AS & A-Level Physics Lecture Notes

(Year 1)

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Contents

Practical Issues

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The latest update can be found via: https://github.com/yuhao-yang-cy/asphysics

About the Notes

The contents of the notes are consistent with the CIE A-Level physics syllabus. I attempt to systematically cover all the key points in the syllabus with brief but sufficient explanations. The notes should be able to serve as a self-contained study guide for the AS CIE course.

I am still working on the notes. The latest release is far from complete as it only covers the first few chapters. I hope I will follow up the other chapters before the end of this year.

If you spot any errors, please let me know.

Literature

I borrow heavily from the following resources:

- Cambridge International AS and A Level Physics Coursebook, by *David Sang, Graham Jones, Richard Woodside* and *Gurinder Chadha*, Cambridge University Press
- International A Level Physics Revision Guide, by *Richard Woodside*, Hodder Education
- Longman Advanced Level Physics, by *Kwok Wai Loo*, Pearson Education South Asia
- Conceptual Physics (10^{th} Edition), by *Paul G. Hewitt*, Pearson International Education
- Fundamentals of Physics, by *Robert Resnick, David Halliday* and *Kenneth S. Krane*, John Wiley & Sons
- Past Papers of Cambridge Internation A-Level Physics Examinations
- HyperPhysics Website: http://hyperphysics.phy-astr.gsu.edu/hbase/index.html
- Wikipedia Website: https://en.wikipedia.org

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Chapter 1
Physical Quantities

1.1 units of measurement

any physical quantity contains a numerical value and its associated unit

a system of units of measurement used throughout the scientific world is the **SI units**[1]

1.1.1 SI base units

SI defines seven units of measure as a basic set, known as the **SI base units**

base quantity	base unit	symbol
mass	kilogram	kg
length	metre	m
time	second	s
electric current	ampere	A
temperature	kelvin	K
amount of substance	mole	mol
luminous intensity	candela	cd

1.1.2 derived units

the seven[2] SI base units are building blocks of the SI system

all other quantities are derived from the base units

Example 1.1 Give the SI base units of (a) speed, (b) acceleration, (c) force, (d) work done.

[1]SI units, abbreviated from the French *Système Internationale d'Unités*, means the International System of Units. Those who are interested in the history and evolution of the SI can check out the Wikipedia article: https://en.wikipedia.org/wiki/International_System_of_Units

[2]Luminous intensity is beyond the scope of the AS-Level syllabus. You are only required to know the other six SI base quantities and their units.

$$\text{speed} = \frac{\text{distance}}{\text{time}} \quad \Rightarrow \quad [v] = \frac{[s]}{[t]} = \frac{\text{m}}{\text{s}} = \text{m s}^{-1}$$
$$\text{acceleration} = \frac{\text{speed}}{\text{time}} \quad \Rightarrow \quad [a] = \frac{[v]}{[t]} = \frac{\text{m s}^{-1}}{\text{s}} = \text{m s}^{-2}$$
$$\text{force} = \text{mass} \times \text{acceleration} \quad \Rightarrow \quad [F] = [m][a] = \text{kg m s}^{-2}$$
$$\text{work} = \text{force} \times \text{distance} \quad \Rightarrow \quad [W] = [F][s] = \text{kg m s}^{-2} \times \text{m} = \text{kg m}^2\text{s}^{-2}$$

□

1.1.3 metric prefixes

prefixes are used to indicate multiples and sub-multiples of original units

name	symbol	meaning	name	symbol	meaning
pico	p	10^{-12}	hecto	h	10^{2}
nano	n	10^{-9}	kilo	k	10^{3}
micro	μ	10^{-6}	mega	M	10^{6}
milli	m	10^{-3}	giga	G	10^{9}
centi	c	10^{-2}	tera	T	10^{12}
deci	d	10^{-1}			

1.1.4 dimensional analysis

if an equation is correct, then the units on both sides must be the same. Such an equation with consistent units is said to be **homogeneous**.

dimensional analysis is widely used as a rough guide to check for the correctness of equations

there are times when the dependence of one physical quantity on various other quantities cannot not be seen easily, but it might give us helpful hints by merely investigating their units

➢ there are *unit free*, or *dimensionless* quantities that do not have units

examples are real numbers (2, $\frac{4}{3}$, π, etc.), coefficient of friction (μ), refraction index (n), etc.

➢ a correct equation must be homogeneous, but the converse may not be true

possible problems include an incorrect coefficient, extra term, an incorrect sign, etc.

Example 1.2 A ball falls in vacuum, all its gravitational potential energy converts into kinetic energy. This is expressed by the equation: $mgh = \frac{1}{2}mv^2$. Show that this equation is homogeneous.

✍ LHS: $[mgh] = [m][g][h] = \text{kg} \times \text{m s}^{-2} \times \text{m} = \text{kg m}^2\text{s}^{-2}$

RHS: $\left[\frac{1}{2}mv^2\right] = [m][v]^2 = \text{kg} \times (\text{ m s}^{-1})^2 = \text{kg m}^2\text{s}^{-2}$

so we see the equation $mgh = \frac{1}{2}mv^2$ is homogeneous □

Example 1.3 The speed of a wave travelling along an elastic string is determined by three things: the tension T in the string, the length L of the string, and the mass m of the string. Let's assume $v = T^aL^bm^c$, where a, b, c are some numerical constants. Find the values of a, b and c.

✍ RHS: $[T]^a[L]^b[m]^c = (\text{kg m s}^{-2})^a\text{m}^b\text{kg}^c = \text{kg}^{a+c}\text{m}^{a+b}\text{s}^{-2a}$

for the equation to be homogeneous, we must have:

$$\text{kg}^{a+c}\text{m}^{a+b}\text{s}^{-2a} = \text{m s}^{-1} \quad \Rightarrow \quad \begin{cases} \text{kg}: & a+c=0 \\ \text{m}: & a+b=1 \\ \text{s}: & -2a=-1 \end{cases} \quad \Rightarrow \quad \begin{cases} a=\frac{1}{2} \\ b=\frac{1}{2} \\ c=-\frac{1}{2} \end{cases}$$

so wave speed is given by: $v = T^{1/2}L^{1/2}m^{-1/2}$, or $v = \sqrt{\frac{TL}{m}}$

this happens to be the correct formula for the wave speed on a string □

1.2 scalars & vectors

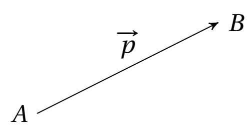
physical quantities come in two types: scalars and vectors

> a **scalar** quantity has magnitude only
>
> a **vector** quantity has magnitude and direction

- a scalar can be described by a single number

  examples of scalars are time, distance, speed, mass, temperature, energy, density, volume, etc.
- a vector is usually represented by an arrow in a specific direction

  a vector $\vec{p}$ pointing from A to B is shown

  length of the arrow shows the magnitude of the vector

  direction of the arrow gives the direction of the vector

  examples of vectors are displacement, velocity, acceleration, force, field strength, etc.
- scalar algebra is just ordinary algebra

  one can add and subtract scalar quantities in the same way as if they were ordinary numbers

  for example, a set of objects with mass $m_1, m_2, \cdots, m_n$ have a total mass of $M = m_1 + m_2 + \cdots + m_n$
- vector algebra is more complicated, since we need keep track of the direction

1.2.1 multiplication of vectors

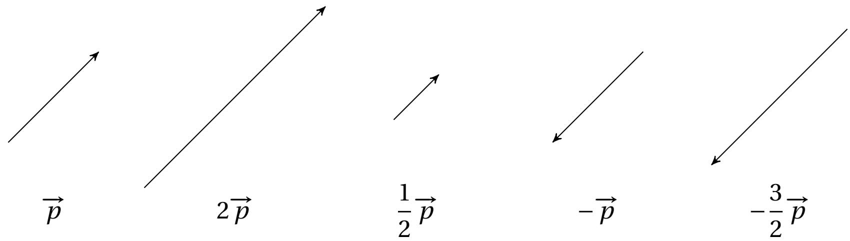
vectors can be multiplied by scalars [3]

when being multiplied by a scalar number, magnitude of the vector changes

if this number is positive, the vector becomes longer or shorter, but still points in same direction

if the number to be multiplied is negative, the operation reverses the vector's direction

Example 1.4 Given a vector $\overrightarrow{p}$, the graphical representations of $2\overrightarrow{p}$, $\frac{1}{2}\overrightarrow{p}$, $-\overrightarrow{p}$, $-\frac{3}{2}\overrightarrow{p}$ are:



1.2.2 addition of vectors

vectors can be added to form a **resultant** vector

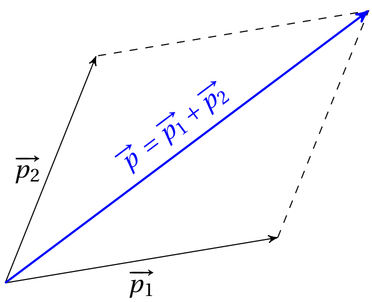
to deal with vector sums, we need take the directions of vectors into account

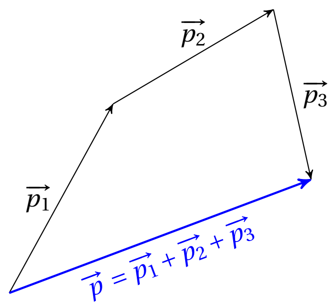
let's consider the sum of two vectors $\overrightarrow{p_1}$ and $\overrightarrow{p_2}$

resultant vector $\overrightarrow{p} = \overrightarrow{p_1} + \overrightarrow{p_2}$ lies on the diagonal of the parallelogram subtended by $\overrightarrow{p_1}$ and $\overrightarrow{p_2}$

this is called the **parallelogram rule** for vector addition

if the resultant of several vectors $\overrightarrow{p} = \overrightarrow{p_1} + \overrightarrow{p_2} + \cdots \overrightarrow{p_n}$ is to be found, one can join these vectors head-to-tail, the resultant is given by the arrow connecting the tail of $\overrightarrow{p_1}$ to the head of $\overrightarrow{p_n}$

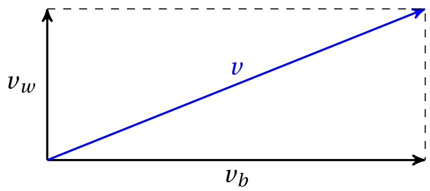




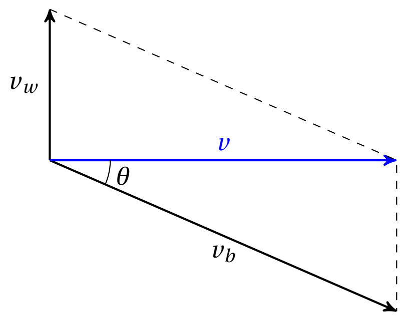
[3]It is also possible to multiply vectors with vectors, and there are basically two ways of doing vector multiplication: the *dot product* and the *cross product*. Both vector products are useful in physics, but we will not go into the details. You may learn more about vector multiplication in the A-Level mathematics course.

Example 1.5 A river flows from south to north with a speed of 2.0 m s^{-1} and the speed of a boat with respect to the water flow is 5.0 m s^{-1}. (a) Suppose the boat leaves the west bank heading due east, what is the resultant velocity of the boat? (b) If the boat is to reach the exact opposite bank across the river, what is the resultant velocity and in what direction should the boat be headed?

✍ vector diagrams for resultant velocity of the boat is illustrated below



(a) boat heading due south



(b) boat aiming at exact opposite bank

(a) magnitude of resultant velocity: $v = \sqrt{v_b^2 + v_w^2} = \sqrt{5.0^2 + 2.0^2} \approx 5.4$ m s^{-1}

in this case, the boat reaches the opposite bank in shortest time but will drift downstream

(b) magnitude of resultant velocity: $v = \sqrt{v_b^2 + v_w^2} = \sqrt{5.0^2 - 2.0^2} \approx 4.6$ m s^{-1}

in this case, the boat reaches the opposite bank in shortest distance

but the boat is headed slightly upstream: $\theta = \sin^{-1}\dfrac{v_w}{v_b} = \sin^{-1}\dfrac{2.0}{5.0} \approx 24°$ □

1.2.3 resolving vectors

one can also resolve a single vector into two (or more) components [4]

let's place a general 2D vector $\vec{p}$ in Cartesian coordinates

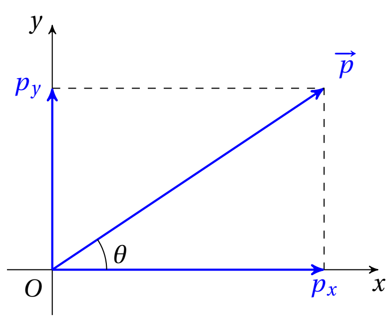
vector $\vec{p}$ can be split into two perpendicular components

– a horizontal component p_x

– a vertical component p_y

if $\vec{p}$ forms an angle θ to the x-axis, then:

$$p_x = p\cos\theta, \quad p_y = p\sin\theta$$
$$p = |\vec{p}| = \sqrt{p_x^2 + p_y^2}, \quad \tan\theta = \frac{p_y}{p_x}$$



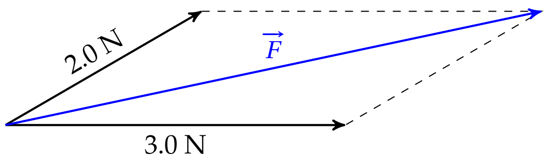
Example 1.6 A force of 3.0 N towards east and a force of 2.0 N towards 30° north of east act on

[4]This depends on the number of dimensions of space we are working with.

an object. Find the magnitude and the direction of the resultant force.

✍ suppose an arrow of length 1 cm represents a force of 1 N

one can draw a *scale diagram* with a ruler and a protractor as shown



one can find length of the resultant vector is about 4.8 cm

also it forms an angle of about 12° to the 3.0 N force

so resultant force is of 4.8 N acting towards 12° north of east

alternatively, one can find components of the resultant as the sum of individual components

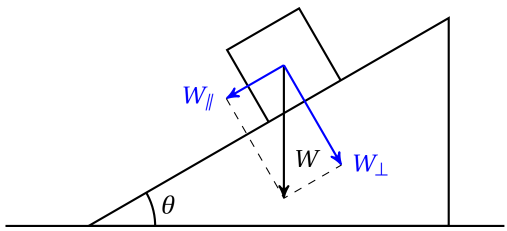
$$F_x = 3.0 + 2.0\cos 30^\circ \approx 4.73\text{ N}, \quad F_y = 2.0\sin 30^\circ = 1.0\text{ N}$$

magnitude and direction of the resultant can then be found from its components

$$F = \sqrt{F_x^2 + F_y^2} = \sqrt{4.73^2 + 1.0^2} \approx 4.84\text{ N}, \theta = \tan^{-1}\frac{F_y}{F_x} = \tan^{-1}\frac{1.0}{4.73} \approx 11.9^\circ$$

this of course agrees with scale diagram method, but resolving gives more precise results □

Example 1.7 A box of weight $W = 20.0$ N is resting on an inclined slope at 30° to the horizontal. Find the components of weight parallel to the slope and normal to the slope.



✍ the vector diagram is shown

component of weight parallel to slope: $W_{\parallel} = W\sin\theta = 20.0 \times \sin 30^\circ = 10.0$ N

component of weight normal to slope: $W_{\perp} = W\cos\theta = 20.0 \times \cos 30^\circ \approx 17.3$ N □

1.3 end-of-chapter questions

SI units

Question 1.1 What are the SI base units of (a) density, (b) pressure, (c) energy, (d) electric charge?

Question 1.2 For a substance of mass m, the heat energy Q needed to change its temperature by ΔT is given by: $Q = cm\Delta T$. Find the SI base units of the constant c.

dimensional analysis

Question 1.3 The resistive force F on a metal ball falling at low speeds in water is given by the equation $F = krv$, where r is the radius of the metal ball, v is its speed and k is a constant. Find the base units of k in the SI system.

Question 1.4 The speed of sound in air can be given by $c = \sqrt{\frac{\gamma p}{\rho}}$, where p is the pressure of the air and ρ is the air density. Show that γ is unit free.

Question 1.5 The effective power output from a wind turbine is given by the equation $P = \frac{1}{2}\eta\rho A v^n$, where ρ is the air density, A is the area of the turbine blades, and v is the wind speed. Given that η is a constant with no units, what is the value of n?

vector algebra

Question 1.6 An aircraft, which has a speed of 35 m s^{-1} in still air, is flying from south to north at a speed of 32 m s^{-1} with respect to a stationary observer on the ground. Find the magnitude and the possible directions of wind velocity.

Question 1.7 Three forces of 5.0 N, 12 N and 13 N act at one point on an object. The angles at which the forces act can vary. What is the maximum and the minimum resultant force?

Chapter 2
Measurements

2.1 uncertainties

physics is a practical science, any law of physics must be evidenced by experimental facts

any meaningful physical quantity is measured either directly or indirectly

but repeated readings may not give a consistent value, instead they show a *spread* of data

uncertainty gives the *range* of values in which *true value* of the measurement is asserted to lie

measurement of a particular quantity is usually reported as $x \pm \Delta x$, where reported value x is the average of repeated readings, and Δx is its uncertainty

2.1.1 absolute uncertainty

Δx measures the size of the range of values where true value probably lies

therefore Δx is called the **absolute uncertainty**

- ➢ absolute uncertainty Δx carries the same unit as quantity x
- ➢ absolute uncertainty can be worked out from *range* of readings

  range of a data set $x_1, x_2, x_3, \cdots$ is the difference between greatest and smallest value

  absolute uncertainty is given by: $\boxed{\Delta x = \frac{1}{2}(x_{\max} - x_{\min})}$
- ➢ absolute uncertainty is usually kept to one significant figure only [5]

[5] In some cases where the uncertainty of a quantity is not stated explicitly, the uncertainty is indicated by the number of significant figures in the stated value. If the height of a person is measured to be 1.75 m, this means the first two digits (1 and 7) are certain, while the last digit (5) is uncertain.

When you add or subtract numbers, the number of significant figures is determined by the location of the decimal place. For example, $1.11 + \underline{4.2} + 0.563 = 5.873$, the result should be written as 5.9. When you multiply or divided numbers, the result can have no more significant figures than the term with the fewest significant figures. For example, $1.35 \times 462 \times \underline{0.27} = 168.399$, the result should be written as 170.

However, in AS & A-Level physics, apart from the problems regarding uncertainties, it is allowed to give since Δx indicates where the readings start to get problematic

measured quantity x is kept to the same decimal place as Δx

for example, if value for the speed of an athlete is found to be $v = (8.16 \pm 0.27)$ m s^{-1}, the result, to an appropriate number of significant figures, should be kept as: $v = (8.2 \pm 0.3)$ m s^{-1}.

2.1.2 fractional & percentage uncertainty

ratio of absolute uncertainty to reported value, i.e., $\dfrac{\Delta x}{x}$, gives the **fractional uncertainty**

recording this ratio as a percentage number, this is known as the **percentage uncertainty**

➢ fractional and percentage uncertainty have no unit

➢ $\dfrac{\Delta x}{x}$ gives relative measure of spread of data, so it is also called the relative uncertainty

Example 2.1 A students measures the diameter of a cylindrical bottle with a vernier calliper. The measurements are taken from several different positions and along different directions. The readings she obtained are: 4.351 cm, 4.387 cm, 4.382 cm, 4.372 cm, 4.363 cm. What is the percentage uncertainty of her measurements?

✍ average value: $d = \frac{1}{5}(4.351 + 4.387 + 4.382 + 4.372 + 4.363) = 4.371$ cm

absolute uncertainty: $\Delta d = \frac{1}{2}(d_{\text{max}} - d_{\text{min}}) = \frac{1}{2}(4.387 - 4.351) = 0.036$ cm

result of measurement should be recorded as: $d = 4.37 \pm 0.04$ cm

percentage uncertainty: $\dfrac{\Delta d}{d} = \dfrac{0.036}{4.371} \approx 0.082\%$ □

2.1.3 propagation of uncertainties

in many situations, the quantity that we want to find cannot be measured directly

the quantity of interest has to be computed from other quantities

uncertainty of this calculated quantity would depend on two things:

– uncertainties of the raw data from which it is calculated,

– how calculated quantity is related to those original quantities

suppose quantities A and B are two measurables with uncertainty ΔA and ΔB

X is a quantity to be computed by taking their sum, difference, product or quotient

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one more significant figure that what is required. So in other sections of my notes where we do not keep track of the uncertainties, I could be a bit sloppy with the issue of significant figures when numerical values are worked out.

to evaluate uncertainty in X, we estimate the worst scenario, i.e., the greatest deviation from its reported value

addition: $S = A + B$

$$S_{\text{max}} = A_{\text{max}} + B_{\text{max}} = (A+\Delta A) + (B+\Delta B) = (A+B) + (\Delta A+\Delta B) = S + (\Delta A+\Delta B) \quad \Rightarrow \quad \boxed{\Delta S = \Delta A + \Delta B}$$

subtraction: $D = A - B$

$$D_{\text{max}} = A_{\text{max}} - B_{\text{min}} = (A+\Delta A) - (B-\Delta B) = (A-B) + (\Delta A+\Delta B) = D + (\Delta A+\Delta B) \quad \Rightarrow \quad \boxed{\Delta D = \Delta A + \Delta B}$$

multiplication: $P = AB$

$$P_{\text{max}} = A_{\text{max}} B_{\text{max}} = (A+\Delta A)(B+\Delta B) = AB + B\Delta A + A\Delta B + \Delta A\Delta B \quad \Rightarrow \quad \Delta P = B\Delta A + A\Delta B + \Delta A\Delta B$$

divide both sides by $P = AB$, we get

$$\frac{\Delta P}{P} = \frac{\Delta A}{A} + \frac{\Delta B}{B} + \frac{\Delta A}{A} \cdot \frac{\Delta B}{B} \quad \Rightarrow \quad \boxed{\frac{\Delta P}{P} = \frac{\Delta A}{A} + \frac{\Delta B}{B}}$$

percentage uncertainty of a measurable is usually a few percent, so $\frac{\Delta A}{A} \cdot \frac{\Delta B}{B} \approx 0$

so this piece is dropped from the last expression

division: $Q = \frac{A}{B}$

one can show that $\boxed{\frac{\Delta Q}{Q} = \frac{\Delta A}{A} + \frac{\Delta B}{B}}$

the derivation is left as an exercise for the reader

power & roots: $Q = A^l B^m C^n \cdots$

percentage uncertainty in Q is: $\frac{\Delta Q}{Q} = l\frac{\Delta A}{A} + m\frac{\Delta B}{B} + n\frac{\Delta C}{C} + \cdots$

this can be thought of as a generalization for multiplication and division operations

the proof is also left as an exercise to the reader

brief summary

– for addition and subtraction, *absolute uncertainties* add up

– for multiplication, division and powers, *percentage uncertainties* add up

➢ notice that uncertainties always add

Example 2.2 The resistance of a resistor is measured in an experiment. The current through the resistor is 1.8 ± 0.1 A and the potential difference across is 7.5 ± 0.2 V. What is the resistance of the resistor and its uncertainty?

value of resistance: $R = \frac{V}{I} = \frac{7.5}{1.8} \approx 4.17\ \Omega$

percentage uncertainty in resistance: $\frac{\Delta R}{R} = \frac{\Delta V}{V} + \frac{\Delta I}{I} = \frac{0.2}{7.5} + \frac{0.1}{1.8} \approx 8.2\%$

absolute uncertainty in resistance: $\Delta R = 8.2\% \times 4.17 \approx 0.34\ \Omega$

so we find resistance of the resistor: $R = 4.2 \pm 0.3\ \Omega$ □

Example 2.3 The density of a liquid is found by measuring its mass and its volume. The following is a summary of the measurements: mass of empty beaker = (20±1)g, mass of beaker and liquid = (100 ± 1)g, and volume of liquid = $(10.0 \pm 0.5)\mathrm{cm}^3$. What is the density of this liquid and the uncertainty in this value?

✍ mass of liquid: $m = m_2 - m_1 = 100 - 20 = 80$ g

uncertainty in mass: $\Delta m = \Delta m_2 + \Delta m_1 = 1 + 1 = 2$ g

density of liquid: $\rho = \frac{m}{V} = \frac{80}{10.0} = 8.00\ \mathrm{g\ cm^{-3}}$

percentage uncertainty in density: $\frac{\Delta \rho}{\rho} = \frac{\Delta m}{m} + \frac{\Delta V}{V} = \frac{2}{80} + \frac{0.5}{10.0} = 7.5\%$

absolute uncertainty in density: $\Delta \rho = 7.5\% \times 8.00 = 0.60\ \mathrm{g\ cm^{-3}}$

so density of this liquid is recorded as: $\rho = 8.0 \pm 0.6\ \mathrm{g\ cm^{-3}}$ □

Example 2.4 The period of simple pendulum is given by $T = 2\pi\sqrt{\frac{L}{g}}$. In an experiment, the length of string is measured to be $L = 100.0 \pm 0.5$ cm, and the time taken for 10 full oscillations is $t = 20.0 \pm 0.2$ s. What is the value for acceleration of free fall g and its uncertainty?

✍ period of one oscillation: $T = \frac{1}{10}t = 2.00 \pm 0.02$ s

let's rearrange $T = 2\pi\sqrt{\frac{L}{g}}$ into $g = \frac{4\pi^2 L}{T^2}$

acceleration of free fall: $g = \frac{4\pi^2 \times 100.0}{2.00^2} \approx 987\ \mathrm{cm\ s^{-2}}$

percentage uncertainty: $\frac{\Delta g}{g} = \frac{\Delta L}{L} + 2\frac{\Delta T}{T} = \frac{0.5}{100.0} + 2 \times \frac{0.02}{2.00} = 2.5\%$ [6]

absolute uncertainty: $\Delta g = 2.5\% \times 987 \approx 24.7\ \mathrm{cm\ s^{-2}}$

[6]Starting from the formula $T = 2\pi\sqrt{\frac{L}{g}}$, it is attempting to write $\frac{\Delta T}{T} = \frac{1}{2}\frac{\Delta L}{L} + \frac{1}{2}\frac{\Delta g}{g}$. But this would mean that T is a calculated quantity whose uncertainty is determined by the uncertainty in L and the uncertainty in g, which is incorrect. The right way to do it is to rearrange the formula so that calculated quantity of

so result of this measurement is: $g = 990 \pm 20 \text{ cm s}^{-2}$ □

2.2 errors of measurement

difference between the measured value and the true value is called **error**

total error is usually a combination of two components: systematic error and random error

2.2.1 systematic & random errors

systematic errors cause the readings to be greater or smaller than the true value by the same amount

➢ faulty equipments, biased observers, calibration errors could produce systematic errors

examples of systematic errors include:

– a vernier calliper does not read zero when fully closed, this introduces **zero error**

– one always reads a measuring cylinder from a higher angle, this introduces **parallax error**

– spring of force meter becomes weaker over time, so force meter always gives overestimates

➢ systematic errors can be reduced by using better equipments or methods

– one can check for zero error before taking readings with a micrometer

– one can *calibrate* a balance with a known mass before using it to measure mass of an object

random errors cause the readings to fluctuate above or below the actual value

➢ deviations caused by random error are unpredictable

➢ insensitive equipments, lack of observer precision, changes in environment, imprecise definitions could produce random errors

examples of causes of random errors include:

– **human reaction errors** when measuring a time quantity on a stop-watch

– electronic noise due to thermal vibrations of atoms when measuring an electric current

– when measuring length of a crack, different people could pick different end points

➢ random errors can be reduced by averaging the results from repeated measurements

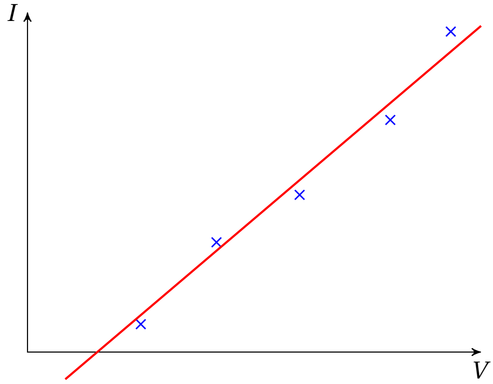
for example, diameter of a sphere can be measured along different directions and averaged

interest is made the subject of the working equation, the propagation of uncertainties then becomes explicit.

➢ random errors can also be reduced by using better equipment and better technique

for example, time for an object to fall can be measured with a light gate, instead of a stopwatch

Example 2.5 An experiment is carried out to measure the resistance of a metallic resistor, which is known to be constant throughout the experiment. A set of readings for voltage V across the resistor and the corresponding current I are obtained. A graph of I against V is plotted as shown. What can you say about the errors of the experiment?



✍ one can first draw a *best fit line* to see the distribution of data points

constant resistance means I should be directly proportional to V

so the best fit should be a straight line through the origin

but the best fit does not pass through origin, so there exists systematic error

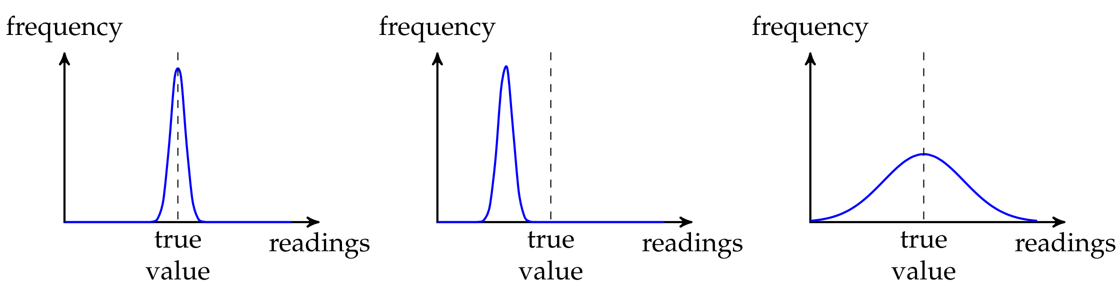
also data points scatter above and below the best fit, so random errors are present □

2.2.2 accuracy & precision

to analyse the result of an experiment, two important aspects are accuracy and precision

> measurement is said to be **accurate** if the result is close to the true value

> measurement is said to be **precise** if repeated readings are close to each other



(a) precise and accurate (b) precise but not accurate (c) accurate but not precise

distribution of readings with different precision and accuracy

➢ accuracy of a measurement is closely related to systematic errors

large systematic errors mean the results must be inaccurate

➢ precision of a measurement is closely related to random errors

large random errors cause repeated readings to spread, so the result must be imprecise

➢ precision is usually indicated by the percentage uncertainty of the measurement

similarly, precision is also indicated by the number of significant figures in a measurement

for example, metre rule has a precision of 0.1 cm, while micrometer has a precision of 0.001 cm

Example 2.6 The value for the acceleration of free fall is determined in an experiment. The result is reported to be $g = 14 \pm 5$ m s^{-2}. Is this result accurate? Is it precise?

✍ true value for g is around 9.8 m s^{-2}

stated value is not close to the true value, so the result is not accurate

percentage uncertainty in this result is $\frac{5}{14} \approx 36\%$, which is quite large

so the result is not precise either □

2.3 end-of-chapter questions

propagation of uncertainties

Question 2.1 A thermometer can measure to a precision of 0.5°C , what is the temperature rise from 20.0°C to 50.0°C and its uncertainty?

Question 2.2 The radius of a sphere is measured to be 5.0 cm with an uncertainty of 1%. What is the volume of this sphere and the uncertainty? (Volume of a sphere is given by: $V = \frac{4}{3}\pi R^3$.)

Question 2.3 The power radiated from a star of radius R and surface temperature T is given by the formula: $P = 4\pi\sigma R^2 T^4$, where σ is the Stefan–Boltzmann constant known to have the value of 5.67×10^{-8} W m^{-2} K^{-4}. If the sun is measured to have a surface temperature of 5800 ± 200 K and a diameter of $(1.40 \pm 0.03) \times 10^9$ m. (a) Find the radiation power P of the sun and its absolute uncertainty. (b) Suggest which measurement has a larger effect on the uncertainty in P.

Chapter 3
Kinematics

Kinematics is the study of motion. In this chapter, we define three useful kinematic quantities, displacement, velocity and acceleration, and use these terms to discuss the motion of an object.

3.1 kinematic quantities

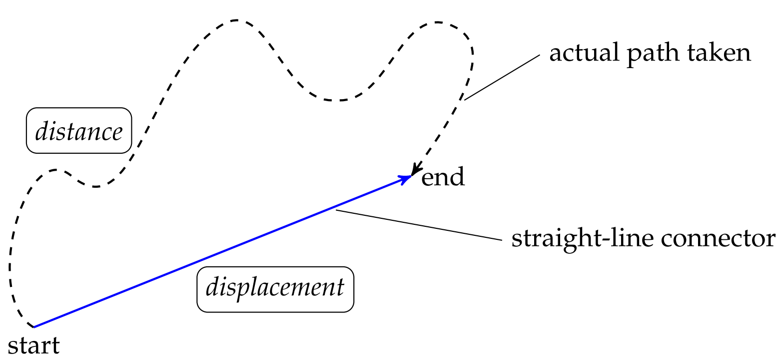
3.1.1 displacement & distance

in everyday language, we talk about the **distance** travelled by an object, which usually refers to the length travelled by an object without considering in what direction it moves

to fully describe position of an object, we also need specify where it moves

> we define **displacement** as the distance moved by an object in a specific direction

- displacement and distance are measured in metres, or any reasonable length units
- displacement is a vector quantity, while distance is a scalar
- displacement is the *straight-line* distance pointing from starting point towards end point
  even if actual path taken is curved, displacement is always the straight-line distance



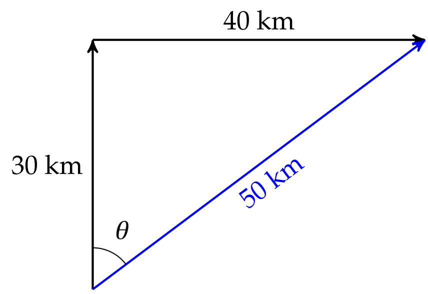
difference between displacement and distance

Example 3.1 An athlete is running around a circular track of radius 60 m. When he completes one lap, what is the distance moved out? What about his displacement?

✍ distance moved is the perimeter of the circle: $s = 2\pi r = 2\pi \times 60 \approx 380$ m

athlete returns to same starting point after one lap, so displacement is zero □

Example 3.2 A ship travels 30 km north, takes a right, and then travels 40 km east to reach its destination. Compare the distance and the displacement travelled.



✍ sum of all lengths gives distance: $30 + 40 = 70$ km

displacement vector is shown on the graph

magnitude of displacement $= \sqrt{30^2 + 40^2} = 50$ km

it is at an angle of $\theta = \tan^{-1}\frac{40}{30} \approx 53°$ east of north □

3.1.2 velocity & speed

displacement of a moving object may change with respect to time

an object is moving fast if it has a large change in displacement during a given time interval

change in displacement per unit time is called the **velocity** of the object: $\boxed{v = \frac{\Delta s}{\Delta t}}$

- SI unit of measurement for velocity: $[v] =$ m s^{-1}
- velocity is a vector quantity

  this follows from the fact that displacement is a vector quantity
- for *linear* motion, one shall pick a specific direction as the positive direction

  then a negative velocity implies motion in the opposite direction
- it is also common to use **speed** to describe how fast an object moves

  speed is defined as the change of the distance travelled per unit time

  velocity can be thought as speed in a particular direction
- defining equation $v = \frac{\Delta s}{\Delta t}$ gives the *average* value for velocity or speed over an interval Δt

  more precisely: $\boxed{\text{average velocity} = \frac{\text{total displacement}}{\text{time taken}}}$, and $\boxed{\text{average speed} = \frac{\text{total distance}}{\text{time taken}}}$

  this should be distinguished from the notion of *instantaneous velocity*

  instantaneous velocity is defined as the rate of change in displacement at a particular instant if we take a very short interval Δt, as $\Delta t \to 0$, average velocity tends to instantaneous velocity

this is expressed in a compact differential form: $v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} \Rightarrow \boxed{v = \frac{ds}{dt}}$

Example 3.3 A cyclist travels a distance of 3.0 km in 20 minutes. She rests for 15 minutes. She then covers a further distance of 5.1 km in a time of 40 minutes. Calculate the average speed of the cyclist in m s^{-1}: (a) during the first 20 minutes of the journey, (b) for the whole journey.

✍ for the first 20 minutes: $v = \frac{3.0 \times 10^3}{20 \times 60} = 2.5 \text{ m s}^{-1}$

for whole journey: $v = \frac{(3.0 + 0 + 5.1) \times 10^3}{(20 + 15 + 40) \times 60} = 1.8 \text{ m s}^{-1}$ □

Example 3.4 A man walks along a straight road for a distance of 800 m in 5.0 minutes. He then turns around, and walks along the same road for a distance of 280 m in 3.0 minutes. What is the average speed and the average velocity of this man during the 8.0 minutes?

✍ total distance travelled = 800 + 280 = 1080m, so average speed: $v = \frac{1080}{8.0 \times 60} = 2.25 \text{ m s}^{-1}$

change of displacement = 800 + (−280) = 520m, so average velocity: $v = \frac{520}{8.0 \times 60} \approx 1.08 \text{ m s}^{-1}$ □

Example 3.5 A maglev train travels at an average speed of 60 m s^{-1} from the city centre to the airport, and at 40 m s^{-1} on its return journey over the same distance. What is the average speed of the round-trip? What about the average velocity?

✍ suppose the distance between airport and city centre is S

average speed: $v = \frac{2S}{t_1 + t_2} = \frac{2S}{\frac{S}{60} + \frac{S}{40}} = 48 \text{ m s}^{-1}$

for a round-trip, train returns to same staring position

change in displacement is zero, so average velocity is zero □

3.1.3 acceleration

velocity of a moving object may change as well, i.e., objects can speed up or slow down

> change in velocity per unit time is defined as the **acceleration**: $\boxed{a = \frac{\Delta v}{\Delta t}}$

- unit of measurement for acceleration: $[a] =$ m s^{-2}
- acceleration is a vector quantity, it has both magnitude and direction

  this is because of vector nature of velocity, change in velocity must also have direction
- for *linear* motion, one usually pick direction of initial velocity as positive direction

  $a > 0$ would imply acceleration in the normal sense, i.e., motion with an increasing speed

$a < 0$ would imply deceleration, i.e., motion with a decreasing speed

➢ when velocity changes, it could be change in magnitude or/and change in direction [7]

for example, for an object moving along a curved path, its velocity is constantly changing direction, so it must have a non-zero acceleration

no acceleration would imply no change in speed and no change in direction of motion

➢ defining equation $a = \dfrac{\Delta v}{\Delta t}$ gives average acceleration over time interval Δt

we can likewise introduce *instantaneous acceleration* as the rate of change in velocity

taking the limit where the time interval $\Delta t \to 0$. we have: $a = \lim\limits_{\Delta t \to 0} \dfrac{\Delta v}{\Delta t} \Rightarrow \boxed{a = \dfrac{\mathrm{d}v}{\mathrm{d}t}}$

Example 3.6 A ball hits a barrier at right angles with a speed of 15 m s^{-1}. It makes contact with the barrier for 30 ms and then rebounds with a speed of 12 m s^{-1}. What is the average acceleration during the time of contact?

✍ note that direction of velocity changed during rebound, so $\Delta v = 15 - (-12) = 27$ m s^{-1}

average acceleration: $a = \dfrac{\Delta v}{\Delta t} = \dfrac{27}{30 \times 10^{-3}} = 900$ m s^{-2} □

3.2 motion graphs

how one physical quantity changes with another quantity can be visually shown on a *graph*

changes in displacement, velocity or acceleration over time can be shown on *motion graphs*

as we will see, s-t graph, v-t graph and a-t graphs are closely interrelated to one another

3.2.1 displacement-time graphs

a displacement-time graph shows an object's position at any given time

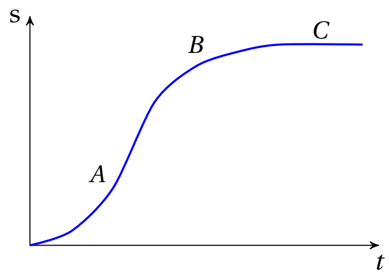
➢ gradient of tangent gives rate of change in displacement

but this is instantaneous velocity, which can be given by $v = \dfrac{\mathrm{d}s}{\mathrm{d}t}$, so we have:

$$\boxed{\text{velocity} = \text{gradient of } s\text{-}t \text{ graph}}$$

[7] Acceleration of an object can be considered as the combination of two components. One component is known as the *normal* acceleration or the *centripetal* acceleration, which is at right angle to the velocity and is responsible for the change in direction of motion. The other component is called the *tangential* acceleration, which is parallel to the direction of motion and causes change in magnitude of object's velocity. You will learn more about these in further mechanics.

Example 3.7 Describing the motion from the displacement-time graph shown.

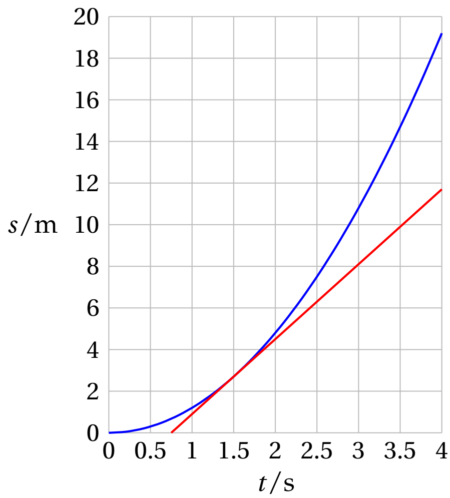


stage A: gradient of the graph is increasing, showing that the object is speeding up

stage B: gradient starts to decrease, so the object gradually slows down

stage C: curve becomes horizontal, gradient becomes zero, means that the object eventually comes to a stop □

Example 3.8 The diagram shows the displacement-time graph for a vehicle travelling along a straight road. Use the graph to find, (a) the average velocity during the first 4.0 s of the motion, (b) the velocity of the vehicle *at* time $t = 1.5$ s.



✍ during first 4.0 s, average velocity is

$$v = \frac{\Delta s}{\Delta t} = \frac{19.2}{4.0} \approx 4.8 \text{ m s}^{-1}$$

to find velocity at $t = 1.5$ s, a tangent is drawn

gradient of tangent gives instantaneous velocity:

$$v = \frac{11.6 - 0}{4.0 - 0.75} \approx 3.6 \text{ m s}^{-1}$$ □

3.2.2 velocity-time graphs

a velocity-time graph shows the velocity of a moving object at any instant

➢ since the rate of change of velocity gives the acceleration, so

$$\boxed{\text{acceleration} = \text{gradient of } v\text{-}t \text{ graph}}$$

➢ v-t graph also gives information about the change in displacement

$$\boxed{\text{change in displacement} = \text{area under } v\text{-}t \text{ graph}}$$

in very short time interval Δt_i, change in velocity is small so $v(t_i) \approx$ constant during this time

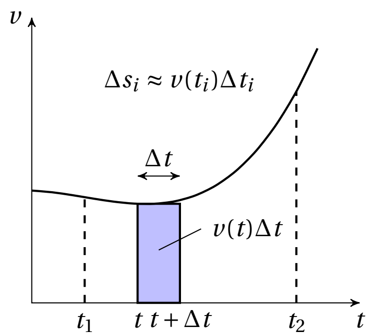
displacement moved out $\Delta s_i \approx v(t_i)\Delta t_i$, which corresponds to area of a thin rectangle

sum of all these small Δs_i's gives total change in displacement over a period of time

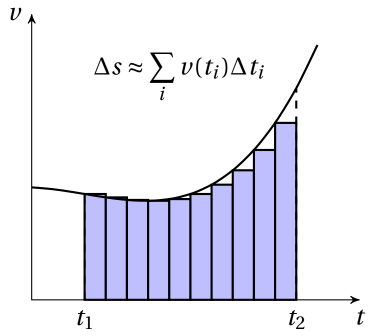
now consider the limit where each of the time interval $\Delta t_i \to 0$

total area of these rectangles approximates area bounded by the v-t curve and time axis [8]

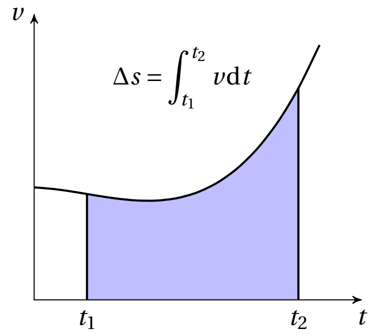
[8] Mathematically, integration is the inverse operation of taking derivatives. By definition $v = \frac{\mathrm{d}s}{\mathrm{d}t}$, then it follows naturally that $\Delta s = \int v\,\mathrm{d}t$. While the derivative of a given function gives the gradient of tangent at



(a) displacement Δs in short interval Δt



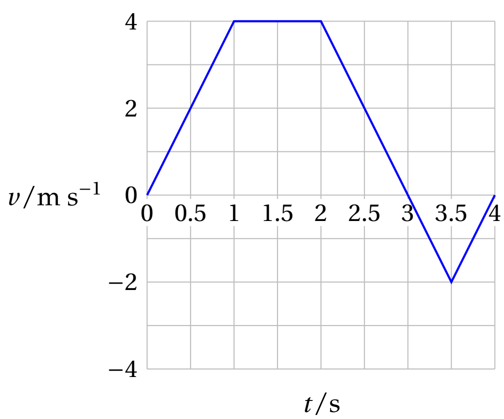
(b) total displacement estimated by summing the many Δs_i's



(c) total displacement as area under v-t graph

calculating change in displacement by finding the area under velocity-time graph

Example 3.9 The velocity of a toy car is shown. For the journey shown on the graph, use the graph to find (a) the total distance travelled, and (b) the total displacement travelled.



✍ distance is estimated using area under v-t graph

0~3.0 s: $s_1 = \frac{1}{2} \times 1.0 + 3.0 \times 4.0 = 8.0$ m

3.0~4.0 s: $s_2 = \frac{1}{2} \times 1.0 \times 2.0 = 1.0$ m

total distance $= 8.0 + 1.0 = 9.0$ m

total displacement $= (+8.0) + (-1.0) = 7.0$ m □

3.2.3 acceleration-time graphs

one can similarly plot an acceleration-time graph to give the changes in acceleration

➢ a-t graphs can give information about changes in velocity

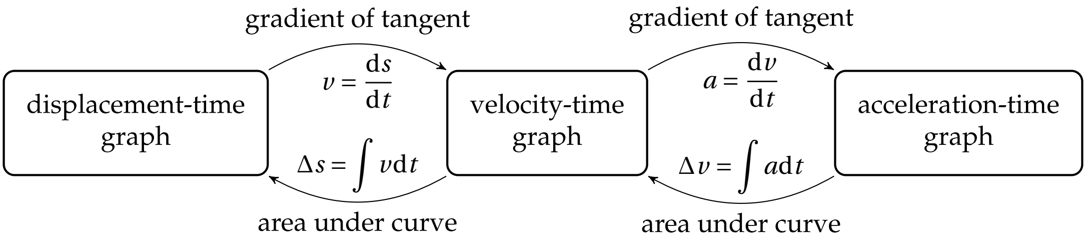
similar discussions lead to the following conclusion:[9]

$$\boxed{\text{change in velocity} = \text{area under } a\text{-}t \text{ graph}}$$

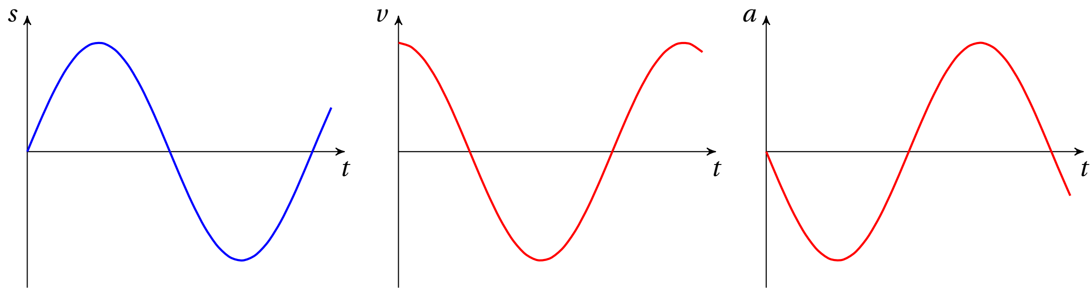
each point on its graph, integrating a function gives the signed area bounded by the graph. The reader may find the formal treatment of this relationship in any calculus textbook.

[9] Using area under a-t graph to find changes in velocity is not required in the AS-Level physics syllabus. I am putting this in the notes mainly for the completeness of the discussions on motion graphs.

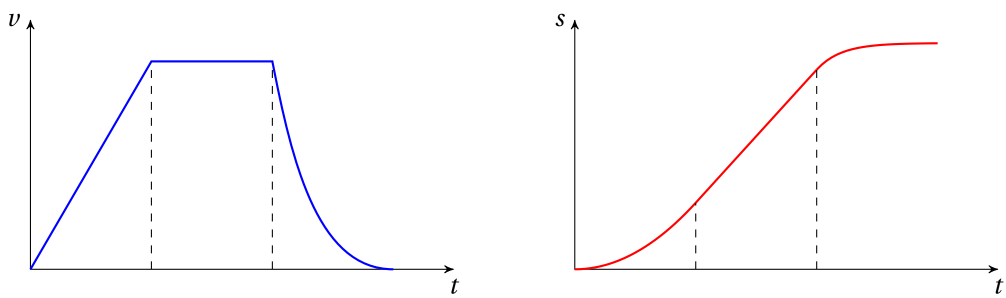
relationships between displacement, velocity and acceleration graphs are summarised below



Example 3.10 Given the displacement-time graph as shown, check yourself that this s-t graph leads to the velocity-time graph and the acceleration-time graph shown.



Example 3.11 Given the velocity-time graph as shown, check yourself that this v-t graph leads to the displacement-time graph as shown.



3.3 linear motion with constant velocity

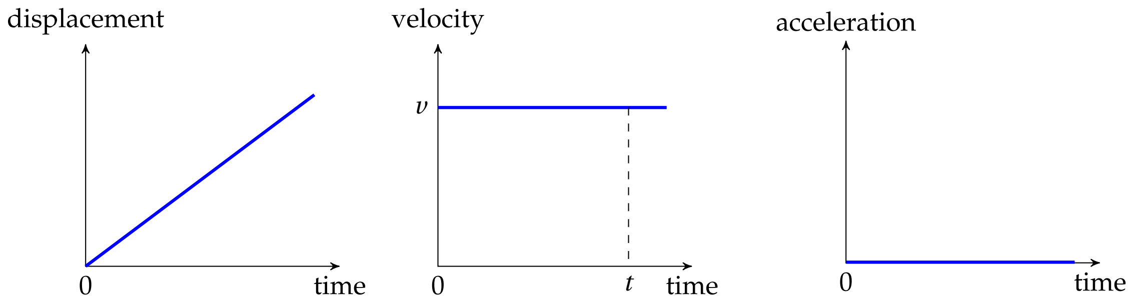
let's look at the simplest kind of motion

that is, an object moving at constant speed in a straight line: $v = \text{constant}$

the equations of motion are straightforward:

$$a = 0 \qquad \boxed{s = vt}^{[10]} \tag{3.1}$$

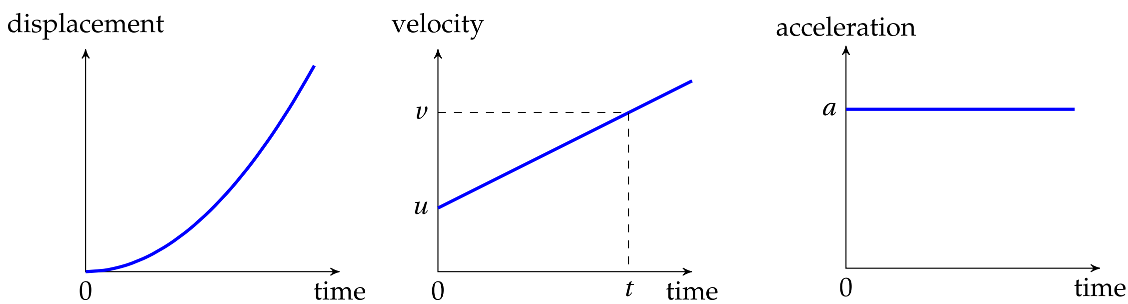
[10]It is implicitly assumed that the motion starts from the origin with respect to which displacement is



motion graphs for linear motion at constant velocity

3.4 linear motion with constant acceleration

the second simplest type of motion is a linear motion with acceleration $a = \text{constant}$



motion graphs for linear motion at constant acceleration

3.4.1 equations of motion

during a time interval t, suppose velocity changes from initial value u to final value v

from the defining equation of acceleration $a = \frac{\Delta v}{\Delta t} = \frac{v-u}{t}$, we get

$$\boxed{v = u + at}\text{[11]} \tag{3.2}$$

to find total displacement travelled, we compute the area under the v-t graph

$$\boxed{s = \frac{1}{2}(u+v)t} \tag{3.3}$$

for which we can interpret $\bar{v} = \frac{1}{2}(u+v)$ as the average velocity during that time

defined. More generically, we should write: $s = s_0 + vt$, where s_0 is the initial displacement.

[11]Proof with calculus: $\mathrm{d}v = a\mathrm{d}t \Rightarrow \Delta v = \int_u^v \mathrm{d}v = \int_0^t a\mathrm{d}t \Rightarrow v - u = at \Rightarrow v = u + at$

plug (3.2) into (3.3), we find an expression for the displacement travelled in terms of u and a:

$$\boxed{s = ut + \frac{1}{2}at^2} \text{[12][13]} \tag{3.4}$$

this shows the displacement s is a quadratic function in time t

this is consistent with the parabolic shape of the s-t graph shown

we can also eliminate the time variable t to derive one last equation

from (3.2) we have $t = \dfrac{v-u}{a}$, substitute this into (3.3), we find

$$s = \frac{1}{2}(u+v) \times \frac{v-u}{a} \quad \Rightarrow \quad \boxed{2as = v^2 - u^2} \tag{3.5}$$

Example 3.12 A car starts from rest and accelerates uniformly at 5.0 m s^{-2}for 6.0 s. (a) How fast is the car travelling at $t = 8.0$ s? (b) What is the distance travelled by the car in this time?

✍
$$v = u + at \quad \Rightarrow \quad v = 0 + 5.0 \times 6.0 = 30 \text{ m s}^{-1}$$
$$s = ut + \frac{1}{2}at^2 \quad \Rightarrow \quad s = 0 + \frac{1}{2} \times 5.0 \times 6.0^2 = 90 \text{ m}$$
□

Example 3.13 A car is travelling at 30 m s^{-1}. A hazard appears in front of the car, and the driver takes immediate action to stop the car. When brakes are applied, deceleration of the car is 5.0 m s^{-2}. What is the braking distance?

✍
$$2as = v^2 - u^2 \quad \Rightarrow \quad s = \frac{v^2 - u^2}{2a} = \frac{0^2 - 30^2}{2 \times (-5.0)} = 90 \text{ m}$$
□

Example 3.14 At the instant the traffic light turns green, a motorcycle waiting at the stop line starts with a constant acceleration of 2.0 m s^{-2}. At the same instant, a truck at a constant speed of 16 m s^{-1} overtakes and passes the motorcycle. How far beyond the stop line will the motorcycle overtake the truck?

✍ suppose overtake occurs at time t after motorcycle starts to accelerate

distance travelled by motorcycle: $s_m = u_m t + \frac{1}{2}at^2 \quad \Rightarrow \quad s_m = 0 + \frac{1}{2} \times 2.0 \times t^2$

distance travelled by truck: $s_t = v_t t \quad \Rightarrow \quad s_t = 16t$

overtake when $s_m = s_t \quad \Rightarrow \quad \frac{1}{2} \times 2.0 \times t^2 = 16t \quad \Rightarrow \quad t = 16$ s

substitute t into either s_m or s_t, one finds distance travelled: $s = 256$ m □

[12]Proof with calculus: $\mathrm{d}s = v\mathrm{d}t \Rightarrow \Delta s = \int_0^s \mathrm{d}s = \int_0^t v\mathrm{d}t \Rightarrow s = \int_0^t (u+at)\mathrm{d}t = \left(ut + \frac{1}{2}at^2\right)\Big|_0^t \Rightarrow s = ut + \frac{1}{2}at^2$

[13]Equation (3.4) assumes a zero initial displacement at $t = 0$. If there is a non-zero initial displacement, one should write $s = s_0 + ut + \frac{1}{2}at^2$. Similar discussion applies to equation (3.3).

3.4.2 free fall

a typical example of uniformly accelerated motion is the free fall

everything has the tendency to fall towards ground due to earth's gravity

in this section, we assume that effects of air resistance are negligible

acceleration of free fall is then a constant $a = g$, regardless of mass of falling object[14]

➢ near surface of earth, value of acceleration of free fall: $g \approx 9.81 \text{ m s}^{-2}$

value of g could be different on a different planet

➢ for a freely-falling object released from rest, its velocity increases with time as

$$v = u + at = 0 + gt \quad \Rightarrow \quad v = gt$$

the distance it has fallen from the point of release is

$$s = ut + \frac{1}{2}at^2 = 0 \cdot t + \frac{1}{2}gt^2 \quad \Rightarrow \quad s = \frac{1}{2}gt^2$$

Example 3.15 An object is released from rest from a height of $h = 24$ m and falls freely under gravity. Air resistance is negligible. (a) How long does it take to hit the ground? (b) What is its speed when hitting the ground?

✍

$$h = \frac{1}{2}gt^2 \quad \Rightarrow \quad t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2 \times 24}{9.81}} \approx 2.21 \text{ s}$$

$$v = gt = 9.81 \times 2.21 \approx 21.7 \text{ m s}^{-1}$$

□

Example 3.16 A photograph is taken for a small particle falling from rest. The photograph is taken at 0.400 s after the object is released. Since the particle is still moving when the photograph is being taken, the image is blurred. The blurred part is found to have a length of 20.8 cm. What is time of exposure for the photograph?

✍ from $t = 0$ to right before photo is taken:

$$s_1 = \frac{1}{2}gt_1^2 = \frac{1}{2} \times 9.81 \times 0.400^2 \approx 0.785 \text{ m}$$

from $t = 0$ to right after photo has been taken:

$$s_2 = s_1 + \Delta s = \frac{1}{2}gt_2^2 \quad \Rightarrow \quad t_2 = \sqrt{\frac{2(s_1 + \Delta s)}{g}} = \sqrt{\frac{2 \times (0.785 + 0.208)}{9.81}} \approx 0.450 \text{ s}$$

time of exposure: $\Delta t = t_2 - t_1 = 0.450 - 0.400 \approx 0.050$ s

□

[14]The reason for this constant acceleration of free fall will be elaborated in §4.3.1.

3.4.3 upward projection

like a freely-falling object, an object tossed upwards experiences the same constant downward acceleration $a = g \approx 9.81$ m s^{-2} as long as resistive forces can be ignored

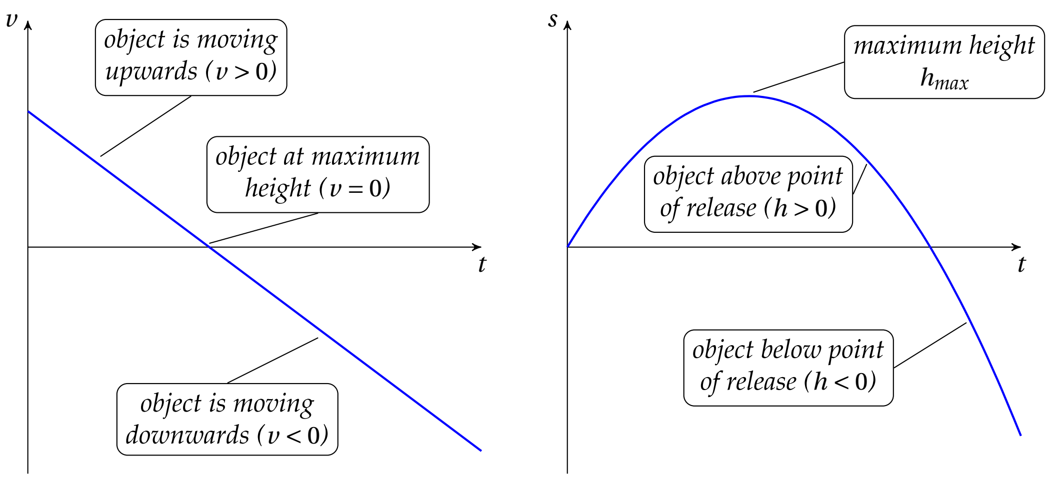
note that initial velocity u is upwards, but acceleration a is downwards so we will have different signs for u and a in the equations

conventionally, positive direction is taken as same direction as initial velocity

in our case, positive direction is upwards, the acceleration is then negative $a = -g$

so the velocity-time relation and displacement-time relation are

$$v = u - gt \qquad s = ut - \frac{1}{2}gt^2$$



v-t graph and s-t graph for upward projectile motion

➢ sign of v now gives direction of motion

$v > 0$ means object is moving upwards, $v < 0$ means it has reversed direction and starts falling

in particular, object attains greatest height when $v = 0$

➢ sign of s gives whether object is at a higher or lower position with respect to point of release

$s > 0$ means the object is above the position from which it is projected

$s < 0$ means it is now below the point of release

Example 3.17 A ball is projected vertically upwards at 12 m s^{-1}. Air resistance is negligible. (a) Find the time taken for the ball to reach the highest position. (b) Find the greatest height.

✍ maximum height is reached when $v = 0$, so

$$v = u - gt = 0 \quad \Rightarrow \quad t = \frac{u}{g} = \frac{12}{9.81} \approx 1.22 \text{ s}$$

$$H_{max} = ut - \frac{1}{2}gt^2 = 12 \times 1.22 - \frac{1}{2} \times 9.81 \times 1.22^2 \approx 7.34 \text{ m}$$

it is also possible to use $v^2 - u^2 = 2as$ to find H_{max}, this is:

$$0^2 - u^2 = 2(-g)H_{max} \quad \Rightarrow \quad H_{max} = \frac{u^2}{2g} = \frac{12^2}{2 \times 9.81} \approx 7.34 \text{ m}$$ □

Example 3.18 A stone is thrown vertically upwards with an initial velocity of 14.0 m s^{-1} from the edge of a cliff that is 35 m from the sea below. (a) Find the speed at which it hits the sea. (b) Find the time taken for the stone to hit the sea.

✍ take positive direction to point upwards, we use $v^2 - u^2 = 2as$ to find

$$v^2 = 14.0^2 + 2 \times (-9.81) \times (-35) \approx 883 \text{ m}^2 \text{ s}^{-2} \quad \Rightarrow \quad v \approx -29.7 \text{ m s}^{-1}$$[15]

to find time, we can use $v = u - gt$, hence: $t = \dfrac{v-u}{-g} = \dfrac{-29.7 - 14.0}{-9.81} \approx 4.46$ s

one can also attempt $s = ut - \frac{1}{2}gt^2$, this leads to the equation: $-35 = 14.0t - \frac{1}{2} \times 9.81t^2$

this quadratic equation in t gives two roots: $t_1 \approx 4.46$ s, and $t_2 \approx -1.60$ s

negative root should be discarded since it means stones hits the sea below it is thrown

so time taken for stone to hit the sea is $t \approx 4.46$ s □

3.5 motion in two dimensions – projectile motion

a **projectile** is an object whose motion is only affected by gravity

for projectile motion, we assume no air resistance and no other forces

gravity causes a constant acceleration of free fall that acts vertically downwards

➢ curved path of a projectile is the combination of its *horizontal* and *vertical* motion

– horizontally: no acceleration, so horizontal component of velocity v_x = constant

– vertically: constant acceleration, vertical component of velocity v_y varies over time

as a consequence, a projectile would follows a *parabolic* path as it travels[16]

let's consider a projectile launched at initial velocity u at angle θ to the horizontal

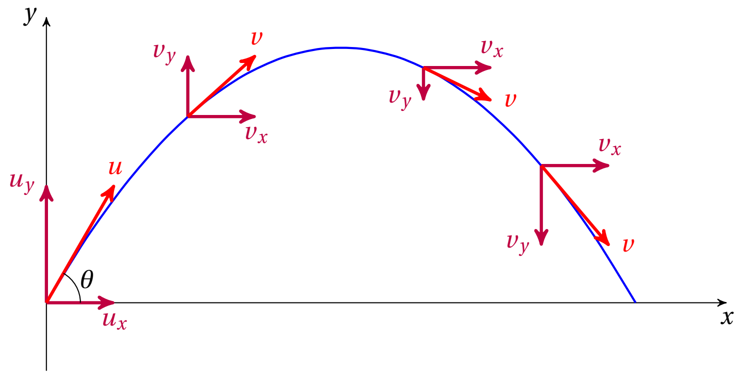
➢ horizontally, projectile maintains a constant velocity, so

$$v_x = u_x \qquad x = u_x t$$

where $u_x = u\cos\theta$ is horizontal component of initial velocity

[15]Note that we have substituted $a = -g$ since acceleration of free fall always points downwards, and $s = -35$ m since sea is below point of release. Also final velocity when hitting water is downwards, which should take a negative sign, so we discarded the positive solution for v.

[16]You may be able to prove this statement in Question 3.24.



components of the velocity of a projectile at different points along its path

- vertically, if upward direction is taken to be positive, then acceleration $a = -g$, so
$$v_y = u_y + at = u_y - gt \qquad y = u_y t + \frac{1}{2}at^2 = u_y t - \frac{1}{2}gt^2$$
where $u_y = u\sin\theta$ is vertical component of initial velocity
- components can be combined to give resultant velocity or resultant displacement:
$$v = \sqrt{v_x^2 + v_y^2} \qquad s = \sqrt{x^2 + y^2}$$

maximum height reached by an projectile

when a projectile reaches the highest position, its instantaneous vertical velocity becomes zero

we can then find the time it takes to attain this maximum height.
$$v_y = u\sin\theta - gt = 0 \quad \Rightarrow \quad t = \frac{u\sin\theta}{g}$$
to find $H_{\max}$, one can use either equation (3.3) or (3.4)
$$H_{\max} = \frac{1}{2}(u_y + v_y)t = \frac{1}{2}u_y t = \frac{1}{2} \times u\sin\theta \times \frac{u\sin\theta}{g} = \frac{u^2\sin^2\theta}{2g}$$
$$H_{\max} = u_y t - \frac{1}{2}gt^2 = u\sin\theta \times \frac{u\sin\theta}{g} - \frac{g}{2} \times \left(\frac{u\sin\theta}{g}\right)^2 = \frac{u^2\sin^2\theta}{2g}$$
- for the same initial speed u, the greater the angle of projection, the higher the object can get

  in the extremal case where $\theta = 90°$, it simply becomes an upward projection motion

airborne time and horizontal range of an projectile

a ball projected from the ground will first rise in height

but it will eventually fall to the ground due to the gravitational pull after a period of time T

when it lands, its vertical displacement is zero, so
$$Y = u_y T - \frac{1}{2}gT^2 = u\sin\theta T - \frac{1}{2}gT^2 = 0 \quad \Rightarrow \quad T = \frac{2u\sin\theta}{g}$$

the horizontal range is given by

$$X = u_x T = u\cos\theta \times \frac{2u\sin\theta}{g} = \frac{2u^2\sin\theta\cos\theta}{g} \quad \Rightarrow \quad X = \frac{u^2\sin 2\theta}{g}$$

where in the last step the trigonometric identity $\sin 2\alpha = 2\sin\alpha\cos\alpha$ has been used

➢ for same initial speed u, projectile launcher at greater angle stays in air for longer time

greatest airborne time is obtained if object is projected straight up, i.e., $\theta = 90^\circ$

➢ for same initial speed, horizontal range of projectile depends on angle θ of projection

to obtain the greatest horizontal range, two things are required

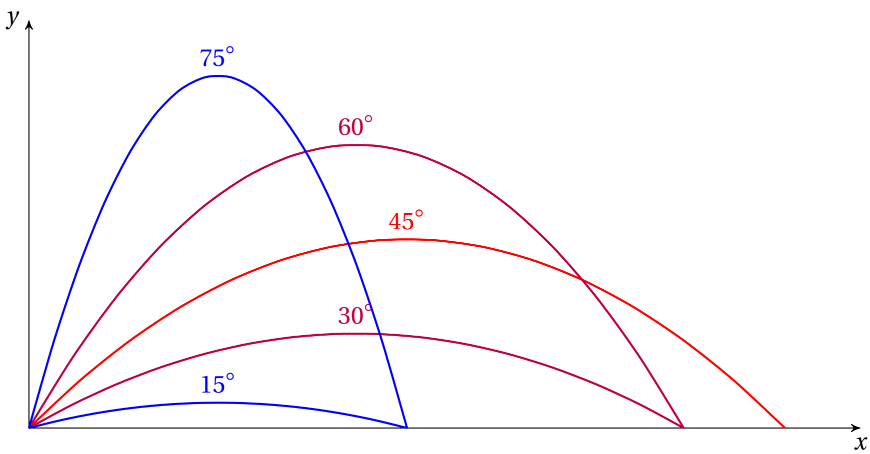
– sufficiently large horizontal velocity v_x

– sufficiently long time T staying in the air

however, a larger v_x requires a smaller θ, hence a shorter airborne time T

therefore, there is a compromise between the two

optimal angle should be neither be too large nor too small, which can be shown to be 45°



trajectories of projectiles launched at the same speed but different angles

Example 3.19 A ball is thrown from a point O at 15 m s^{-1} at an angle of 40° to the horizontal. The ball reaches its highest position at point P. Ignore the effects of air resistance. (a) How long does it take to reach P? (b) What is the magnitude of the displacement OP?

✍ at highest point: $v_y = u_y - gt = 0 \quad \Rightarrow \quad t = \frac{u\sin\theta}{g} = \frac{15\times\sin 40^\circ}{9.81} \approx 0.983$ s

vertical displacement: $y = u_y t - \frac{1}{2}gt^2 = 15\sin 40^\circ \times 0.983 - \frac{1}{2}\times 9.81 \times 0.983^2 \approx 4.74$ m

horizontal displacement: $x = u_x t = 15\cos 40^\circ \times 0.983 \approx 11.3$ m

resultant displacement: $|OP| = \sqrt{x^2+y^2} = \sqrt{11.3^2+4.74^2} \approx 12.2$ m □

Example 3.20 A small object is horizontally projected at 7.20 ms^{-1} from a surface at a height of $h = 1.2$ m above the ground. Assume there is no air resistance. (a) What is the time taken for the object to hit the ground? (b) What is the horizontal range? (c) Find the velocity at which the object hits the ground.

✍ vertically, take downward as positive: $h = \cancelto{0}{u_y t} + \frac{1}{2}gt^2 \Rightarrow t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2 \times 1.2}{9.81}} \approx 0.495$ s

horizontal range: $x = u_x t = 7.20 \times 0.495 \approx 3.56$ m

final vertical velocity: $v_y = \cancelto{0}{u_y} + gt = 9.81 \times 0.495 \approx 4.85\ \mathrm{m\,s^{-1}}$

magnitude of resultant velocity: $v = \sqrt{v_x^2 + v_y^2} = \sqrt{7.20^2 + 4.85^2} \approx 8.68\ \mathrm{m\,s^{-1}}$

angle to which resultant velocity makes with horizontal: $\phi = \tan^{-1}\frac{v_y}{v_x} = \tan^{-1}\frac{4.85}{7.20} \approx 34°$ □

3.6 end-of-chapter questions

kinematic quantities

Question 3.1 What is the distance covered for a car that travels half a lap along a circular path of radius of 200 m. What about the displacement?

Question 3.2 A ball is released from a height of 2.0 m above the ground. It bounces vertically for quite a number of times before coming to rest. (a) State the change of displacement for the ball. (b) Explain how the distance travelled is different from the change in displacement.

Question 3.3 For an athlete running around a track for many laps, suggest how his average velocity could be zero?

Question 3.4 A car travels 2400 m east in 3.0 minutes, then takes a left turn, and then travels 700 m north in 1.5 minutes. What is the average speed and the average velocity for this journey?

Question 3.5 Is it possible for an object moving at a steady speed to have acceleration?

motion graphs

Question 3.6 For the v-t graph given in Example 3.11, sketch the a-t graph for this motion.

Question 3.7 If the tangent of a displacement-time graph at one particular instant is sloping downwards, what does that imply about the velocity at that instant?

Question 3.8 A vehicle initially travels at a steady speed of 15 $\mathrm{m\,s^{-1}}$. It accelerates uniformly for 10 s to reach a higher speed of 20 $\mathrm{m\,s^{-1}}$. It maintains at this speed for 20 s, and then decelerates uniformly to a stop in the last 10 s. (a) Sketch the velocity-time graph for this motion. (b) Sketch the acceleration-time graph. (c) Find the distance travelled during the 40 s.

linear motion with constant velocity

Question 3.9 Sonar is a technique that uses sound waves to detect objects. It can be used to measure the depth of the seabed. Given that speed of sound in water is 1500 m s^{-1}, and reflected waves sent from a submarine are detected 0.50s after they are transmitted. How deep is the water below the submarine?

Question 3.10 Given that the speed of sound in air is 340 m s^{-1} and the speed of light in air is 3.0×10^8 m s^{-1}. If a person hears the sound of a thunder 5.0 seconds after seeing a lightning flash, how far away from this person is did the lightning strike?

linear motion with constant acceleration

Question 3.11 A train initially travels at a speed of 40 m s^{-1}. It starts to decelerate at 0.50 m s^{-2}. (a) What is the distance travelled in 50 s? (b) When it comes to a stop, how far out has it travelled?

Question 3.12 A vehicle moving at 14 m s^{-1} accelerate uniformly to 26 m s^{-1} in 6.0 s. (a) What is the average velocity during this time? (b) What is the acceleration during this time (c) What is the distance travelled by the vehicle? (d) The vehicle then braked with constant deceleration to stop in another 8.0 s. What is the distance travelled during the time when brakes are applied?

free fall

Question 3.13 Two balls are dropped from rest from the same height. The second ball is released 0.80 s after the first one. What is their separation 1.5 s after the second ball is dropped?

Question 3.14 A golf ball is dropped from the top of a tower of height 30 m. The ball falls from rest and air resistance is negligible. What time is taken for the ball to fall (a) the first 10 m from rest, (b) the last 10 m to the ground?

Question 3.15 The acceleration of free fall on Pluto is about one-fifteenth of that on Earth. If it takes a time of T for a rock to fall from rest a distance of S, what is the time taken, in terms of T, for a rock to fall from rest through the same distance S on Pluto?

Question 3.16 In an experiment is carried out to determine the acceleration of free fall g using a falling body. The body is released from rest from a height of h, the time taken t for it to hit the floor is measured. (a) Find the expression that can be used to calculate the value of g? (b) Suggest what could lead to an overestimation for the value of g.

upward projection

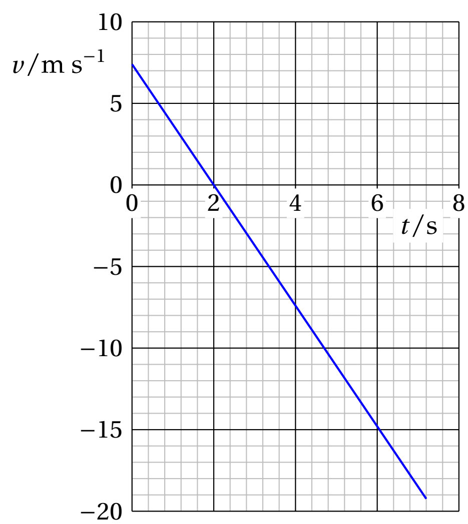
Question 3.17 A ball is tossed upwards with a speed of $9.0\ \mathrm{m\ s^{-1}}$. (a) How long does it take to return to the same point if air resistance is negligible? (b) How does the return velocity compare with its initial velocity?

Question 3.18 Someone wants to toss a ball onto a platform that is at a height of 20 m above him. What is the minimum initial velocity needed to launch the ball?

Question 3.19 Someone standing at the top of a high building throws a ball straight up and another ball straight down with the same initial speed. Assume that air drag is negligible, which ball will have a greater speed when it hits the ground?

Question 3.20 In basketball games, hang time refers to the length of time a player stays in the air after jumping from the floor. (a) It is reported that Michael Jordan is able to stay in the air for $T = 1.0$ s to do his slam-dunk tricks, estimate how high he can jump. (b) The acceleration of fee fall on the Moon is about one-sixth of that on Earth. What is the hang time of Michael Jordan if he takes off from the surface of the Moon?

Question 3.21 Mark Watney[17] stands at the edge of a cliff on the Mars and throws a rock vertically upwards with a speed of $7.4\ \mathrm{m\ s^{-1}}$. The graph shows the variation with the time t of the rock's velocity v. (a) What is the acceleration of free fall on the Mars? (b) When does the rock reach the maximum height? (c) What is the height above the base of the cliff the moment when the rock is thrown? (d) What is the maximum height above the base of the cliff to which the rock rises? (e) What is the total distance travelled by the rock before it strikes the ground?



[17] A fictional character in the science fiction movie *The Martian* (2015) based on the novel of the same name written by *Andy Weir*.

projectile motion

Question 3.22 A ball rolls off a table and lands at a position of a horizontal distance of 1.2 m from the table. The table is 0.95 m high. Find the speed at which the ball leaves the table.

Question 3.23 A ball is kicked from the ground towards a vertical barrier. The barrier is at a horizontal distance of 18 m from the initial position of the ball. The ball strikes the barrier after 1.5 s at a height of 2.5 m above the ground. (a) Find the magnitude and the direction of the initial velocity. (b) Find the magnitude and the direction of the velocity at which the ball hits the barrier.

Question 3.24 When a particle is launched from the origin at an angle θ with the horizontal at a speed of u, show that its trajectory is a parabola given by the equation: $y = x\tan\theta - \dfrac{gx^2}{2u^2\cos^2\theta}$.

Question 3.25 Two golf players each hit a ball at the same speed. One at 30° with the horizontal, the other at 60°. Which ball hits the ground first? Which ball goes farther?

Question 3.26 (a) State the difference between the displacement of a projectile and the distance it travels. (b) Suggest in what situation a projectile's displacement could have the same magnitude as the distance.

Question 3.27 An archer always aims slight higher than the distant target that she wants to hit. Why isn't the bow lined up such that it points exactly at the target?

Chapter 4

Force & Motion

4.1 force & motion: an introduction

in physics, a force appears when two bodies interact with one another

➢ you will encounter various types of forces in this course, some of which are

- *weight*: the gravitational attraction acting on any object exerted by the earth
- *tension*: a force in a string, a rope, a chain, etc. when it is being pulled
- *normal contact*: when a body's surface is compressed, there reacts a normal contact force[18]
- *friction*: a force that resists relative motion when two surfaces tend to slide over one another
- *resistance*: also called drag force, this is experienced when a body travels through a medium
- *upthrust*: an upward force acting on an object immersed in a fluid
- *electric force*: an attractive or repulsive interaction between electrically charged objects

detailed features of these forces follow later in the notes.

➢ a force can produce various effects to the object, the effect could be

- an increase/decrease in speed
- a change in the direction of motion
- causing the object to rotate
- a change in shape of the object

in this and the next few chapters, we will be looking into each of these aspects

➢ when more than one force act on a body, it is useful to find their *resultant*, or the *net force*

resultant force, or **net force**, is a single force that has the same effect as all forces acting on an object combined

vector sum of all of the individual forces gives the resultant force

[18] Examples of normal contact force are support force that stops a desk from sinking into the ground, and the impact on a football when you kick it, etc.

➢ in this chapter, we will study the dynamics of *point masses*

point mass is an idealization that the object has a mass but does not take up any space

position of an object treated as a point mass is specified with a geometric point in space

this is a simplification when size, shape, rotation, or structure of object are not important

4.2 Newton's laws of motion

Newton's laws of motion[19] are three laws that form the basis of classical mechanics

they describe the relationships between motion of an object and forces acting on it

4.2.1 first law

> **Newton's first law** states that an object continues in its state of rest or uniform motion at constant velocity if there is no resultant force acting

➢ any object 'dislikes' any change to its state of motion, uniform motion tends to persist forever

this tendency to resist changes in motion is called the **inertia**

Newton's first law is also called *the law of inertia*

➢ if there is no change in state of motion, the object is said to be in **equilibrium**

equilibrium could be either *static* (being at rest) or *dynamic* (steadily moving in a straight line)

both cases require zero resultant force

➢ Newton's law of inertia is placed to establish frames of reference

it is in an reference frame that notions of displacement, velocity and acceleration can be defined

an **inertial reference frame** is one in which Newton's laws hold [20]

[19] These three laws were first addressed by *Isaac Newton* in his famous work *Mathematical Principles of Natural Philosophy*, or simply the *Principia*. The three-volume work was first published in 1687, and was soon recognised as one of the most important works in the history of science. Apart the from the three laws that laid the foundations for classical mechanics, the *Principia* also stated *the law of gravitation*, and accounted for planetary orbits and tides and other phenomena.

[20] Inertial frame is not unique. An observer moving at constant velocity to an inertial observer is in a different inertial frame, since constant velocity of object added to a constant relative velocity is still a constant velocity. Two inertial observers would disagree on a body's velocity, but they would agree that the body maintains its velocity in absence of net force, i.e., they will observe the same physics phenomena. This is

4.2.2 second law

if resultant force is non-zero, velocity of the object will change, i.e., force produces acceleration

Newton's second law states that the acceleration is proportional to the resultant force and inversely proportional to the mass of the object

- symbolically, we write $a \propto \dfrac{F_{\text{net}}}{m}$

  with consistent units of measure, this proportionality can be written as an exact equation:

$$a = \frac{F_{\text{net}}}{m} \quad \text{or} \quad \boxed{F_{\text{net}} = ma} \tag{4.1}$$

- SI unit of measurement for force F is **newton** (N)

  a force of one newton acting a body of 1 kg produces an acceleration of 1 m s^{-2}
- note that the force in the equation $F = ma$ is the resultant force

  to determine change in motion for a body, you should always ask what the resultant force is
- acceleration produced is always in same direction of the net force
- for same force, an object of greater mass has a smaller acceleration

  hence mass is a measure of the *inertia* of this object in response to a net force

  a definition for mass of an object from the point of view of Newton's laws can be stated as[21]:

mass is an intrinsic property of a body to resist any change in its state of motion

Example 4.1 A box of 6.0 kg is being pushed along a horizontal surface with a force of 30 N. The resistive force acting is 21 N. What is the acceleration of the box?

✍ $$F_{\text{net}} = F - f = ma \quad \Rightarrow \quad a = \frac{F - f}{m} = \frac{30 - 21}{6.0} = 1.5 \text{ m s}^{-2}$$ □

Example 4.2 A car of mass 800 kg is travelling at a speed of 20 m s^{-1}. The driver then operates the brake pedal so a braking force of 2000 N gradually brings the car to stop. (a) What is the deceleration for the car? (b) What is the stopping distance?

---

known as the *equivalence principle*.

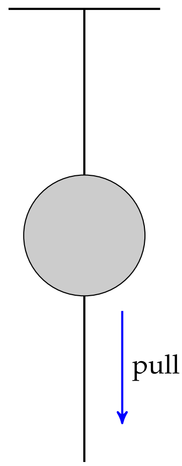
[21]The concept of mass can be defined in many different ways. You might be familiar with the definition for mass as the amount of matter an object possesses. I personally think this definition is a bit vague and does not tell you anything new. Thinking of mass as a measure of inertia surely brings more insights. Mass also tells the strength at which an object interacts with other bodies through the gravitational attraction. As you will see later, from the view of Albert Einstein, it is also possible to think of mass as a form of energy, which is my favourite definition for mass.

✍ using Newton's second law and noticing braking force acts opposite to direction of motion:

$$F_{\text{net}} = ma \quad \Rightarrow \quad -2000 = 800 \times a \quad \Rightarrow \quad a = -2.5 \text{ m s}^{-2}$$

$$2as = v^2 - u^2 \quad \Rightarrow \quad s = \frac{v^2 - u^2}{2a} = \frac{0^2 - 20^2}{2 \times (-2.5)} = 80 \text{ m}$$ □

Example 4.3 A massive ball is suspended on a string. A second string is attached to the bottom of the ball. If one pulls the bottom string with a gradually increasing force, does the top string or the bottom string break first? What if the bottom string is jerked, which string breaks?



✍ when tension gradually increases, system is always in equilibrium

tensions in strings must have $T_{\text{top}} = T_{\text{bottom}} + mg$

top string suffers a greater force, so it breaks first

however, when bottom string is jerked, the ball tends to remain at rest due to its large mass, preventing sudden change to the tension in top string

so in this case bottom string is more likely to snap □

4.2.3 third law

every force is part of a pair of interactions between one body and another

when one body exerts a force on another, the second body also exerts a reaction on the first

> **Newton's third law**, also called the **action-reaction principle**, states that action and reaction are always equal in magnitude, opposite in directions and of the same type

Example 4.4 Suggest the action and reaction force in the following cases: (a) A man stands on a bathroom scale. (b) A helicopter hovers in air. (c) The earth orbits around the sun.

✍ (a) man exerts downward force on scale, scale exerts an upward reaction on man

(b) rotors of helicopter push air downwards, air exerts an upward force on helicopter

(c) sun pulls the earth through gravitational attraction, earth also attracts the sun in return □

4.2.4 force analysis & free-body diagrams

when doing mechanics problems, it is necessary to find all forces applied upon an object

to visualise all these forces, it is helpful to draw a **free-body diagram** (FBD)

an FBD shows a simplified version of the body with arrows indicating forces applied it is recommended to follow the routine stated below when solving a mechanics problem

(1) draw a FBD for the object in the problem

(2) resolve and find the resultant force with aid of the FBD

(3) apply Newton's laws to write down the equation of motion for the object

(4) solve the equation(s) to find acceleration

(5) use kinematic relations to deduce information about motion of the object

4.3 types of forces

4.3.1 weight

all objects exert attractive forces of gravity upon each other[22]

weight of a body is due to the gravitational pull from our planet – the earth

weight W of any object is proportional to its mass m: $\boxed{W = mg}$

g is called the gravitational field strength, or the gravitational acceleration constant

➢ at vicinity of earth's surface, gravitational field is almost uniform: $g \approx 9.81 \text{ N kg}^{-1}$

but this value for g does not hold in a satellite orbit, on Mars, near a black hole, etc.

➢ the concept of weight is different from mass in many aspects

- weight is a force, so it is a vector (always acting downwards still makes a direction)
  mass is a scalar, it has magnitude only
- weight is measured in newtons, mass is measured in kilograms
- weight of object depends on its mass but also strength of gravitational field
  mass is an intrinsic property of object, so does not depend on force fields
  same object can have different weights on different planets, but its mass will be the same[23]

Example 4.5 An astronaut finds that he weighs 300 N on the surface of Mars, where the gravitational field strength is known to be 3.7 N kg^{-1}. Find his mass and hence estimate his weight if he returns to his home on the Earth.

✍ mass of astronaut: $m = \dfrac{W_M}{g_M} = \dfrac{300}{3.7} \approx 81.1 \text{ kg}$

weight on earth: $W_E = mg_E = 81.1 \times 9.81 \approx 795 \text{ N}$ □

[22] You will learn more about gravitational forces at A2 Level.

[23] Here we do not take into account the effects of *relativity*. A clever student who has learned Einstein's theories might suggest the mass of the same object increases with its velocity.

free fall

all things on the earth fall because of the force of gravity

if we ignore the restraints such as air resistance and upthrust force on a falling object, say the object is under the influence of gravity only, then the object is in a state called **free fall**

assuming the object is subject to gravity only, the resultant force is simply its weight

applying the Newton's second law, we have: $F_{\text{net}} = W \quad \Rightarrow \quad ma = mg$

so acceleration of the freely-falling object is: $\boxed{a = g}$[24]

➢ this shows **acceleration due to free fall** is simply equal to field strength g

so any object, regardless of its mass, has same acceleration due to free fall[25]

4.3.2 drag

when a body moves through air, water or any fluid, it experiences resistance called drag force

➢ factors that determine the value of fluid drag include

- relative speed of the object to the fluid ($v \nearrow \Rightarrow f \nearrow$)
- cross section of the object ($A \nearrow \Rightarrow f \nearrow$)
- shape of the object (streamlined shape has smaller drag)
- density of the fluid ($\rho \nearrow \Rightarrow f \nearrow$)

but what determines the drag force is a complicated issue[26]

---

[24]In the derivation, the mass terms cancel out. Rigorously speaking, these are two different masses. One is the measure of inertia, and the other is a measure of gravitational force. It is experimentally found that the inertia mass and the gravitational mass are equal. The fact that the two masses are equal has profound reasons. We have shown here acceleration of free fall equals gravitational field strength, but Albert Einstein's suggests that it is actually impossible to distinguish between a uniform acceleration and a uniform gravitational field. This idea lies at the heart of his *general theory of relativity*. Those who are interested in this topic are recommended to start from here and do some online researches.

[25]In §3.4.2 and §3.5, the statement that acceleration of free fall is constant in absence of air resistance was asserted without further explanation. Now you know why.

[26]There are a few empirical formula for drag force, each of which is accurate under certain conditions.

For an object moving through a fluid at low speeds (*laminar flow*, no turbulence occurs), the resistance it experiences is proportional to its speed: $f = bv$, where b is some constant which depends on fluid viscosity and the effective cross-sectional area of the object.

If objects are moving at relative high speeds through the fluid such that *turbulence* is produced behind the

➢ drag force always acts in a direction to oppose relative motion of object through fluid

free fall through air

let's consider an object falling through air from a very high tower

forces acting are weight and air resistance (shown in the free-body diagram)

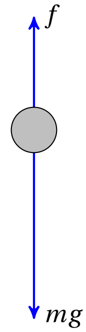
equation of motion for this falling object is:

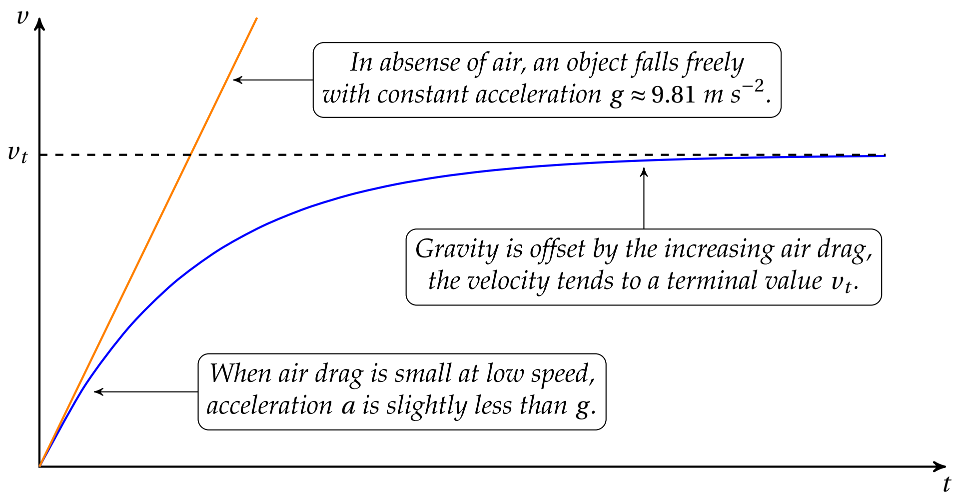
$$F_{\text{net}} = mg - f = ma$$

as v increases, air resistance f increases, so net force F_{net} decreases

this means acceleration a would decrease as object falls

i.e., speed will increase at a decreasing rate during the fall [27]





variation of velocity for a falling object through air

object, drag force is proportional to the speed squared: $f = \frac{1}{2}\rho C_D A v^2$, where ρ is the fluid's density, A is the cross-sectional area, C_D is a dimensionless quantity called the drag coefficient.

[27]The velocity-time relation can be obtained for some simple models. Suppose air resistance is proportional to speed of the falling body, i.e., $f = bv$, then the equation of motion reads: $F_{\text{net}} = m\frac{\mathrm{d}v}{\mathrm{d}t} = mg - bv$, where acceleration is written explicitly as the rate of change in velocity. With the initial conditions $v = 0$ at $t = 0$, we can solve this differential equation to obtain the speed of this falling object at any given time t:

$$\mathrm{d}t = \frac{\mathrm{d}v}{g - \frac{b}{m}v} \quad \Rightarrow \quad \int_0^t \mathrm{d}t = \int_0^v \frac{\mathrm{d}v}{g - \frac{b}{m}v} \quad \Rightarrow \quad t = -\frac{m}{b}\ln\left(g - \frac{b}{m}v\right)\Bigg|_0^v = -\frac{m}{b}\ln\left(1 - \frac{bv}{mg}\right)$$

Rearrange the terms, we find: $v(t) = \frac{mg}{b}\left(1 - \mathrm{e}^{-\frac{bt}{m}}\right)$

➢ after sufficient long time, acceleration gradually decreases to zero

velocity gradually increases and tends to a maximum value

at this stage, equilibrium is restored: $f = mg$, object no longer accelerates

this constant final velocity is known as the **terminal velocity**

➢ at low speeds, air resistance is negligible, so $F_{\text{net}} = ma \approx mg$

acceleration of object at start of the fall is similar to g

but as v increases, acceleration decreases so a becomes less than g

Example 4.6 An object of 5.0 kg falls through the atmosphere from a very high altitude. After some time, it falls at a constant speed of 70 m s^{-1}. Assume there is no significant change in gravitational field during the fall and the air resistance is proportional to speed: $f = bv$. (a) Find the value of the coefficient k. (b) Find the acceleration of the object when it is falling at 30 m s^{-1}.

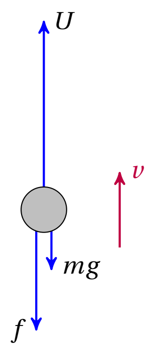
✍ equilibrium between weight and air drag when falling at terminal speed, so

$$mg = bv_t \quad \Rightarrow \quad b = \frac{mg}{v_t} = \frac{5.0 \times 9.81}{70} \approx 0.70 \text{ kg s}^{-1}$$

at any instant, equation of motion is: $F_{\text{net}} = ma = mg - bv$

at 30 m s^{-1}, acceleration is: $a = \dfrac{mg - bv}{m} = \dfrac{5.0 \times 9.81 - 0.70 \times 30}{5.0} \approx 5.6 \text{ m s}^{-2}$ □

bubble rising in a liquid



let's now consider bubbles formed at the bottom of a soda water

forces acting on bubble are weight, water resistance and upthrust

equation of motion for the rising bubble is:

$$F_{\text{net}} = U - mg - f = ma$$

as bubble moves faster, f increases, then F_{net} decreases

so acceleration a would gradually decrease to zero as bubble rises

speed of bubble increases and reaches a maximum value

at terminal speed, $a \to 0$, one has: $U = f + mg$

4.3.3 normal contact

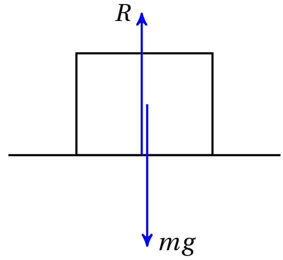
when two objects are in contact, the interaction between them is called the *contact force*

normal contact force is the component of contact force that is perpendicular to contact surface

➢ by definition, normal contact is always at right angle to surface of contact

➢ origin of normal contact is the *electrostatic interaction* between atoms when two objects are pressed against each other, surface atoms get close

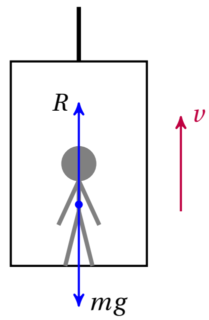
electrostatic repulsion between electron clouds of the atoms prevent them from penetrating through one another



Example 4.7 A box of mass $m = 4.0$ kg is resting on a horizontal ground. What is the normal contact force acting?

✍ equilibrium between weight and normal contact, so

$$R - W = 0 \quad \Rightarrow \quad R = mg = 4.0 \times 9.81 \approx 39.2 \text{ N} \qquad \square$$



Example 4.8 A man of 80 kg stands in a lift. Find his apparent weight, i.e., the contact force, when the lift is (a) moving upwards at steady speed of 2.0 m s^{-1}, (b) accelerating upwards at 2.0 m s^{-2}, (c) moving upwards but slowing down at a deceleration of 1.5 m s^{-2}.

✍ forces acting on man are weight and normal contact

for either case, equation of motion for the man reads:

$$F_{\text{net}} = ma = R - mg$$

so normal contact force: $R = mg + ma$

when rising at steady speed, man is in equilibrium ($a = 0$), so: $R = mg = 80 \times 9.81 \approx 785$ N

when accelerating upwards ($a = +2.0$ m s^{-2}): $R = 80 \times 9.81 + 80 \times 2.0 \approx 945$ N

when decelerating upwards ($a = -1.5$ m s^{-2}): $R = 80 \times 9.81 + 80 \times (-1.5) \approx 665$ N $\square$

Example 4.9 A sleigh of mass 15 kg lies at rest on an icy ground. The surface is frictionless. A force P of 75 N is applied to the sleigh. Find the normal contact force and the acceleration of the sleigh if P is acting (a) horizontally, (b) at an angle α to the horizontal where $\tan\alpha = \frac{3}{4}$.

✍ free-body diagrams for both cases are shown

for both cases, no net force acts in vertical direction

net force in horizontal direction provides acceleration

when P acts horizontally:

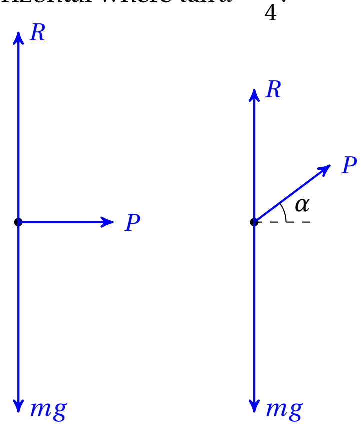
$$R = mg = 15 \times 9.81 \approx 147 \text{ N}$$

$$P = ma \quad \Rightarrow \quad a = \frac{P}{m} = \frac{75}{15} = 5.0 \text{ m s}^{-2}$$

when P acts at angle α:

$$R + P\sin\alpha = mg \quad \Rightarrow \quad R = 15 \times 9.81 - 75 \times \frac{4}{5} \approx 87 \text{ N}$$

$$P\cos\alpha = ma \quad \Rightarrow \quad a = \frac{P\cos\alpha}{m} = \frac{75 \times \frac{3}{5}}{15} = 3.0 \text{ m s}^{-2} \quad \square$$



(a) (b)

4.3.4 friction

friction is the component of contact force that is parallel to contact surfaces

when there is potential or actual sliding between surfaces, frictional force come into action

– for surfaces *tend* to move relative to each other, **static friction** acts to oppose this tendency

– if surfaces are already sliding over one another, then **kinetic friction** opposes this motion

➢ static friction f_S is self-adjusting

an object placed on a rough surface can stay at rest when acted by a small external force F

it can do so because f_S equalises external force to maintain equilibrium

if no external force acts, then $f_S = 0$

➢ there exists a maximum limiting friction f_{lim}

when external force $F < f_{\text{lim}}$, there is sufficient f_S to prevent object from sliding

when $F = f_{\text{lim}}$, object is on the verge of sliding

when $F > f_{\text{lim}}$, object start to move and static friction becomes kinetic friction f_K

➢ factors that determine frictional forces are

– nature of contacting surfaces (for both f_S and f_K)

– normal reaction R (for f_K)

these dependences are usually expressed by a mathematical equation $\boxed{f_K \approx f_{\text{lim}} = \mu R}$

μ is the *coefficient of friction* whose value depends on the nature of the two surfaces[28]

➢ friction, on microscopic level, is an *electromagnetic force* in nature

when two surfaces are in contact, irregularities on the surface touch each other

surface atoms come very close and bonds are formed through electrostatic force

in some sense, surface atoms get *cold welded* to each other

when surfaces try to move relative to each other, this electrostatic weld is origin of friction

4.4 inclined slope

inclined slope is probably the entry ticket into the business of mechanics

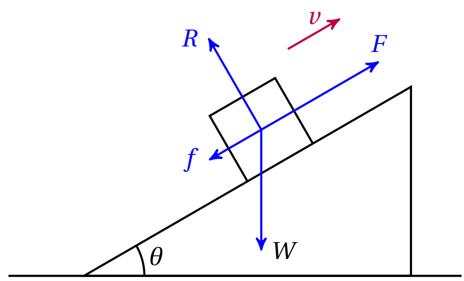
this notorious problem is found in any physics textbook and any exam paper on mechanics

[28]The idea of limiting friction is not required in the AS Physics syllabus, but it is required in *Mechanics 1* of the A-Level Mathematics course.

the problem is about a mass m placed on a plane inclined at angle θ to the horizontal

the mass could sit at rest on, slide down, or get pulled/pushed up the plane

motion of the mass could be affected by weight, friction, normal contact, or other forces



➢ forces can be resolved in directions parallel and perpendicular to the slope

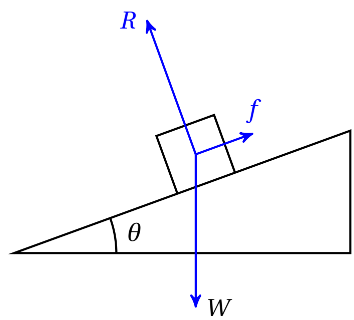
resolving along the slope leads to the equation of motion from which acceleration is found

➢ it is almost inevitable to break weight into two components[29]

– component of weight parallel down the slope is: $W_{\parallel} = mg\sin\theta$

– component of weight perpendicular to slope is: $W_{\perp} = mg\cos\theta$

Example 4.10 A block of mass m stays at rest on an inclined plane. The plane makes an angle θ with the horizontal. Find the normal contact force R and the frictional force f acting on the block.

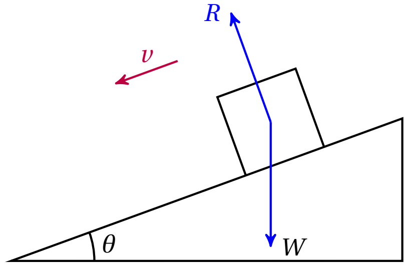


✍ block in equilibrium, so $F_{\text{net}} = 0$ in any direction

parallel to slope: $f = W_{\parallel} \quad\Rightarrow\quad f = mg\sin\theta$

normal to slope: $R = W_{\perp} \quad\Rightarrow\quad R = mg\cos\theta$ □

Example 4.11 A block of mass m slides down a *smooth* slope. The angle of the slope to the horizontal is θ. Find the acceleration of the block.



✍ only force acting along the slope is component of weight down the slope, so:

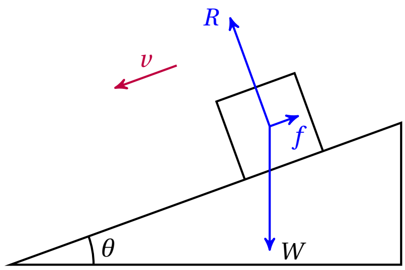
$$F_{\text{net}} = ma = mg\sin\theta \quad\Rightarrow\quad a = g\sin\theta$$

as $\theta \to 0$, $a \to 0$, this shows if plane becomes horizontal, the block simply stays put

as $\theta \to 90^{\circ}$, slope becomes vertical, block would undergo free fall, so naturally $a \to g$ □

[29]We have already done that in Example 1.7.

Example 4.12 A block of mass 2.0 kg slides down a rough slope from rest. The slope is inclined at angle $\theta = 20°$ to the horizontal, and the block experiences a constant friction of 5.0 N. (a) What is the block's acceleration? (b) What is the distance travelled in 2.5 seconds?



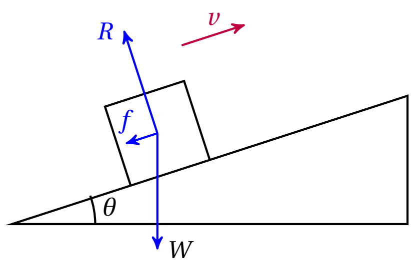
✍ resolving along slope:

$$F_{\text{net}} = mg\sin\theta - f = ma$$

$$a = \frac{2.0 \times 9.81 \times \sin 20° - 5.0}{2.0} \approx 0.855 \text{ m s}^{-2}$$

distance travelled: $s = \cancelto{0}{ut} + \frac{1}{2}at^2 = \frac{1}{2} \times 0.855 \times 2.5^2 \approx 2.67$ m □

Example 4.13 A block of mass 3.0 kg is travelling up an inclined slope at an initial speed of 2.8 m s^{-1}. The slope makes an angle of 18° with the horizontal. A constant friction of 7.5 N acts on the block. (a) What is the block's deceleration? (b) How far does the block travel along the slope before its speed decreases to zero? (c) Suggest whether the block could stay on the slope.



✍ resolving along slope (take direction of initial velocity as positive direction):

$$F_{\text{net}} = -mg\sin\theta - f = ma \quad \Rightarrow \quad a = \frac{-mg\sin\theta - f}{m} = \frac{-3.0 \times 9.81 \times \sin 18° - 7.5}{3.0} \approx -5.53 \text{ m s}^{-2}$$

$$v^2 - u^2 = 2as \quad \Rightarrow \quad s = \frac{v^2 - u^2}{2a} = \frac{0^2 - 2.8^2}{2 \times (-5.5.3)} \approx 0.709 \text{ m}$$

note that component of weight down the slope is: $W_{/\!/} = mg\sin\theta = 3.0 \times 9.81 \times 18° \approx 9.1$ N

$W_{/\!/} > f$, so friction is not enough to prevent block from sliding back down the slope □

4.5 many-body problems

the problems we have been dealing with so far only involve one body

a mechanical system could consist of several objects that mutually interact

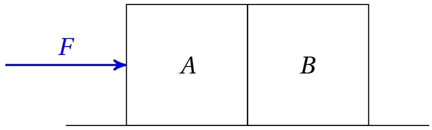
➢ one can take each individual and look into the *internal* forces between the objects of interest

for any force acting between objects *within* system, there is an equal but opposite reaction force

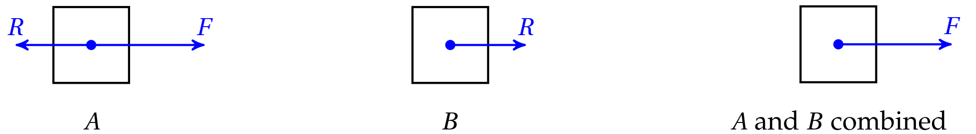
➢ the system can also be treated as a whole

we can analyse *net external force* acting on entire system and work out combined acceleration

Example 4.14 Two boxes A and B are placed on a smooth surface. They are accelerated together by a horizontal force F as shown. Find the acceleration and the contact force between them.



✍ free-body diagrams for A, B, and entire system are given below

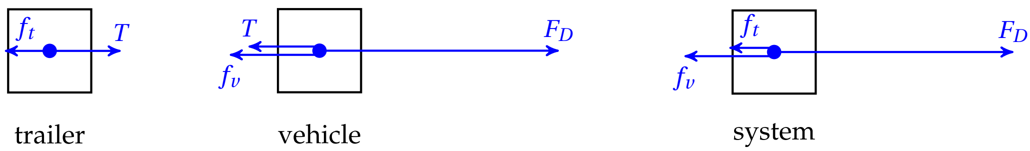


equations of motion can be written down for each free-body diagram and solved[30]

$$\begin{cases} \text{for } A: & F - R = M_A a \\ \text{for } B: & R = M_B a \\ \text{for system:} & F = (M_A + M_B)a \end{cases} \Rightarrow \begin{cases} a = \dfrac{F}{M_A + M_B} \\ R = \dfrac{M_B}{M_A + M_B} F \end{cases}$$ □

Example 4.15 A vehicle of mass 1500 kg is towing a trailer of mass 500 kg by a light inextensible tow-bar. The engine of the vehicle exerts a driving force of 9600 N, and the tractor and the trailer experience resistances of 3600 N and 1800 N respectively. Find the acceleration of the vehicle and the tension in the tow-bar.

✍ free-body diagrams for trailer, vehicle and entire system are given below



equations of motion can be written down for each free-body diagram:

$$\begin{cases} \text{for trailer:} & T - f_t = M_t a \\ \text{for tractor:} & F_D - f_v - T = M_v a \\ \text{for system:} & F_D - f_v - f_t = (M_v + M_t)a \end{cases} \Rightarrow \begin{cases} T - 1800 = 500a \\ 9600 - 3600 - T = 1500a \\ 9600 - 3600 - 1800 = (1500 + 500)a \end{cases}$$

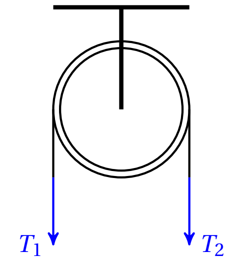
solving simultaneous equations[31], we find

$$a = 2.1 \text{ m s}^{-2}, \quad \text{and} \quad T = 2850 \text{ N}$$ □

[30] In fact, only two of the three equations are independent. You can easily check that adding the equation for A to that for B would produce the equation for the system. To solve the two unknowns for this problem, any two of the three equations shall do the job.

[31] Again, only two of the three equations are independent. You can freely choose your favourite two.

pulleys



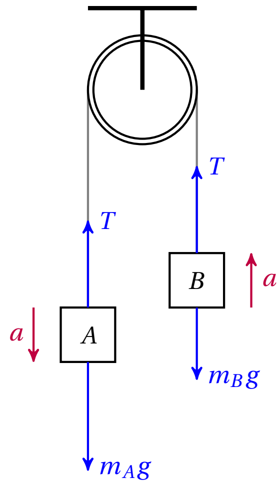
a *pulley* is basically a wheel that carries a string/rope/cable

in this section, we only consider pulleys whose axis of rotation is fixed

such pulleys can be used to change direction of tension in a taut string

we also assume pulleys to be *ideal*: they have no mass and no friction

for an ideal pulley, tensions on both sides are equal: $T_1 = T_2$



Example 4.16 Two blocks of mass m_A and m_B ($m_A > m_B$) are joined together by a light inextensible string. The string passes over a smooth pulley as shown. The two blocks are suddenly released from rest. Find the acceleration of each block and the tension in the string.

✍ apply Newton's second law to each block:

$$\begin{cases} \text{for } A: & m_A g - T = m_A a \\ \text{for } B: & T - m_B g = m_B a \end{cases}$$

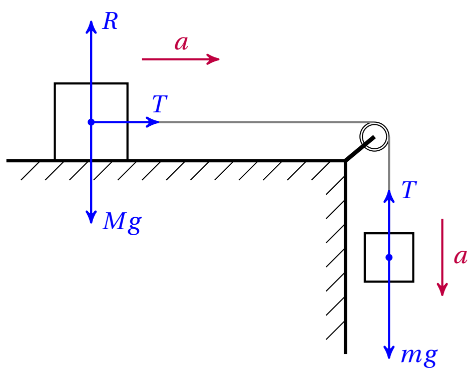
adding the two, one obtains equation of motion for whole system:

$$m_A g - m_B g = (m_A + m_B) a$$

solving these equations, we find

$$a = \frac{m_A - m_B}{m_A + m_B} g \qquad T = \frac{2 m_A m_B g}{m_A + m_B}$$ □

Example 4.17 A mass $M = 4.0$ kg is attached to a block of mass $m = 2.0$ kg through a light string which passes over a frictionless pulley as shown. When both masses are released, find the acceleration and the tension in the string.



✍ apply Newton's second law to each mass:

$$\begin{cases} \text{for } M: & T = Ma \\ \text{for } m: & mg - T = ma \end{cases}$$

adding the two equations, we have:

$$mg = (M + m) a$$

so acceleration is:

$$a = \frac{mg}{M+m} = \frac{2.0 \times 9.81}{4.0 + 2.0} = 3.27 \text{ m s}^{-2}$$

tension in string: $T = Ma = 4.0 \times 3.27 \approx 13.1$ N □

4.6 end-of-chapter questions

Newton's first law

Question 4.1 A little girl tries to lift a luggage bag of mass 25 kg. She pulls upwards with a force of 150 N. The bag does not move. What is the normal reaction from the floor?

Question 4.2 To push a trolley around in a supermarket with constant velocity, you need to exert a steady force. How does this fact agree with Newton's first law, which suggests that motion with constant velocity requires no force?

Question 4.3 A worker is pulling a wagon of mass of 40 kg across a lawn at a constant velocity. He applies a force of 200 N at an angle of 15° above the horizontal. (a) Draw a free-body diagram for the wagon. (b) Find the frictional force. (c) Find the normal contact force.

Newton's second law

Question 4.4 (a) Forces of 3.0 N and 4.0 N act at right angles upon a mass of 160 g. What is the acceleration produced? (b) If the angle between the two forces are allowed to vary, what is the maximum possible acceleration they produce on the same mass? (c) What about the minimum possible acceleration?

Question 4.5 Explain why it becomes increasingly easier for an rocket to accelerate as it travels through space. (Hint: consider the fuel carried by the rocket.)

Question 4.6 Many cars are equipped with airbags which can inflate quickly in case of a collision event. Using Newton's second law, suggest why airbags could protect the driver and the passenger in the car during a car crash.

Question 4.7 A rocket of mass $30,000$ kg is launched vertically upwards at uniform acceleration of 1.6 m s^{-2}. What is the minimum thrust force required?

Question 4.8 A fire-fighter of mass 85 kg slides down a vertical pole. He descends through a distance of 6.0 m in 2.0 seconds. (a) Find the average acceleration. (b) Find the average frictional force acting on the fire-fighter.

Question 4.9 A trolley has mass m. A person needs to push the trolley with force F to produce an acceleration of a, and with force $2F$ to produce an acceleration of $3a$. Find, in terms of m and a, the constant resistive force opposing the trolley's motion.

Question 4.10 A girl stands onto a bathroom scale and finds the reading is 35.0 kg. She then takes the scale into a lift, what mass reading would she observe if the lift (a) is going down at a constant speed, (b) is accelerating downwards at 2.1 m s^{-2}?

Question 4.11 A pirate finds a box of gold coins at the bottom of a lake. The box and its contents have a total mass of 40 kg. The pirate pulls on the box by means of a cable, so that the box is made to rise vertically through the water. Meanwhile, the flow of water creates a constant horizontal force on the box, and the upthrust on the box is known to be 150 N. At one instant, the pirate applies a force of 380 N at an angle of 25° to the upward vertical, and the acceleration of the box is found to be 0.80 m s^{-2}. Assume all the forces acting are coplanar. (a) Draw a free-body diagram for the box. (b) Find the horizontal force due to water flow. (c) Find the drag force on the box.

Newton's third law

Question 4.12 A book placed on your desk experiences two forces: its weight and the support force. Identify the associated reaction forces.

Question 4.13 A student deduces that a rocket travelling in space can never accelerate because the propelling force acting on the rocket is cancelled by an equal and opposite force. Explain why this statement is incorrect.

Question 4.14 A U-shaped magnet lies on a top-pan balance and a mass reading of 180 g is registered. A current-carrying wire is then placed above the magnet. The wire experiences an additional force of 0.30 N that acts upwards. What is the mass reading on the balance?

terminal velocity

Question 4.15 A light ball and a heavy ball of the same size are released from a very high tower, state and explain whether they will reach the ground at the same time.

Question 4.16 A stone is dropped from rest from a high tower. Air resistance is not negligible as the stone reaches terminal speed. Sketch two separate graphs to show the variation of its displacement and acceleration with time.

Question 4.17 How does the terminal speed of a parachutist before opening the parachute compare to that after? Explain your reasons.

Question 4.18 A ball is thrown horizontally from the top of a cliff. Effects of air resistance cannot be neglected. What happens to the horizontal and vertical components of the ball's velocity?

Question 4.19 A small sphere of mass 20.0 g is dropped from rest in a viscous liquid. When the sphere is moving at a speed of v, the viscous drag has a magnitude of $f = \alpha v^2$, where $\alpha = 14\ \mathrm{kg\ m^{-1}}$. (a) What is the sphere's acceleration at the instant when it is released? (b) What is the acceleration when it is moving at $5.0\ \mathrm{cm\ s^{-1}}$? (c) What is the terminal velocity?

Question 4.20 A stone is thrown with some initial velocity at an angle to the horizontal. Sketch on the same graph the path of the stone if (a) air resistance is negligible, (b) air resistance is significant.

inclined slopes

Question 4.21 A 3.0 kg mass is placed on an inclined plane and it does not move. Given that the normal contact force acting on it is 28.0 N. (a) Find the angle of the plane to the horizontal. (b) Find the frictional force acting on the mass.

Question 4.22 A small mass slides down a frictionless slope with an acceleration of $2.8\ \mathrm{m\ s^{-2}}$. Determine the angle that the slope makes with the horizontal.

Question 4.23 A car of mass 1400 kg is moving up a slope at a constant velocity of $13.5\ \mathrm{m\ s^{-1}}$. The slope makes an angle of $6.0°$ to the horizontal. Total resistive force of 650 N acts on the car. What is the driving force required to push the car up the slope?

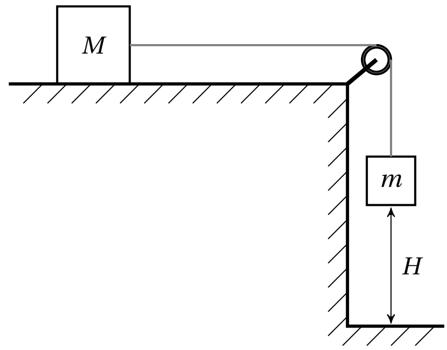
Question 4.24 A shopping trolley somehow loses control and runs down a straight slope from rest. The slope makes an angle of $3.0°$ to the horizontal. The resistive force acting on the trolley is a constant 15 N. The trolley and its contents have a total mass of 40 kg. (a) Find the acceleration of the trolley. (b) Determine the time for the trolley to travel a distance of 4.0 m along the the slope. (c) Suggest why the slope in shopping malls are not made any steeper.

Question 4.25 A heavy log of mass 240 kg is initially placed at a point P at the bottom of a slope. A motor drags the log up the slope through a cable. The slope is inclined at an angle of $16°$ to the horizontal. The motor provides a tension of 1200 N parallel to the slope. The friction that acts on the log is a constant 450 N. (a) Find the acceleration of the log. (b) Find the time taken to pull the log through a distance of 8.0 m to a point Q. (c) Find the velocity of the log at Q. (d) The cable breaks when the log reaches Q, find the distance moved beyond Q until the log's speed becomes zero. (e) The log will then slide back down the slope. Find the time for the log to return to its starting position. (f) Sketch a v-t graph for the log from the start at P until it returns to P.

many-body problems

Question 4.26 Block A of mass 5.0 kg is connected by means of a light string to block B of mass 3.0 kg. The two blocks are placed on a horizontal table. A force of 30 N is applied to pull on block A. Given that the friction on each block is 30% of its own weight. (a) Find the acceleration of the blocks. (b) Find the tension in the string.

Question 4.27 A box of mass $M = 3.6$ kg rests on a horizontal, rough surface. The box is connected to a block of mass $m = 2.0$ kg through a light cord that passes over a frictionless pulley as shown. The box is released from rest. Given that the box experiences a frictional force of 12 N and the block is initially at a height of $H = 0.80m$ above the floor. (a) Find the acceleration of the block. (b) Determine the time taken for the block to hit the floor.



Chapter 5
Mechanical Equilibrium

in this chapter, we will study the mechanical equilibrium of objects

for *point objects*, zero resultant force suffices for equilibrium

but for *rigid bodies*, force many produce *turning effects*

hence rigid bodies must satisfy another condition to stay in equilibrium

this brings forward the notion of moment of a force and the principle of moments

5.1 moment of force

torque, or **moment** of a force, is defined as the product of the force and the perpendicular distance from the pivot to the line of action: $\boxed{\tau = Fd_{\perp}}$

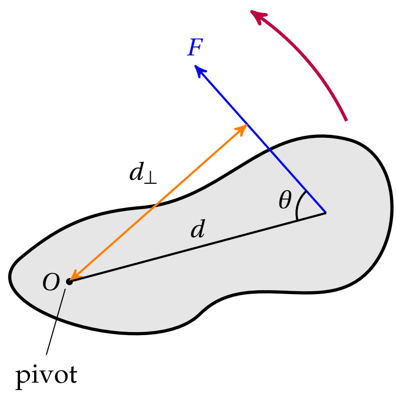
- unit of torque/moment: $[\tau] = \text{N m}$
- perpendicular distance from pivot to the line of action is also called the **lever arm**

  this is the shortest distance between the force applied and axis of rotation

  in the diagram, lever arm $d_{\perp} = d\sin\theta$

  so moment of this force is: $\tau = Fd\sin\theta$
- moment is a vector quantity [32] [33]

  it can act in *clockwise* or *anti-clockwise* direction



[32]Using vector notation, moment of a force can be defined as a *cross product*: $\vec{\tau} = \vec{r} \times \vec{F}$, where $\vec{r}$ is the position vector from the pivot to the point at which the force is applied.

[33]Rigorously speaking, moment is a *pseudovector*, which means that it does not transform quite like a normal vector although it does have a direction. In particular, if an object acted by a force is reflected across a plane, the moment of this force would not be reflected. Instead, it would be reflected and *reversed*.

➢ moment of a force produces turning effects[34]

if there exists a non-zero moment, the object will start to rotate clockwise or anti-clockwise

➢ note that moment of a force depends on choice of pivot

moment of the same force with respect to different points can be very different

5.1.1 torque of couple

let's take a pair of equal but opposite forces acting at different positions on the same object

the two forces produce torques in the same direction

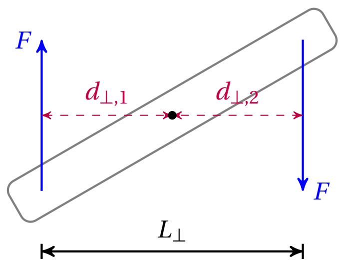
the combined effect is called the torque of a couple

resultant torque due to the couple is

$$\tau = Fd_{\perp,1} + Fd_{\perp,2} = 2F(d_{\perp,1} + d_{\perp,2})$$

$d_{\perp,1} + d_{\perp,2}$ is perpendicular distance $L_\perp$ between the pair, so

$$\boxed{\tau = FL_\perp}$$



torque of a couple can be therefore defined as the product of one force in the couple and the perpendicular distance between the pair

➢ a pair of equal but opposite forces give zero net force

but they can produce rotational effects, so no net force does not necessarily mean equilibrium

➢ torque of couple does not depend on choice of pivot

for same force pair, resultant moment is constant about any point

5.1.2 moment of weight

recall that weight is a force of gravity which is actually experienced by every part of the object

when dealing with moment of weight, we need to sum up torques on each part of this object

this brings a problem since the lever arms can be all different

fortunately, this calculation can be simplified using the idea of centre of gravity

centre of gravity is a point at which the entire weight of an object is considered to act

[34]Moment is like the rotational counterpart of a force: force changes the state of translational motion, moment changes the state of rotation.

➢ there is a similar concept called the centre of mass

centre of mass is the average position of all the mass that makes up the object

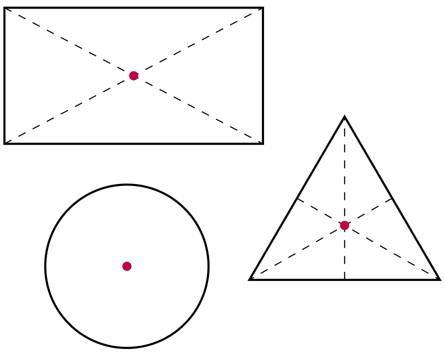
near the surface of the earth, mass and weight are directly proportional to each other

so centre of mass is interchangeable with centre of gravity if we stay on earth

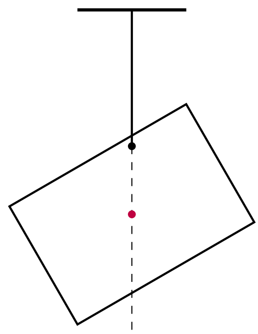
➢ for a regularly-shaped uniform object, the centre of gravity/mass is its geometrical centre

➢ if an object is hung freely, centre of gravity/mass is vertically below the point of suspension

otherwise weight would produce a non-zero torque about the point of suspension, causing the object to rotate until torque becomes zero



centre of mass of uniform lamina is at the geometrical centre



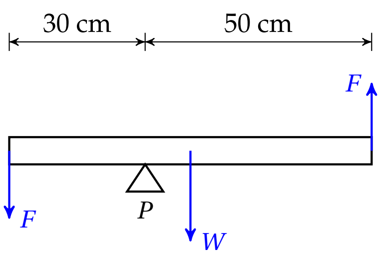
centre of mass is vertically below the point of suspension

➢ to find the centre of gravity/mass, the object of interest is suspended from several positions

each time we draw a plumb-line through the point of suspension

centre of mass/gravity lies where the lines intersect

Example 5.1 The diagram shows a uniform beam of weight $W = 20$ N and length 80 cm pivoted at point P. P is 30 cm from one end. Two equal but opposite forces of magnitude $F = 12$ N are acting at the two ends of the beam as shown. What is the resultant moment about point P?



✍ moment of weight: $\tau_w = Wd_w = 20 \times \left(0.50 - \frac{1}{2} \times 0.80\right) = 2.0$ N m (clockwise)

torque of couple: $\tau_c = FL = 12 \times 0.80 = 9.6$ N m (anti-clockwise)

resultant moment: $\tau_{\text{net}} = \tau_c - \tau_W = 9.6 - 2.0 = 7.6$ N m (anti-clockwise) □

5.2 mechanical equilibrium

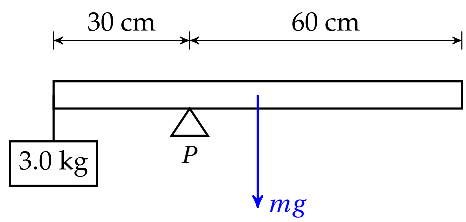
5.2.1 principle of moments

if there is no turning effect for an object, the total moment of all forces must vanish

> for a rigid body in equilibrium, sum of all clockwise moments must be equal to the sum of anti-clockwise moments *about any point*, this is called the **principle of moments**

- an object in equilibrium has no turning effect about any point
  so zero resultant moment about any point [35]

Example 5.2 A uniform rod of length 90 cm is pivoted 30 cm from one end. It is balanced with a 3.0 kg load. Find the mass of the rod.

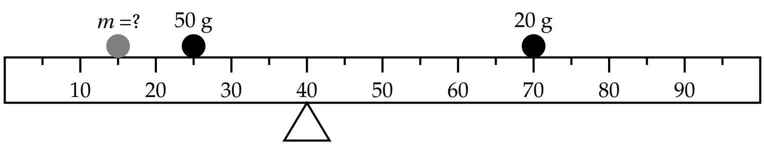


✍ take moments about P:

$$3.0 \times 9.81 \times 0.30 = m \times 9.81 \times \left(0.60 - \frac{1}{2} \times 0.90\right)$$

so we find mass of rod: $m = 1.5$ kg □

Example 5.3 A student balances a metre rule of mass 120 g supported on a fulcrum at the 40 cm mark. She then places a 20 g mass on the 70 cm mark and a 50 g mass on the 25 cm mark as shown. To balance the rule, what mass should she place on the 15 cm mark?



[35] As long as there is no resultant force, then zero resultant moment about any particular point would imply zero resultant moment about any point.

Mathematically, let's take a collection of forces $\vec{F_1}, \vec{F_2}, \cdots, \vec{F_n}$ acting at positions $\vec{r_1}, \vec{r_2}, \cdots, \vec{r_n}$ on an object with respect to some fixed point O. Suppose their resultant moment vanishes, i.e., $\sum \vec{\tau_i} \equiv \sum \vec{r_i} \times \vec{F_i} = 0$, and also their resultant force vanishes, i.e., $\sum \vec{F_i} = 0$. If we focus on a different point P with a relative displacement $\vec{R}$ to point O', then taking moments about O', we will have:

$$\sum \vec{\tau_i'} = \sum \vec{r_i'} \times \vec{F_i} = \sum (\vec{r_i} + \vec{R}) \times \vec{F_i} = \sum \vec{r_i} \times \vec{F_i} + \vec{R} \times \sum \vec{F_i} = 0 + 0 = 0$$

which shows zero resultant moment about one point together with zero resultant force guarantee resultant moment must be zero about any point in space.

✍ take moments about the support:

$$mg \times (40-15) + 0.050g \times (40-25) = 0.12g \times (50-40) + 0.020g \times (70-40) \quad \Rightarrow \quad m = 42\text{ g} \qquad \square$$

Example 5.4 A cylinder of weight 100 N and diameter 50 cm rests against point P of a curb of height 10 cm. What is the minimum force required to cause the cylinder to roll to the left?

✍ to roll the cylinder, force applied must produce a torque no less than that of weight

force is minimum if lever arm is greatest

take moments about P (see diagram):

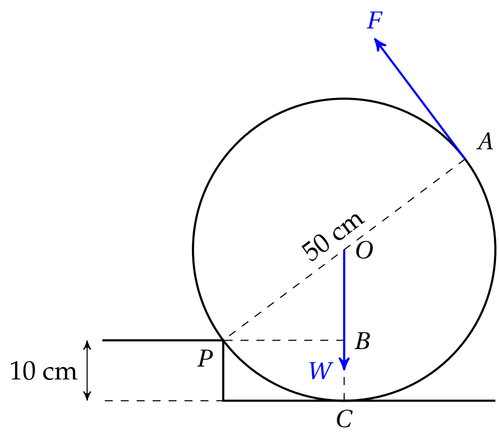
$$F_{\min} \times |PA| = W \times |PB|$$

note that $|PB| = \sqrt{|OP|^2 - |OB|^2}$, so

$$|PB| = \sqrt{0.25^2 - (0.25-0.10)^2} = 0.20\text{ m}$$

plug back into the equation above:

$$F_{\min} \times 0.50 = 100 \times 0.20 \quad \Rightarrow \quad F_{\min} = 40\text{ N} \quad \square$$



5.2.2 mechanical equilibrium

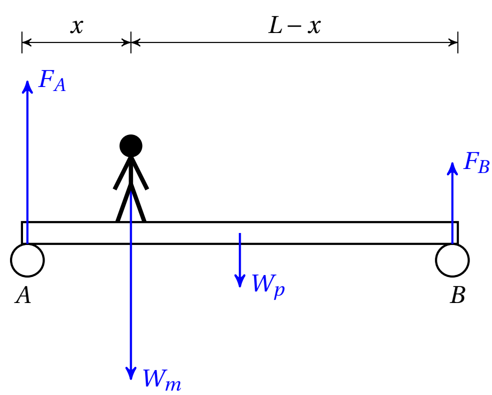
combining Newton's first law and principle of moments, we have the following statement:

for any mechanical system in equilibrium, two conditions must be satisfied:

- resultant force is zero in any direction: $\sum F = 0$
- resultant moment is zero about any point: $\sum \tau = 0$

these two conditions allow for many possible equations that can be written down

Example 5.5 A uniform plank of weight 100 N and length $L = 6.0$ m rests horizontally on two supports A and B. A man of weight 800 N stands a distance of $x = 1.5$ m from end A. Determine the forces acting at the two supports.



✍ take moments about A: $W_m x + W_p \cdot \frac{L}{2} = F_B L$

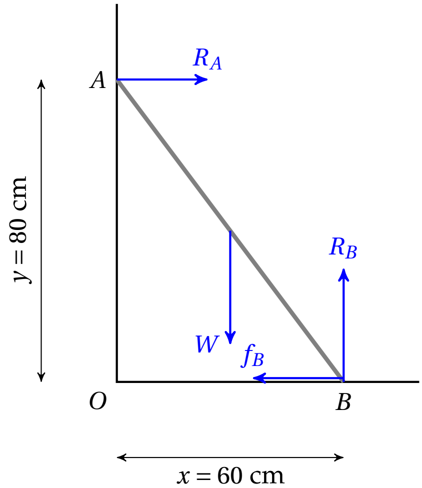
$$800 \times 1.5 + 100 \times 3.0 = F_B \times 6.0 \Rightarrow F_B = 250\text{ N}$$

take moments about B: $W_m(L-x) + W_p\frac{L}{2} = F_A L$

$$800 \times 4.5 + 100 \times 3.0 = F_A \times 6.0 \Rightarrow F_B = 650\text{ N}$$

one can check that: $F_A + F_B = W_m + W_p$, there must be no resultant force in vertical direction $\square$

Example 5.6 A uniform ladder of weight 120 N rests on a rough ground against a smooth wall as shown. The dimensions are labelled on the diagram. (a) What is the contact force acting at B? (b) What is the contact force acting at A? (c) What is the frictional force at B?



✍ we draw free-body diagram as shown

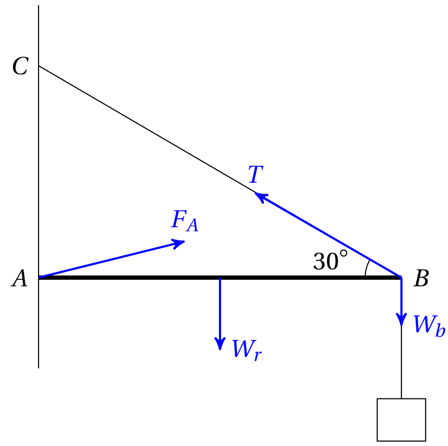
resolve vertically: $R_B = W \quad \Rightarrow \quad R_B = 120$ N

take moments about B: $R_A y = W\dfrac{x}{2}$

$$R_A = \frac{120 \times 0.30}{0.80} = 45 \text{ N}$$

resolve horizontally: $f_B = R_A \quad \Rightarrow \quad f_B = 45$ N □

Example 5.7 The diagram shows a uniform rod AB of weight 60 N that is held horizontally to a vertical wall by means of a light string. The string is attached to the rod at B, where a basket of weight 40 N is suspended. The other end of the string is fixed on the wall at C. The angle between the string and the rod is 30°. (a) Find the tension in the string. (b) Find the force acting on the rod at A.



✍ take moments about A: $TL\sin\theta = W_b L + W_r \dfrac{1}{2}L$

$$T\sin 30° = 40 + 60 \times \frac{1}{2} \quad \Rightarrow \quad T = 140 \text{ N}$$

resolve horizontally: $F_{A,x} = T\cos\theta \quad \Rightarrow \quad F_{A_x} = 140\cos 30° \approx 121$ N

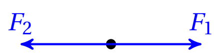
resolve vertically: $F_{A,y} + T\sin\theta = W_r + W_b \quad \Rightarrow \quad F_{A,y} = 60 + 40 - 140\sin 30° = 30$ N

force at A: $F_A = \sqrt{F_{A,x}^2 + F_{A,y}^2} \quad \Rightarrow \quad F_A = \sqrt{121^2 + 30^2} \approx 125$ N □

5.3 two forces in equilibrium

the problem of two balanced forces is trivial

suppose two forces F_1 and F_2 are acting on an object in equilibrium



- to have zero resultant force, F_1 and F_2 must be equal but opposite
- to have zero resultant moment, F_1 and F_2 must act along same line
  otherwise they would produce torque of couple, which causes turning effects

5.4 three forces in equilibrium

5.4.1 force triangle

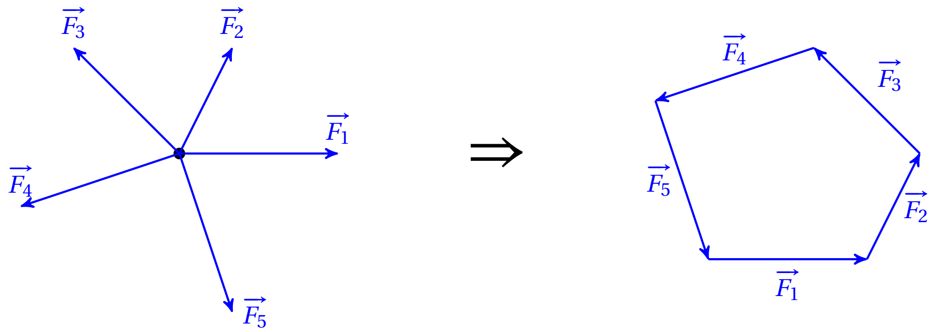
when there are more than two forces, situation becomes more complicated

one can use *vector diagram* to solve the problem

suppose a set of forces $\vec{F_1}, \vec{F_2}, \cdots, \vec{F_n}$ are in equilibrium

no resultant force requires $\vec{F_1} + \vec{F_2} + \cdots + \vec{F_n} = 0$

recall that resultant force is vector sum of all forces acting, and now this sum has to vanish, so if the force vectors are connected head to tail, they should form a closed n-polygon

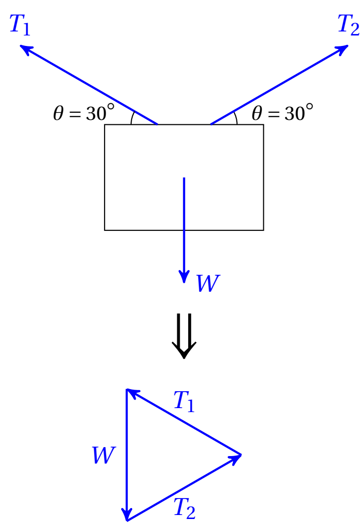


an n-polygon formed by a set of n balanced forces

in the case of three balanced forces, net force is zero means they should form a **force triangle**

unknown forces can then be solved by cracking a geometric problem

Example 5.8 A painting of weight $W = 20$ N is supported by two strings as shown. Both strings form an angle $\theta = 30^\circ$ to the horizontal. Find the tension in the strings.



✍ by resolving forces, we have:

$$\begin{cases} T_1 \cos\theta = T_2 \cos\theta \\ T_1 \sin\theta + T_2 \sin\theta = W \end{cases}$$

we solve the equations to obtain:

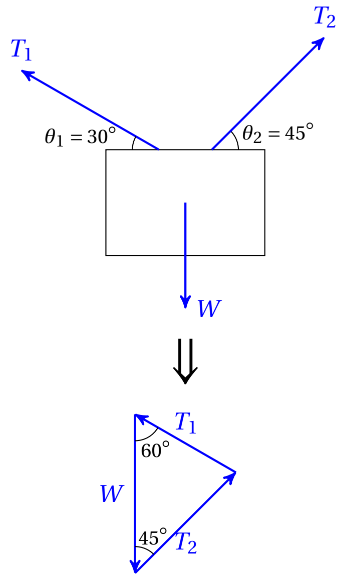
$$T_1 = T_2 = \frac{W}{2\sin\theta} = \frac{20}{2\sin 30^\circ} = 20 \text{ N}$$

alternatively, we can construct the force triangle as shown

T_1, T_2 and W form an equilateral triangle, so

$$T_1 = T_2 = W = 20 \text{ N} \qquad \square$$

Example 5.9 The same painting of weight $W = 20\,\text{N}$ is supported by two strings at different angles $\theta_1 = 30^\circ$ and $\theta_2 = 45^\circ$ as shown. Find the forces in the two strings.



✍ by resolving forces, we have:

$$\begin{cases} T_1 \cos\theta_1 = T_2 \cos\theta_2 \\ T_1 \sin\theta_1 + T_2 \sin\theta_2 = W \end{cases} \Rightarrow \begin{cases} \frac{\sqrt{3}}{2} T_1 = \frac{\sqrt{2}}{2} T_2 \\ \frac{1}{2} T_1 + \frac{\sqrt{2}}{2} T_2 = 20 \end{cases}$$

solving this, we find: $T_1 \approx 14.6\,\text{N}$, $T_2 \approx 17.9\,\text{N}$

alternatively, we construct the force triangle, computation for T_1 and T_2 would be more straightforward

the three forces are related by the the law of sine:

$$\frac{W}{\sin 75^\circ} = \frac{T_1}{\sin 45^\circ} = \frac{T_2}{\sin 60^\circ}$$

from this we get the same result: $T_1 \approx 14.6\,\text{N}$, $T_2 \approx 17.9\,\text{N}$ □

5.4.2 concurrent forces

for three forces in equilibrium, they must produce zero resultant moment about any point

suppose lines of action of F_1 and F_2 meet at point P

moment of F_1 and moment of F_2 about P are both zero

to produce zero resultant moment about P, then moment of F_3 about P must vanish

this suggest line of action of F_3 must pass through point P

therefore lines of action of F_1, F_2 and F_3 must pass through the same point[36]

such three forces are said to be *concurrent*

summary for three forces in equilibrium

for three forces in equilibrium, we can now conclude:

- the three force vectors must be able to form a force triangle

  this is a consequence of zero resultant force
- the lines of action for the three forces must pass through same point

  this is a consequence of zero resultant moment

[36]In the case of three parallel forces in equilibrium, we can introduce the notion of an ideal point at infinity, so that parallel lines could meet at that point.

5.5 end-of-chapter questions

mechanical equilibrium

Question 5.1 Is it possible for an object to be in equilibrium if only one force is acting on it?

Question 5.2 If three forces are in equilibrium, suggest and explain whether the lines of action must lie in the same plane.

Chapter 6

Momentum

6.1 momentum & momentum conservation

6.1.1 momentum

momentum of an object is defined as the product of its mass and its velocity: $\boxed{p = mv}$

- unit of momentum: $[p] = \text{kg m s}^{-1} = \text{N s}$
- momentum is a vector quantity

  momentum is in same direction as the object's velocity

  to find change in momentum of a body, or to find sum of the momenta[37] for a system of several objects, one has to keep track of directions
- momentum is like inertia in motion

  inertia, the property that object resists change in motion, is incorporated in Newton's first law

  we will see very soon that momentum is closely related to Newton's second law

6.1.2 relation to force

suppose a constant net force F is applied on a body, we can write:

$$F = ma = m\frac{\Delta v}{\Delta t} = \frac{m\Delta v}{\Delta t} = \frac{\Delta p}{\Delta t}$$

where we have used Newton's second law and defining equation for momentum

from this derivation, we can give a formal definition for the force

force is defined as the rate of change in momentum: $\boxed{F = \frac{\Delta p}{\Delta t}}$

[37]Momenta is the plural form of momentum.

we can also restate Newton's second law in terms of momentum:

> **Newton's second law** states that resultant force acting on an object equals the rate of change in the object's momentum

- information about force can be extracted from momentum-time graphs

  gradient of p-t graph equals the resultant force acting

- information about *change* in momentum can be deduce from force-time graphs

  area under F-t graph equals the change in object's momentum

Example 6.1 A ball of 120 g strikes a wall at right angle with a speed of 10 m s^{-1}. It rebounds with the same speed. If the time of impact is 25 ms, find the average force exerted on the ball.

✍ change in momentum: $\Delta p = mv - mu = 0.12 \times 10 - 0.12 \times (-10) = 2.4$ kg m s^{-1}

average force: $F = \dfrac{\Delta p}{\Delta t} = \dfrac{2.4}{2.5 \times 10^{-3}} = 96$ N □

Example 6.2 Water is pumped through a hose-pipe. A man is holding the hose-pipe horizontally and water emerges from the hose-pipe with a speed of 16 m s^{-1} at a rate of 45 kg per minute. Find the force required from this man to hold steady the hose-pipe.

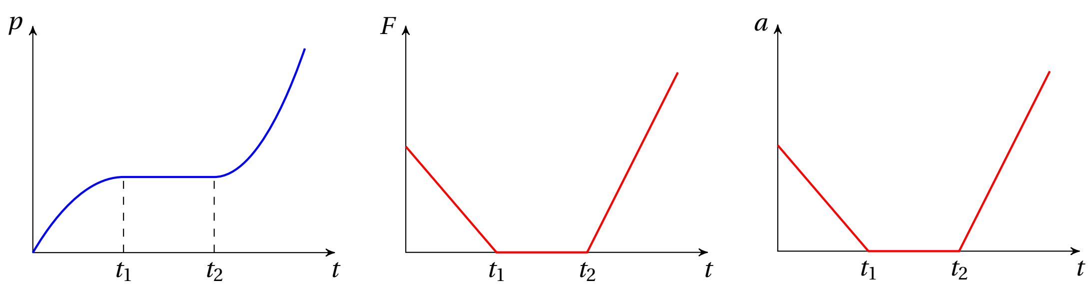
✍
$$F = \frac{\Delta p}{\Delta t} = \frac{\Delta(mv)}{\Delta t} = \frac{\Delta m(v-0)}{\Delta t} = \frac{45 \times 16}{60} \quad \Rightarrow \quad F = 12 \text{ N}$$
□

Example 6.3 A strong wind of speed 30 m s^{-1} blows against a wall of area 10 m^2 at right angles. The density of the air is 1.2 kg m^{-3}. Assume air speed reduces to zero when it hits the wall. What is the approximate force exerted by the air on the wall?

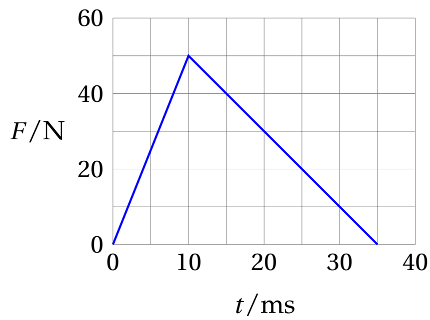
✍
$$F = \frac{\Delta p}{\Delta t} = \frac{\Delta(mv)}{\Delta t} = \frac{\Delta m(v-0)}{\Delta t} \xlongequal{m=\rho V} \frac{\rho \Delta V v}{\Delta t} \xlongequal{V=AL} \frac{\rho A \Delta L v}{\Delta t} \xlongequal{\Delta L = v\Delta t} \rho A v^2$$

$$\Rightarrow \quad F = 1.2 \times 10 \times 30^2 = 10800 \text{ N}$$
□

Example 6.4 Given the variation with time of the momentum of a body as shown in the p-t graph, check yourself that the variation of force acting and the variation of the object's acceleration should be plotted as shown in the F-t graph and the a-t graph.



Example 6.5 An object of mass 70 g is initially at rest. A force that varies with time is exerted on the object. The graph shows the how the force varies during the time of impact. What is the final velocity of the object?



✍ area under F-t graph gives change in momentum

$$\Delta p = \frac{1}{2} \times 50 \times 35 \times 10^{-3} = 0.875 \text{ kg m s}^{-1}$$

final velocity: $v = \dfrac{\Delta p}{m} = \dfrac{0.875}{70 \times 10^{-3}} = 12.5 \text{ m s}^{-1}$ □

6.1.3 principle of momentum conservation

change in an object's momentum is given by: $\Delta p = F\Delta t$ [38]

in particular, if there is zero net force, then object's momentum stays constant

this idea can be generalised to a system of objects

there are external forces from outside and internal forces between objects within the system

let's take two mutually interacting objects within the system, say A and B

by Newton's 3rd law, force on A by B is equal and opposite to force on B by A

change in A's momentum by B is then equal and opposite to change in B's momentum by A

change in total momentum of A and B due to each other is therefore cancelled out

therefore, for the system as a whole, effect of internal forces always cancel out

change of total momentum of the system would only depend on net external force [39]

[38]The relation $\Delta p = F\Delta t$ is valid if we are dealing with a *constant* force. If the object is acted by a varying force, then the change in its momentum is given by: $\Delta p = \int F \mathrm{d}t$.

[39]A more rigorous derivation goes as follows.

Let's consider a system of point objects m_i, each experiences a force F_i where F_i can come from some external source or another object j within the system: $F_i = F_{i,\text{ext}} + \sum_j F_{i,j}$.

Summing over all objects, we can write: $\sum_i F_i = \sum_i F_{i,\text{ext}} + \sum_{i,j} F_{i,j}$

For each pair i and j, the action-reaction principle suggests that the mutual interaction between the two are equal but opposite: $F_{i,j} = -F_{j,i}$, so $\sum_{i,j} F_{i,j} = 0$. Therefore, $\sum_i F_i = \sum_i F_{i,\text{ext}}$.

Multiply both sides by Δt, we can write: $\sum_i F_i \Delta t = \sum_i F_{i,\text{ext}} \Delta t$. Note that $\sum_i F_i \Delta t = \sum_i \Delta p_i$ gives the change in total momentum of system, so this shows the change of total momentum is determined by the net external force: $\boxed{\left(\sum_i F_{i,\text{ext}}\right) \Delta t = \sum_i \Delta p_i}$

if there is no net external force, then no change in total momentum

> for any closed system where net external force is zero, the total momentum of the system remains constant, this is called the **principle of momentum conservation**

Example 6.6 A uranium-238 nucleus disintegrates, emitting an α-particle of mass 4u and producing a thorium-234 nucleus of mass 234u. The uranium nucleus is initially at rest. (a) What is the ratio of the velocities of the product particles $\frac{v_\alpha}{v_{\text{Th}}}$? (b) Explain why the α-particle and the thorium nucleus must be emitted in opposite directions.

✍ no external force involved during decay, so momentum is conserved

zero initial momentum means total momentum of α-particle and thorium nucleus is zero

α-particle and thorium nucleus must carry equal but opposite momenta

equal momentum $\Rightarrow$ $m_\alpha v_\alpha = m_{\text{Th}} v_{\text{Th}}$ $\Rightarrow$ $\frac{v_\alpha}{v_{\text{Th}}} = \frac{m_{\text{Th}}}{m_\alpha} = \frac{234}{4} = 58.5$

opposite momentum so they move off in exactly opposite directions □

6.2 collision problems

for two bodies colliding together, external forces are negligible during the time of contact

hence total momentum is considered to be conserved for any collision process

6.2.1 collision in one dimension

suppose two masses m_1 and m_2 are restricted to move in one dimension only

they move at initial velocities u_1 and u_2 before they collide

after collision, their velocities become v_1 and v_2, as depicted in the diagram

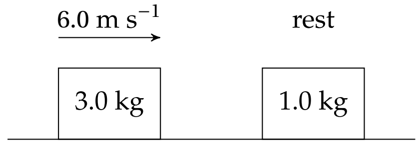
total momentum is conserved, so: $\boxed{m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2}$

➢ recall the vector nature of momentum and velocity

we normally choose objects moving to the right to have positive momentum/velocity

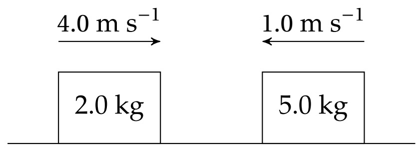
then object travelling to the left would have negative momentum/velocity

Example 6.7 A 3.0 kg mass moving at 6.0 m s^{-1} has a head-on collision with a stationary 1.0 kg mass. The two masses stick together on impact. What is the final velocity of the two masses?



✍ $m_1 u_1 + \cancelto{0}{m_2 u_2} = (m_1 + m_2) v \quad \Rightarrow \quad 3.0 \times 6.0 = (3.0 + 1.0) \times v \quad \Rightarrow \quad v = 4.5 \text{ m s}^{-1}$ □

Example 6.8 A 2.0 kg mass moving at 4.0 m s^{-1} collides head on with a 5.0 kg mass moving at 1.0 m s^{-1}. After the collision, speed of the 5.0 kg mass is unchanged but its direction is reversed. What is the velocity of the 2.0 kg mass after the collision?



✍ $m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2 \Rightarrow 2.0 \times 4.0 + 5.0 \times (-1.0) = 2.0 \times v_1 + 5.0 \times 1.0 \Rightarrow v_1 = -1.0 \text{ m s}^{-1}$

minus sign means the 2.0 kg mass reverses direction after collision □

6.2.2 elastic & inelastic collisions

elastic collisions

collision can be either elastic or inelastic[40]

> for **elastic collisions**, there is no loss of kinetic energy

we hereby derive a condition that must be satisfied by two objects colliding elastically

since momentum and kinetic energy are both conserved, we can write two equations[41]

$$\boxed{m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2}$$

$$\boxed{\frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2}$$

rearranging both equations, we have

$$m_1 (u_1 - v_1) = m_2 (v_2 - u_2) \tag{1}$$

$$m_1 (u_1^2 - v_1^2) = m_2 (v_2^2 - u_2^2) \tag{2}$$

[40]Here we assume you already have some knowledge about *kinetic energy*. Kinetic energy of a moving body is given by the formula: $E_k = \dfrac{1}{2} m v^2$. You might have learned about it in an GCSE course or elsewhere. We will talk about kinetic energy in §7.2.

[41]For simplicity, we consider two-body collision in one dimension only, that is the two bodies move along the same straight line before and after the collision. For a two-body collision problem in two dimension, the conservation of momentum can be broken into two independent component equations.

equation (4) can be further rewritten as

$$m_1(u_1 - v_1)(u_1 + v_1) = m_2(v_2 - u_2)(v_2 + u_2) \tag{2'}$$

comparing equation (2') to equation (1), one has: $u_1 + v_1 = v_2 + u_2$

rearrange the equation, we find: $\boxed{v_2 - v_1 = u_1 - u_2}$

both side of the equation now represent a *relative speed* between the two colliding bodies

> for an elastic collision process between two bodies, the relative velocity of separation after collision equals the relative velocity of approach before collision

inelastic collisions

> for **inelastic collisions**, part of kinetic energy is lost due to change in object's shape

➢ for an inelastic process, the following will hold:

– K.E. after collision is less than K.E. before collision

– relative speed after collision is less than relative speed before collision

brief summary

discussions on elastic and inelastic collisions are summarised in the table below

	elastic collision	inelastic collision
conservation of momentum	✓	✓
conservation of kinetic energy	✓	✗
conservation of total energy	✓	✓
relative speed stays unchanged	✓	✗

Example 6.9 A sphere of mass m moves on a smooth horizontal surface at speed v and collides *elastically* with an identical ball at rest. What are the final velocities of the two spheres?

before collision after collision

✍ $$\begin{cases} mv = mv_1 + mv_2 & \text{(momentum conservation)} \\ v = v_2 - v_1 & \text{(relative speed unchanged)} \end{cases} \Rightarrow \begin{cases} v_1 = 0 \\ v_2 = v \end{cases}$$

the two spheres simply exchange velocities during the collision, as shown in diagram[42] □

Example 6.10 A 4.2 kg mass A and a 1.5 kg mass B are travelling towards each other on a frictionless horizontal plane. Mass A and B move at 5.40 m s^{-1} and 1.80 m s^{-1} respectively before they strike, as shown below. Mass B moves to the right at 4.50 m s^{-1} after the collision, (a) find the velocity of A after the impact, and (b) suggest whether the collision is elastic.

before collision after collision

✍ momentum conserved: $m_A u_A + m_B u_B = m_A v_A + m_B v_B$

$$4.2 \times 5.40 + 1.5 \times (-1.80) = 4.2 \times v + 1.5 \times 4.50 \quad \Rightarrow \quad v_A = 3.15 \text{ m s}^{-1}$$

K.E. before: $E_{k,i} = \frac{1}{2} m_A u_A^2 + \frac{1}{2} m_A u_B^2 = \frac{1}{2} \times 4.2 \times 5.40^2 + \frac{1}{2} \times 1.5 \times 1.80^2 \approx 63.7$ J

K.E. after: $E_{k,f} = \frac{1}{2} m_A v_A^2 + \frac{1}{2} m_A v_B^2 = \frac{1}{2} \times 4.2 \times 3.15^2 + \frac{1}{2} \times 1.5 \times 4.50^2 \approx 36.0$ J

there is K.E. loss, so collision is inelastic

alternatively, we can compare the relative speed before and after the collision

$$u_A - u_B = 5.40 - (-1.80) = 7.20 \text{ m s}^{-1} \qquad v_B - v_A = 4.50 - 3.15 = 1.35 \text{ m s}^{-1}$$

relative speed changed after the collision, so collision must be inelastic □

6.2.3 collision in two dimensions

when objects collide on a horizontal plane, they can possibly move off in any direction

negligible net external force is present, total momentum is still conserved

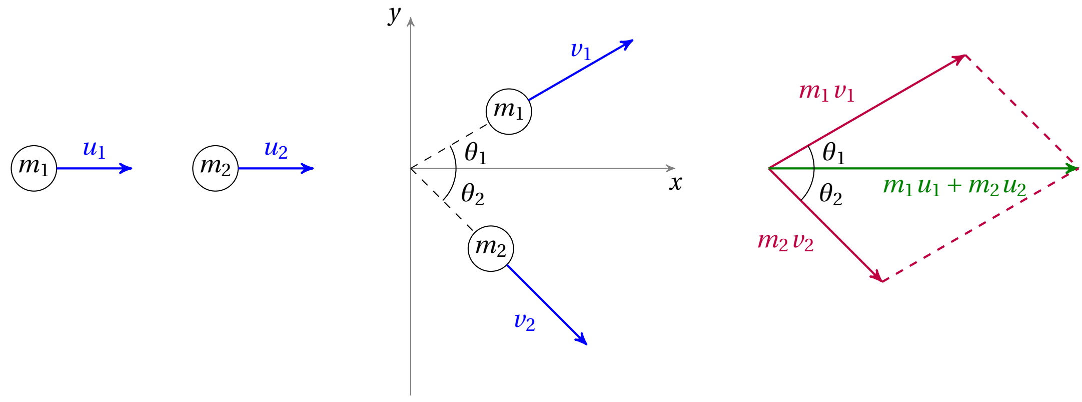
recall that momentum is a vector quantity, so momentum should be conserved in any direction

let's consider the collision between two masses m_1 and m_2

for simplicity, assume their initial velocities u_1 and u_2 are in same direction

final velocities v_1 and v_2 after the collision are shown

[42]If two objects of equal mass collide elastically with one another, one can actually show that their velocities would exchange regardless of their initial velocities.



before collision after collision momentum conservation

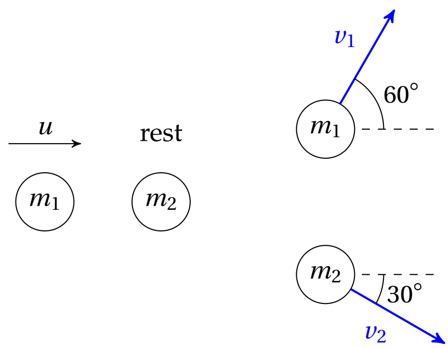
equations of momentum conservation can be written for two perpendicular directions

$$m_1 u_1 + m_2 u_2 = m_1 v_1 \cos\theta_1 + m_2 v_2 \cos\theta_2 \qquad \text{(in } x\text{-direction)}$$

$$0 = m_1 v_1 \sin\theta_1 - m_2 v_2 \sin\theta_2 \qquad \text{(in } y\text{-direction)}$$

one can also construct a vector triangle to transform the problem into a geometry problem

Example 6.11 A ball of mass $m_1 = 1.0$ kg travelling with a speed of $u = 6.0\ \mathrm{m\ s^{-1}}$ in the x-direction strikes a stationary ball of mass $m_2 = 2.0$ kg. The direction of the balls' velocities v_1 and v_2 after the collision are shown in the diagram. Find v_1 and v_2.



before collision after collision

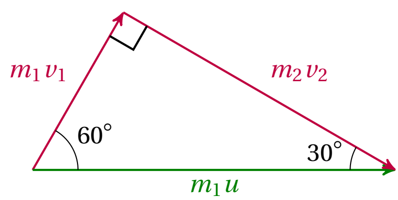
✍ start with momentum conservation equations:

$$\begin{cases} m_1 u = m_1 v_1 \cos 60^\circ + m_2 v_2 \cos 30^\circ \\ 0 = m_1 v_1 \sin 60^\circ - m_2 v_2 \sin 30^\circ \end{cases}$$

$$\begin{cases} 1.0 \times 6.0 = 1.0 \times v_1 \times \dfrac{1}{2} + 2.0 \times v_2 \times \dfrac{\sqrt{3}}{2} \\ 0 = 1.0 \times v_1 \times \dfrac{\sqrt{3}}{2} - 2.0 \times v_2 \times \dfrac{1}{2} \end{cases}$$

simplify and solve the equations:

$$\begin{cases} \dfrac{1}{2} v_1 + \sqrt{3} v_2 = 6 \\ v_2 = \dfrac{\sqrt{3}}{2} v_1 \end{cases} \Rightarrow \begin{cases} v_1 = 3.0\ \mathrm{m\ s^{-1}} \\ v_2 \approx 2.6\ \mathrm{m\ s^{-1}} \end{cases}$$



one can also draw and use the vector triangle

for this question, this happens to be a right-angled triangle, so things become much easier

$$\begin{cases} m_1 v_1 = m_1 u \cos 60^\circ \\ m_2 v_2 = m_1 u \cos 30^\circ \end{cases} \Rightarrow \begin{cases} 1.0 \times v_1 = 1.0 \times 6.0 \times \dfrac{1}{2} \\ 2.0 \times v_2 = 1.0 \times 6.0 \times \dfrac{\sqrt{3}}{2} \end{cases} \Rightarrow \begin{cases} v_1 = 3.0\ \mathrm{m\ s^{-1}} \\ v_2 \approx 2.6\ \mathrm{m\ s^{-1}} \end{cases}$$

□

6.3 end-of-chapter questions

From this definition, we see that the force acting depends on the magnitude of the change in momentum, but also depends on how long this change occurs. , it would be silly to land on the ground with stiff legs. You naturally bend your knees. Acrobats in a circus fall on a soft mat or a safety net. All these actions does not change the impulse, but they increase the time of contact, and therefore reduce the force that might cause harmful injuries.

force & momentum

Question 6.1 When speed cars run out of control in a racing game, why are they stopped by haystacks instead of concrete walls? When you jump from an elevated position and land on the ground, you naturally bend your knees instead of keeping your legs stiff. How does that reduce the chance of causing harmful injuries?

Question 6.2 Automobiles were manufactured to be as rigid as possible, but nowadays many cars are designed to crumple upon impact. Can you explain why?

principle of momentum conservation

Question 6.3 How does conservation of momentum apply to a ball bouncing off a wall?

Question 6.4 In the comic hero series, Superman hurls an asteroid in outer space, and he is seen at rest after the throw. What law of physics is violated here? If the asteroid is 100 times as massive as the superhero, and it is thrown at 50 m s^{-1}. What is Superman's velocity right after the throw?

collision problems

Question 6.5 When a piece of putty falls and hits the floor without bouncing, what becomes of its momentum before impact? What becomes of its kinetic energy?

Chapter 7

Work & Energy

We have considered in the previous chapter the accumulative effect of a force over a period of time and have seen how this gives rise to the idea of impulse and momentum. In this chapter, we consider the effect of a force over a certain displacement, and you will learn how this is related to the concept of work done and energy.

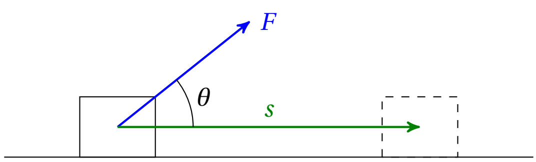
Energy is a concept central to all of physical sciences. The entire universe is made up by energy and matter. In this section we study the energy changes during various physical processes.

7.1 work

work done by a force is defined as the product of the force and the displacement moved out in the direction of the force: $\boxed{W = Fs}$

- unit for work done: $[W] = [F][s] = \mathrm{N}\cdot\mathrm{m} \equiv \mathrm{J}$ (joule)

  if a one newton force makes an object move out by one metre, then it does work of one **joule**
- if force acts at angle θ to the displacement travelled, then: $\boxed{W = Fs\cos\theta}$



- work is a *scalar* quantity[43], i.e., it has no direction
- work done can be either positive or negative

  resistive forces, such as friction and air drag, act in the opposite direction to motion

[43] Although work is defined as the product of two vectors, work carries no information about direction. This vector product is called as a *scalar product* or a *dot product*, which can be written explicitly as: $W = \vec{F}\cdot\vec{s} = |F||s|\cos\theta$. You might have seen this operation in the A-Level course in Mathematics.

so work done by resistive forces is negative[44], we say this is work *against* resistance

the minus sign will be crucial in energy calculations in later discussions

- a gas can do work to/against the surroundings

  if pressure stays constant, then work by/on gas: $W = F\Delta s = pA\Delta s \quad \Rightarrow \quad \boxed{W_{\text{gas}} = p\Delta V}$

- work done by a varying force[45] is found by integration or using a F-s graph

  if force varies with position, its change over small displacements is still considered small

  work done over an infinitesimal displacement is therefore $\mathrm{d}W = F\mathrm{d}s$

  integrate from initial position to final position, total work done is given by: $\boxed{W = \int_i^f F\mathrm{d}s}$

  if one plots force against displacement, then $\boxed{\text{area under } F\text{-}s \text{ curve gives work done}}$

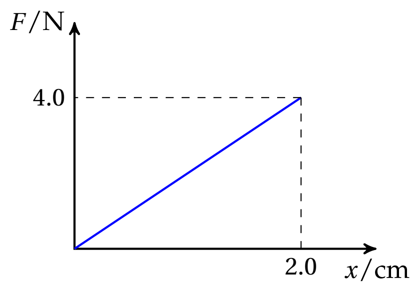
Example 7.1 A 20 N force is applied at 60° to the horizontal to move a 1.0 kg object at a constant speed of 2.0 m s^{-1} for 30 s. How much work is done by the force?

✍ $$W = Fs\cos\theta = Fvt\cos\theta = 20 \times 2.0 \times 30 \times \cos 60^\circ \quad \Rightarrow \quad W = 600 \text{ J}$$ □

Example 7.2 A piston in a gas pump has an area of 600 cm^2. During one stroke, the pump moves a distance of 30 cm against a constant pressure of 8000 Pa. How much work is done?

✍ $$W = p\Delta V = pA\Delta s = 8000 \times 600 \times 10^{-4} \times 30 \times 10^{-2} \quad \Rightarrow \quad W = 144 \text{ J}$$ □

Example 7.3 When a spring is compressed by 2.0 cm, the force applied increases uniformly from zero to 4.0 N. How much work is done by this force?



✍ F-x graph for the force is plotted as shown

work to compress spring equals area under F-x graph:

$$W = \frac{1}{2} \times 4.0 \times 2.0 \times 10^{-2} = 0.040 \text{ J}$$ □

7.2 types of energies

Energy is something acquired by an object that enables it to do work. A moving vehicle, water stored in a reservoir, a compressed spring, separated magnets, all of these objects can do work to other objects. In this section, we will look at various situations where work on an object causes a change in some form of energy.

[44]You might take $\theta = 180^\circ$, then $\cos\theta = -1$, giving rise to a negative work done.

[45]The equation $W = Fs$ is valid only if the force acting is constant.

Similarly, the equation $W_{\text{gas}} = p\Delta V$ holds for constant pressure processes only.

7.2.1 kinetic energy

suppose a constant force F is acting over a distance s, we have:

$$W = Fs \xlongequal{F=ma} mas \xlongequal{v^2=u^2+2as} m\frac{v^2-u^2}{2} \quad \Rightarrow \quad W = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$$

this shows work is transformed into change in some quantity associated with object's motion

this is recognised as the gain in kinetic energy of the object: $\Delta E_k = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$

> **kinetic energy** (K.E.) is the energy possessed by an object due to its motion

- an object of mass m moving with speed v has K.E.: $\boxed{E_k = \frac{1}{2}mv^2}$

Example 7.4 Estimate the kinetic energy of a running man.

✍ suppose the man has a mass of 75 kg and is running at 5 m s^{-1} (any reasonable value will do)

$$E_k = \frac{1}{2}mv^2 = \frac{1}{2} \times 75 \times 5^2 \approx 940 \text{ J}$$ □

7.2.2 gravitational potential energy

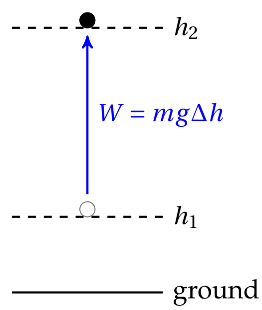
consider a body being slowly pulled from a height of h_1 to h_2

work done for this process is: $W = Fs = mg\Delta h = mgh_2 - mgh_1$

this shows work is transformed into change in some quantity associated with object's position

we say this is the gain in gravitational potential energy:

$$\Delta E_p = mgh_2 - mgh_1$$



> **gravitational potential energy** (G.P.E.) is the energy possessed by a body due to its position in a gravitational field

- a body of mass m at a height of h has G.P.E.: $\boxed{E_p = mgh}$
- G.P.E. is a *relative* quantity, only its change is important in physical processes [46]

[46] The formula $E_p = mgh$ implies that we have defined G.P.E. at the ground level to be zero. But this is purely conventional. Here I would like to point out that one can freely choose any zero potential energy level as he/she wishes, but no matter what point is picked as reference, we will always agree on the quantity of physical significance, that is, the *change* in G.P.E between two fixed points.

7.2.3 elastic potential energy

let's now consider a spring being stretched or compressed

work is done by external forces to cause the change in shape

this becomes of the elastic energy stored in the body

> **elastic potential energy**, also called **strain energy**, is the energy possessed by an elastic body due to deformation

this topic will be revisited in details in §**??**

7.2.4 other types of energies

apart from those we have mentioned above, there are many other types of energies

- *electric potential energy*: energy of a charged object due to its position in an electric field
- *chemical energy*: ability to do work due to potential energy between atoms and molecules
- *nuclear energy*: ability to do work due to potential energy of subatomic particles in the nuclei
- *internal energy*: sum of random kinetic and potential energies of molecules in a substance
- *electromagnetic energy*: energy carried by light/electromagnetic waves

7.2.5 work & energy transformations

from previous discussions, we have seen doing work is a way of transferring energy

for examination purposes[47], we can say the following:

> the change in the total energy of an object equals the net work done by all external forces (excluding those associated with potential energies): $\boxed{W = \Delta E}$

Example 7.5 A racing car of 800 kg starts off from rest. If the driving force is 5000 N, and the car experiences a constant resistive force of 1500 N, what is its speed after it has travelled 50 m?

✍ gain in K.E. equals work by driving force plus *negative* work against resistance

$$W_{\text{total}} = \Delta E_k \quad \Rightarrow \quad Fs - fs = \frac{1}{2}mv^2 - 0 \quad \Rightarrow \quad (5000-1500)\times 50 = \frac{1}{2}\times 800\times v^2 \quad \Rightarrow \quad v \approx 20.9 \text{ m s}^{-1} \quad \square$$

Example 7.6 A concrete cube of side 0.50 m and density 2400 kg m^{-3} is lifted 4.0 m by a crane. How much work is done?

[47]More rigorous discussions (which go beyond the syllabus) are given in §7.3.

✍ work by crane equals gain in G.P.E. of cube

$$W = \Delta E_p \quad \Rightarrow \quad W = mg\Delta h = \rho V g \Delta h = 2400 \times 0.50^3 \times 9.81 \times 4.0 \approx 1.18 \times 10^4 \text{ J} \qquad \square$$

7.3 conservation of energy

7.3.1 work-energy theorem (⋆)

more generally, net work done due to several forces $F_1, F_2, \ldots,$ acting on an object is

$$W_{\text{total}} = \sum W_i = \sum \left(\int_i^f F_i \mathrm{d}s \right) = \int_i^f \left(\sum F_i \right) \mathrm{d}s = \int_i^f F_{\text{net}} \mathrm{d}s = \int_i^f ma \mathrm{d}s$$

recall in kinematics, $a = \frac{\mathrm{d}v}{\mathrm{d}t}$ and $\mathrm{d}s = v\mathrm{d}t$, so we have

$$W_{\text{total}} = \int_i^f m \frac{\mathrm{d}v}{\mathrm{d}t} v \mathrm{d}t = \int_i^f mv \mathrm{d}v$$

we are integrating over velocity from initial value u to final value v, wo we find

$$W_{\text{total}} = \int_i^f mv \mathrm{d}v = \frac{1}{2} mv^2 \Big|_u^v = \frac{1}{2} mv^2 - \frac{1}{2} mu^2$$

identify kinetic energy $E_k = \frac{1}{2} mv^2$, now we come to the **work-energy theorem**

> the net work done on a body equals the change in the body's kinetic energy: $\boxed{W_{\text{total}} = \Delta E_k}$

7.3.2 conservative forces & potential energy (⋆)

if work by a force does not depend on which specific path is taken, this force is **conservative**

put differently, when an object is moved from one place to another under a conservative force, work done depends on initial and final positions only

we can split W_{total} into two parts: contributions from conservative and non-conservative forces

for conservative part, let's define: $\Delta E_p = -W_c = -\int_i^f F_c \mathrm{d}s$

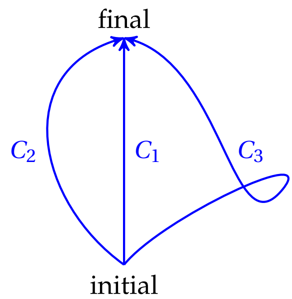
this means work by conservative forces can be interpreted as change in its **potential energy**

let's call the sum of kinetic and potential energy of a body its **mechanical energy**: $E_m = E_k + E_p$

work-energy theorem can then be rewritten as: $W_{\text{nc}} = \Delta E_k + \Delta E_p = \Delta E_m$

> the change in mechanical energy equals the net work done by all non-conservative forces

gravitational potential energy revisited (⋆)



work by gravitational force is path independent

for example, same work is done via path C_1, C_2 and C_3

so gravitational force is conservative, G.P.E. can be defined

as a consequence, gain or loss in G.P.E. only depends on the difference in initial and final position of the object

if there is no other conservative force acting on a body apart from gravitational force, it follows that change in total energy of an object equals the net work done by all forces excluding gravity

7.3.3 conservation of energy

> **the law of conservation of energy** states that energy cannot be created or destroyed, but can only transform from one form into another while the total amount is always constant

in absence of any non-conservative force, sum of a body's kinetic energy and potential energies, or the total mechanical energy, is constant, this is also a law of conservation

Example 7.7 For an object falling from rest due to gravity, if air resistance is negligible, what is its speed when it has fallen through a distance of h?

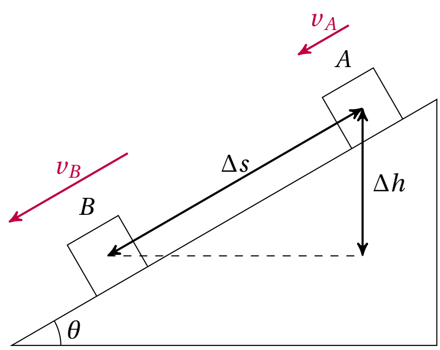
✍ $$\text{G.P.E loss} = \text{K.E gain} \quad \Rightarrow \quad mgh = \frac{1}{2}mv^2 \quad \Rightarrow \quad v = \sqrt{2gh}$$ □

Example 7.8 A marble is projected vertically upwards with an initial velocity u. The average resistive force acting is f. How do you determine the maximum height reached by the marble?

✍ K.E. loss = G.P.E. gain + energy loss due to resistance

$$\frac{1}{2}mu^2 - 0 = mgH_{\max} + fH_{\max} \quad \Rightarrow \quad H_{\max} = \frac{mu^2}{2(mg+f)}$$ □

Example 7.9 A box of mass m slides down along a slope that is inclined at an angle θ to the horizontal. There is a constant friction f acting on the box. When the box has moved through a distance of Δs down the slope from A to B, write down an equation relating its velocities v_A and v_B by applying the law of conservation of energy.



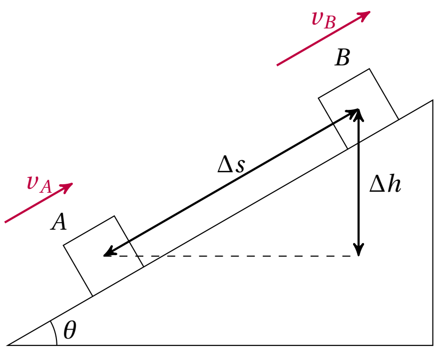
✍
$$\begin{aligned}
\text{work done against friction} &= \text{change in total energy}\\
-W_f &= \Delta E_k + \Delta E_p\\
-fs &= \left(\frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2\right) + (mgh_B - mgh_A)\\
-fs &= \left(\frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2\right) - mg\Delta h\\
v_B^2 &= v_A^2 + 2g\Delta s\sin\theta - \frac{2fs}{m}
\end{aligned}$$

or equivalently, we can write

$$\begin{aligned}
\text{G.P.E loss} &= \text{K.E gain} + \text{energy loss due to friction}\\
mg\Delta h &= \left(\frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2\right) + fs\\
v_B^2 &= v_A^2 + 2g\Delta s\sin\theta - \frac{2fs}{m}
\end{aligned}$$

note that the two alternative ways of thinking produce the same result □

Example 7.10 A slope is inclined at an angle θ to the horizontal. A box of mass m is pushed up the slope with a constant force F parallel to the slope, and the box experiences a constant frictional force f. When the box has moved through a distance of Δs along the slope from A to B, find an equation relating its velocities v_A and v_B.



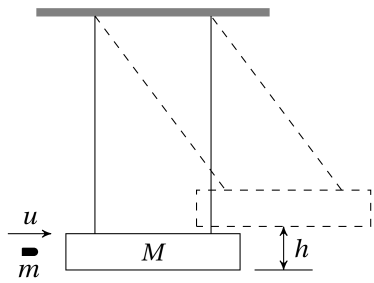
✍
$$\begin{aligned}
\text{work by } F + \text{work against friction} &= \text{change in total energy}\\
W_F - W_f &= \Delta E_k + \Delta E_p\\
Fs - fs &= \left(\frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2\right) + (mgh_B - mgh_A)\\
Fs - fs &= \left(\frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2\right) + mg\Delta h\\
v_B^2 &= v_A^2 + \frac{2(F-f)s}{m} - 2g\Delta s\sin\theta
\end{aligned}$$

or equivalently, we can write

$$\begin{aligned}
\text{work by } F &= \text{K.E gain} + \text{G.P.E gain} + \text{energy loss due to friction}\\
Fs &= \left(\frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2\right) + mg\Delta h + fs\\
v_B^2 &= v_A^2 + \frac{2(F-f)s}{m} - 2g\Delta s\sin\theta
\end{aligned}$$

again the two approaches produce the same expression for final velocity □

Example 7.11 A *ballistic pendulum* is a device used to measure the speeds of fast-moving bullets. It consists of a large block of wood of mass M, suspended from two long light strings. A bullet of mass m is fired into the block, and the block and bullet combination swings upward. If the centre of mass rises a vertical distance h, what is the initial speed u of the bullet?



✍ as bullet enters block, combined momentum is conserved:

$$mu = (M+m)v \quad \Rightarrow \quad v = \frac{mu}{M+m}$$

when the system swings upward, K.E. transforms into G.P.E. but total energy is conserved:

$$\frac{1}{2}(M+m)v^2 = (M+g)gh \quad \Rightarrow \quad v = \sqrt{2gh}$$

putting the two equations together, we find: $u = \left(1 + \frac{M}{m}\right)\sqrt{2gh}$ □

7.4 power

to describe how fast work is done, we introduce the notion of power

power is defined as the work done per unit time: $\boxed{P = \frac{\Delta W}{\Delta t}}$

➢ unit of power: $[P] = \frac{[W]}{[t]} = \mathrm{J\,s^{-1}} = \mathrm{W}$ (watt)

if one joule of work is done in one second, the power is one **watt**

➢ $P = \frac{\Delta W}{\Delta t}$ gives the *average* power during in a period of time Δt

to find the *instantaneous* power at a particular moment, we have

$$P = \frac{\Delta W}{\Delta t} = \frac{F\Delta s}{\Delta t} \quad \Rightarrow \quad \boxed{P = Fv}$$

Example 7.12 There are 150 steps to the top of a tower, and the average height of each step is 25 cm. It takes a man of 72 kg two minutes to run up all the steps. What is his average power?

✍ $$P = \frac{\Delta E_p}{\Delta t} = \frac{mg\Delta h}{\Delta t} = \frac{72 \times 9.81 \times (150 \times 0.25)}{120} \quad \Rightarrow \quad P \approx 221\ \mathrm{W}$$ □

Example 7.13 A turbine is used to generate electrical power from the wind. Given that the blades of the turbine sweep an area of 500 $\mathrm{m^2}$, the density of air is 1.3 kg $\mathrm{m^{-3}}$, and the wind speed is 10 m $\mathrm{s^{-1}}$. Assume no energy loss, find the power available from the wind.

✍ $$P = \frac{\Delta E_k}{\Delta t} = \frac{\frac{1}{2}\Delta m v^2}{\Delta t} = \frac{\frac{1}{2}\rho \Delta V v^2}{\Delta t} = \frac{\frac{1}{2}\rho A \Delta x v^2}{\Delta t} \quad \Rightarrow \quad P = \frac{1}{2}\rho A v^3$$

$$P = \frac{1}{2} \times 1.3 \times 500 \times 10^3 \approx 3.25 \times 10^5 \text{ W} \qquad \square$$

Example 7.14 A ship is cruising a a constant speed of 15 m s^{-1}. The total resistive force acting is 9000 N. What is the output power of this ship?

✍ constant speed so equilibrium between driving force and resistive force

$$P = Fv = fv = 9000 \times 15 \quad \Rightarrow \quad P = 1.35 \times 10^5 \text{ W} \qquad \square$$

Example 7.15 A car of mass 800kg accelerates from rest on a horizontal road. Suppose the engine provides a constant power of 24000 W, and the resistive force can be given by $f = 16v$, where v is the speed of the car in m s^{-1}. (a) What is the acceleration of the car when it is travelling at 15 m s^{-1}? (b) What happens to the car if it maintains this driving power?

✍ equation of motion for the car is: $F_{\text{net}} = F - f = ma \quad \Rightarrow \quad \frac{P}{v} - \alpha v = ma$

at $v = 15$ m s^{-1}: $\frac{24000}{15} - 16 \times 15 = 800a \quad \Rightarrow \quad a = 1.7$ m s^{-2}

as car's velocity v increases, driving force $F = \frac{P}{v}$ decreases, resistive force $f = \alpha v$ increases

so resultant force will decrease, the car will accelerate at a decreasing acceleration

eventually it reaches an equilibrium state where $F = f$, the car then travels at constant speed v_t

$$F = f \quad \Rightarrow \quad \frac{P}{v_t} = \alpha v_t \quad \Rightarrow \quad \frac{24000}{v_t} = 16 v_t \quad \Rightarrow \quad v_t \approx 38.7 \text{ m s}^{-1} \qquad \square$$

7.5 efficiency

efficiency of a system is given by: efficiency $= \frac{\text{useful energy output}}{\text{total energy input}}$, or $\boxed{\eta = \frac{W_{\text{useful}}}{W_{\text{total}}}}$

since $\Delta W = P\Delta t$, efficiency can also be evaluated in terms of power: $\boxed{\eta = \frac{P_{\text{useful}}}{P_{\text{total}}}}$

Example 7.16 A water pumping system uses 3.0 kW of electrical power to raise water from a well. The pump lifts 1500 kg of water per minute through a vertical height of 8.0 m. What is the efficiency of the sysem?

✍
$$\eta = \frac{\Delta E_p}{\Delta E_{\text{in}}} = \frac{\Delta mgh}{P_{\text{in}}\Delta t} = \frac{1500 \times 9.81 \times 8.0}{3000 \times 60} \quad \Rightarrow \quad \eta \approx 65.4\% \qquad \square$$

Example 7.17 Water flows into a turbine from a reservoir at a vertical distance of 70 m above. The water flows through the turbine at a rate of 2500 kg per minute. What is the output power of the turbine if it is 85% efficient?

✍
$$P_{\text{out}} = \eta P_{\text{in}} = \eta \frac{\Delta E_p}{\Delta t} = \eta \frac{\Delta mgh}{\Delta t} = 85\% \times \frac{2500 \times 9.81 \times 70}{60} \quad \Rightarrow \quad P_{\text{out}} \approx 2.43 \times 10^4 \text{ W}$$

7.6 end-of-chapter questions

work

Question 7.1 A trolley is pushed through a distance of 2.0 m with a force of $F = 5.0$ N along a track. The trolley experiences a constant frictional force of 3.0 N. What is the work done by F?

Question 7.2 A child of mass 40 kg slides down a slope from a height of 2.0 m above the ground. The slide is of a length of 6.0 m. How much work is done by gravity?

Question 7.3 A dog pulls on a lead with a force of 20 N at an angle of 20° to the horizontal. As the dog moves 10 m along the playground, find the work done (a) by the dog, (b) by the person holding the lead.

Question 7.4 A satellite is orbiting around the earth in a circular orbit due to gravitational attraction. The gravitational force on the satellite always acts towards the centre of the earth. Does this force do any work?

Question 7.5 A fixed mass of gas at a pressure of 1.50×10^5 Pa and initial volume of 2.80×10^{-4} m^3 is heated. The gas expands at a constant pressure to a final volume of 8.40×10^{-4} m^3. Find the work done by the gas.

kinetic energy

Question 7.6 Estimate the kinetic energy of a family car travelling at 40 km per hour.

Question 7.7 Object B has double the mass and moves at twice the speed of object A. If the kinetic energy of object A is K, what is the kinetic enery of B?

gravitational potential energy

Question 7.8 Estimate how much gravitational potential energy you gain when you get to the top of the highest building in your country?

Question 7.9 Four uniform bricks, each of mass m and thickness H, are laid out on a table. In order to stack them on top of one another, how much work has to be done on the bricks?

work & energy transformations

Question 7.10 A stone is projected vertically upwards at a speed of 16 m s^{-1} from the ground. Air resistance is negligible. (a) What is the greatest height reached by the stone? (b) When the stone is at a height of 5.0 m, what is its speed? (c) If the stone is projected at an angle from the upward vertical with the same initial speed, what is its speed when it reaches a height ob 5.0 m?

Question 7.11 A hammer of mass 600 g hits a nail at a velocity of 10 m s^{-1}. The nail is pushed into a plank by 3.0 mm. What is the average frictional force acting on the nail?

Question 7.12 A block of mass m is pushed through a fixed distance along a horizontal frictionless surface by a constant force. Show that the final speed v of the block has: $v \propto \frac{1}{\sqrt{m}}$.

Question 7.13 A block of 3.0 kg is released from rest on a slope at an angle $\theta = \sin^{-1}\left(\frac{1}{5}\right)$ to the horizontal. As the block travels 6.0 m down the slope, it experiences a frictional force of 5.0 N. What is the final speed of the block?

Question 7.14 A cyclist is travelling up a hill at a constant speed. The cyclist uses 640 J of energy to travel a distance of 25 m. If the total resistive force opposing the motion is 9.0 N, what is the increase in gravitational potential energy? To determine the increase in the height, what further information do you need?

Question 7.15 A pendulum of mass 120 g is released from a position that is 1.6 cm above its equilibrium position. (a) At what point in its motion is the kinetic energy of a pendulum bob at its maximum? (b) At what point is the gravitational potential energy at a maximum? (c) When the kinetic energy is half its greatest value, how much gravitational potential energy does it have?

Question 7.16 A particle of mass m is initially at a height of h above the ground. It is then released from rest. Just before hitting the ground, the particle gains a speed of v. What is the average resistive force acting on the particle during the fall?

Question 7.17 Men's pole vault (a athletic event in which a person jumps over a bar with the aid of a long, flexible pole) world record is about 6 m. Could this record be raised to, say, 10 m by using a longer pole? If not, why is it impossible?

Question 7.18 This question concerns the design of a roller coaster. One designer says each summit must be lower than the previous one. Another designer suggests that it does not matter what heights the summits are as long as the first one is the highest. What do you think?

conservative forces ($\star$)

Question 7.19 Explain why gravitational force is conservative.

Question 7.20 A charged object is acted by an electric force in an electric field. State and explain whether the electric force is conservative.

Question 7.21 Is friction a conservative force? Explain your reasons.

power

Question 7.22 A car engine exerts an average force of 400 N in moving the car 900 m in 200 s. What is the average power developed?

Question 7.23 During a human heart beat, about 20 g of blood is pushed into the main arteries. This blood is accelerated from a speed of 0.20 m s^{-1} to 0.35 m s^{-1}. For a heart pulsing at 75 beats per minute, what is the average power developed by the heart?

Question 7.24 Estimate your body power when you run upstairs at full speed.

Question 7.25 A truck of mass 2700 kg is travelling at a constant speed of 9.0 m s^{-1} up a road that is inclined at 8.0° to the horizontal. Assume that the resistive forces are negligible. What is the useful power from the engine of the truck?

Question 7.26 Given that the forces resisting the motion of a racing car is proportional to the square of the car's speed. The car has an output power of 200 kW when travelling at a steady speed of 50 m s^{-1}. What output power is required to maintain a speed of 80 m s^{-1}?

Question 7.27 A car of mass 800 kg travels in a straight line up a slope. The total resistive force f_R can be modelled by the equation: $f_R = kv^2$, where constant $k = 5.0$ kg m^{-1} and v is the car's speed. When the car travels at a steady speed of 10 m s^{-1}, the engine exerts a force of 2.1 kN up the slope. (a) Find the component of the car's weight down the slope, and hence find the angle that the slope makes with the horizontal. (b) Find the power output from the engine. (c) If the car then travels onto a horizontal road. The engine's output power is unchanged and the resistive force obeys the same model as before. Find the acceleration of the car when its speed is 15 m s^{-1}. (d) The car eventually reaches a constant speed on the horizontal road. Find this terminal speed.

efficiency

Question 7.28 An electric motor is used to lift a mass. When operating at full power, the current in the motor is 2.0 A and the voltage is 5.0 V. If the motor is 50% efficient, what is the time take to lift a mass of 400 g through a height of 2.5 m?

Question 7.29 A turbine at a hydroelectric power station is designed to drive a generator to produce electrical energy. The water falls through a vertical distance of 8.0 m at a rate of 150 kg per second as it passes through the turbine. The generator supplies a current of 24 A at a voltage of 220 V. What is the efficiency of the turbine system?

Question 7.30 When power to the grid is not required during the night, 200 MW of electrical power is used to pump water from a reservoir up to a lake 300 m higher. The pumping system operates at an efficiency of 90%. What mass of water can be pumped to the lake in three hours?

Question 7.31 Given that the petrol engine for a real car is 25% efficient. The fuel consumption for the engine is 16 litres per hour, and the energy density of the fuel is 44 MJ per litre. What is the power output from the engine?

Question 7.32 A bow shoots an arrow of mass 150 g vertically upwards with an initial speed of 30 m s^{-1}. The potential energy stored in the bow before release is 200 J. The arrow reaches a height of 30 m above the point of release. (a) What is the energy loss just after the arrow is released? (b) What is the efficiency of the bow for converting its potential energy into useful kinetic energy? (c) What is the energy loss due to air resistance? (d) What is the average force due to air resistance?

Chapter 8
Solids

in this chapter, we study how a force changes the shape of an object

an important notion is the elasticity of materials

for **elastic** materials, when external force is removed, it can return to its original shape

if the material cannot restore to original shape, it is said to be **inelastic**, or **plastic**

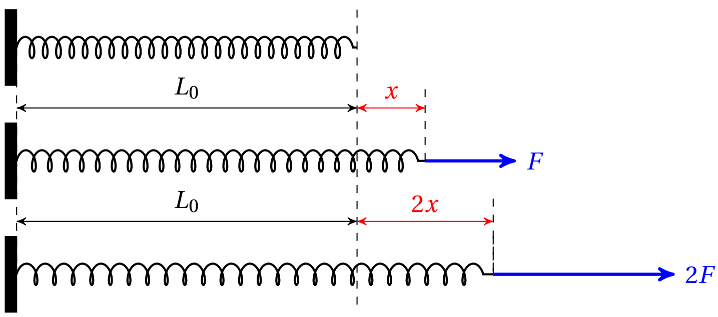
8.1 springs

8.1.1 Hooke's law

when a force F is applied to a spring, it is stretched from original length L_0 to some length L

extension of a spring, $x = L - L_0$, is dependent on the force applied

> extension of an ideal spring is directly proportional to the load applied (within a certain range), this is called **Hooke's law**



- Hooke's law can be summarised by the equation: $\boxed{F = kx}$
- the proportionality constant k is called the **spring constant**

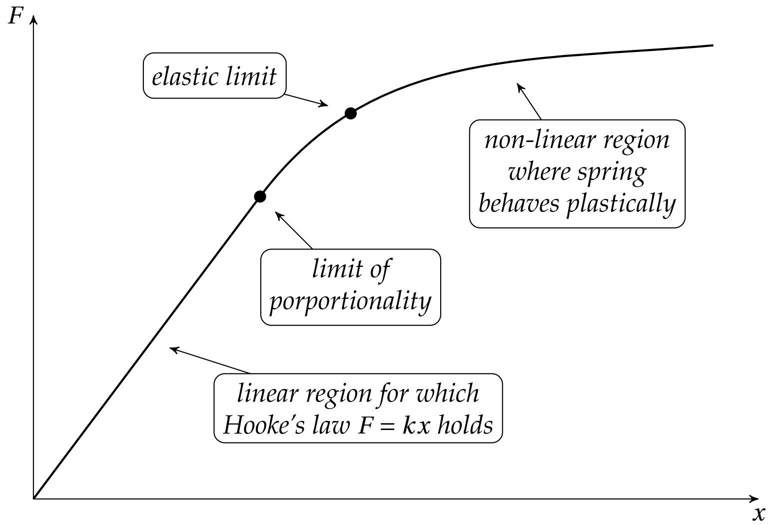
  larger k means a greater force is required to extend the spring by same amount

  a spring with a large k is said to *stiff*
- linear relationship between F and x is only true up to a certain range

the limit at which Hooke's law no longer holds is called the **limit of proportionality**

➢ if load is too large, spring might be overstretched and no longer exhibits elastic behaviour

the point beyond which spring cannot return to original length is called the **elastic limit** [48]



force-extension graph for a typical spring under load

Example 8.1 A spring has a natural length of 20.0 cm. When a mass of 250 g is suspended from the spring, the new length of the spring is 26.0 cm. Find the spring constant.

✍ $$k = \frac{F}{x} = \frac{mg}{L - L_0} = \frac{0.250 \times 9.81}{(26.0 - 20.0) \times 10^{-2}} \quad \Rightarrow \quad k \approx 40.9\ \mathrm{N\ m^{-1}}$$ □

Example 8.2 A spring has a spring constant of 270 N m^{-1}. A mass of 1.2 kg is hung from the spring. When the mass is released from a position where the spring has an extension of 5.0 cm, what is the acceleration of the mass?

✍ $$F_{\text{net}} = F - mg = kx - mg = ma \quad \Rightarrow \quad a = \frac{kx - mg}{m} = \frac{270 \times 5.0 \times 10^{-2} - 1.2 \times 9.81}{1.2} \approx 1.44\ \mathrm{m\ s^{-2}}$$ □

8.1.2 elastic potential energy in a spring

to stretch or compress a spring, work must be done

this becomes of *elastic potential energy* stored in the spring

[48]The elastic limit of a spring and the limit of proportionality are two different but always confused concepts. These two limits are usually very close to one another, but they are conceptually different.

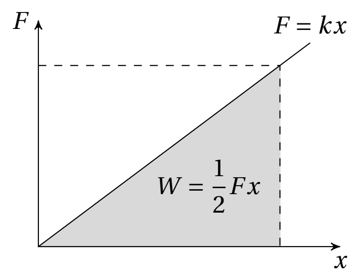
note that force in spring varies as spring is stretched

to find work in stretching a spring by x, we compute area under F-x graph[49], which is a right-angled triangle

$$W=\frac{1}{2}Fx=\frac{1}{2}kx^2$$

so elastic potential energy stored in a spring is:

$$\boxed{E_p=\frac{1}{2}kx^2}$$



Example 8.3 A steel spring has a spring constant of 20 N cm^{-1}. How much work is needed to stretch it from an extension of 3.0 cm to an extension of 5.0 cm?

✍ work done needed equals the increase in elastic potential energy:

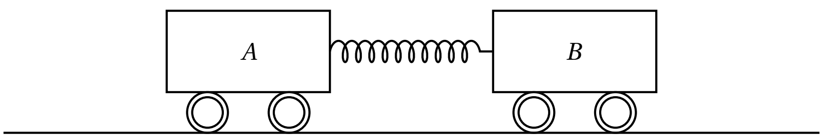
$$W=\Delta E_p=\frac{1}{2}kx_2^2-\frac{1}{2}kx_1^2=\frac{1}{2}\times 2000\times(0.050^2-0.030^2)=1.6\text{ J}$$ □

Example 8.4 A trolley of 400 g can freely travel along a horizontal surface. It is pushed against a spring buffer. Suppose the spring is initially compressed by 5.0 cm under a 20 N force. When the trolley is released, it accelerates until it becomes detached. What is the trolley's final speed?

✍ elastic potential energy in spring transforms into kinetic energy of trolley

$$\frac{1}{2}Fx=\frac{1}{2}mv^2 \quad\Rightarrow\quad \frac{1}{2}\times 20\times 0.050=\frac{1}{2}\times 0.40\times v^2 \quad\Rightarrow\quad v\approx 1.58\text{ m s}^{-1}$$ □

Example 8.5 The same spring is now set between two trolleys A and B of mass 400 g and 600 g. Initially the spring is again compressed by 5.0 cm under a force of 20 N. After both trolleys are released, what are their final speeds?



✍ elastic potential energy in spring transforms into kinetic energy of the two trolleys:

$$\frac{1}{2}Fx=\frac{1}{2}m_Av_A^2+\frac{1}{2}m_Bv_B^2 \quad\Rightarrow\quad 20\times 0.050=0.40v_A^2+0.60v_B^2$$

total momentum for trolley A and B as a whole is conserved:

$$m_Bv_B-m_Av_A=0 \quad\Rightarrow\quad 0.40v_A=0.60v_B$$

solving the simultaneous equations, we find: $v_A\approx 1.22\text{ m s}^{-1}$, $v_B\approx 0.82\text{ m s}^{-1}$ □

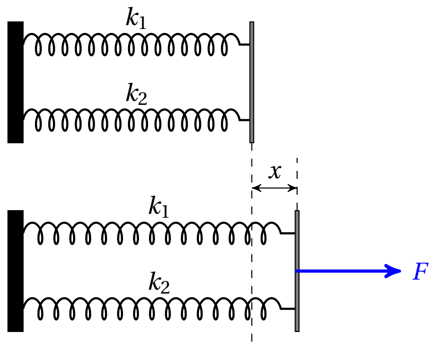
[49]Mathematically, we can also integrate over the total extension to find this work done $W=\int_0^x F\mathrm{d}x=\int_0^x kx\mathrm{d}x=\frac{1}{2}kx^2$, which of course gives the same result.

8.1.3 spring combinations

so far we have discussed the properties of a single spring

next we investigate how a set of different springs respond to a given load

parallel springs



let's take two springs connected in parallel

when the combination is stretched under a load of F, extension in each spring should be the same:

$$x_1 = x_2 = x$$

the forces in each spring will in general be different, but sum of these must be equal to load:

$$F = F_1 + F_2$$

divide both sides by x, we have: $\dfrac{F}{x} = \dfrac{F_1}{x_1} + \dfrac{F_2}{x_2}$.

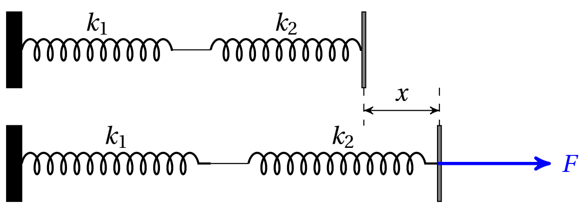
recall the Hooke's law, this becomes: $k = k_1 + k_2$

generalize for n springs in parallel connection, the combined spring constant is

$$\boxed{k = k_1 + k_2 + \cdots + k_n}$$

series springs

let's now take two springs in series



force in each spring is the same: $F_1 = F_2 = F$

but total extension is the sum of individual extensions: $x = x_1 + x_2$.

divide by the same F, we have: $\dfrac{x}{F} = \dfrac{x_1}{F_1} + \dfrac{x_2}{F_2}$.

for combined spring constant, we find: $\dfrac{1}{k} = \dfrac{1}{k_1} + \dfrac{1}{k_2}$

for n springs connected in series, the combined spring constant is therefore given by

$$\boxed{\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2} + \cdots \frac{1}{k_n}}$$

some brief remarks

➢ if we set up n springs in parallel, we actually make it thicker

it then requires a stronger force to stretch it by the same extension

so we indeed see the combined spring constant is greater than that of any individuals

➢ if we set up n springs in series, we make it longer

instead of pulling one spring at a time, the same force now stretches n springs simultaneously

this gives rise to a greater total extension

so the combined spring constant must be less than that of any individual

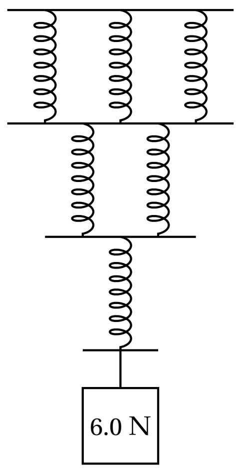
Example 8.6 (a) A spring with $k_1 = 20\ \text{N cm}^{-1}$ is connected in series with a second spring with $k_2 = 30\ \text{N cm}^{-1}$. When a force of 60 N is applied, what is the total extension of the combination? (b) The same two springs are now connected in parallel. When a force of 50 N is applied on the combination, what is the extension?

✍ for series connection: $k = \left(\frac{1}{k_1} + \frac{1}{k_1}\right)^{-1} = \left(\frac{1}{20} + \frac{1}{30}\right)^{-1} = 12\ \text{N cm}^{-1} \quad \Rightarrow \quad x = \frac{F}{k} = \frac{60}{12} = 5.0\ \text{cm}$

or sum up extension of each spring: $x = x_1 + x_2 = \frac{F}{k_1} + \frac{F}{k_2} = \frac{60}{20} + \frac{60}{30} = 5.0\ \text{cm}$

for parallel connection: $k = k_1 + k_2 = 20 + 30 = 50\ \text{N cm}^{-1} \quad \Rightarrow \quad x = \frac{F}{k} = \frac{50}{50} = 1.0\ \text{cm}$ □

Example 8.7 A set of identical springs are set up as shown. Each individual spring extends by 1.0 cm under a load of 1.0 N. Assume the limit of proportionality is not exceeded, what is the total extension for this combination when a load of 6.0 N is applied?



✍ for a single spring: $k = 1.0\ \text{N cm}^{-1}$

combined spring constant: $k_{\text{total}} = \left(\frac{1}{3k} + \frac{1}{2k} + \frac{1}{k}\right)^{-1}$

$$k_{\text{total}} = \left(\frac{1}{3.0} + \frac{1}{2.0} + \frac{1}{1.0}\right)^{-1} = \frac{11}{6} \approx 1.83\ \text{N cm}^{-1}$$

total extension: $x = \frac{F}{k_{\text{total}}} = \frac{6.0}{1.83} = 11\ \text{cm}$

alternatively, we can find and add extensions of each layer

in particular, the top layer withstands a force of 6.0 N shared by three springs, so each spring has a force of 2.0 N, extension is 2.0 cm

similarly, the other two layers extend by 3.0 cm and 6.0 cm respectively

hence, total extension: $x = 2.0 + 3.0 + 6.0 = 11\ \text{cm}$ □

8.2 stress, strain & Young modulus

from daily experience, it is easier to stretch a longer wire than a shorter one

same tensile force also produce greater effects on a thinner material than on a thicker one

so we need better quantities to describe the amount of deformation and the amount of action

to study how a material responds to a tensile force, several new quantities are to be introduced

8.2.1 stress & strain

tensile strain is defined as the ratio of the extension to the natural length of a wire: $\boxed{\epsilon = \frac{x}{L}}$

tensile stress is defined as the force applied per unit cross-sectional area: $\boxed{\sigma = \frac{F}{A}}$

- strain is the ratio of two lengths, so it is unit free

  strain is usually expressed in terms of a percentage number
- units of stress: $[\sigma] = \mathrm{N\,m^{-2}} = \mathrm{Pa}$ (pascal)
- elastic behaviour of a wire is more or less like that of a spring

  Hooke's law says $F \propto x$, it then follows that $\frac{F}{A} \propto \frac{x}{L}$, so $\sigma \propto \epsilon$

  i.e., stress and strain should also be proportional to each other within certain limit

Example 8.8 A copper wire has a cross-sectional area of $1.5 \times 10^{-6}\ \mathrm{m^2}$. The breaking stress of the wire is 2.0×10^{8} Pa. Find the breaking force.

✍
$$\sigma = \frac{F}{A} \quad \Rightarrow \quad F = \sigma A = 2.0 \times 10^{8} \times 1.5 \times 10^{-6} = 3.0 \times 10^{2}\ \mathrm{N}$$
□

8.2.2 Young modulus

ratio of stress to strain of a material is called the **Young modulus**: $\boxed{E = \frac{\sigma}{\epsilon}}$

- unit of measurement: $[E] = \mathrm{Pa}$ (pascal)
- Young modulus is a property of the material

  for the same material, Young modulus is a constant, no matter in what shape it takes i.e., it does not depend on the length or the cross section of the object

➢ typical value of Young's modulus for metals: $E_{\text{metal}} \sim 10^{11}$ Pa

➢ Young modulus is a measure of the stiffness of a material

to produce same strain, greater stress is required for a material with greater Young modulus

➢ Young modulus E is related to the force constant k

from $E = \frac{\sigma}{\epsilon} = \frac{F/A}{x/L} = \frac{FL}{xA}$, we rearrange to get: $F = \frac{EA}{L}x$

compare with Hooke's law, we can identify the force constant to be given by: $k = \frac{EA}{L}$

– $E \nearrow \Rightarrow k \nearrow$, stiffer material makes stiffer springs

– $A \nearrow \Rightarrow k \nearrow$, more difficult to stretch a thick spring (think about parallel springs)

– $L \nearrow \Rightarrow k \searrow$, easier to stretch a long spring (think about series springs)

Example 8.9 A 200 N tensile force is applied on a steel wire of 1.5 m, the wire extends by 5.0 mm. The diameter of the cross section is 0.60 mm. What is the Young modulus of the steel wire?

✍ $$E = \frac{\sigma}{\epsilon} = \frac{F/A}{x/L} = \frac{FL}{Ax} = \frac{200 \times 1.5}{\pi \times (0.30 \times 10^{-3})^2 \times 5.0 \times 10^{-3}} \quad \Rightarrow \quad E \approx 2.1 \times 10^{11} \text{ Pa}$$ □

Example 8.10 A copper wire of length 2.0 m is under a stress of 7.8×10^7 Pa. Given that the Young modulus of copper is 1.2×10^{11} Pa, what is (a) the strain of the wire, (b) the extension of the wire?

✍ strain: $\epsilon = \frac{\sigma}{E} = \frac{7.8 \times 10^7}{1.2 \times 10^{11}} \quad \Rightarrow \quad \epsilon = 6.5 \times 10^{-4} = 0.065\%$

extension: $x = \epsilon L = 6.5 \times 10^{-4} \times 2.0 \quad \Rightarrow \quad x = 1.3 \times 10^{-3}$ m □

Example 8.11 Several blocks of steel are used to support a bridge. Each block has a height of 30 cm and a cross section of 15 cm × 15 cm. The steel block is designed to compress 2.0 mm when the maximum load is applied. Given that the Young modulus of steel is 2.1×10^{11} Pa, what is the maximum load that can be supported by one block?

✍ $$E = \frac{FL}{Ax} \quad \Rightarrow \quad F = \frac{EAx}{L} = \frac{2.1 \times 10^{11} \times (0.15 \times 0.15) \times 2.0 \times 10^{-3}}{0.30} \quad \Rightarrow \quad F \approx 3.15 \times 10^7 \text{ N}$$ □

Example 8.12 Two metal wires A and B are of the same length and they extend by the same amount under the same load. Given that Young modulus of wire A is twice of B, what is the ratio of their diameters?

✍ $$E = \frac{FL}{Ax} \quad \Rightarrow \quad A = \frac{1}{4}\pi d^2 = \frac{FL}{Ex} \quad \Rightarrow \quad d^2 \propto \frac{1}{E} \quad \Rightarrow \quad \frac{d_A}{d_B} = \sqrt{\frac{E_B}{E_A}} = \frac{1}{\sqrt{2}}$$ □

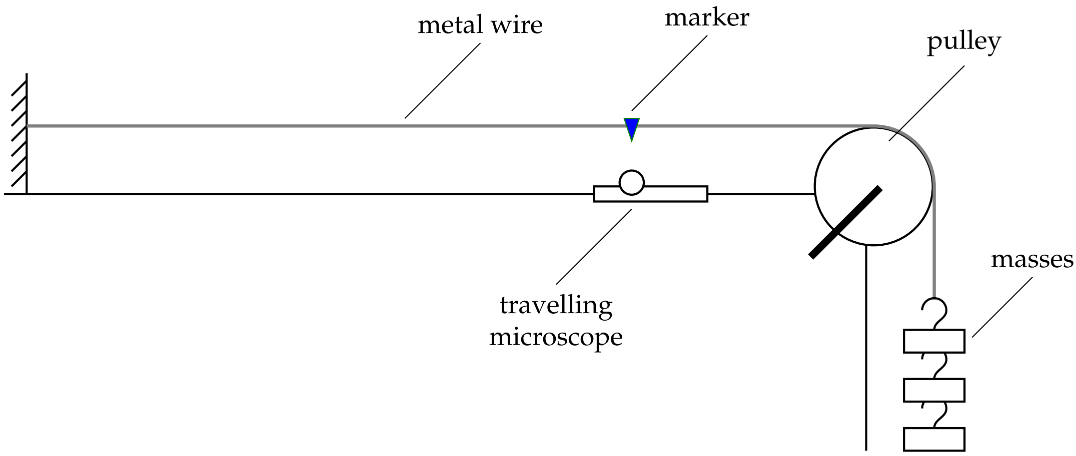
Example 8.13 A full-size crane is ten times greater than a model crane in all linear dimensions. If they are made of the same materials, what is the ratio of the cable's extension?

✍ same material means same density ρ and same Young's modulus E, so:

$$E = \frac{FL}{Ax} \quad \Rightarrow \quad x = \frac{FL}{AE} = \frac{mgL}{\pi r^2 E} = \frac{\rho V g L}{\pi r^2 E} \quad \Rightarrow \quad x \propto \frac{VL}{r^2} \propto \frac{l^3 l}{l^2} \propto l^2 \quad \Rightarrow \quad \frac{x_{\text{full}}}{x_{\text{model}}} = 10^2 = 100$$ □

8.2.3 measurement of Young modulus

to measure Young's modulus of a metal wire, experimental setup can be laid out as shown



➢ method of data collection

– original length L (up to the marker) of the wire is measured with *metre rule*

– diameter d of the wire is measured with *micrometer*, then cross-sectional area is: $A = \frac{1}{4}\pi d^2$

– record mass m attached to the wire, then force applied is $F = mg$

– extension $x$of the wire is taken to be distance moved out by the marker

this can be measured with a *travelling microscope*[50] or a *vernier calliper*

➢ analysis of data

stress can be calculated by $\sigma = \frac{F}{A}$, and strain can be calculated by $\epsilon = \frac{x}{L}$

a graph of stress against strain can be plotted, a best fit curve can be drawn

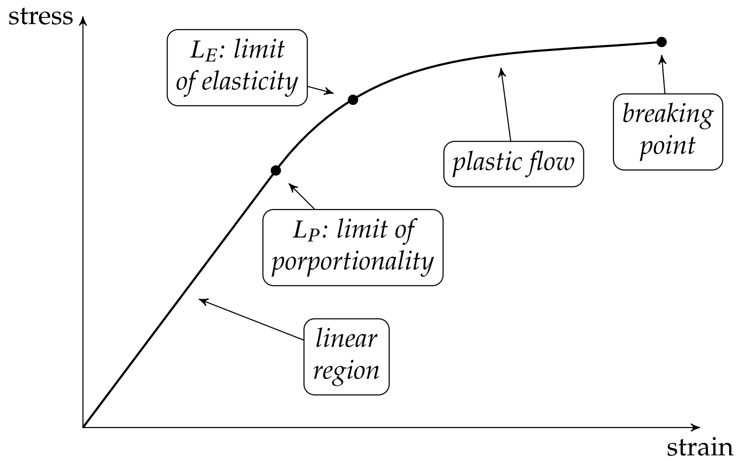
gradient of the straight-line section gives Young modulus

8.2.4 stress-strain curves

stress-strain curve for a material can be obtained using the methods in §8.2.3

in AS-Level, you are only supposed to know the behaviour of a metal under stress

[50]A travelling microscope is basically a microscope that can move back and forth along a rail. The position of the microscope can be varied by turning a screw. This position can be read off a vernier scale. So in short, a travelling microscope can be used to measure the change in length with a very high resolution (typically to a precision of 0.01mm or 0.02 mm).



stress-strain graph for a typical metal

- ➢ up to limit of proportionality L_E, stress is proportional to strain

  Young modulus can be given by the gradient of the curve before L_P
- ➢ consider the product of stress and strain: $\sigma \cdot \epsilon \sim \frac{F}{A}\frac{x}{L} \sim \frac{Fx}{AL} \sim \frac{W}{V}$

  so area under stress-strain curve gives the work done per unit volume to stretch the wire
- ➢ up to limit of elasticity L_E, metal wire can return to original length when it relaxes
- ➢ if a wire is stretched beyond the elastic limit, it follows a different path when force is removed

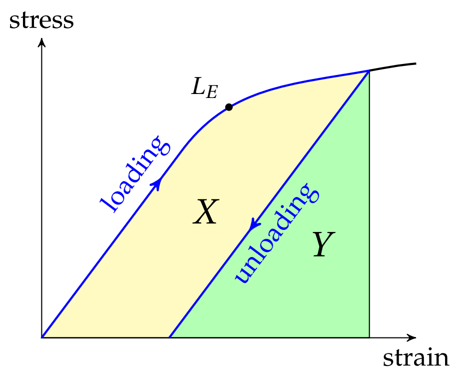
  recall area under stress-strain graph relates to energy

  work to stretch wire is given by $X + Y$

  energy goes out when wire contracts is Y

  the difference X gives energy loss during one cycle

  this energy difference becomes heat produced in wire



8.3 end-of-chapter questions

springs & Hooke's law

Question 8.1 A spring has a force constant of 500 N m^{-1}. The spring has a stretch length of 30 cm when a 40 N weight is hung from it. Find the natural length of the spring.

Question 8.2 Two springs, one with spring constant $k_1 = 8.0$ N cm^{-1} and the other with $k_2 =$ 12 N cm^{-1}. (a) When they are connected in parallel, what is the total extension when the combination supports a load of 60 N? (b) If they are connected in series, what is the total extension when they support the same load?

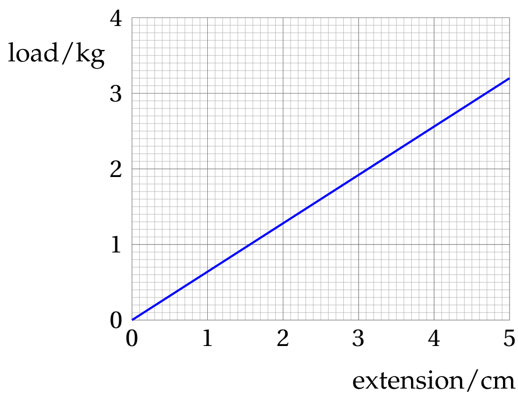
Question 8.3 A weight of 100 N is placed on top of a spring. The spring is compressed by 2.0 cm. Assume the spring obeys Hooke's law, how much strain energy is stored in the spring?

Question 8.4 A spring with a force constant 800 N m^{-1} is supported and stands vertically. A ball of mass 60 g falls vertically onto it. The ball has a speed of 3.0 m s^{-1} as it makes contact with the spring. Assume no energy loss, what is the maximum compression of the spring?

Question 8.5 A trolley of mass 160 g is placed on a frictionless track. The trolly pushed against a spring buffer of a force constant 90 N m^{-1}. The spring buffer is compressed by 7.0 cm. The trolley is then released from rest. (a) What is the initial acceleration of the trolley? (b) Assume no loss of energy, what is the final speed of the trolley along the track?

Question 8.6 Spring A is stiffer than spring B, i.e., $k_A > k_B$. On which spring is more elastic potential energy stored if they are stretched by (a) the same extension, (b) the same force?

Question 8.7 A load is attached to the lower end of a spring. The extension of the spring is measured when the load increases. The variation with extension of the load is shown. (a) Suggest whether the spring obeys Hooke's law. (b) Find the spring constant. (c) How much elastic potential energy is stored in the spring when the extension is 5.0 cm? (d) How much work is done to increase the extension from 2.0 cm to 4.0 cm?



Question 8.8 Suppose you have a spring, a ruler, a mass hanger and a set of masses. Suggest how the apparatus may be used to determine the load on the spring at (a) the limit of elasticity, (b) the limit of proportionality?

stress, strain, & Young modulus

Question 8.9 A cable of diameter 1.5 mm is under a tension of 200 N. Find the stress in this cable.

Question 8.10 Estimate the stress in your neck when it supports your head in a vertical position.

Question 8.11 A metal wire of natural length 1.8 m and diameter 0.70 mm is fixed to the ceiling at one end. When a mass of 6.5 kg is hung from the lower end, the wire extends by 2.7 mm. (a) Find the strain of the wire. (b) Find the Young modulus of the metal.

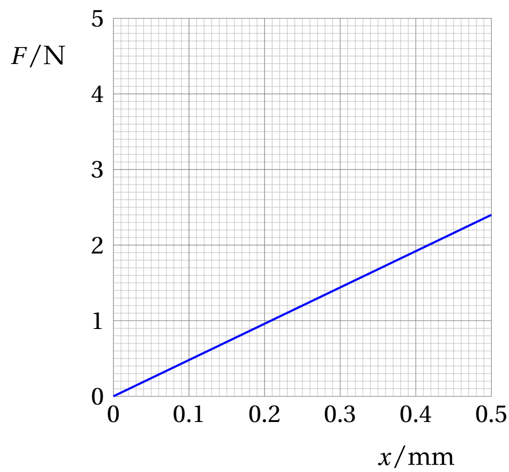
Question 8.12 A wire has a diameter of 0.50 mm with a Young modulus of 0.18 TPa. The length of the wire is increased by 0.20% by a force F. (a) Find the stress in the wire. (b) Find the force F.

Question 8.13 Two metal wires A and B are made of different materials. The diameter of wire A is twice that of wire B, and the Young modulus of wire A is three times that of wire B. If the wires are extended by the same strain, what is the ratio of the tension in wire A to tension in wire B?

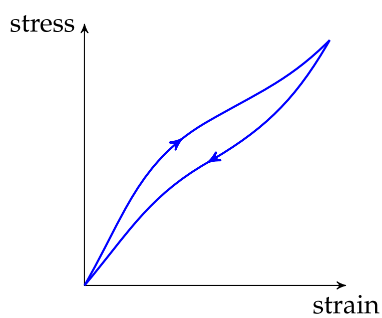
Question 8.14 Two rods A and B of the same diameter are joined end to end and hug vertically. Rod B is twice as long as rod A but has half the Young modulus. When a mass is hung from the combination, what is the ratio of the extension of rod A to the extension of rod B?

Question 8.15 Bones of different animals have very similar compositions. Suggest why heavier animals appear to have thicker bones?

Question 8.16 A load F is suspended from a copper wire. The graph shows the load-extension relation. (a) State two quantities other than F and x that are required to determine the Young modulus of copper. (b) Suggest how the two quantities may be measured. (c) Use the graph to find the energy stored in the wire when a load of 2.0 N is applied. (d) Given that steel has about twice the Young modulus as copper, sketch the variation with x of F for a steel wire that has the same dimensions as the copper wire.



Question 8.17 In an experiment, a speciman of a rubber compound is being stretched and relaxed. The stress-strain curve is plotted. (a) State and explain whether this rubber compound behaves elastically. (b) The tyres on a vehicle are made of this rubber compound. Explain why the tyres become warm as the car travels on a road.



Chapter 9

Fluids

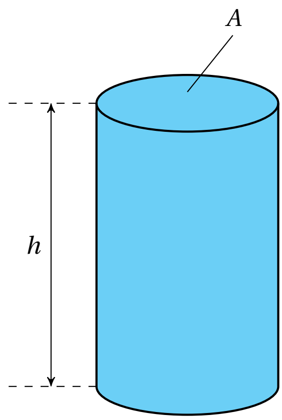
a fluid, such as a liquid or a gas, is a substance that has no fixed shape

unlike a solid, a fluid can flow and yield easily under external force

in this chapter, we will study several aspects of a fluid

9.1 pressure in a fluid

at a depth of h below surface of a fluid, self-weight of the fluid could produce a pressure



$$p=\frac{F}{A}=\frac{mg}{A}=\frac{\rho Vg}{A} \quad \Rightarrow \quad \boxed{p=\rho gh}$$

➢ pressure in a liquid depends on depth

for different positions in a liquid, as long as they are at same depth, pressure is the same

i.e., pressure does not depend on volume or shape of container

➢ atmospheric pressure also accounts for total pressure in a liquid

atmosphere presses on surface of a liquid, so total pressure at depth h is: $P=\rho gh+P_{\text{atm}}$

nevertheless, change in pressure still satisfies: $\Delta p=\rho g\Delta h$

Example 9.1 The atmospheric pressure is about 1.0×10^5 Pa. Given that the density of sea water is 1020 kg m^{-3}, what is the total pressure 50 m below the surface of the sea?

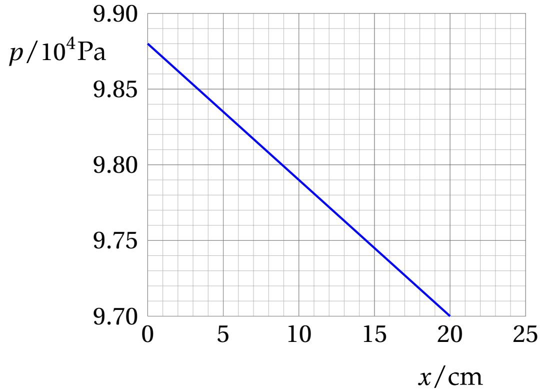
✍ $$P=P_{\text{atm}}+\rho gh=1.0\times10^5+1020\times9.81\times50 \quad \Rightarrow \quad P\approx6.0\times10^5\text{ Pa}$$ □

Example 9.2 A vertical column of liquid of height 10 m contains both oil and water. The pressure due to the liquids at the bottom of the column is 89 kPa. Given that the density of water is 1000 kg m^{-3} and the density of the oil is 840 kg m^{-3}. What is the depth of the oil?

✍ $$P=P_{\text{oil}}+P_{\text{water}}=\rho_{\text{o}}gh_{\text{o}}+\rho_{\text{w}}gh_{\text{w}}$$

$$840\times9.81\times x+1000\times9.81\times(10-x)=89\times10^3 \quad \Rightarrow \quad x=5.8\text{ m}$$ □

Example 9.3 The pressure p of a liquid in a container varies with the height x above the base of the container as shown. The total depth of the liquid is 20 cm. (a) What is the atmospheric pressure? (b) What is the density of the liquid?



✍ surface of liquid at height $x = 20$ cm, so

$$P_{\text{atm}} = 9.60 \times 10^4 \text{ Pa}$$

density of liquid: $\rho = \dfrac{\Delta p}{g\Delta h} = \dfrac{(9.88 - 9.60) \times 10^4}{9.81 \times 0.20} \quad \Rightarrow \quad \rho \approx 917 \text{ kg m}^{-3}$ □

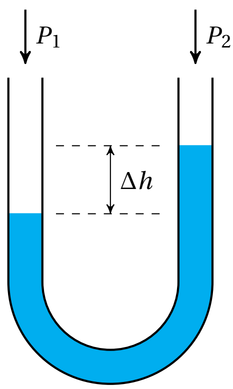
9.2 pressure meters

there are many types of instruments for pressure measurement

we would only focus on simple manometers and barometers [51]

they both use the fact that $\Delta p = \rho g \Delta h$ within a liquid

9.2.1 manometers



a **manometer** consists of a U-shaped tube filled with some liquid

any pressure difference between the two ends of the tube could cause a height difference between liquid levels

for the situation shown, at equilibrium, one has: $P_1 - P_2 = \rho g \Delta h$

if P_2 is a reference pressure, then P_1 can be calculated

➢ though any fluid can be used in a manometer, *mercury* is preferred because of its high density ($\rho_{\text{Hg}} = 1.36 \times 10^4 \text{ kg m}^{-3}$)

Example 9.4 A manometer is used to measure the pressure of a gas supply. Side A of the tube is connected to the gas pipe, and the other side B of the tube is open to the atmosphere. If the

[51]Some examples of many other types of pressure gauges include

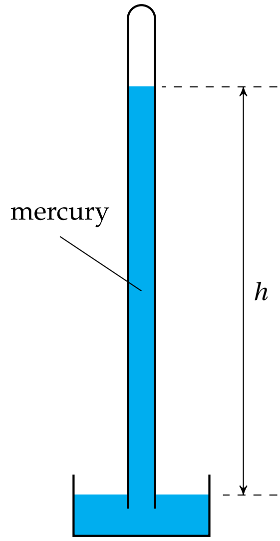
- mechanical gauges based on metallic pressure-sensing elements
- electronic gauges based on piezo-resistive effect
- hot-filament ionization gauges based on ion currents from a gas

Those who are interested are welcome to research into their functions and principles.

mercury on side A is higher than on side B by 14 cm, what is the pressure of the gas? (density of mercury: 1.36×10^4 kg m^{-3}; atmospheric pressure: 1.01×10^5 Pa)

✍ $P_{atm} - P_{gas} = \rho g h \quad \Rightarrow \quad P_{gas} = 1.01 \times 10^5 - 1.36 \times 10^4 \times 9.81 \times 0.14 \quad \Rightarrow \quad P_{gas} \approx 8.23 \times 10^4$ Pa □

9.2.2 barometers



take a long glass tube and fill it with mercury

let it stand upside down in a basin

there is atmospheric pressure pushing down on surface of mercury, so a height of mercury is supported up the tube

we can then compute atmospheric pressure by: $P_{atm} = \rho g h$

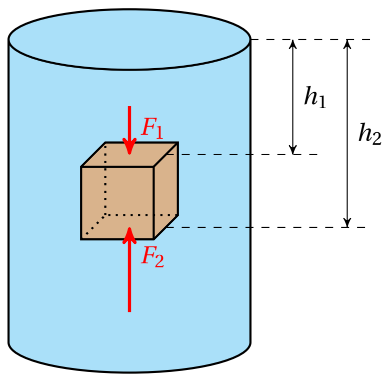
this instrument makes a **mercury barometer**

Example 9.5 If a mercury barometer supports a height of 760 mm of mercury above the fluid level in the container, what is the atmospheric pressure? If water is used as the barometric liquid, what is the minimum length of the tube required for the same atmospheric pressure? (density of mercury: 1.36×10^4 kg m^{-3}; density of water: 1.00×10^3 kg m^{-3};)

✍ atmospheric pressure: $P_{atm} = \rho g h = 1.36 \times 10^4 \times 9.81 \times 0.760 \approx 1.01 \times 10^5$ Pa

if mercury is replaced by water: $h' = \dfrac{P_{atm}}{\rho' g} = \dfrac{1.01 \times 10^5}{1000 \times 9.81} \approx 10.3$ m □

9.3 upthrust



now consider a rectangular block immersed in a fluid

top and bottom surface are at different depths, so they experience different pressures

this gives rise to an overall upward force on the cylinder

this force is called the **upthrust**:

$$F_U = F_2 - F_1 = \rho g (h_2 - h_1) \times A \quad \Rightarrow \quad \boxed{F_U = \rho g V}$$

therefore upthrust exerted on an immersed object equals the weight of the fluid displaced

this is known as the **Archimedes' principle**

➢ origin of upthrust: pressure difference between top and bottom surfaces

➢ for an object of density ρ_o immersed in a liquid of density ρ_l

take force in downward direction to be positive, then resultant force acting is:

$$F_{net} = W - F_U = \rho_o g V - \rho_l g V = (\rho_o - \rho_l) g V$$

– if $\rho_o > \rho_l$, then $F_{net} > 0$, resultant force acts downwards, object will sink

– if $\rho_o < \rho_l$, then $F_{net} < 0$, resultant force acts upwards, object will rise

– if $\rho_o = \rho_l$, then $F_{net} = 0$, object is in equilibrium, it can float at that level

Example 9.6 A block of mass 80 g and volume 50 cm^3 is suspended from a string into water. When the block is fully immersed and kept at rest, what is the tension in the string?

✍ $T + F_U = W \quad \Rightarrow \quad T = mg - \rho g V = 0.080 \times 9.81 - 1000 \times 9.81 \times 50 \times 10^{-6} \quad \Rightarrow \quad T \approx 0.29\ \text{N}$ □

9.4 end-of-chapter questions

Data for the questions below where applicable:

– density of water: $1.00 \times 10^3 \text{kg m}^{-3}$

– density of mercury: $13.6 \times 10^3 \text{kg m}^{-3}$

– atmospheric pressure: 1.0×10^5 Pa

pressure in a fluid

Question 9.1 (a) A dam holds a depth of 50 m of water. What is the water pressure at the base of the dam? (b) Suggest why the walls of a dam must be made thicker near the bottom?

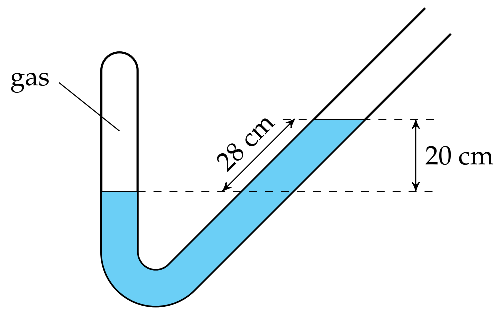
Question 9.2 The deepest trench on Earth is The Mariana Trench located in the Pacific Ocean. The maximum known depth is about 11 km. (a) Assume the sea water has a uniform density of about 1030 kg m^{-3}, estimate the pressure at the base of the trench. (b) The density of sea water actually increases slightly with pressure, suggest how this affects the result you have found.

Question 9.3 Instead of a large viewing window, the window of a submarine is usually of only a few centimetres in diameter. Why is the window made so small?

Question 9.4 If you punch several holes in the bottom of a container filled with water, water will spurt out due to the pressure. Now drop the container, suggest what will happen as it falls freely and defend your explanation.

Question 9.5 Air trapped inside a cylinder is attached to a U-shaped manometer containing mercury. The other side of the manometer is open to atmosphere. The mercury column on the side open to atmosphere is found to be 45 mm higher, what is the pressure of the trapped air?

Question 9.6 The figure shows a pipe closed at one end and open at the other end. Some gas is trapped by a column of mercury as shown. Find the pressure of the gas.



Question 9.7 A water manometer is used to measure the pressure created in a flexible container. Initially, the water columns on each side of the manometer is at the same level. When a girl stands on a 30 cm by 30 cm platform placed on top of the container, a height difference of 50 cm is observed in the manometer. (a) Find the pressure created by the girl. (b) Find the mass of the girl.

upthrust

Question 9.8 A lead block and an aluminium block of identical size are immersed in water. Upon which block is the upthrust greater?

Question 9.9 If somehow the gravitational field on the earth is increased, does a fish sink, float to the surface, or otherwise?

Question 9.10 A diver holds a cube of side 0.30 m and density 800kg m^{-3} near the seabed. (a) What is the upthrust on the cube? (b) What is its initial acceleration when the cube is released from rest?

Question 9.11 A solid sphere of radius 18 cm and density 2.4g cm^{-3} is fully submerged in water. A string pulls on the sphere so that the sphere does not sink. (a) Find the tension in the string. (b) If the string is cut, what is the instantaneous acceleration. (c) Describe and explain the acceleration of the sphere as it sinks.

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