Recall

Common continuous distributions

Uniform random variable. with parameter (a, b) where $a < b$. Denote $X \sim U(a, b)$.

(1) X is equally likely to be near each value in the interval (a, b).

(2) PDF: $f(x) = \begin{cases} \frac{1}{b-a} & x \in [a, b] \\ 0 & \text{otherwise.} \end{cases}$ and CDF: $F(t) = \begin{cases} 0 & t \in (-\infty, a) \\ \frac{t-a}{b-a} & t \in [a, b] \\ 1 & t \in (b, +\infty). \end{cases}$

$E[X] = \frac{a+b}{2}$ and $\mathrm{Var}(X) = \frac{(a-b)^2}{12}$.
In particular, if $Y \sim U(0, 1)$, then for Y,

PDF: $f(y) = \begin{cases} 1 & y \in [0, 1] \\ 0 & \text{otherwise.} \end{cases}$ and CDF: $F(t) = \begin{cases} 0 & t \in (-\infty, 0) \\ t & t \in [0, 1] \\ 1 & t \in (1, +\infty). \end{cases}$

Normal random variable. with parameter (μ, σ^2) where $\sigma > 0$. Denote $X \sim N(\mu, \sigma^2)$.

(2) PDF: $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ $, \forall x \in \mathbb{R}$ and CDF: $F(t) = \int_{-\infty}^{t} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$ $, \forall t \in \mathbb{R}$.
$E[X] = \mu$ and $\mathrm{Var}(X) = \sigma^2$.
Let $a, b \in \mathbb{R}$ with $a \neq 0$. Then $Y = aX + b$ is also a normal random variable. In particular, $Y = \frac{X-\mu}{\sigma} \sim N(0, 1)$ is called the *standard* normal random variable.
The CDF of Y is conventionally denoted by Φ. Recall $\Phi(t) := \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$ for $t \in \mathbb{R}$.

(1) Binomial random variable $Bin(n, p)$ when n large $\approx$ normal r.v.. Later we will discuss about this fact when the *central limit theorem* is introduced.

Theorem (DeMoivre-Laplace)**.** *Let $S_n \sim Bin(n, p)$ and $Y \sim N(0, 1)$. Then for $a < b \in \mathbb{R}$,*

$$P\left\{a \leq \frac{S_n - np}{\sqrt{np(1-p)}} \leq b\right\} \to P\{a \leq Y \leq b\} = \Phi(b) - \Phi(a) \quad \text{as } n \to \infty.$$

Exponential random variable. with parameter $\lambda > 0$. Denote $X \sim \mathrm{Exp}(\lambda)$.

(2) PDF: $f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0. \end{cases}$ and CDF: $F(t) = \begin{cases} 1 - e^{-\lambda t} & t \geq 0 \\ 0 & t < 0. \end{cases}$
$E[X] = \frac{1}{\lambda}$, $E[X^n] = \frac{n}{\lambda} E[X^{n-1}]$ for $n \geq 2$ and $\mathrm{Var}(X) = \frac{1}{\lambda^2}$.

(1) In practice, X arises as the distribution of the amount of time until some specific event occurs (see e.g., Example 3). By $P\{X > t\} = 1 - F(t) = e^{\lambda t}$ for $t > 0$, there is a key property (*memoryless*) of X that

$$P\{X > s + t | X > s\} = P\{X > t\} \quad \forall s, t > 0.$$

Examples about the above random variables

Example 1 (Standard uniform r.v. is universal)**.** Consider the random variable $U \sim U(0,1)$. Suppose F is a strictly increasing continuous CDF. Then the following statements hold:

(i) Define $X := F^{-1}(U)$. Then the CDF of X is F.

(ii) If the CDF of X is F, then $F(X) \sim U(0,1)$.

Proof. (i) Let F_X denote the CDF of X. Then for $t \in \mathbb{R}$, since $F(t) \in [0,1]$ for all $t \in \mathbb{R}$,

$$F_X(t) = P\{X \leq t\} = P\{F^{-1}(U) \leq t\} = P\{U \leq F(t)\} = F(t).$$

Hence the CDF of X is F.

(ii) Let $F_{F(X)}$ denote the CDF of $F(X)$. Then for $t \in \mathbb{R}$,

$$F_{F(X)}(t) = P\{F(X) \leq t\} = \begin{cases} 0 & t \leq 0, \\ P\{X \leq F^{-1}(t)\} = F(F^{-1}(t)) = t & 0 < t < 1, \\ 1 & t \geq 1. \end{cases}$$

Hence $F(X) \sim U(0,1)$.

□

Remark. It follows from (i) of Example 1 that we can generate samples that satisfy the desired distribution F by assigning F^{-1} to the samples with distribution $U(0,1)$.

Example 2. Let $X \sim N(0,1)$. Find a PDF of $Y = X^2$.

Solution. Let F denote the CDF of Y. Then for $t \in \mathbb{R}$,

$$F(t) = P\{Y \leq t\} = P\{X^2 \leq t\}.$$

If $t < 0$, then $F(t) = 0$ and $f(t) = 0$ by differentiation.

If $t < 0$, then $F(t) = P\{-\sqrt{t} \leq X \leq \sqrt{t}\} = P\{-\sqrt{t} < X \leq \sqrt{t}\} = \Phi(\sqrt{t}) - \Phi(-\sqrt{t})$. By chain rule,

$$f(t) = \frac{dF(t)}{dt} = \frac{1}{\sqrt{2\pi}} e^{-t/2} \cdot \frac{1}{2\sqrt{t}} - \frac{1}{\sqrt{2\pi}} e^{-t/2} \cdot \frac{-1}{2\sqrt{t}} = \frac{1}{\sqrt{2\pi t}} e^{-t/2}.$$

Define

$$f(t) := \begin{cases} \frac{1}{\sqrt{2\pi t}} e^{-t/2} & t > 0 \\ 0 & t \leq 0. \end{cases}$$

Hence f is a PDF of Y. □

Example 3. For $t > 0$, let N_t be the number of emails that we receive during time $[0,t]$. Suppose $N_t \sim Poisson(\lambda t)$ with $\lambda > 0$. Let T be the time when the first email come. Find the CDF of T.

Solution. Let F denote the CDF of T. If $t < 0$, then $F(t) = 0$. If $t > 0$, then

$$F(t) = P\{T \leq t\} = 1 - P\{T > t\}.$$

Since the event $\{T > t\}$ that the first email comes after time t is equivallent to the event that there is no emails during the time $[0, t]$, we have

$$F(t) = 1 - P\{N_t = 0\} = 1 - \frac{e^{-\lambda t}(\lambda t)^0}{0!} = 1 - e^{-\lambda t}.$$

Hence by differentiation, we define

$$f(t) := \begin{cases} \lambda e^{-\lambda t} & t > 0 \\ 0 & t \leq 0 \end{cases}.$$

Thus T has PDF f and $T \sim \mathrm{Exp}(\lambda)$. $\square$

A flash card about Φ to feel the concentration of the probability around the expectation:

The 68–95–99.7 rule for $X \sim N(\mu, \sigma^2)$:

- $P\{|X - \mu| \leq \sigma\} = 2\Phi(1) - 1 \approx 0.68.$
- $P\{|X - \mu| \leq 2\sigma\} = 2\Phi(2) - 1 \approx 0.95.$
- $P\{|X - \mu| \leq 3\sigma\} = 2\Phi(3) - 1 \approx 0.997.$