Recall

Let $X, \widetilde{X}, Y$ and Z be random variables.

Conditional expectation

Given $y \in \mathbb{R}$, $E[X|Y = y]$ is the expectation of X with respect to the conditional probability $P(X \in \cdot|Y = y)$. As y varies, we obtain a function $f\colon y \mapsto E[X|Y = y]$. Then the *conditional expectation* $E[X|Y]$ is the random variable $f(Y)$.

Some properties of $E[\,\cdot\,|Y]$ which maps a random variable to another **random variable**:

(1) ($\mathbb{R}$-linear) $\forall\, \alpha, \beta \in \mathbb{R},\ E[\alpha X + \widetilde{X} + \beta|Y] = \alpha E[X|Y] + E[\widetilde{X}|Y] + \beta$.

(2) (monotone) If $X \le Z$, then $E[X|Y] \le E[Z|Y]$.

(3) ($g(Y)$-scaling) In most cases, for a function $g\colon \mathbb{R} \to \mathbb{R}$ we have $E[g(Y)X|Y] = g(Y)E[X|Y]$ since $E[g(y)X|Y = y] = g(y)E[X|Y = y]$, $\forall\, y \in \mathbb{R}$.

(4) In particular, $E[E[X|Y]|Y] = E[X|Y]$ by (3).

(5) (towering property) $E[X] = E[E[X|Y]]$, i.e., compute expectations by conditioning.

(6) We take X in (5) to be the indicator variable χ_E for an event E. Note that $E[\chi_E] = P(E)$ and $E[\chi_E|Y = y] = P(E|Y = y)$. Then we can compute probabilities by conditioning,

$$P(E) = \begin{cases} \sum_y P(E|Y = y)P(Y = y) & \text{if } Y \text{ discrete} \\ \int_{-\infty}^{\infty} P(E|Y = y)f_Y(y)dy & \text{if } Y \text{ continuous.} \end{cases}$$

In particular, if $Y = \sum_{i=1}^{n} i\chi_{F_i}$ for some partition $F_1, \ldots, F_n$ of the sample space, then the law of total probability is recovered.

Moment generating functions

For a random variable X, the *moment generating function* (MGF) is $M_X(t) := E[e^{tX}]$ for $t \in \mathbb{R}$ whenever $E[e^{tX}]$ exists. Note $M_X(t) > 0$. The following facts make MGF useful:

- (generate moments) $E[X^n] = M_X^{(n)}(0)$ for $n \in \mathbb{N}$ (if $E[X^n] < \infty$).
- (determine distributions) If there exists $t_0 > 0$ such that $M_X(t) = M_Y(t)$ for $t \in (-t_0, t_0)$, then $F_X = F_Y$.
- (multiplicative under independent sums) If X, Y are independent, then $M_{X+Y}(t) = M_X(t)M_Y(t)$.

A table about MGFs of common distributions can be found in the textbook [Ross, Ch. 7-Sec. 7].

Examples

Example 1. Let X, Y be random variables and $g\colon \mathbb{R} \to \mathbb{R}$ be a function. Show that

(i) $\operatorname{Cov}(X, E[Y|X]) = \operatorname{Cov}(X, Y)$.

(ii) $E[(X - E[X|Y])^2] = E[X^2] - E[E[X|Y]^2]$.

(iii) $E[(X - g(Y))^2] \geq E[(X - E[X|Y])^2]$.

Proof. (i) It follows from (3) that $XE[Y|X] = E[XY|X]$. Then by (5),

$$\begin{aligned}\operatorname{Cov}(X, E[Y|X]) &= E[XE[Y|X]] - E[X]E[E[Y|X]] \\ &= E[E[XY|X]] - E[X]E[Y] \\ &= E[XY] - E[X]E[Y] \\ &= \operatorname{Cov}(X, Y).\end{aligned}$$

(ii) By (5) and (3), we have

$$E[XE[X|Y]] = E[E[XE[X|Y]|Y]] = E[E[X|Y]E[X|Y]] = E[E[X|Y]^2].$$

Hence

$$\begin{aligned}E[(X - E[X|Y])^2] &= E[X^2] - 2E[XE[X|Y]] + E[E[X|Y]^2] \\ &= E[X^2] - 2E[E[X|Y]^2] + E[E[X|Y]^2] \\ &= E[X^2] - E[E[X|Y]^2].\end{aligned}$$

(iii) By (5), it suffices to prove $E[(X - g(Y))^2|Y] \geq E[(X - E[X|Y])^2|Y]$.

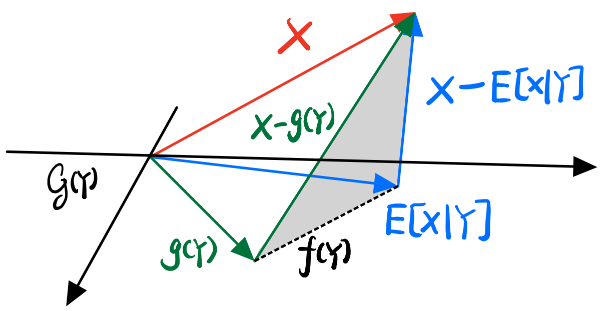

Figure 1: A possible intuition about (iii)

Based on the above intuition, we first establish that $X - E[X|Y]$ is 'orthogonal' to the plane. For any function $f\colon \mathbb{R} \to \mathbb{R}$, by (3) and (4) we have

$$\begin{aligned}E[(X - E[X|Y])f(Y)|Y] &= f(Y)E[X - E[X|Y]|Y] \\ &= f(Y)(E[X|Y] - E[E[X|Y]|Y]) \\ &= f(Y)(E[X|Y] - E[X|Y]) \\ &= 0.\end{aligned}$$

Next we focus on the shaded 'right triangle'. By viewing $E[X|Y] - g(Y)$ as $f(Y)$,

$$\begin{aligned}
&E[(X - g(Y))^2|Y] \\
&= E[(X - E[X|Y] + E[X|Y] - g(Y))^2|Y] \\
&= E[(X - E[X|Y])^2|Y] + 2E[(X - E[X|Y])(E[X|Y] - g(Y))|Y] + E[(E[X|Y] - g(Y))^2|Y] \\
&= E[(X - E[X|Y])^2|Y] + 0 + E[(E[X|Y] - g(Y))^2|Y] \\
&\geq E[(X - E[X|Y])^2|Y],
\end{aligned}$$

where the last inequality follows from (2).

□

Remark. Similar to (iii) in Example 1, it is also intuitive to use the *correlation* to indicate the linear relationship between X and Y. If $\mathrm{Var}(X), \mathrm{Var}(Y) > 0$, we normalize X, Y (as we often do in Central Limit Theorem) to

$$\widetilde{X} := \frac{X - E[X]}{\sqrt{\mathrm{Var}(X)}} \quad \text{and} \quad \widetilde{Y} := \frac{Y - E[Y]}{\sqrt{\mathrm{Var}(Y)}}.$$

Then the *correlation coefficient* is defined as

$$\rho(X, Y) := \mathrm{Cov}(\widetilde{X}, \widetilde{Y}) = \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}},$$

where in the last equality recall that $\mathrm{Cov}(\alpha, Z) = 0$ for all $\alpha \in \mathbb{R}$ and r.v. Z.

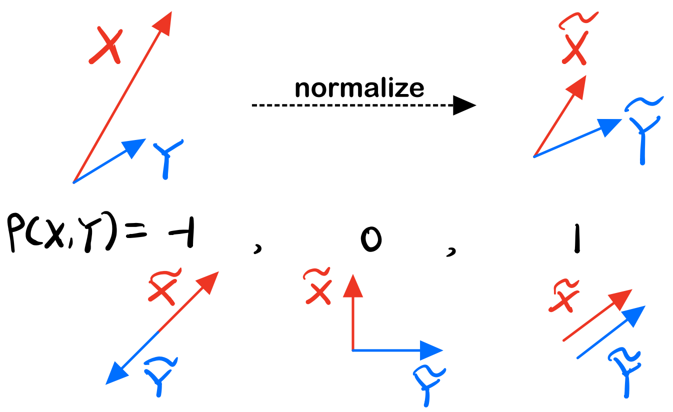

Figure 2: Special cases for $\rho(X, Y)$

If interested, we may refer to the arguments preceding [Ross, Ch. 7-Example 4d] for a basic analysis of $\rho(X, Y)$.

Example 2. Let $X \sim N(0, 1)$ and $I \sim \mathit{Bern}(1/2)$. Suppose that X, Y are independent. Define

$$Y = \begin{cases} X & \text{if } I = 0 \\ -X & \text{if } I = 1. \end{cases}$$

Find $\mathrm{Cov}(X, Y)$.

Solution. Since X, I are independent, we have X^2, I are independent. Thus $E[X^2|I = i] = E[X^2]$ for $i = 0, 1$. Note $E[X] = 0$. Then

$$\begin{aligned}
\mathrm{Cov}(X, Y) &= E[XY] - E[X]E[Y] \\
&= E[E[XY|I]] - 0 && \text{by (5)} \\
&= P(I = 0)E[XY|I = 0] + P(I = 1)E[XY|I = 1] \\
&= \frac{1}{2}E[X^2|I = 0] + \frac{1}{2}(-E[X^2|I = 1]) && \text{by def. of } Y \\
&= \frac{1}{2}(E[X^2] - E[X^2]) && \text{by independence} \\
&= 0.
\end{aligned}$$

□

Remark. By checking X, Y are NOT independent, we find another example showing $\mathrm{Cov}(X, Y) = 0 \not\Longrightarrow$ independence.

Example 3. Let $(U_i)_{i=1}^{\infty}$ be an i.i.d. sequence of random variables with common distribution $U(0, 1)$. For $x \in [0, 1]$, define $N(x) := \min\{n : \sum_{i=1}^{n} U_i > x\}$. Show that $E[N(x)] = e^x$.

Proof. Notice that $N(x)$ is a non-negative random variable. We will compute $E[N(x)]$ by layer-cake (see e.g., [Tutorial 4, Example 2]). First we prove by induction that for $n \in \mathbb{Z}_{\geq 0}$,

$$P(N(x) \geq n + 1) = \frac{x^n}{n!}. \tag{$*$}$$

When $n = 0$, we have $P(N(x) \geq 1) = P(U_1 \leq x) = x$. When $n \geq 1$, suppose ($*$) holds for $n - 1$, i.e., $P(N(x) \geq n) = \frac{x^{n-1}}{(n-1)!}$. We check ($*$) for n by conditioning on U_1,

$$\begin{aligned}
P(N(x) \geq n + 1) &= \int_{-\infty}^{\infty} P(N(x) \geq n + 1|U_1 = y)f_{U_1}(y)dy && \text{by (6)} \\
&= \int_0^1 P(y + \sum_{i=2}^{n} U_i \leq x|U_1 = y)dy && \text{by def. of } N(x),\ f_{U_1} \\
&= \int_0^1 P(\sum_{i=2}^{n} U_i \leq x - y|U_1 = y)dy \\
&= \int_0^1 P(\sum_{i=2}^{n} U_i \leq x - y)dy && \text{by } \sum_{i=2}^{n} U_i,\ U_1 \text{ independent} \\
&= \int_0^1 P(\sum_{i=1}^{n-1} U_i \leq x - y)dy && \text{by i.i.d.} \\
&= \int_0^x P(\sum_{i=1}^{n-1} U_i \leq x - y)dy && \text{by vanished integrand when } y > x \\
&= \int_0^x P(N(x - y) \geq n)dy && \text{by def. of } N(x) \\
&= \int_0^x \frac{(x - y)^{n-1}}{(n - 1)!}dy && \text{by induction hypothesis}
\end{aligned}$$

$$= \int_0^x \frac{t^{n-1}}{(n-1)!} dt \qquad \text{by change of variable } t = x - y$$
$$= \frac{x^n}{n!}.$$

Hence

$$E[N(x)] = \sum_{n=1}^{\infty} P(N(x) \geq n) = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = e^x.$$

□

It's good to stop here.

Limit theorems

We include this section only for a relatively complete content but without examples.

Inequalities

Proposition 4 (Markov inequality). *Let X be a non-negative random variable. Then for $\varepsilon > 0$,*

$$P(X \geq \varepsilon) \leq \frac{E[X]}{\varepsilon}.$$

Proposition 5 (Chebyshev inequality). *Let X be a random variable with finite mean μ and variance σ^2. Then for $\varepsilon > 0$,*

$$P(|X - \mu| \geq \varepsilon) \leq \frac{\sigma^2}{\varepsilon^2}.$$

Limit theorems

Theorem 6 (Weak law of large numbers WLLN). *Let $(X_i)_{i=1}^{\infty}$ be an i.i.d. sequence of random variables with finite mean μ. Then for $\varepsilon > 0$,*

$$P\left(|\frac{X_1 + \cdots + X_n}{n} - \mu| \geq \varepsilon\right) \to 0 \quad \text{as } n \to \infty.$$

Theorem 7 (Strong law of large numbers SLLN). *Let $(X_i)_{i=1}^{\infty}$ be an i.i.d. sequence of random variables with finite mean μ. Then*

$$P\left(\lim_{n\to\infty} \frac{X_1 + \cdots + X_n}{n} = \mu\right) = 1.$$

Theorem 8 (Central limit theorem CLT). *Let $(X_i)_{i=1}^{\infty}$ be an i.i.d. sequence of random variables with finite mean μ and variance σ^2. Then for $t \in \mathbb{R}$,*

$$P\left(\frac{X_1 + \cdots + X_n - n\mu}{\sqrt{n}\sigma} \leq t\right) \to \Phi(t) \quad \text{as } n \to \infty.$$

Some simulation experiments about the limit theorems can be played interactively by clicking here (in major browsers). It might take a 7 – 20 mins to initialize. The static PDF version is attached below for convenience.

MATH3280 Tutorial 13

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Strong Law of Large number

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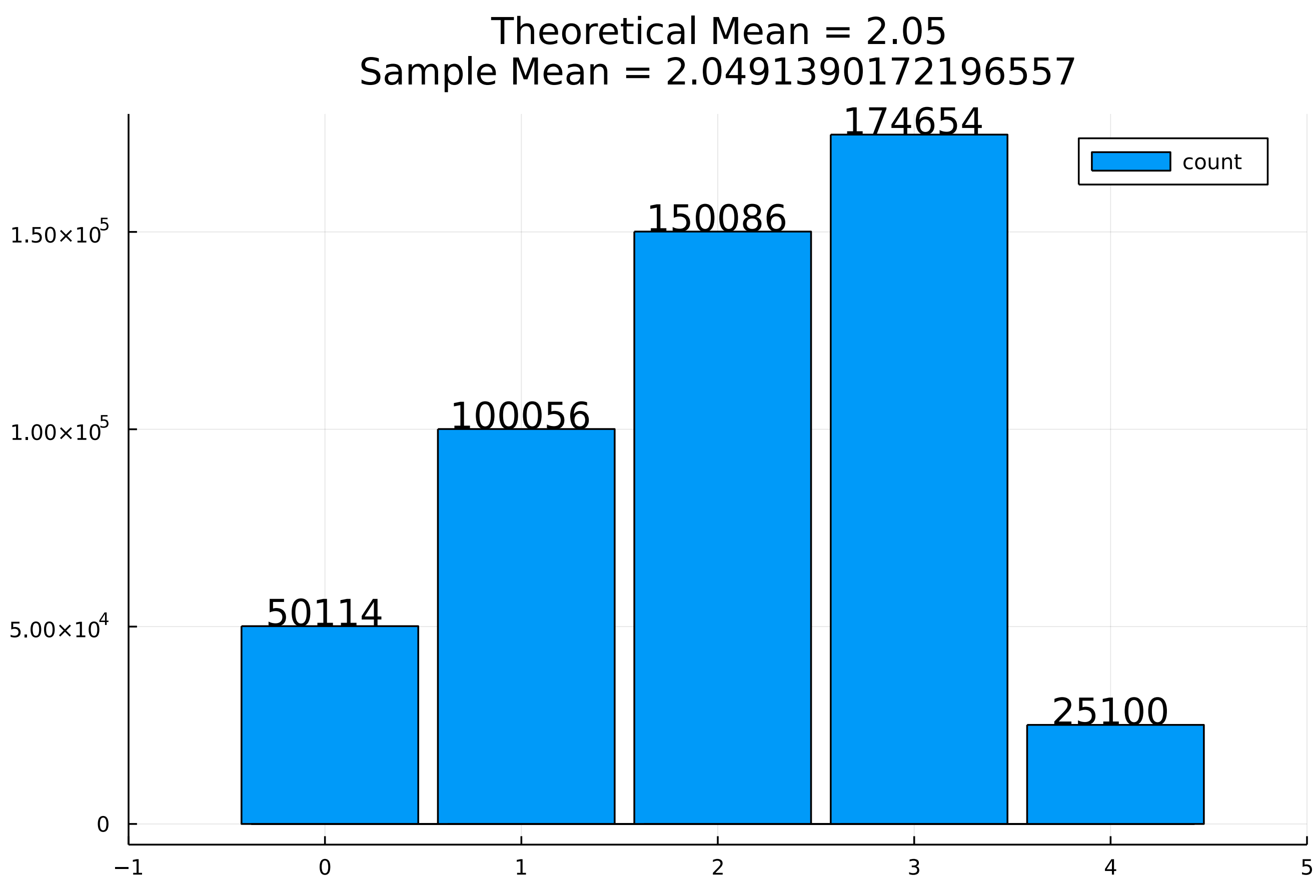
Strong Law of Large number

```
p = [0.1, 0.2, 0.3, 0.35, 0.05]; # make sure p is a probability vector
```

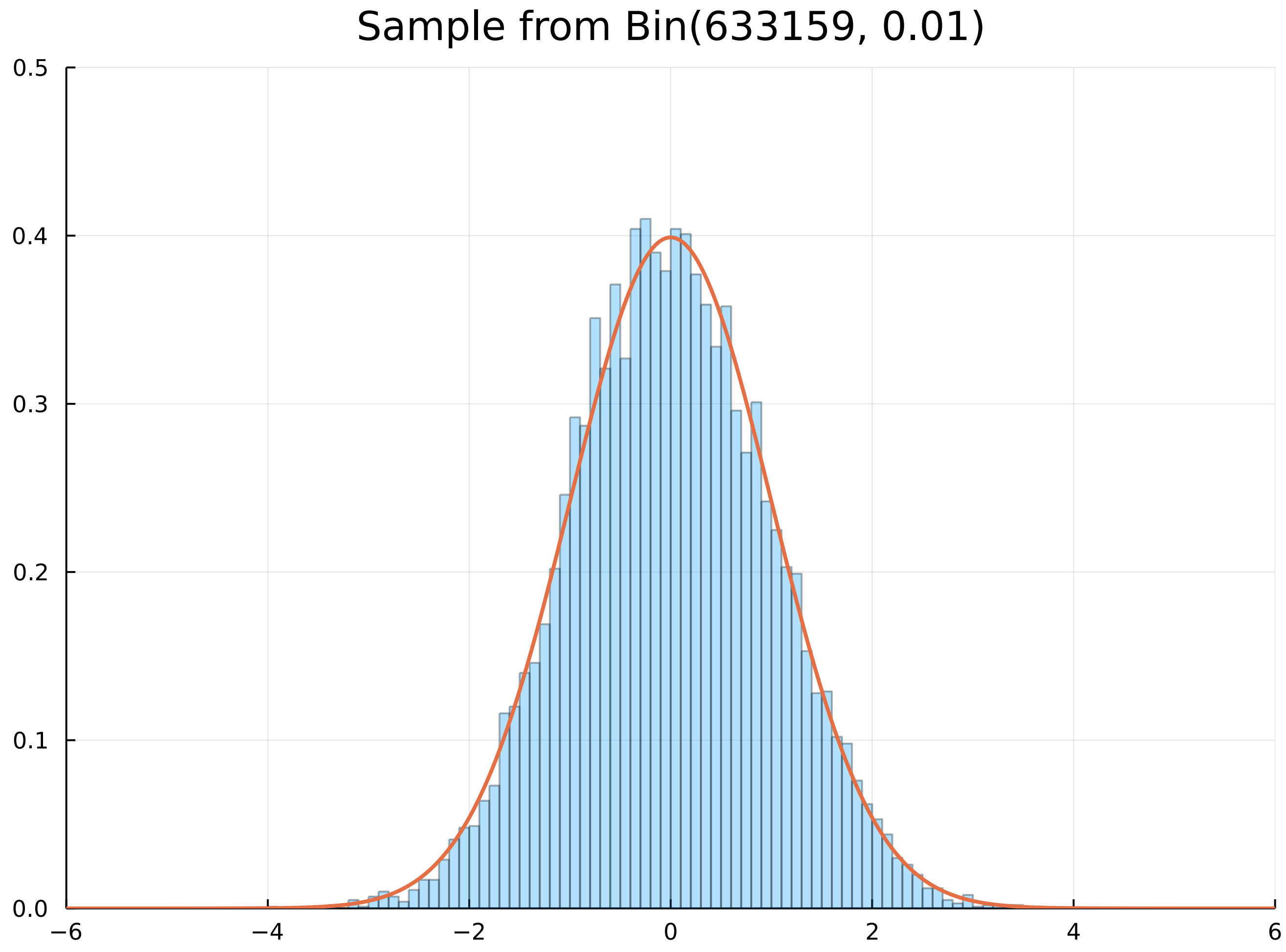
$n =$ 1000 Run! 🚀



Central Limit Theorem

$Bin(n, p)$ with $n =$ 633159 $p =$ 0.01

Sample size = 10000 Run! 🚀



Normal Distributions

$\mu =$ 0.14 $\sigma =$ 0.71