Recall

Independence of events

- E and F are *independent* if $P(EF) = P(E)P(F) \iff P(E|F) = P(E)$ if $P(F) > 0$.
- $\{E_1, \dots, E_n\}$ are *independent* if for **every** $\{i_1, \dots, i_r\} \subset \{1, \dots, n\}$,
$$P(E_{i_1} \cdots E_{i_r}) = P(E_{i_1}) \cdots P(E_{i_r}).$$
- An infinite family of events are *independent* if every finite subset of events from that family are independent.

Example 1 (Pairwise independence $\not\Rightarrow$ independence)**.** Roll a die twice. Consider the events

$$\begin{aligned} A &= \{ \text{ sum of the two numbers is 7 } \}, \\ B &= \{ \text{ the first number is 7 } \}, \\ C &= \{ \text{ the second number is 7 } \}. \end{aligned}$$

Then

$$P(A) = P(B) = P(C) = \frac{1}{6} \quad \text{and} \quad P(A \cap B) = P(A \cap C) = P(B \cap C) = \frac{1}{36}.$$

However,

$$P(A \cap B \cap C) = \frac{1}{36} \quad \text{while} \quad P(A)P(B)P(C) = \frac{1}{216}.$$

Hence the events $\{A, B, C\}$ are pairwise independent but NOT independent.

Remark. It follows from Example 1 that we should check all the 'product' equations in the definition to assure the independence of a finite family of events.

Discrete random variables

A random variable X is a function from the sample space S to the real numbers $\mathbb{R}$. The randomness comes from the probability $P(\cdot)$ on the sample space. If the range of X is a countable set in $\mathbb{R}$, then X is called *discrete* random variable. The *probability mass function* (PMF) of a discrete random variable is defined as

$$p(x) \coloneqq P(X = x).$$

Expectation / expected value / mean.

$$E[X] \coloneqq \sum_{x\colon p(x)>0} xp(x).$$

Let $g\colon \mathbb{R} \to \mathbb{R}$. Then it is proved that (*law of the unconscious statistician*)

$$E[g(X)] = \sum_{x\colon p(x)>0} g(x)p(x).$$

In particular, for any $a, b \in \mathbb{R}$,

$$E[aX + b] = aE[X] + b.$$

Variance.

$$\mathrm{Var}(X) := E[(X - E[X])^2] = E[X^2] - (E[X])^2,$$

And for $a, b \in \mathbb{R}$,

$$\mathrm{Var}(aX + b) = a^2 \,\mathrm{Var}(X).$$

Computing expectations

Example 2 (discrete layer-cake)**.** Let X be a random variable with non-negative integral values, i.e., $X\colon S \to \mathbb{Z}_{\geq 0}$ where S denotes the sample space. Then

$$E[X] = \sum_{k=1}^{\infty} P(X \geq k).$$

Proof. By countable additivity and change of the order of summation,

$$\begin{aligned}
\sum_{k=1}^{\infty} P(X \geq k) &= \sum_{k=1}^{\infty}\sum_{i=k}^{\infty} P(X = i) \\
&= \sum_{k=1}^{\infty}\sum_{i=1}^{\infty} \chi_{\{(x,y)\colon x \geq y\}}(i,k) P(X = i) = \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} \chi_{\{(x,y)\colon y \leq x\}}(i,k) P(X = i) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i} P(X = i) \\
&= \sum_{i=1}^{\infty} iP(X = i) = E[X],
\end{aligned}$$

where the blue line contains the formal details about the change of the order of summation. Recall the indicator function for a set $E \subset \mathbb{R}^2$ is defined as

$$\chi_E(i,k) := \begin{cases} 1 & \text{if } (i,k) \in E, \\ 0 & \text{if } (i,k) \notin E. \end{cases}$$

□

Remark. The blue line is not only a safe trick for us to change the order of summation or integration, but also a simple example showing the usefulness of indicator functions.

Example 3. Randomly choose 3 numbers from $\{1, \ldots, 10\}$. Let X be the smallest number among the 3 chosen numbers. Find $E[X]$.

Solution. First determine the probability mass function

$$p(k) = P(X = k) = \begin{cases} \dfrac{\binom{10-k}{2}}{\binom{10}{3}} & , k = 1, \ldots, 8; \\ 0 & , k = 9, 10, \end{cases}$$

where the first case is obtained by choosing the other 2 numbers that are greater than k, and the second case results from the observation that $9, 10$ can never be the smallest numbers among the 3 chosen numbers. Then

$$E[X] = \sum_{k=1}^{10} kp(k) = \sum_{k=1}^{8} k\frac{\binom{10-k}{2}}{\binom{10}{3}} = \frac{11}{4}.$$

□

Example 4. In a game show, there are 3 doors:

$$\boxed{+\$30},\quad \boxed{-\$10},\quad \boxed{-\$10}.$$

To start the game, you have to pay \$5 to randomly open a door from these 3 doors which look the same to you. You will get \$30 if opening $\boxed{+\$30}$ and lose \$10 if opening $\boxed{-\$10}$.

Assume you are 'reasonable': if you win \$30, then you quit the game; if you lose \$10, then you flip a coin to decide whether you should continue, i.e., if the coin shows tail then you quit the game; if the coin shows head then you pay another \$5 to randomly open a door from the remaining 2 doors. Assume you are only allowed to play at most 2 rounds.

Let X be the winnings when you leave. Find $E[X]$ and $\mathrm{Var}(X)$.

Solution. First determine the sample space of the door results each of which is represented by a vector

$$S = \{(+30), (-10), (-10, +30), (-10, -10)\}.$$

Let G be the random variable labeling the above outcomes with $1, 2, 3, 4$ from left to right, e.g. the event $\{G = 2\}$ is the outcome (-10). Then the r.v. X is a function of the r.v. G, i.e. $X = f(G)$ for some function $f : \mathbb{R} \to \mathbb{R}$ which assigning the winnings to outcomes. Hence

$$\begin{aligned}
E[X] &= E[f(G)] \\
&= \sum_{k=1}^{4} P(G = k)f(k) \\
&= \frac{1}{3} \times (-5 + 30) + \frac{2}{3} \times \frac{1}{2} \times (-5 - 10) + \\
&\quad + \frac{2}{3} \times \frac{1}{2} \times \frac{1}{2} \times (-5 - 10 - 5 + 30) + \frac{2}{3} \times \frac{1}{2} \times \frac{1}{2} \times (-5 - 10 - 5 - 10) \\
&= 0,
\end{aligned}$$

And since $E[X] = 0$,

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2 = E[(f(G))^2] = \sum_{k=1}^{4} P(G = k)\,(f(k))^2 = 450.$$

□