THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH3280 Introductory Probability 2022-2023 Term 1
Suggested Solutions of Homework Assignment 5

Q1

(a) Let $p(x_1, x_2)$ be the joint probability mass X_1 and X_2.

$$
\begin{aligned}
p(0,0) &= \frac{8}{13} \cdot \frac{7}{12} = \frac{14}{39} \\
p(0,1) &= \frac{8}{13} \cdot \frac{5}{12} = \frac{10}{39} \\
p(1,0) &= \frac{5}{13} \cdot \frac{8}{12} = \frac{10}{39} \\
p(1,1) &= \frac{5}{13} \cdot \frac{4}{12} = \frac{5}{39}
\end{aligned}
$$

(b) Let $q(x_1, x_2, x_3)$ be the joint probability mass of X_1, X_2 and X_3.

$$
\begin{aligned}
q(0,0,0) &= \frac{8}{13} \cdot \frac{7}{12} \cdot \frac{6}{11} = \frac{28}{143} & q(0,1,1) &= \frac{8}{13} \cdot \frac{5}{12} \cdot \frac{4}{11} = \frac{40}{429} \\
q(0,0,1) &= \frac{8}{13} \cdot \frac{7}{12} \cdot \frac{5}{11} = \frac{70}{429} & q(1,1,0) &= \frac{5}{13} \cdot \frac{4}{12} \cdot \frac{8}{11} = \frac{40}{429} \\
q(0,1,0) &= \frac{8}{13} \cdot \frac{5}{12} \cdot \frac{7}{11} = \frac{70}{429} & q(1,0,1) &= \frac{5}{13} \cdot \frac{8}{12} \cdot \frac{4}{11} = \frac{40}{429} \\
q(1,0,0) &= \frac{5}{13} \cdot \frac{8}{12} \cdot \frac{7}{11} = \frac{70}{429} & q(1,1,1) &= \frac{5}{13} \cdot \frac{4}{12} \cdot \frac{3}{11} = \frac{5}{143}
\end{aligned}
$$

Q2

(a) Since f is non-negative and

$$
\iint_{\mathbb{R}^2} f(x,y)\,dxdy = \int_0^1 \int_0^2 \frac{6}{7}\left(x^2 + xy/2\right) dydx = 1
$$

it follows that f is a joint density function.
(b) The density of X is given by

$$
f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^2 \frac{6}{7}\left(x^2 + xy/2\right) dy = \frac{6}{7}\left(2x^2 + x\right), \quad x \in (0,1)
$$

and $f_X(x) = 0$ elsewhere.
(c)

$$P(X > Y) = \iint_{\{(x,y):x>y\}} f(x,y)dxdy = \int_0^1 \int_0^x \frac{6}{7}\left(x^2 + xy/2\right) dydx = \frac{15}{56}$$

(d)

$$\begin{aligned} P(Y > 1/2 \mid X < 1/2) &= \frac{P(X < 1/2, Y > 1/2)}{P(X < 1/2)} \\ &= \frac{\displaystyle\int_0^{1/2} \int_{1/2}^2 \frac{6}{7}\left(x^2 + xy/2\right) dydx}{\displaystyle\int_0^{1/2} \int_0^2 \frac{6}{7}\left(x^2 + xy/2\right) dydx} \\ &= \frac{\frac{6}{7} \cdot \frac{23}{128}}{\frac{6}{7} \cdot \frac{5}{24}} \\ &= \frac{69}{80} \end{aligned}$$

(e)

$$E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x)dx = \int_0^1 x \cdot \frac{6}{7}\left(2x^2 + x\right) dx = \frac{5}{7}$$

(f) The density of Y is given by

$$f_Y(y) = \int_0^1 \frac{6}{7}\left(x^2 + xy/2\right) dx = \frac{6}{7}\left(\frac{1}{3} + \frac{y}{4}\right), \quad y \in (0,2)$$

and $f_Y(y) = 0$ elsewhere. Thus

$$E(Y) = \int_{-\infty}^{\infty} y \cdot f_Y(y)dy = \int_0^2 y \cdot \frac{6}{7}\left(\frac{1}{3} + \frac{y}{4}\right) dy = \frac{8}{7}$$

Q3

(a)

$$P(X < Y) = \iint_{((z,y):x<y)} f(x,y)dxdy = \int_0^{\infty} \int_x^{\infty} e^{-(x+y)}dydx = \frac{1}{2}$$

(b)

$$P(X < a) = \begin{cases} \displaystyle\int_0^a \int_0^{\infty} e^{-(x+y)}dydx = 1 - e^{-a} & a > 0 \\ 0 & a \le 0 \end{cases}$$

Q4

(a) The density of X and Y are given by

$$f_X(x) = \begin{cases} \int_0^1 (x+y)dy = x + \frac{1}{2} & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} \int_0^1 (x+y)dx = y + \frac{1}{2} & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Since $f \neq f_X \cdot f_Y$, we conclude that X and Y are not independent.
(b) See $f_X(x)$ in (a).
(c)

$$P(X+Y<1) = \iint_{\{(z,y):x+y<1\}} f(x,y)dxdy = \int_0^1 \int_0^{1-x} (x+y)dydx = \frac{1}{3}$$

Q5

(a) The joint density of A, B and C is given by $f(a,b,c) = f_A(a) \cdot f_B(b) \cdot f_C(c)$. Thus the joint cumulative distribution of A, B and C is

$$F(a,b,c) = \int_{-\infty}^a \int_{-\infty}^b \int_{-\infty}^e f_A(a) \cdot f_B(b) \cdot f_C(c)dcdbda = F_A(a) \cdot F_B(b) \cdot F_C(c)$$

where

$$F_A(t) = F_B(t) = F_C(t) = \begin{cases} 1, & t \geq 1 \\ t, & 0 < t < 1 \\ 0, & t \leq 0 \end{cases}$$

(b) Note that all roots of $Ax^2 + Bx + C$ are real if and only if $B^2 \geq 4AC$.

$$\begin{aligned} P\left(B^2 \geq 4AC\right) &= \iiint_{\{(a,b,c)\in[0,1]^3:b^2\geq 4ac\}} f(a,b,c)dadbdc \\ &= \int_0^{1/4} \int_0^1 \int_{\sqrt{4ac}}^1 dbdcda + \int_{1/4}^1 \int_0^1 \int_0^{\frac{1}{4a}b^2} dcdbda \\ &= \frac{5}{36} + \frac{1}{6}\ln 2 \end{aligned}$$

where the second equality is derived by the following argument:
- If $0 \le a \le 1/4$, then $4ac \le 1$ always hold for $0 \le c \le 1$, thus $\sqrt{4ac} \le b \le 1$
- If $1/4 \le a \le 1$, then $b^2/4a \le 1$ always hold for $0 \le b \le 1$, thus $0 \le c \le b^2/4a$

Alternatively,

$$
\begin{aligned}
P\left(B^2 \ge 4AC\right) &= 1 - \iiint_{\{(a,b,c)\in[0,1]^3 : b^2 \le 4ac\}} f(a,b,c)\,dadbdc \\
&= 1 - \int_0^1 \int_{b^2/4}^1 \int_{b^2/4a}^1 dcdadb \\
&= \frac{5}{36} + \frac{1}{6}\ln 2
\end{aligned}
$$

Q6

(a) Let $g_1(x,y) = x + y$ and $g_2(x,y) = x/y$. Then

$$
|J(x,y)| = \left| \begin{array}{cc} \frac{\partial g_1(x,y)}{\partial x} & \frac{\partial g_1(x,y)}{\partial y} \\ \frac{\partial g_2(x,y)}{\partial x} & \frac{\partial g_2(x,y)}{\partial y} \end{array} \right| = \left| \begin{array}{cc} 1 & 1 \\ 1/y & -x/y^2 \end{array} \right| = \left| \frac{x+y}{y^2} \right|
$$

Note that

$$
\left\{ \begin{array}{l} U = X + Y \\ V = \frac{X}{Y} \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} X = \frac{UV}{V+1} \\ Y = \frac{U}{V+1} \end{array} \right.
$$

Hence the joint density of U and V is

$$
\begin{aligned}
f_{U,V}(u,v) &= f(x,y) \cdot |J(x,y)|^{-1} \\
&= f\left(\frac{uv}{v+1}, \frac{u}{v+1}\right) \cdot \left| J\left(\frac{uv}{v+1}, \frac{u}{v+1}\right) \right|^{-1} \\
&= \begin{cases} \frac{u}{(v+1)^2} & 0 < \frac{u}{v+1} < 1, 0 < \frac{uv}{v+1} < 1 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

(b) Let $g_1(x,y) = x$ and $g_2(x,y) = x/y$. Then

$$
|J(x,y)| = \left| \begin{array}{cc} 1 & 0 \\ 1/y & -x/y^2 \end{array} \right| = \left| \frac{x}{y^2} \right|
$$

Note that

$$
\left\{ \begin{array}{l} U = X \\ V = \frac{X}{Y} \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} X = U \\ Y = \frac{U}{V} \end{array} \right.
$$

Hence the joint density of U and V is

$$\begin{aligned}
f_{U,V}(u,v) &= f(x,y)\cdot|J(x,y)|^{-1} \\
&= f\left(u,\frac{u}{v}\right)\cdot\left|J\left(u,\frac{u}{v}\right)\right|^{-1} \\
&= \begin{cases} \frac{u}{v^2} & 0<u<1, 0<\frac{u}{v}<1 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

(c) Let $g_1(x,y) = x+y$ and $g_2(x,y) = \frac{x}{x+y}$. Then

$$|J(x,y)| = \left| \begin{array}{cc} 1 & 1 \\ \frac{y}{(x+y)^2} & \frac{-x}{(x+y)^2} \end{array} \right| = \left|\frac{1}{x+y}\right|$$

Note that

$$\begin{cases} U = X+Y \\ V = \frac{X}{X+Y} \end{cases} \Leftrightarrow \begin{cases} X = UV \\ Y = U-UV \end{cases}$$

Hence the joint density of U and V is

$$\begin{aligned}
f_{U,V}(u,v) &= f(x,y)\cdot|J(x,y)|^{-1} \\
&= f(uv, u-uv)\cdot|J(u,u-uv)|^{-1} \\
&= \begin{cases} u & 0<uv<1, 0<u-uv<1 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

Q7

(a) We have $P(X=j, Y=k) = P(Y=k \mid X=j)P(X=j) = \frac{1}{5j}$. Since given $X=j$ we know Y is uniform in $\{1,2,\ldots,j\}$. Here we have $k \le j$. Thus

$$P(Y=1) = \sum_{j=1}^{5} \frac{1}{5j} = \frac{137}{300} = c$$

(b) We use Bayes formula to get

$$P(X=j \mid Y=1) = P(Y=1 \mid X=j)\frac{P(X=j)}{P(Y=1)} = \frac{1}{5j}\frac{1}{c}$$

where c as in (1).

(c) Not independent: $P(X=1, Y=1) = \frac{1}{5}$. But $P(X=1)P(Y=1) = \frac{c}{5} \neq \frac{1}{5}$.

Q8

(a) For $x > 0$, we have that

$$f_X(x) = \int_0^\infty f(x,y)dy = \int_0^\infty xe^{-x(y+1)}dy = -\, e^{-x(y+1)}\Big|_{y=0}^{y=\infty} = e^{-x}$$

and for $y > 0$ we have that

$$f_Y(y) = \int_0^\infty f(x,y)dx = \int_0^\infty xe^{-x(y+1)}dx$$

In order to solve this integral, use integration by parts. Define $u = x$ and $dv = e^{-x(y+1)}dx$. Thus

$$f_Y(y) = -\frac{x}{y+1} \cdot e^{-x(y+1)}\Big|_0^\infty + \frac{1}{y+1}\int_0^\infty e^{-x(y+1)}dx = \frac{1}{(y+1)^2}$$

Now, for $x > 0$, the conditional density of X given $Y = y$ is

$$f_{X|Y}(x \mid y) = \frac{f(x,y)}{f_Y(y)} = \frac{xe^{-x(y+1)}}{\frac{1}{(y+1)^2}} = x(y+1)^2 \cdot e^{-x(y+1)}$$

If $x \leq 0$, then $f_{X|Y}(x \mid y) = 0$. And similarly, for $y > 0$,

$$f_{Y|X}(y \mid x) = \frac{f(x,y)}{f_X(x)} = \frac{xe^{-x(y+1)}}{e^{-x}} = xe^{-xy}$$

For $y \leq 0$, $f_{Y|X}(y \mid x) = 0$.
(b) If $z \leq 0$, then $F_Z(z) = 0$, which implies $f_Z(z) = 0$. If $z > 0$, we have

$$P(Z \leq z) = P(XY \leq z) = \int_0^\infty \int_0^{\frac{z}{x}} f(x,y)dydx$$

Take the derivative with respect to z, we then get

$$f_Z(z) = \int_0^\infty \frac{1}{x} f(x, \frac{z}{x})dx = \int_0^\infty e^{-(x+z)}dx = e^{-z}$$

Q9

Let $f(x), g(y)$ be the densities of X, Y. (a) Since the cumulative distribution of Z is

$$F_Z(z) = P(X/Y \leq z) = P(X \leq Yz) = \int_0^\infty \int_0^{yz} f(x)g(y)dxdy$$

then the density of Z is

$$h(z)=\frac{dF_Z(z)}{dz}=\int_0^\infty \frac{d}{dz}\int_0^{yz} f(x)g(y)dxdy=\int_0^\infty yf(yz)g(y)dy$$

When $z\leq 0, h(z)=0$. (b) Since the cumulative distribution of Z is

$$F_Z(z)=P(XY\leq z)=P(X\leq z/Y)=\int_0^\infty\int_0^{z/y} f(x)g(y)dxdy$$

then the density of Z is

$$h(z)=\frac{dF_Z(z)}{dz}=\int_0^\infty \frac{d}{dz}\int_0^{z/y} f(x)g(y)dxdy=\int_0^\infty \frac{1}{y}f\left(\frac{z}{y}\right)g(y)dy$$

When $z\leq 0, h(z)=0$ If $f(x)=\lambda\exp(-\lambda x), x>0$ and $g(y)=\eta\exp(-\eta y), y>0$ for some $\lambda,\eta>0$, then (a)

$$h(z)=\begin{cases}\lambda\eta\displaystyle\int_0^\infty ye^{-(\lambda z+\eta)y}dy=\frac{\lambda\eta}{(z\lambda+\eta)^2} & z>0\\ 0 & \text{otherwise}\end{cases}$$

(b)

$$h(z)=\begin{cases}\lambda\eta\displaystyle\int_0^\infty \frac{1}{y}e^{-(\lambda z/y+\eta y)}dy & z>0\\ 0 & \text{otherwise}\end{cases}$$

Q10

$$\begin{aligned}P(X=n,Y=m)&=P\left(X_1+X_2=n,X_2+X_3=m\right)\\&=\sum_{k=0}^{\min\{n,m\}}P\left(X_1=n-k,X_2=k,X_3=m-k\right)\\&=\sum_{k=0}^{\min\{n,m]}P\left(X_1=n-k\right)P\left(X_2=k\right)P\left(X_3=m-k\right)\\&=\sum_{k=0}^{\min\{n,m\}}\frac{e^{-\lambda_1}\lambda_1^{n-k}}{(n-k)!}\cdot\frac{e^{-\lambda_2}\lambda_2^k}{k!}\cdot\frac{e^{-\lambda_3}\lambda_3^{m-k}}{(m-k)!}\\&=e^{-(\lambda_1+\lambda_2+\lambda_3)}\sum_{k=0}^{\min\{n,m\}}\frac{\lambda_1^{n-k}\lambda_2^k\lambda_3^{m-k}}{(n-k)!\cdot k!\cdot(m-k)!}\end{aligned}$$