Recall

Conditional probability.

Suppose $P(F) > 0$. Define $P(E|F) := \dfrac{P(EF)}{P(F)}$.

Multiplicative rule.

$$\begin{cases} P(E_1E_2) = P(E_1)P(E_1|E_2), \\ P(E_1 \cdots E_n) = P(E_1)P(E_2|E_1) \cdots P(E_n|E_1 \cdots E_{n-1}). \end{cases}$$

Law of total probability.

The special case of $\{F, F^c\}$: $P(E) = P(F)P(E|F) + P(F^c)P(E|F^c)$.

Suppose $F_1, \ldots, F_n$ satisfy $(*)$ $\begin{cases} F_i \bigcap F_j = \emptyset & \text{mutually exclusive,} \\ \bigcup_{i=1}^n F_i = S & \text{exhaustive.} \end{cases}$ Then

$$P(E) = \sum_{i=1}^{n} P(F_i)P(E|F_i).$$

$\Longrightarrow$ *Bayes' formula.*

Assume $F_1, \ldots, F_n$ satisfy $(*)$. For each $1 \leq i \leq n$,

$$P(F_i|E) = \frac{P(F_iE)}{P(E)} = \frac{P(E|F_i)P(F_i)}{\sum_{i=1}^n P(F_i)P(E|F_i)}.$$

Notice that $(*)$ implies that $\{F_1, \ldots, F_n\}$ forms a partition of the total sample space S.

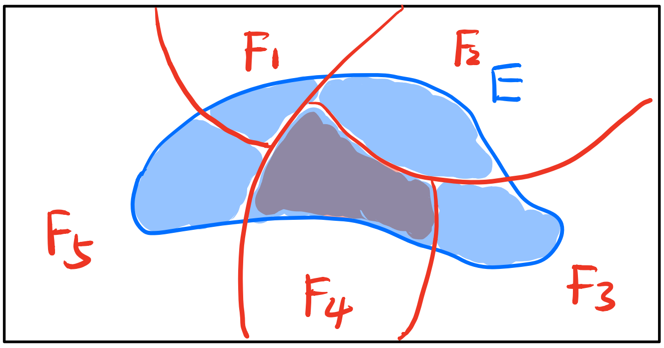

Figure 1: A logic diagram of Bayes' formula

Conditional probability

Q: Why should we care about conditional probability?

A: – As a concept, conditional probability provides a way to use the partial information and precisely express the desired probability involving the information.

– By intentionally conditioning on a partition, we can determine the probability of a desired event separately on each set in the partition with the additional information provided by the corresponding set. Then sum them up with weights to obtain the desired probability. E.g. Law of total probability.

Example 1. Randomly choose 6 balls from a box containing $\begin{cases} \text{Red: } 8 \\ \text{Blue: } 10 \\ \text{Green: } 12 \end{cases}$ balls. **Given that** no red balls are chosen, what's the probability that there are exactly 2 green balls?

Solution. Denote the events

$$\begin{aligned} E &= \{ \text{ no red balls } \}, \\ F &= \{ \text{ exactly 2 green balls } \}. \end{aligned}$$

The desired probability

$$P(F|E) = \frac{P(EF)}{P(E)} = \frac{|EF|}{|E|} = \frac{\binom{12}{2}\binom{10}{4}}{\binom{22}{6}},$$

where in the second equality we have used the equally-likely assumption on the outcomes. □

Note that in many questions, it is often convenient (and important) for us to interpret the statements of questions and use notations to denote the concerned information.

Law of total probability & Bayes' formula

Example 2. Randomly choose 3 **ordered** cards from a deck of 52 cards without replacements. What's the probability that the 2nd and 3rd cards are both heart ♡?

Solution. We will apply the law of total probability by conditioning on the suit of the 1st card. Denote the events

$$\begin{aligned} E &= \{ \text{ the 2nd and 3rd cards are } \heartsuit \}, \\ F &= \{ \text{ the 1st card is } \heartsuit \}. \end{aligned}$$

By sequentially determing the choices of 2nd and 3rd cards, we have

$$P(E|F) = \frac{12 \times 11}{51 \times 50} \text{ and } P(E|F^c) = \frac{13 \times 12}{51 \times 50}.$$

By law of total probability, the desired probability

$$P(E) = P(F)P(E|F) + P(F^c)P(E|F^c) = \frac{1}{4} \times \frac{12 \times 11}{51 \times 50} + \frac{1}{4} \times \frac{13 \times 12}{51 \times 50} = \frac{1}{17}.$$

□

Example 3. Denote head by H and tail by T.

There are 4 coins $\begin{cases} 1 \text{ of type } a: & \text{Heads on both sides,} \\ 2 \text{ of type } b: & \frac{1}{2}H \text{ and } \frac{1}{2}T, \\ 1 \text{ of type } c: & \frac{2}{3}H \text{ and } \frac{1}{3}T. \end{cases}$ Randomly choose 1 coin out of these 4 coins and flip the chosen coin. It shows head.

What's the probability that the chosen coin is of type c?

Solution. Denote the events

$$\begin{aligned} A &= \{ \text{ the chosen coin is of type } a \}, \\ B &= \{ \text{ the chosen coin is of type } b \}, \\ C &= \{ \text{ the chosen coin is of type } c \}, \\ H &= \{ \text{ the flipped result is head } \} \end{aligned}$$

Hence the desired probability is $P(C|H)$. Then prepare the ingredients

- $P(A) = \frac{1}{4}$; $P(B) = \frac{1}{2}$; $P(C) = \frac{1}{4}$,
- $P(H|A) = 1$; $P(H|B) = \frac{1}{2}$; $P(H|C) = \frac{2}{3}$.

By Bayes' formula

$$P(C|H) = \frac{P(C)P(H|C)}{P(A)P(H|A) + P(B)P(H|B) + P(C)P(H|C)} = \frac{\frac{1}{4} \times \frac{2}{3}}{\frac{1}{4} \times 1 + \frac{1}{2} \times \frac{1}{2} + \frac{1}{4} \times \frac{2}{3}} = \frac{1}{4}.$$

□