THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH3280 Introductory Probability 2022-2023 Term 1
Suggested Solutions of Homework Assignment 7

Q1

Compute the second derivative of Φ.

$$\Phi''(t) = \frac{M''(t)}{M(t)} - \left(\frac{M'(t)}{M(t)}\right)^2.$$

Hence

$$\Phi''(t)|_{l=0} = \frac{M''(0)}{M(0)} - \left(\frac{M'(0)}{M(0)}\right)^2 = E\left(X^2\right) - E(X)^2 = \mathrm{Var}(X).$$

Q2

Since $\mu = \sigma^2 = 20$, by Chebyshev's inequality, we have

$$\begin{aligned} P(0 < X < 40) &= P(-20 < X - 20 < 20) = P(|X - 20| < 20) \\ &= 1 - P(|X - 20| \geq 20) \geq 1 - \frac{\sigma^2}{20^2} = \frac{19}{20}. \end{aligned}$$

Q3

(a) By Markov's inequality,

$$P\left(\sum_{i=1}^{20} X_i > 15\right) \leq \frac{E\left(\sum_{i=1}^{20} X_i\right)}{15} = \frac{20}{15} = \frac{4}{3}.$$

(b) By the central limit theorem,

$$\begin{aligned}
P\left(\sum_{i=1}^{20} X_i > 15\right) &= P\left(\sum_{i=1}^{20} X_i > 15.5\right) \\
&= P\left(\frac{\sum_{i=1}^{20} X_i - 20}{\sqrt{20}} > \frac{15.5 - 20}{\sqrt{20}}\right) \\
&\approx 1 - \Phi(-1.01) \\
&= \Phi(1.01) \\
&\approx 0.8438.
\end{aligned}$$

Q4

For $\varepsilon > 0$, let $\delta > 0$ be such that $|g(x) - g(\mathrm{c})| < \varepsilon$ whenever $|x - c| \leq \delta$. Also, let B be such that $|g(x)| < B$. Then,

$$\begin{aligned}
E\left[g\left(Z_n\right)\right] &= \int_{|x-c|\leq\delta} g(x) dF_n(x) + \int_{|x-c|>\delta} g(x) dF_n(x) \\
&\leq (\varepsilon + g(\mathrm{c})) P\left\{\left|Z_n - c\right| \leq \delta\right\} + B \cdot P\left\{\left|Z_n - c\right| > \delta\right\}
\end{aligned}$$

In addition, the same equality yields that

$$E\left[g\left(Z_n\right)\right] \geq (g(c) - \varepsilon) P\left\{\left|Z_n - c\right| \leq \delta\right\} - B \cdot P\left\{\left|Z_n - c\right| > \delta\right\}$$

Upon letting $n \to \infty$, we obtain that

$$\begin{aligned}
\limsup E\left[g\left(Z_n\right)\right] &\leq g(c) + \varepsilon \\
\liminf E\left[g\left(Z_n\right)\right] &\geq g(c) - \varepsilon
\end{aligned}$$

The result now follows since ε is arbitrary.

Q5

Let $X_1, X_2, \ldots$ be independent Bernoulli random variables with mean x. Define

$$Z_n = \frac{X_1 + \cdots X_n}{n}$$

By the weak law of large numbers, for each $\varepsilon > 0$,

$$P\left\{\left|Z_n - x\right| > \varepsilon\right\} \to 0 \text{ as } n \to \infty$$

(Alternatively, using central limit theorem to compute the probability $P\{|Z_n - x| > \varepsilon\} = 2\Phi\left(-\frac{\varepsilon\sqrt{n}}{\sigma}\right) \to 0$ as $n \to \infty$.)
Since f defined on $[0,1]$ is continuous, f is bounded. Applying Problem 4 with $c = x$ and $g = f$, we have

$$E\left[f\left(Z_n\right)\right] \to f(x) \text{ as } n \to \infty$$

(Alternatively, set $h = |f - f(x)|$ on $[0,1]$, then h is continuous and bounded above by some contant M. For each $\varepsilon > 0$. By the continuity of $h, \exists \delta > 0, \forall\, |Z_n - x| \leq \delta, h \leq \frac{\varepsilon}{2}$. By the weak law of large numbers, $\exists N \in \mathbb{N}$ such that $\forall n \geq N, P\left(|Z_n - x| > \delta\right) \leq \frac{\varepsilon}{2M}$. Hence

$$E\left[h\left(Z_n\right)\right] \leq \frac{\varepsilon}{2} P\left(|Z_n - x| \leq \delta\right) + MP\left(|Z_n - x| > \delta\right) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \leq \varepsilon$$

It follows that $E\left[h\left(Z_n\right)\right] \to 0$, thus $E\left[f\left(Z_n\right)\right] \to f(x)$ as $n \to \infty$.) On the other hand,

$$\begin{aligned} E\left[f\left(Z_n\right)\right] &= \sum_{k=0}^{n} f\left(\frac{k}{n}\right) P\left(X_1 + \cdots + X_n = k\right) \\ &= \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k} = B_n(x) \end{aligned}$$

Hence

$$\lim_{n \to \infty} B_n(x) = f(x).$$

Q6

Firstly, for $i > \lambda$, we can apply the Chernoff bound to get

$$P(X \geq i) = P\left(e^{tX} \geq e^{ti}\right) \leq e^{-ti} M_X(t) = e^{-ti} e^{\lambda(e^t - 1)}, \quad t > 0$$

Let $f(t) = e^{\lambda e^t - ti - \lambda}, t > 0$. $f(t)$ obtains its minimal value at $t = \log(\frac{i}{\lambda}) > 0$. Then put $t = \log(\frac{i}{\lambda})$, we get

$$P(X \geq i) \leq \left(\frac{\lambda}{i}\right)^i e^{i-\lambda}.$$

Secondly, for $i < \lambda$,

$$
\begin{aligned}
P(X \leq i) &= \sum_{n=0}^{i} \frac{e^{-\lambda}\lambda^n}{n!} \\
&= \frac{e^{-\lambda}\lambda^i}{i^i} \sum_{n=0}^{i} \frac{i^n}{n!} \left(\frac{i}{\lambda}\right)^{i-n} \\
&\leq \frac{e^{-\lambda}\lambda^i}{i^i} \sum_{n=0}^{\infty} \frac{i^n}{n!} \\
&= \frac{e^{i-\lambda}\lambda^i}{i^i}.
\end{aligned}
$$

Alternatively, we may still use the Chernoff bound to obtain

$$
P(X \leq i) = P\left(e^{tX} \geq e^{ti}\right) \leq e^{-ti} M_X(t) = e^{-ti} e^{\lambda(e^t - 1)}, \quad t < 0
$$

Putting $t = \log(i/\lambda)$, we have

$$
P(X \leq i) \leq \left(\frac{\lambda}{i}\right)^i e^{i-\lambda}.
$$