CSE 546 Homework #0 -B

Lu Yu

Spring 2020

Probability and Statistics

B.1 *[1 points]* Let $X_1, \dots, X_n$ be n independent and identically distributed random variables drawn unfiromly at random from $[0, 1]$. If $Y = \max\{X_1, \dots, X_n\}$ then find $\mathbb{E}[Y]$.
$X \sim uniform(0, 1)$
Given any $u \in (0, 1)$,

$$F_U(u) = P(U \leq u) = P(max\{X_1, X_2, \dots, X_n\} \leq u) = P(X_1 \leq u)P(X_2 \leq u) \dots P(X_n \leq u) = u^n$$

$$f_U(u) = [F_U(u)]' = nu^{n-1}$$

$$E(U) = \int_0^1 u f_U(u) du = \int_0^1 nu^n du = \frac{n}{n+1}$$

So, $\mathbb{E}[Y_n] = \frac{n}{n+1}$.

Linear Algebra and Vector Calculus

B.2 *[1 points]* The *trace* of a matrix is the sum of the diagonal entries; $Tr(A) = \sum_i A_{ii}$. If $A \in \mathbb{R}^{n \times m}$ and $B \in \mathbb{R}^{m \times n}$, show that $Tr(AB) = Tr(BA)$.
Notice that AB is a $n \times n$ matrix, BA is a $m \times m$ matrix. $A = (a_{ij})$, and $B = (b_{ij})$.

$$(AB)_{ii} = \sum_{j=1}^{m} a_{ij} b_{ji}$$

$$Tr(AB) = \sum_{i=1}^{n} (AB)_{ii} = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} b_{ji}$$

$$(BA)_{ii} = \sum_{j=1}^{n} a_{ij} b_{ji}$$

$$Tr(BA) = \sum_{i=1}^{m} (BA)_{ii} = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} b_{ji}$$

So, $Tr(AB) = Tr(BA)$

B.3 *[1 points]* Let $v_1, \dots, v_n$ be a set of non-zero vectors in $\mathbb{R}^d$. Let $V = [v_1, \dots, v_n]$ be the vectors concatenated.

a. What is the minimum and maximum rank of $\sum_{i=1}^{n} v_i v_i^T$?

$$v_i = \begin{bmatrix} v_{i1} & v_{i2} & \dots & v_{id} \end{bmatrix}$$

$$\sum_{i=1}^{n} v_i v_i^T = \begin{bmatrix} \sum_{i=1}^{n} v_{i1}v_{i1} & \sum_{i=1}^{n} v_{i1}v_{i2} & \dots & \sum_{i=1}^{n} v_{i1}v_{id} \\ \sum_{i=1}^{n} v_{i2}v_{i1} & \sum_{i=1}^{n} v_{i2}v_{i2} & \dots & \sum_{i=1}^{n} v_{i2}v_{id} \\ \dots & \dots & \dots & \dots \\ \sum_{i=1}^{n} v_{id}v_{i1} & \sum_{i=1}^{n} v_{id}v_{i2} & \dots & \sum_{i=1}^{n} v_{id}v_{id} \end{bmatrix}$$

If all vectors are equal, the rank will be 1 which is the minimum rank.
If all vectors are linearly independent, then the maximum rank is $min(n, d)$.

b. What is the minimum and maximum rank of V?
V is a $d \times n$ matrix, same to part a, if all vectors are equal, the rank will be 1 which is the minimum rank. While if all vectors are linearly independent, then the maximum rank is $min(n, d)$.

c. Let $A \in \mathbb{R}^{D\times d}$ for $D > d$. What is the minimum and maximum rank of $\sum_{i=1}^{n}(Av_i)(Av_i)^T$?
$(Av_i)(Av_i)^T$ is a $D \times D$ matrix. If A is a zero matrix, the rank will be 0 which is the minimum rank. While if all vectors in matrix are linearly independent, then the maximum rank is D.

d. What is the minimum and maximum rank of AV? What if V is rank d?
A is a $D \times d$ matrix, V is a $d \times n$ matrix, AV is a $D \times n$ matrix.

$$rank(AV) \leq min(rank(A), rank(V)) = min(D, n, d) = min(d, n)$$

If A is a zero matrix, the rank will be 0 which is the minimum rank. While if all vectors in matrix are linearly independent, then the maximum rank is $min(D, n, d)$.
If $rank(V) = d$, then $n > d$. The minimum keeps the same. The maximum will be $min(D, d) = d$.