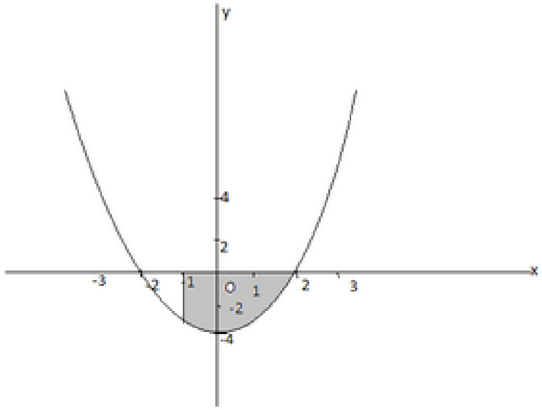
$$A(x) = \int_a^x f(x)dx$$

1. Find the area bounded by $y = x^2 - 4$, the x-axis and the lines $x = -1$ and $x = 2$.

Sol:

The required area is totally below the

x-axis in this example, so we need to use absolute value signs.



$$\text{Area} = \left|\int_a^b f(x)\,dx\right|$$

$$= \left|\int_{-1}^{2} (x^2 - 4)\,dx\right|$$

$$= \left|\left[\frac{x^3}{3} - 4x\right]_{-1}^{2}\right|$$

$$= \left|\left[\left(\frac{8}{3} - 8\right) - \left(-\frac{1}{3} + 4\right)\right]\right|$$

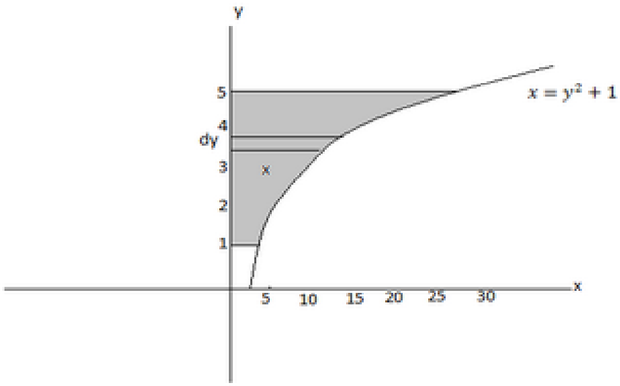
$$= |-9| = 9 \text{ units}^2$$

2. Find the area of the region bounded by the curve $y = \sqrt{x-1}$, the y-axis and the lines $y = 1$ and $y = 5$.

Sol:

In this case, we express x as a function of y:

$$y = \sqrt{x-1}$$
$$y^2 = x - 1$$
$$x = y^2 + 1$$

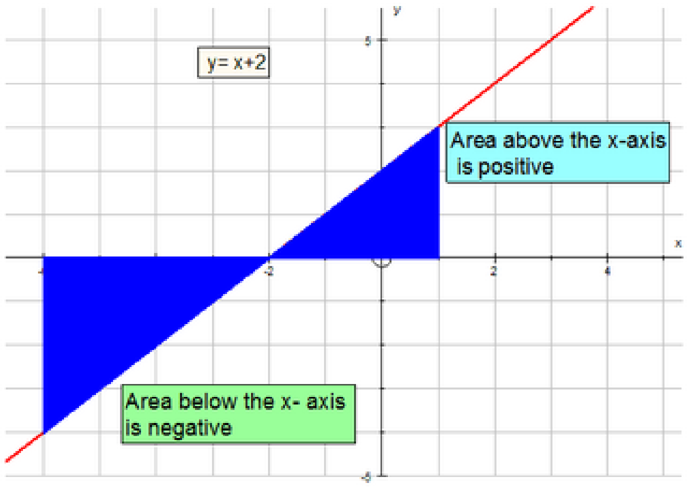


So the area is given by:

$$A = \int_1^5 (y^2+1)\,dy = \left[\frac{y^3}{3} + y\right]_1^5$$
$$= 45\frac{1}{3} \text{ sq units}$$

3. Calculate the area enclosed by the graph of y=x+2 and the x-axis for $-6 \leq x \leq 1$

Sol: Area below the x-axis



$$\int_{-6}^{-2}(x+2)dx = \left[\frac{x^2}{2}+2x\right]_{-6}^{-2}$$

$$= \left(\frac{(-2)^2}{2}+2\times(-2)\right)-\left(\frac{(-6)^2}{2}+2\times(-6)\right)$$

$$= \left(\frac{4}{2}-4)\right)-\left(\frac{36}{2}-12\right)$$

$$= -2-6$$

$$= -8 \text{ units}^2$$

Area above the x-axis

$$\int_{-2}^{1}(x+2)dx = \left[\frac{x^2}{2}+2x\right]_{-2}^{1}$$

$$= \left(\frac{1^2}{2}+2\times 1\right)-\left(\frac{(-2)^2}{2}+2\times(-2)\right)$$

$$= \left(\frac{1}{2}+2)\right)-\left(\frac{4}{2}-4\right)$$

$$= 2\frac{1}{2}+2$$

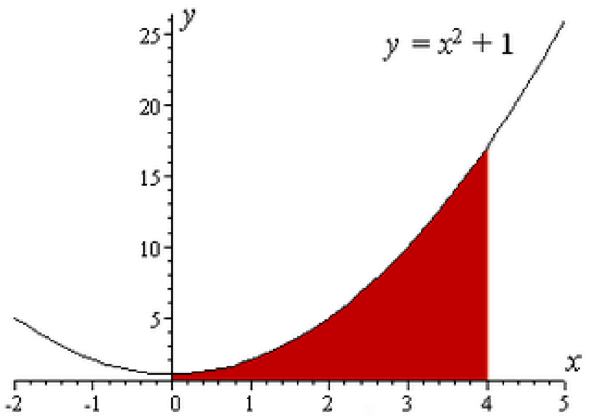
$$= 4\frac{1}{2}\,\text{units}^2$$

$$\textit{Total area} = 8 + 4\frac{1}{2} = 12\frac{1}{2}\ \textit{units}^2$$

4. Find the (exact) area under the curve $y = x^2 + 1$ between $x = 0$ and $x = 4$ and the x-axis.

Sol:

This is the area we need to find:



The area is given by:

$$\int_0^4 (x^2+1)\,dx = \left[\frac{x^3}{3} + x\right]_0^4$$

$$= \left(\frac{4^3}{3} + 4\right) - \left(\frac{0^3}{3} + 0\right)$$

$$= \frac{76}{3}\text{ units}^2$$

$$\approx 25.3\text{ units}^2$$

Centre of gravity and moment of inertia by integration method

Centre of gravity

Consider of system of particles of masses m_1, m_2, m_3, ... having their centres of gravity G_1, G_2, G_3, ... at points (x_1, y_1), (x_2, y_2), (x_3, y_3),.............Let (x, y) be the coordinates of their common centre of gravity G. It is clear that the sum of the moment of m_1, m_2, m_3, ... about any line in the plane is same as the moment about the same line of the whole mass $m_1 + m_2 + m_3 +$ .... of the entire system.

By taking moments about x-axis($m_1 + m_2 + m_3 +$ ....) x= m_1 $+ m_2 + m_3 +$ ....

Moment of inertia

The moment of inertia of a particle about an axis is the product of its mass and the square of its perpendicular distance from the axis. If "m" is the mass of the particle and 'r' its perpendicular distance from the axis then MI of the particle about the axis = mr^2

In case of rigid body, we consider an elementary mass * whose distance from the axis is r. then MI of the whole body about the axis is * evaluated over the whole mass.

Introduction

Differential Equations: An equation involving derivatives of dependent variable with respect to independent variable(s) is called differential equation.

Example: For a given function g function f such that $\frac{dy}{dx} = g(x)$, where $y = f(x)$ is known as differential equation.

1. $\left(\frac{d^2y}{dx^2}\right) = 6x$
2. $\left(\frac{d^2y}{dx^2}\right)^2 + 4\left(\frac{dy}{dx}\right)^3 = y$

These equation have variety of applications in geology, anthropology, physics etc. In this chapter we shall study some basic concepts related to differential equation, formation of differential equations, methods to solve first order, first degree differential equations, general solutions of differential equations.

Order of a differential equation is the order of the highest order derivative occurring in the differential equation.

Degree of a differential equation is the highest power (exponent) of the highest order derivative in it.

Example:

1. $\left(\frac{dy}{dx}\right)^2 - 10\left(\frac{dy}{dx}\right)^5 + 2 = 0$, order of this differential equation is 1 and degree is 5
2. $\left(\frac{d^2y}{dx^2}\right) - 10x\left(\frac{dy}{dx}\right)^2 + 5y = 1$ order of this differential equation is 2 and degree is 1.
3. $\left(\frac{d^6y}{dx^6}\right)^2 + 3\left(\frac{d^3y}{dx^3}\right) - 5 = 0$ order of this differential equation is 6 and degree is 2.
4. $y_3 + x^2{y_2}^3 = 0$ order of this differential equation is 3 and degree is 1.
5. $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 2y = 0$, order =2, degree =1.
6. $\left(\frac{d^2y}{dx^2}\right)^2 + 3\left(\frac{dy}{dx}\right)^3 + 5y = 0$, order = 2, degree =2.

Activity:

Find the order and degree of each of the following equations:

1. $\left(\frac{dy}{dx}\right)^2 + y^3 - 1 = 0$
2. $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^4 + 8y = 0$
3. $x\left(\frac{dy}{dx}\right) + \frac{2}{\left(\frac{dy}{dx}\right)} = y^2$
4. $t^2\frac{d^2s}{dt^2} - st\frac{ds}{dt} = s$
5. $\left(\frac{dy}{dx}\right)^2 + \left(\frac{d^2y}{dx^2}\right)^4 + 5\left(\frac{d^3y}{dx^3}\right)^3 - 8y^3 = 0$

Formation of differential equation by eliminating arbitrary constants up to second order

Formation of differential equation by eliminating arbitrary constants up to second order

Properties

- The order of differential equation is equal to the number of arbitrary constants in the given relation
- The differential equation is free from arbitrary constants.

1. If $y = a\cos x + b\sin x$ by eliminating arbitrary constants a and b

Solution: Given $y = a\cos x + b\sin x$ --------(1)

$$y' = -a\sin x + b\cos x \text{ ----------(2)}$$

$$y'' = -a\cos x - b\sin x$$

$$y'' = -(a\cos x + b\sin x) = -y \qquad y'' + y = 0$$

2 Eliminate the arbitrary constants c_1 and c_2 from the relation $y = c_1 e^{-3x} + c_2 e^{2x}$

Solution: $y = c_1 e^{-3x} + c_2 e^{2x}$--------------(1)

$$y' = -3c_1 e^{-3x} + 2c_2 e^{2x} \text{---------(2)}$$

$$y'' = 9c_1 e^{-3x} + 4c_2 e^{2x} \text{------------(3)}$$

(2) + (3)

$$y' + y'' = \left(-3c_1 e^{-3x} + 2c_2 e^{2x}\right) + \left(9c_1 e^{-3x} + 4c_2 e^{2x}\right)$$

$$= 6c_1 e^{-3x} + 6c_2 e^{2x} = 6\left(c_1 e^{-3x} + c_2 e^{2x}\right) = 6y$$

$$y' + y'' = y$$

3. $y = A\cos(x + B)$ where A and B are arbitrary constants

solution: $y = A\cos(x + b)$

$$y' = -A\sin(x + B)$$

$$y'' = -A\cos(x + B)$$

$$y'' = -y$$

$$y'' + y = 0$$

4. *Eliminate the constant c from the equation below.*

$$(x-c)^2 + y^2 = c^2$$

Differentiating above, we get

which leads to,

$$c = x + yy'$$

squaring above,

$$c^2 = (x + yy')^2$$

and then equating to our original equation, we get

then reduced to

$$y^2 = x^2 + 2xyy'.$$

Since $y' = dy/dx$, so the equation above can be rewritten as,

$$(x^2 - y^2)dx = -2xydy.$$

5. Eliminate β *and* α from following relation, (w is a parameter, not a constant)

$$x = \beta\cos(\omega t + \alpha). \quad (1)$$

We differentiate x with respect to t twice,that is,

$$\frac{dx}{dt} = -\omega\beta\sin(\omega t + \alpha),$$

$$\frac{d^2x}{dt^2} = -\omega^2\beta\cos(\omega t + \alpha). \quad (2)$$

Substituting (1) to (2) eliminates the arbitrary constant β, thus

$$\frac{d^2x}{dt^2} + \omega^2 x = 0.$$

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Substituting (1) to (2) eliminates the arbitrary constant β, thus

$$\frac{d^2x}{dt^2} + \omega^2 x = 0.$$

6. Eliminate b from the equation below.

$$bxy + b^2x + 4 = 0. \qquad (1)$$

Direct differentiation results to

$$b(y + xy') + b^2 = 0. \qquad (2)$$

Solving for b,

$$b = -(y + xy') \qquad (3)$$

Substituting b to equation (1), we get

$$x^3(y')^2 + x^2yy' + 4 = 0. \qquad (4)$$

Problems

Form the d.e from the following equations by the eliminating the arbitrary/ parameters a *and* b

1. $y = ax^2 + bx$
2. $\frac{x}{a} + \frac{y}{b} = 1$
3. $yx = ae^x + be^{-x} + x^2$
4. $y = e^{2x}(a + bx)$
5. $ax^2 + by^2 = 1$
6. $y^2 = 4ax$
7. $y = ax^2 + bx + c$
8. $y = c_1 e^{ax} cosbx + c_2 e^{ax} sinbx;$
9. $y = nx + h/n$
10. $x = asin(wt + b)$

Solution of O.D.E of first degree and first order by variable separable method

A first order-first degree differential equation is of the form

$$\frac{dy}{dx} = F(x,y) \text{----------}(1)$$

If $F(x,y)$ can be expressed as a product $g(x)$, $h(y)$ where $g(x)$ is a function of x and $h(y)$ is a function of y then d.e (1) is said to be variable separable type. The d.e (1) then has the form

$$\frac{dy}{dx} = h(y).g(x) \text{-------}(2)$$

If $h(y) \neq 0$, separating the variables, (2) can be written as

$$\frac{1}{h(y)} dy = g(x)dx \text{------}(3)$$

Integrating both sides we get,

$$\int \frac{1}{h(y)} dy = \int g(x)dx \text{-----}(4)$$

Thus, equation (4) provides the solutions of given differential equation in the form

$$H(y) = G(x) + c$$

Here $H(y)$ *and* $G(x)$ are the anti derivative of $\frac{1}{h(y)}$ *and* $g(x)$ respectively and c is arbitrary constant.

Examples

1.Solve $\frac{dy}{dx} = \frac{x+1}{2-y} (y \neq 2)$

solution: $\frac{dy}{dx} = \frac{x+1}{2-y}$ ----------------(1)

separating the variables in equation (1) we get

$$(2-y)dy = (x+1)dx$$

Integrate on both sides, we get

$$\int (2-y)dy = \int (x+1)dx$$

$$2y - \frac{y^2}{2} = \frac{x^2}{2} + x + c$$

$$\frac{x^2}{2} + \frac{y^2}{2} - 2y + x + c_1 = 0$$

$$x^2 + y^2 + 2x - 4y + 2c_1 = 0$$

$x^2 + y^2 + 2x - 4y + C = 0$ where $C = 2c_1$

which is the general solution of equation (1).

2. Solve $\frac{dy}{dx} = -4xy^2$

Solution: Given $\frac{dy}{dx} = -4xy^2$

Separating the variables, we get

$$\frac{dy}{y^2} = -4x\,dx$$

Integrating on both sides, we get

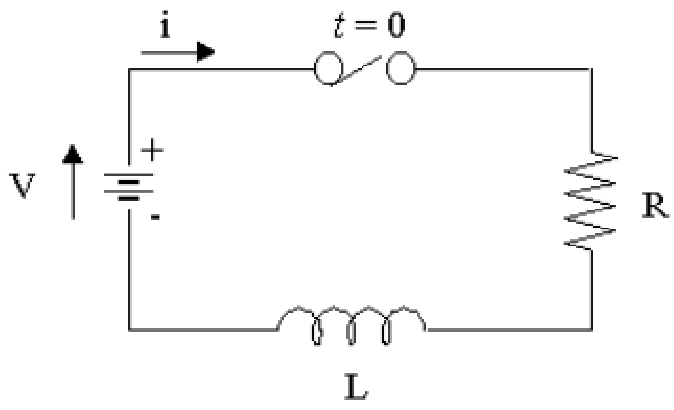
$$\int \frac{dy}{y^2} = \int -4x\,dx$$

$$-\frac{1}{y} = -4\frac{x^2}{2} + c$$

$$-\frac{1}{y} = -2x^2 + c$$

$2x^2 + \frac{1}{y} + c = 0$ which the general solution of given d.e (1).

3. **RL Circuit Application**



In an RL circuit, the differential equation formed using Kirchhoff's law, is

$$Ri + L\frac{di}{dt} = V$$

Solve this DE, using separation of variables, given that

R = 10 Ω, L = 3 H and V = 50 volts, and i(0) = 0.

Sol: Substituting R = 10, L = 3 and V = 50 gives:

$$10i + 3\frac{di}{dt} = 50$$

$$3\frac{di}{dt} = 50 - 10i$$

First, we separate the variables.

$$\frac{di}{50-10i} = \frac{dt}{3}$$

Integrate,

$$\frac{1}{10}\int \frac{di}{5-i} = \frac{1}{3}\int dt$$

$$-\frac{1}{10}\log(5-i) = \frac{t}{3} + c$$

since i(0)=0,

$$\frac{-1}{10}\log(5-0) = 0 + c$$

$$c = -\frac{\log 5}{10}$$

So, substituting for C:

$$-\frac{1}{10}\log(5-i)=\frac{t}{3}-\frac{\log 5}{10}$$

Put log parts together.

$$-\frac{t}{3}=\frac{log(5-i)}{10}-\frac{log5}{10}$$

$$-\frac{10t}{3}=log(5-i)-log5$$

$$-\frac{10t}{3}=log\frac{(5-i)}{5}$$

$$e^{-\frac{10t}{3}}=\frac{(5-i)}{5}$$

$$5e^{-10t/3}=5-i$$

$$i=5-5e^{-10t/3}=5(1-e^{-10t/3})$$

Problems

Solve the differential equation by variable separable form

1. $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$
2. $\frac{dy}{dx} = \frac{1+y}{1+x}$
3. $\frac{dr}{dt} = -4rt$ when $r = r_0,\ t = 0$
4. $2xyy' = 1 + y^2$ when x=2 and y=3.
5. $(1-x)y' = y^2$
6. $xy^3 dx + e^{x^2} dy = 0$
7. $xydx - (x+2)dy = 0$
8. $x\cos^2 y\ dx + \tan y\ dy = 0$
9. $2ydx = 3xdy$
10. $\frac{dV}{dP} = -\frac{V}{P}$

Linear differential equations and its solution using integrating factor

Linear differential equations and its solution using integrating factor

A differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

where, P and Q are constants or functions of x only, is known as a first order linear differential equation.

Ex:1) $\frac{dy}{dx} + \sin x . y = x^5$

2) $\frac{dy}{dx} + \frac{y}{x} = x^3$ **etc**

Such equations are solved using an integrating factor. The integrating factor of the equation is

IF=$e^{\int P dx}$

The equation is then multiplied by the integrating factor.

Then we get

$$e^{\int P dx}\left[\frac{dy}{dx} + P(x).y\right] = e^{\int P dx}.Q(x)$$

The LHS is clearly the derivative of $y.e^{\int P dx}$

Therefore we can rewrite equation (1) as

$$\frac{dy}{dx}\left[y.e^{\int Pdx}\right]=e^{\int Pdx}.Q(x)$$

integrating both sides we get

$ye^{\int Pdx}=\int\left[e^{\int Pdx}.Q(x)\right]dx+c$ **, as the required solution.**

Solve the following differential equations

1. Solve $\frac{dy}{dx}+y=e^{-x}$ $\frac{dy}{dx}+y=e^{-x}$ $\frac{dy}{dx}+y=e^{-x}$ "> with y(0) = 1.

Sol: We know that $\frac{dy}{dx}+p(x)y=q(x)$ $\frac{dy}{dx}+p(x)y=q(x)$ $\frac{dy}{dx}+p(x)y=q(x)$ "> gives $p(x)=1$ $p(x)=1$ $p(x)=1$ "> **and** $q(x)=e^{-x}$ $q(x)=e^{-x}$ $q(x)=e^{-x}$ ">.

Integrating factor $e^{\int 1\,dx}=e^{x}$ $e^{\int 1\,dx}=e^{x}$ $e^{\int 1\,dx}=e^{x}$ ">

Multiplying through by the integrating factor, we get

$e^{x}\frac{dy}{dx}+ye^{x}=1$ $e^{x}\frac{dy}{dx}+y\,e^{x}=1$ $e^{x}\frac{dy}{dx}+y\,e^{x}=1$ "> .

Rewrite $e^{x}\frac{dy}{dx}+ye^{x}$ $e^{x}\frac{dy}{dx}+ye^{x}$ $e^{x}\frac{dy}{dx}+ye^{x}$ "> as the derivative of ye^{x} ye^{x} ye^{x} "> **i.e.** $\frac{d}{dx}[ye^{x}]$ $\frac{d}{dx}\left[ye^{x}\right]$

$\frac{d}{dx}\left[ye^{x}\right]$ **" width="52" height="35" align="absmiddle"> .**

Integrate both sides with respect to x and get

$ye^{x}=x+c$ $ye^{x}=x+c$ $ye^{x}=x+c$ "> .

Use the initial condition to find c. $x=0$ $x=0$ $x=0$ ">, $y=1$ $y=1$ $y=1$ "> **gives** $c=1$ $c=1$ $c=1$ ">.

Hence the solution to the problem is $ye^{x}=x+1$ $ye^{x}=x+1$ $ye^{x}=x+1$ "> which is

$y=(x+1)e^{-x}$ $y=(x+1)e^{-x}$ $y=(x+1)e^{-x}$ ">

2. Solve $\frac{dy}{dx} - \frac{1}{x}.y = 2x$

solution: given equation is of the form $\frac{dy}{dx} + P(x).y = Q(x)$

where $P(x) = \frac{1}{x}$ and $Q(x) = 2x^2$

Its integrating factor will be :

$$e^{\int P dx} = e^{\int \frac{-1}{x} dx} = e^{\log[x]} = \frac{1}{x}$$

hence the general solution is

$$y.e^{\int P dx} = \int \left[e^{\int P dx}.Q(x)\right] dx + c$$

$$y.\frac{1}{x} = \int \frac{1}{x}.2x^2 dx + c$$

$\frac{y}{x} = x^2 + c$ as the required solution.

3. Solve $x\frac{dy}{dx}+2y=x^2(x\neq 0)$

Solution: Given $x\frac{dy}{dx}+2y=x^2$------------(1)

Dividing both sides of the equation (1) by x, we get

$$\frac{dy}{dx}+2\frac{y}{x}=x$$

which is linear d.e of the type $\frac{dy}{dx}+Py=Q$ where $P=\frac{2}{x}$ and $Q=x$

so, I.F $=e^{\int\frac{2}{x}dx}=e^{2\log x}=e^{\log x^2}=x^2$ (as $e^{\log f(x)}=f(x)$)

Therefore, solution of the equation given by

$$y.x^2=\int(x)(x^2)dx+c$$

$$y.x^2=\int x^3dx+c$$

$y.x^2=\frac{x^4}{4}+c$ is the required solution.

4. Find the particular solution of:

$$y' + \tan(x)y = \cos^2(x), \quad y(0) = 2.$$

Solution: Let us use the steps:

Step 1: There is no need for rewriting the differential equation. We have

$$p(x) = \tan(x) \quad \text{and} \quad q(x) = \cos^2(x)\ .$$

Step 2: Integrating factor

$$u(x) = e^{\int \tan(x)dx} = e^{-\ln(\cos(x))} = e^{\ln(\sec(x))} = \sec(x).$$

Step 3: We have

$$\int \sec(x)\cos^2(x)dx = \int \cos(x)dx = \sin(x).$$

Step 4: The general solution is given by

$$y = \frac{\sin(x) + C}{\sec(x)} = \left(\sin(x) + C\right)\cos(x).$$

Step 5: In order to find the particular solution to the given IVP, we use the initial condition to find C. Indeed, we have

$$y(0) = C = 2.$$

Therefore the solution is

$$y = \left(\sin(x) + 2\right)\cos(x).$$

Problems

1. $(x^5 + 3y)dx - xdy = 0$

2. $\cos x . \frac{dy}{dx} + y = \sin x$

3. $\frac{dy}{dx} + y\cot x = 4x\operatorname{cosec} x$

4. $\frac{dy}{dx} + y = \cos x - \sin x$

5. $\frac{dy}{dx} + \frac{2}{x}y = \frac{2}{x}$