General information

- Tutor: Zhou Feng; Email: zfeng@math.cuhk.edu.hk;
- Tutorial time and venue: Mon. 11:30–12:15, WMY 408;
- Course webpage: `https://www.math.cuhk.edu.hk/~math3280/`
- Textbook: Sheldon Ross, *A first course in probability.* 8th edition. Pearson Education International.
- Structure of tutorials:
  1. Review learned concepts in the previous week/ give outline and topic of the tutorial.
  2. Present models/ explain examples/ solve problems.
  3. Q & A.
- All the suggestions and feedback are welcome.

Basic counting models

Why should we count? First, it gives us the total number of the outcomes of some experiment, i.e., the size of (discrete) sample space, which is the basic of further study. Second, during the process of counting, we can have a better understanding of the considered experiment. Third, countings arise frequently in our daily lives. And etc.

Setup: In this semester, we have **47** students in session **A** and **50** students in session **B**.

Example 1. How many possible selections of two students, one from session A and the other from session B, do we have?

Solution. 47×50. Because there are 47×50 vectors (a, b) with $a \in A$ and $b \in B$ and each selection can be represented by such a vector while any two different vectors represent different selections. □

Model 1 (Basic counting principle)**.** *Suppose the experiment A has m possible outcomes and for each outcome of A there are n outcomes of experiment B. Then together there are mn outcomes.*

Note that we can visualized this process in our mind and this principle underlies many counting arguments.

Example 1′. What if some students $\alpha \in A, \beta \in B$ are best friends and they only want to selected together?

Solution. $47 \times 50 - 49 - 46 = 46 \times 49 + 1$. LHS is followed by removing the unsatisfied pairs in which α and β are not matched together. RHS is obtained by first arranging all the other students except α and β, then plus the one pair (α, β). □

In general, more than one reasoning or counting methods can work. We can pick the ones that we are comfortable with or collect them all. Like above, two of the major ways of thinking: 1. follow the description of the experiments, count in the respective cases, and finally sum them up. 2. find a larger set of possible outcomes and then 'remove' the over-counting ones.

Example 2. In a classroom with 47 seats, how many seat plans does session A have?

Solution. $47 \times 46 \times \cdots \times 1 =: 47!$. Determine the possible selections one by one from the first seat to the last seat. □

Model 2 (Permutation). *# permutations of n elements* $= n(n-1)\cdots 1 =: n!$ *with convention* $0! = 1$.

Example 2′. What if we only want to know who are sitting in the 1st, 2nd and 3rd seats?

Solution. 'Forward' reasoning: $47 \times 46 \times 45$. The 47 possible students in the 1st seat, multiplies the 46 students in 2nd seat (after fixing the student in 1st seat) and similar for the 3rd seat.

'Exclusion' reasoning; $47!/44!$. We have $47!$ permutations of 47 students. However, no matter how we permute the 44 students in the last 44 seats, the first three students remain the same to us. Hence dividing the over-counting $44!$ permutations yields the answer. □

Model 2′. # ways to choose k **ordered** elements from n elements $= n(n-1)\cdots(n-(k-1)) = n!/(n-k)!$.

Example 3. What if we just want to know who are sitting in the first three seats and do **not** care the order?

Solution. $(47!/44!)/3!$. Further divide the over-counted $3!$ permutations of 3 students. □

Model 3 (Combination). *# ways to choose k elements from n elements* $= \frac{n!}{k!(n-k)!} =: \binom{n}{k}$.

Example 3′. After the deadline of HW1, the three TAs (F: 12, X: 10, and W: 25) are supposed to mark the corresponding number of HWs. How many HW marking plans do TAs have?

Solution. $\binom{47}{12}\binom{35}{10}\binom{25}{25} = \frac{47!}{12!10!25!}$. Similarly, we can also reason in two ways. □

Model 3′. # ways to choose $n_1, \ldots, n_k$ elements with $n_1 + \cdots + n_k = n$ into k **distinct** groups from n elements $= \frac{n!}{n_1! \cdots n_k!}$.

Until now, we have reviewed the basic counting principle, permutations and combinations which are the core concepts in basic counting.

——— It's good to stop here. ———

Example 4 (Caret inserting method). In a 'very long' classroom with 100 seats for the midterm exam, how many seat plans do we have for session A such that any two students do not sit next to each other?

There are three choices:

(a) $\binom{53}{47}$ (b) $\binom{54}{47}$ (c) $54!/7!$

Consider the 47 students and 53 empty seats. Use $\bigcirc$ to denote empty seat and $\wedge$ to represent the positions that students can take.

$$\wedge \bigcirc \wedge \bigcirc \wedge \cdots \wedge \bigcirc \wedge \bigcirc \wedge$$

where there are 53 $\bigcirc$ and $53 + 1 = 54$ $\wedge$.

Solution. Choosing 47 positions from 54 $\wedge$ will satisfy the condition. Notice that the order matters since each student is unique. Hence the correct choice is (c). □

From this example, we know that several factors can affect the counting results. Two common factors are: 1. whether the order matters? 2. whether it allows repetitions?

What if the classroom is shaped in a circle $\bigcirc$? We see the counting result will change a little bit.

Example 5. Supposed the marked HWs are divided into 3 groups: F: 12; X: 10; W: 25.

Take 5 HWs out of 47 HWs. Write down f the number of HWs marked by F; x the number of HWs marked by X; w the number of HWs marked by W ($f + x + w = 5$). How many possible different vectors (f, x, w) are there?

Solution.

$$\bigcirc | \bigcirc | \bigcirc \bigcirc \bigcirc$$

It is equivallent to divide 5 balls in to 3 groups using $2 = 3 - 1$ red carets (e.g., the above diagram represents $(1, 1, 3)$). Then the answer is the number of ways to choose 2 positions for red carets from the total $7 = 5 + 2$ positions. Thus $\binom{5+2}{2}$ is the final answer. □

Model 5. *Restated as number of non-negative integer solutions like our textbook:*

 # vectors $(n_1, \ldots, n_k)$ *with integers* $n_i \geq 0$ *and* $n_1 + \cdots n_k = n$ $= \binom{n+k-1}{k-1}$.

Remark. Notice that Example 5 is more interesting than Model 5 since we have the upper-bound conditions $f \leq 12, x \leq 10, w \leq 25$. So what if we take 11 HWs out of all the HWs? How about 23 HWs? Is there a general formula for the number of vectors that we are interested in?

After some computer experiments, we can believe the formula may exist. To obtain the formula, we have reasoned based on the principle that 1. remove the unsatisfied (illegal) outcomes. 2. compensate the outcomes that we have over-removed. Please feel free to get in touch if you are interested in the details.

You may interactively play the experiments by clicking here (or opening it) in major browsers like Chrome, Safari, Firefox or Edge. It may takes a while (up to 20 minutes?) to start but after that everything works smoothly. You can take a cup of coffee instead of waiting. (Source available at here). For convenience, we attach the static PDF in the below.

MATH3280A Tutorial 1

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A continuation of Example 5

Let $k \geq 2$ an integer and $t = (t_1, \ldots, t_k)$ be a vector of non-negative integers.

For each $0 \leq s \leq (t_1 + \cdots + t_k)$, how many vectors $(n_1, \ldots, n_k)$ are there such that

$$\begin{cases} n_1 + \cdots + n_k = s \\ 0 \leq n_i \leq t_i \quad , \forall\, 1 \leq i \leq k \end{cases} \qquad ?$$

We denote the total number of such vectors by $N_t(s)$ as a function of the target sum s depending on the treshold vector t.

A theoretical formula: For $0 \leq i \leq k$, define the multisets (i.e., allowing repeating elements)

$$T_i := \left\{ t \cdot b : b = (b_1, \ldots, b_k) \in \{0,1\}^k \text{ and } b_1 + \cdots + b_k = i \right\},$$

where $\cdot$ denotes the standard inner product, then

$$N_t(s) = \sum_{i=0}^{k} (-1)^i \sum_{x \in T_i} \binom{s - (x+i) + k - 1}{k-1} \chi_{[x+i, +\infty)}(s) \quad .$$

Specific setup

```
t =  [12, 10, 25]
```

```
t = [12, 10, 25] .|> Int
```

$s =$ 18

Count by enumeration: $C_t(s) = 133$.

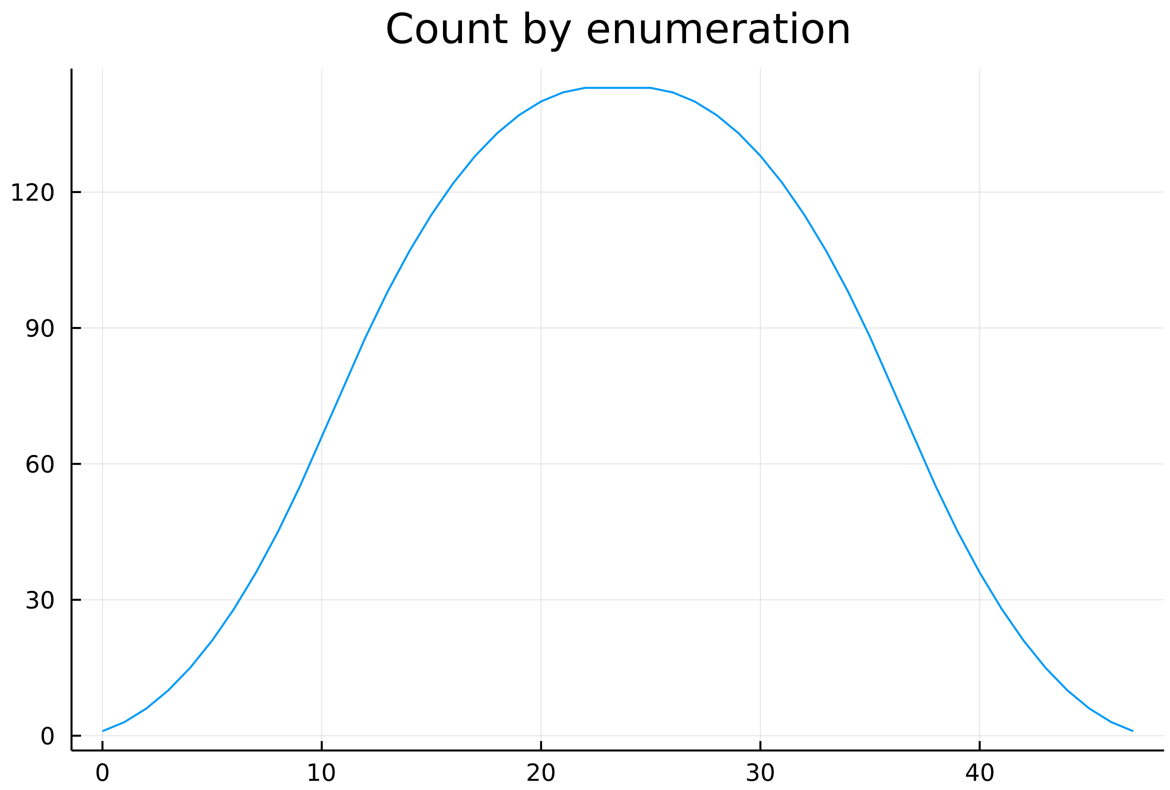
Output by formula: $N_t(s) = 133$.

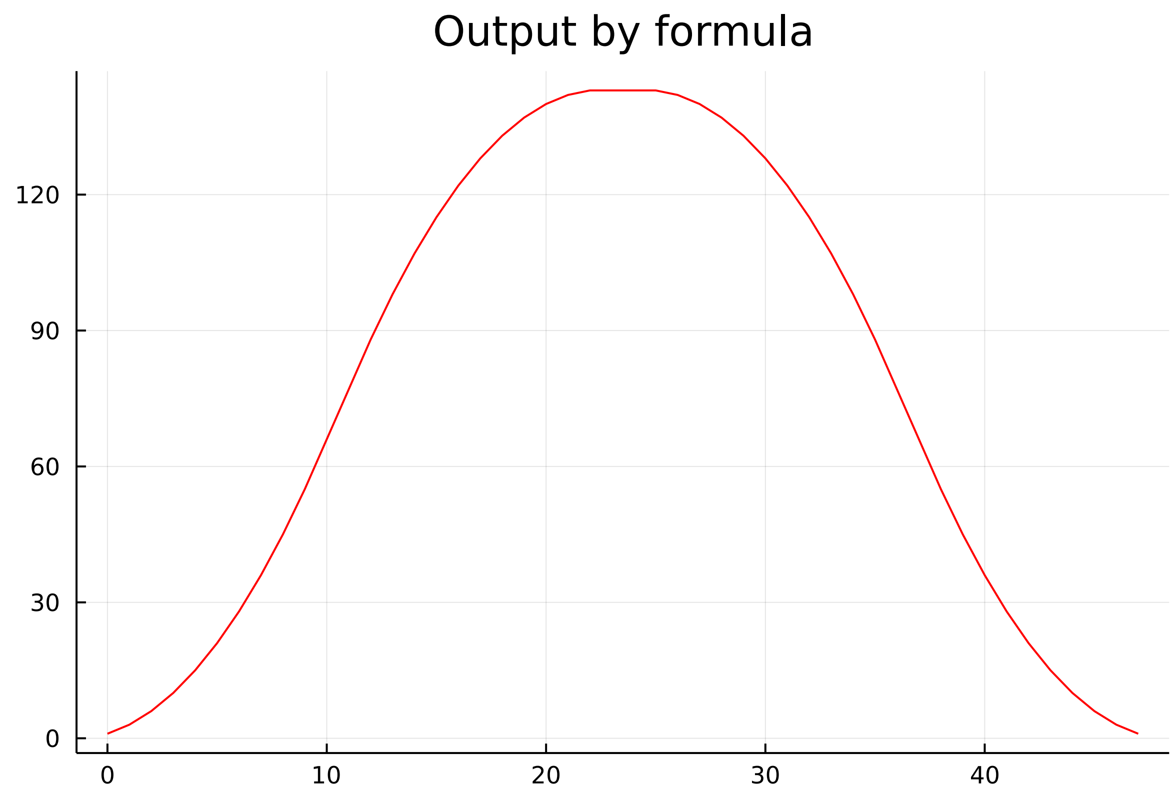
All outcomes:

```
countOutcomes =
  [1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 77, 88, 98, 107, 115, 122, 128, 133, 137,    mor
```

```
formulaOutcomes =
  [1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 77, 88, 98, 107, 115, 122, 128, 133, 137,    mor
```

Plot the outcomes





Appendix

`count_outcome` (generic function with 1 method)

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`formula_outcome` (generic function with 1 method)

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