

Recall

Common discrete distributions

Usually, there are several equivalent ways to characterize a common r.v. X . (1) The story/ background/ definition of X . (2) The explicit PMF/ CDF/ (PDF) of X . (3) Express X in terms of other r.v.s. The story shows the specialty of X and hints us the type of examples that we can use X to model. The PMF or CDF way is concise and allows us to do the computations.

Bernoulli r.v. with parameter $p \in [0, 1]$. Denote $X \sim \text{Bern}(p)$.

- (1) X is the outcome of a trial that succeeds with probability p and fails with probability $1 - p$.
- (2) The PMF of X is $p(1) = P\{X = 1\} = p$ and $p(0) = P\{X = 0\} = 1 - p$.

Note $E[X] = p$ and $\text{Var}(X) = p(1 - p)$.

Binomial r.v. with parameter (n, p) . Denote $X \sim \text{Bin}(n, p)$.

- (1) X is the number of successes that occur in the n independent Bernoulli trials with parameter p .
- (2) The PMF of X is $p(k) = P\{X = k\} = \sum_{k=0}^n \binom{n}{k} p^k (1 - p)^{n-k}$ for $k = 0, \dots, n$.
- (3) Let X_1, \dots, X_n be independent Bernoulli r.v.s with parameter p . Then $X = \sum_{k=1}^n X_k$.

Note $E[X] = np$ and $\text{Var}(X) = np(1 - p)$.

Poisson r.v. with parameter $\lambda > 0$. Denote $X \sim \text{Poisson}(\lambda)$.

- (2) The PMF of X is $p(k) = P\{X = k\} = \frac{e^{-\lambda} \lambda^k}{k!}$ for $k = 0, 1, 2, \dots$.
- (1) Approximate binomial r.v. with parameter (n, p) when n large and p small such that np moderate. This story is not precise but flexible to model many examples.

Note $E[X] = \text{Var}(X) = \lambda$.

Geometric random variable & computing examples

Example 1 (Geometric r.v. with parameter p). Denote $X \sim \text{Geom}(p)$.

- (1) X is the number of independent Bernoulli trials with parameter $p \in (0, 1)$ such that first success occur.
- (2) By independence, the PMF is $p(k) = (1 - p)^{k-1} p$ for $k = 1, 2, 3, \dots$.

Then we show $E[X] = \frac{1}{p}$ and $\text{Var}(X) = \frac{1 - p}{p^2}$.

By definition, $E[X] = \sum_{k=1}^{\infty} k(1-p)^{k-1}p$. Note that for $x \in (-1, 1)$, (by a result about the uniform convergence of power series) we have

$$\sum_{k=1}^{\infty} kx^{k-1} = \sum_{k=0}^{\infty} (x^k)' = \left(\sum_{k=0}^{\infty} x^k \right)' = \left(\frac{1}{1-x} \right)' = \frac{1}{(1-x)^2}.$$

Then setting $x = 1 - p$ leads to $E[X] = p \sum_{k=1}^{\infty} k(1-p)^{k-1} = p \frac{1}{[1 - (1-p)]^2} = \frac{1}{p}$.

To obtain $\text{Var}(X)$, it suffices to compute $E[X^2]$

$$\begin{aligned} E[X^2] &= \sum_{k=1}^{\infty} k^2(1-p)^{k-1}p \\ &= \sum_{k=1}^{\infty} (k-1+1)^2(1-p)^{k-1}p \\ &= \sum_{k=1}^{\infty} (k-1)^2(1-p)^{k-1}p + \sum_{k=1}^{\infty} 2(k-1)(1-p)^{k-1}p + \sum_{k=1}^{\infty} (1-p)^{k-1}p \\ (\text{let } n = k-1) \quad &= (1-p) \sum_{n=1}^{\infty} n^2(1-p)^{n-1}p + 2(1-p) \sum_{k=1}^{\infty} n(1-p)^{n-1}p + 1 \\ &= (1-p)E[X^2] + 2(1-p)E[X] + 1 \end{aligned}$$

Then substitute $E[X] = \frac{1}{p}$ and solve the equation to get $E[X^2] = \frac{2-p}{p^2}$.

$$\text{Hence } \text{Var}(X) = E[X^2] - (E[X])^2 = \frac{1-p}{p^2}.$$

Remark. Observe that we have used two different ways to compute $E[X]$ and $\text{Var}(X)$ in [Example 1](#) both of which can be recursively extended to deal with series like $\sum_{k=1}^{\infty} k^p x^k$ with $p \in \mathbb{N}$.

Example 2. Let $X \sim \text{Bin}(n, p)$. Prove

$$E \left[\frac{1}{1+X} \right] = \frac{1 - (1-p)^{n+1}}{(n+1)p}.$$

Proof. Recall the PMF of $\text{Bin}(n, p)$ and by the formula of $E[g(X)]$,

$$\begin{aligned} E \left[\frac{1}{1+X} \right] &= \sum_{k=0}^n \frac{1}{k+1} \binom{n}{k} p^k (1-p)^{n-k} \\ &= \frac{1}{(n+1)p} \sum_{k=0}^n \frac{1}{k+1} \frac{(n+1)!}{k![(n+1)-(k+1)]!} \cdot p^{k+1} (1-p)^{[(n+1)-(k+1)]} \\ &= \frac{1}{(n+1)p} \sum_{k=0}^n \binom{n+1}{k+1} p^{k+1} (1-p)^{[(n+1)-(k+1)]} \\ (\text{let } j = k+1) \quad &= \frac{1}{(n+1)p} \left[\sum_{j=0}^n \binom{n+1}{j} p^j (1-p)^{[(n+1)-j]} - (1-p)^{n+1} \right] \\ (\text{by Binomial Thm}) \quad &= \frac{1}{(n+1)p} [1 - (1-p)^{n+1}]. \end{aligned}$$

□

Example 3. Let X be a r.v. with non-negative integral values. Prove

$$\sum_{k=0}^{\infty} kP(X \geq k) = \frac{1}{2}(E[X^2] + E[X]).$$

Proof.

$$\begin{aligned} \sum_{k=0}^{\infty} kP(X \geq k) &= \sum_{k=0}^{\infty} \sum_{i=k}^{\infty} kP(X = i) \\ &= \sum_{i=0}^{\infty} \sum_{k=0}^i kP(X = i) \\ &= \sum_{i=0}^{\infty} \frac{i(i+1)}{2} P(X = i) \\ &= \frac{1}{2}E[X^2] + \frac{1}{2}E[X], \end{aligned}$$

where in the second equality we have changed the order of summation (see e.g. [Tutorial 4, Example 2] for details). \square

Remark. Recall for the r.v. X in [Example 3](#) we also have the layer-cake $E[X] = \sum_{k=0}^{\infty} P(X \geq k)$ (see e.g. [Tutorial 4, Example 2]). Together we can express $E[X^2]$ in terms of $P(X \geq k)$ if $E[X] < \infty$, thus $\text{Var}(X)$. This process can recursively continue to express $E[X^p]$, $p \in \mathbb{N}$ in terms of $P(X \geq k)$.