Recall

Continuous random variable

A random variable X is (*absolutely*) *continuous* if there exists a function, called *probability density function* (PDF), such that

$$P(X \in B) = \int_B f(x)dx,$$

where B is a 'measurable' set in $\mathbb{R}$. Fortunately, countable unions of intervals are 'measurable'.

Some facts about **continuous** random variable X:

*Unit integral of **a** PDF.* $\int_{-\infty}^{+\infty} f(x)dx = 1$.

Zero probability at any point. $\forall\, x \in \mathbb{R}, P(X = x) = 0$.

Cumulative distribution function (CDF). $F(t) \coloneqq \int_{-\infty}^{t} f(x)dx,\ \forall\, t \in \mathbb{R}$.

For $t \in \mathbb{R}$, it follows from $F(t) = P(X \le t) = P(X < t) = \lim_{x\to t-} F(x)$ that $F(t)$ is left-continuous, hence continuous, at t. In conclusion, the CDF of a continuous r.v. is continuous.

Expectation. $E[X] \coloneqq \int_{-\infty}^{+\infty} xf(x)dx$.

Continuous layer-cake. If X is continuous and non-negative, then $E[X] = \int_0^{+\infty} P(X > t)dt$.

LOTUS. Let $g\colon \mathbb{R} \to \mathbb{R}$. Then $E[g(X)] = \int_{-\infty}^{+\infty} g(x)f(x)dx$.

Variance. $\mathrm{Var}(X) \coloneqq E[(X - E[X])^2] = E[X^2] - (E[X])^2$.

Affine transform. For $a, b \in \mathbb{R}$, $\begin{cases} E[aX+b] = aE[X] + b; \\ \mathrm{Var}(aX+b) = a^2\,\mathrm{Var}(X). \end{cases}$

Relation between PDF f and CDF F. If f is continuous at $x \in \mathbb{R}$, then $F(x)' = \frac{dF(x)}{dx} = f(x)$.

Some computations about continuous random variables

Example 1. Let X be a random variable with PDF

$$f(x) = \begin{cases} c(1-x^2) & -1 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Find the value of c and the CDF of X.

Solution. Since f is a PDF,

$$\begin{aligned} 1 = \int_{-\infty}^{\infty} f(x)dx &= \int_{-1}^{1} c(1-x^2)dx \\ &= c(x - \frac{x^3}{3})\Big|_{-1}^{1} \end{aligned}$$

$$= \frac{4}{3}c.$$

Hence $c = \frac{3}{4}$. Recall for $t \in \mathbb{R}$, the CDF $F(t) := \int_{-\infty}^{t} f(x)dx$.

$$\text{If } t \le -1, \text{ then } F(t) = \int_{-\infty}^{t} f(x)dx = \int_{-\infty}^{t} 0dx = 0,$$
$$\text{If } 1 < t \le 1, \text{ then } F(t) = \int_{-\infty}^{t} f(x)dx = \int_{-1}^{t} \frac{3}{4}(1-x^2)dx = \frac{3}{4}(t - \frac{t^3}{3} + \frac{2}{3}) = -\frac{t^3}{4} + \frac{3t}{4} + \frac{1}{2},$$
$$\text{If } t > 1, \text{ then } F(t) = P(X \le t) = 1 - P(X > t) = 1 - \int_{t}^{\infty} 0dx = 1.$$

Thus

$$F(t) = \begin{cases} 0 & t \in (-\infty, -1] \\ -\frac{t^3}{4} + \frac{3t}{4} + \frac{1}{2} & t \in (-1, 1] \\ 1 & t \in (1, \infty). \end{cases}$$

□

Example 2. Let X be a random variable with PDF f_X. Find a PDF of random variable $Y = aX + b$ where $0 \neq a \in \mathbb{R}, b \in \mathbb{R}$.

Solution. Let F_X and F_Y denote the CDFs of X and Y respectively. For $t \in \mathbb{R}$,

$$F_Y(t) = P(Y \le t) = P(aX + b \le t).$$

If $a > 0$, then $F_Y(t) = P(X \le \frac{t-b}{a}) = F_X(\frac{t-b}{a})$. When F_X is differentiable at $\frac{t-b}{a}$, by chain rule

$$f_Y(t) = \frac{dF_Y(t)}{dt} = \frac{1}{a} f_X(\frac{t-b}{a}).$$

When F_X is NOT differentiable at $\frac{t-b}{a}$, define $f_Y(t) = \frac{1}{a} f_X(\frac{t-b}{a})$. Together, when $a > 0$, a possible PDF of Y is

$$f_Y(t) = \frac{1}{a} f_X(\frac{t-b}{a}) \quad , \forall\, t \in \mathbb{R}.$$

If $a < 0$, then $F_Y(t) = P(X \ge \frac{t-b}{a}) = 1 - P(X < \frac{t-b}{a}) = 1 - P(X \le \frac{t-b}{a}) = 1 - F_X(\frac{t-b}{a})$. We omit the similar discussion about differentiability. By differentiation, when $a < 0$, a PDF of Y is

$$f_Y(t) = \frac{dF_Y(t)}{dt} = -\frac{1}{a} f_X(\frac{t-b}{a}) \quad , \forall\, t \in \mathbb{R}.$$

□

Remark. Observe that in Example 2, the rigorus arguments about the differentiability of CDF are given when $a > 0$, which is the right way to think about it. However, in practice, we **omit** the discussion because we can prove a CDF is differentiable at **most** points. Then as in Example 2, we adjust the values on the **tiny** part of non-differentiable points to simplify the final results. As is discussed in the live tutorial, a **tiny** change of a PDF is still a PDF.

Remark. Let X be a continuous random variable and $g\colon \mathbb{R} \to \mathbb{R}$. The following example shows that we are not even sure whether $g(X)$ has a PDF. Actually, in Example 2 the first thing we should do is to prove that $Y = aX + b$ is indeed continuous with a PDF, which is omitted either. In practice, when the question asks for a PDF, it is implicitly assumed that a PDF exists like Example 2 and Example 4.

Example 3. Let $g(x) = 0$ for all $x \in \mathbb{R}$. Then for any random variable X (including the continuous ones), $g(X)$ is the discrete random variable such that $P(g(X) = 0) = 1$.

Proof. Let F denote the CDF of $g(X)$. Then for $t \in \mathbb{R}$,

$$F(t) \coloneqq P(g(X) \le t) = P(0 \le t) = \begin{cases} 1 & t \ge 0 \\ 0 & t < 0. \end{cases}$$

Hence $g(X)$ is the discrete random variable such that $P(g(X) = 0) = 1$. $\square$

Example 4. Suppose the CDF of X is

$$F(t) = \begin{cases} 1 - e^{-t^2} & t > 0 \\ 0 & t \le 0. \end{cases}$$

Find $P(X > 2)$ and a PDF of X.

Solution. First

$$P(X > 2) = 1 - P(X \le 2) = 1 - F(2) = e^{-4}.$$

Then

$$\text{If } x > 0, \text{ then } \frac{dF(x)}{dx} = 2xe^{-x^2}.$$
$$\text{If } x < 0, \text{ then } \frac{dF(x)}{dx} = 0.$$

Define

$$f(x) = \begin{cases} 2xe^{-x^2} & x > 0, \\ 0 & x \le 0. \end{cases}$$

Hence $f(x)$ is a PDF of X. $\square$