THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH3280A Introductory Probability 2022-23 Term 1
Solutions to Course Examination

1 (10 pts)

Let X be a continuous random variable, having a density function given by

$$f(x) = \begin{cases} ax + bx^2, & \text{if } 0 < x < 1; \\ 0, & \text{otherwise.} \end{cases}$$

Suppose that $E[X] = 0.6$.

(i) Determine a and b.

(ii) Calculate $\text{Var}(X)$.

Solution.

(i) Since

$$1 = \int f(x)\, dx = \int_0^1 ax + bx^2\, dx = \frac{a}{2} + \frac{b}{3}$$

and

$$0.6 = E[X] = \int x f(x)\, dx = \frac{a}{3} + \frac{b}{4},$$

we solve that

$$a = 3.6 \quad \text{and} \quad b = -2.4.$$

(ii) By computation,

$$\begin{aligned} \text{Var}(X) &= E[X^2] - (E[X])^2 \\ &= \int x^2 f(x)\, dx - (0.6)^2 \\ &= \int_0^1 (3.6x^3 - 2.4x^4)\, dx - 0.36 \\ &= 0.42 - 0.36 \\ &= 0.06. \end{aligned}$$

□

2 (10 pts)

In a certain community, 40 percent of the families own a dog and 25 percent of the families that own a dog also own a cat. In addition, 30 percent of the families own a cat.

(i) Find the probability that a randomly selected family owns both a dog and a cat.

(ii) Find the conditional probability that a randomly selected family owns a dog given that it owns a cat.

Solution. Let A denote the collection of families owning a dog and B denote the collection of families owning a cat. By assumption,

$$P(A) = 0.4, \quad P(B \mid A) = 0.25, \quad P(B) = 0.3.$$

(i) By description, the target probability is

$$P(A \cap B) = P(B \mid A)P(A) = 0.25 \times 0.4 = 0.1.$$

(ii) By description,

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{0.1}{0.3} = \frac{1}{3}.$$

$\square$

3 (10 pts)

Let X, Y be independent Poisson random variables with respectively parameters λ_1 and λ_2.

(i) Calculate the probability mass function of $X + Y$.

(ii) Calculate the conditional cumulative distribution function of X given that $X + Y = n$.

Solution. Let f, g denote the probability mass functions of X, Y respectively. Then for $k \in \mathbb{Z}_{\geq 0}$,

$$f(k) = e^{-\lambda_1} \frac{\lambda_1^k}{k!} \quad \text{and} \quad g(k) = e^{-\lambda_2} \frac{\lambda_2^k}{k!}.$$

Write $Z = X + Y$ with the probability mass function h .

(i) For $k \in \mathbb{Z}_{\geq 0}$, by the convolution formula for independent sums,

$$\begin{aligned} h(k) = f * g(k) &= \sum_{i=0}^{k} f(k-i) g(i) \\ &= e^{-(\lambda_1+\lambda_2)} \lambda_1^k \sum_{i=0}^{k} \frac{1}{(k-i)!\, i!} \left(\frac{\lambda_2}{\lambda_1}\right)^i \\ &= e^{-(\lambda_1+\lambda_2)} \frac{\lambda_1^k}{k!} \sum_{i=0}^{k} \frac{k!}{(k-i)!\, i!} \left(\frac{\lambda_2}{\lambda_1}\right)^i \\ &= e^{-(\lambda_1+\lambda_2)} \frac{\lambda_1^k}{k!} \left(1 + \frac{\lambda_2}{\lambda_1}\right)^k \\ &= e^{-(\lambda_1+\lambda_2)} \frac{(\lambda_1+\lambda_2)^k}{k!} \end{aligned}$$

where the second last equality is by Binomial Theorem. Hence

$$h(k) = \begin{cases} e^{-(\lambda_1+\lambda_2)} \frac{(\lambda_1+\lambda_2)^k}{k!} & \text{if } k \in \mathbb{Z}_{\geq 0} \\ 0 & \text{otherwise.} \end{cases}$$

(ii) For $i \in \{0, \ldots, n\}$, by the independence of X, Y and (i),

$$\begin{aligned} P\{X = i \mid X + Y = n\} &= \frac{P\{X = i,\, X + Y = n\}}{P\{X + Y = n\}} \\ &= \frac{P\{X = i,\, Y = n - i\}}{P\{X + Y = n\}} \\ &= \frac{P\{X = i\}\, P\{Y = n - i\}}{P\{X + Y = n\}} \\ &= \frac{e^{-\lambda_1} \frac{\lambda_1^i}{i!} \times e^{-\lambda_2} \frac{\lambda_2^{n-i}}{(n-i)!}}{e^{-(\lambda_1+\lambda_2)} \frac{(\lambda_1+\lambda_2)^n}{n!}} \\ &= \binom{n}{i} \frac{\lambda_1^i \lambda_2^{n-i}}{(\lambda_1+\lambda_2)^n}. \end{aligned}$$

Let F denote the conditional cumulative distribution function of X given that $X + Y = n$. Then for $k \in \{0, \dots, n\}$,

$$F(k) = P\{X \le k \mid X + Y = n\} = \sum_{i=0}^{k} P\{X = i \mid X + Y = n\} = \sum_{i=0}^{k} \binom{n}{i} \frac{\lambda_1^i \lambda_2^{n-i}}{(\lambda_1 + \lambda_2)^n}$$

Hence

$$F(t) = \begin{cases} 0 & \text{if } t < 0 \\ \sum_{i=0}^{\lfloor t \rfloor} \binom{n}{i} \frac{\lambda_1^i \lambda_2^{n-i}}{(\lambda_1+\lambda_2)^n} & \text{if } 0 \le t \le n \\ 1 & \text{if } t > n. \end{cases}$$

$\square$

4 (10 pts)

Let X be the number of 2's and Y be the number of 3's that occurs in n rolls of a fair die.

(i) Find the probability mass function of $X + Y$.

(ii) Compute $\mathrm{Cov}(X, Y)$.

Solution.

(i) By definition, $X + Y$ is the number of 2 or 3's that occurs in n rolls. Note that a fair die shows a number in $\{2, 3\}$ with probability $2/6 = 1/3$. Then $X + Y \sim \mathrm{Bin}(n, 1/3)$ and the probability mass function of $X + Y$ is

$$f(k) = \begin{cases} \binom{n}{k}\left(\frac{1}{3}\right)^k\left(\frac{2}{3}\right)^{n-k} & \text{if } k \in \{0, \dots, n\} \\ 0 & \text{otherwise.} \end{cases}$$

(ii) By the bilinearity of covariance,

$$\mathrm{Var}(X + Y) = \mathrm{Cov}(X + Y, X + Y) = \mathrm{Var}(X) + 2\,\mathrm{Cov}(X, Y) + \mathrm{Var}(Y).$$

Note that $\mathrm{Var}(Z) = p(1 - p)n$ if $Z \sim \mathrm{Bin}(n, p)$. Since $X, Y \sim \mathrm{Bin}(n, 1/6)$ and $X + Y \sim \mathrm{Bin}(n, 1/3)$, we have

$$\begin{aligned} \mathrm{Cov}(X, Y) &= \frac{\mathrm{Var}(X + Y) - \mathrm{Var}(X) - \mathrm{Var}(Y)}{2} \\ &= \frac{(1/3)(2/3)n - 2 \times (1/6)(5/6)n}{2} \\ &= -\frac{n}{36}. \end{aligned}$$

$\square$

5 (10 pts)

A fair die is successfully rolled until all 6 sides appears at least once. Let X denote the number of rolls. Calculate $E[X]$.

Solution. For $\ell \in \mathbb{N}$, let Y_ℓ be the number showed in ℓ-th roll. Fix any $k \geq 6$. For $i \in \{1, \dots, 6\}$, define the event

$$\begin{aligned} B_i &:= \bigcap_{\ell=1}^{k-1} \Big\{ Y_\ell \in \{1, \dots, 6\} \setminus \{i\} \Big\} \\ &= \Big\{ \text{ the numbers of the first } k-1 \text{ rolls are in } \{1, \dots, 6\} \setminus \{i\} \Big\} \end{aligned}$$

and

$$\begin{aligned} A_i &:= B_i \bigcap \bigcap_{j \in \{1,\dots,6\} \setminus \{i\}} \bigcup_{\ell=1}^{k-1} \{Y_\ell = j\} \\ &= \Big\{ \text{ the numbers of the first } k-1 \text{ rolls are in } \{1, \dots, 6\} \setminus \{i\}; \\ &\qquad \text{each of } \{1, \dots, 6\} \setminus \{i\} \text{ appears at least once in first } k-1 \text{ rolls} \Big\}. \end{aligned}$$

Next we compute $P(A_6)$ and the computation for $P(A_i)$ is the same. For $j \in \{1, \dots, 5\}$, define the event

$$\begin{aligned} E_j &= \bigcap_{\ell=1}^{k-1} \Big\{ Y_\ell \in \{1, \dots, 5\} \setminus \{j\} \Big\} \\ &= \Big\{ \text{ the numbers of the first } k-1 \text{ rolls are in } \{1, \dots, 5\} \setminus \{j\} \Big\}. \end{aligned}$$

By the inclusion-exclusion principle,

$$\begin{aligned} P\left(\bigcup_{j=1}^{5} E_j \right) &= \sum_{1 \leq j_1 \leq 5} P(E_{j_1}) - \sum_{1 \leq j_1 < j_2 \leq 5} P(E_{j_1} E_{j_2}) + \sum_{1 \leq j_1 < j_2 < j_3 \leq 5} P(E_{j_1} E_{j_2} E_{j_3}) \\ &\quad - \sum_{1 \leq j_1 < j_2 < j_3 < j_4 \leq 5} P(E_{j_1} E_{j_2} E_{j_3} E_{j_4}) + P(E_1 E_2 E_3 E_4 E_5) \\ &= \frac{1}{6^{k-1}} \left(5 \times 4^{k-1} - \binom{5}{2} 3^{k-1} + \binom{5}{3} 2^{k-1} - \binom{5}{4} \times 1 + 0 \right) \\ &= 5(\frac{2}{3})^{k-1} - 10(\frac{1}{2})^{k-1} + 10(\frac{1}{3})^{k-1} - 5(\frac{1}{6})^{k-1}. \end{aligned}$$

Since $B_6 \setminus A_6 = \bigcup_{j=1}^5 E_j$ and $P(B_6) = (5/6)^{k-1}$,

$$P(A_6) = P(B_6) - P\left(\bigcup_{j=1}^{5} E_j \right) = (\frac{5}{6})^{k-1} - 5(\frac{2}{3})^{k-1} + 10(\frac{1}{2})^{k-1} - 10(\frac{1}{3})^{k-1} + 5(\frac{1}{6})^{k-1}.$$

By the descriptions of X, A_i and the independence of rolls,

$$P\{X = k \mid Y_k = i\} = P\{A_i \mid Y_k = i\} = P(A_i) = P(A_6).$$

Hence by the law of total probability,

$$\begin{aligned}
P\{X=k\} &= \sum_{i=1}^{6} P\{X=k \mid Y_k=i\}P\{Y_k=i\} \\
&= \sum_{i=1}^{6} P(A_6)\times\frac{1}{6} \\
&= P(A_6) \\
&= (\frac{5}{6})^{k-1} - 5(\frac{2}{3})^{k-1} + 10(\frac{1}{2})^{k-1} - 10(\frac{1}{3})^{k-1} + 5(\frac{1}{6})^{k-1}.
\end{aligned} \tag{1}$$

Note that for $x \in (0,1)$, we have

$$f(x) := \sum_{k=6}^{\infty} kx^{k-1} = \left(\sum_{k=6}^{\infty} x^k\right)' = \left(\frac{x^6}{1-x}\right)' = \frac{x^5(6-5x)}{(1-x)^2}.$$

Finally, by (1),

$$\begin{aligned}
E[X] &= \sum_{k=6}^{\infty} k\,P\{X=k\} \\
&= f\left(\frac{5}{6}\right) - 5f\left(\frac{2}{3}\right) + 10f\left(\frac{1}{2}\right) - 10f\left(\frac{1}{3}\right) + 5f\left(\frac{1}{6}\right) \\
&= \frac{147}{10} = 14.7.
\end{aligned}$$

□

6 (10 pts)

Let Z be a standard normal random variable. Let

$$W = \begin{cases} 2Z & \text{if } Z < 2, \\ -Z & \text{if } Z \geq 2. \end{cases}$$

Find $E[W]$.

Solution. Let f denote the probability density function of Z. Then

$$\begin{aligned} E[W] &= E[W\chi_{Z<2}] + E[W\chi_{Z\geq 2}] \\ &= E[2Z\chi_{Z<2}] + E[-Z\chi_{Z\geq 2}] \\ &= \int_{-\infty}^{2} 2xf(x)\,dx - \int_{2}^{\infty} xf(x)\,dx \\ &= \frac{2}{\sqrt{2\pi}}\int_{-\infty}^{2} xe^{-x^2/2}\,dx - \frac{1}{\sqrt{2\pi}}\int_{2}^{\infty} xe^{-x^2/2}\,x \\ &= -\frac{3e^{-2}}{\sqrt{2\pi}}. \end{aligned}$$

□

7 (10 pts)

The joint density function of X and Y is given by

$$f(x,y) = \begin{cases} e^{-x-y} & \text{if } x > 0 \text{ and } y > 0, \\ 0 & \text{otherwise.} \end{cases}$$

(i) Compute the conditional expectation $E[X+Y \mid X-Y=a]$.

(ii) Compute $E[\min(X,Y)]$, where $\min(X,Y)$ represents the minimum element in $\{X,Y\}$.

Solution.

(i) Let

$$\begin{cases} U := X+Y \\ V := X-Y. \end{cases}$$

Then

$$\begin{cases} X = \frac{U+V}{2} \\ Y = \frac{U-V}{2}. \end{cases}$$

and for $u > 0, -u < v < u$, the joint PDF of U and V is

$$\begin{aligned} f_{U,V}(u,v) &= f\left(\frac{u+v}{2}, \frac{u-v}{2}\right) \left|\det\left(\frac{\partial(U,V)}{\partial(X,Y)}\right)\right|^{-1} \\ &= e^{-u} \times \frac{1}{2} = \frac{e^{-u}}{2}. \end{aligned}$$

Hence

$$f_{U,V}(u,v) = \begin{cases} \frac{e^{-u}}{2} & \text{if } u > 0 \text{ and } -u < v < u \\ 0 & \text{otherwise.} \end{cases}$$

Then the PDF of V is

$$f_V(v) = \int f_{U,V}(u,v)\, du = \int_{|v|}^{\infty} \frac{e^{-u}}{2}\, du = \frac{e^{-|v|}}{2} \qquad \text{for } v \in \mathbb{R}.$$

and the conditional PDF of U given $V=a$ is

$$f_{U|V=a}(u) = \frac{f_{U,V}(u,a)}{f_V(a)} = e^{|a|}e^{-u} \qquad \text{for } u > |a|.$$

Finally,

$$\begin{aligned} E[X+Y \mid X-Y=a] &= E[U \mid V=a] \\ &= \int u f_{U|V=a}(u)\, du \\ &= e^{|a|} \int_{|a|}^{\infty} u e^{-u}\, du \\ &= 1 + |a|. \end{aligned}$$

(ii) Since X, Y are joint continuous,

$$\begin{aligned}
E[\min(X,Y)] &= \iint_{\{x<y\}} xf(x,y)\,dxdy + \iint_{\{x>y\}} yf(x,y)\,dxdy \\
&= \iint_{\{x<y\}} xf(x,y)\,dxdy + \iint_{\{x>y\}} yf(x,y)\,dxdy \\
&= \int_0^\infty \int_0^y xe^{-x}e^{-y}\,dxdy + \int_0^\infty \int_0^x ye^{-x}e^{-y}\,dydx \\
&= \frac{1}{4} + \frac{1}{4} \\
&= \frac{1}{2}.
\end{aligned}$$

$\square$

8 (10 pts)

Let $X_1, \ldots, X_{100}$ be independent random variables, each uniformly distributed over $(1, 2)$. Use the central limit theorem to compute an approximation for

$$P\left\{\prod_{i=1}^{100} X_i > a^{100}\right\}, \quad a \in (1,2),$$

the answer is in terms of the distribution function of standard normal variable $\Phi(x), x \geq 0$.

Solution. For $i \in \{1, \ldots, 100\}$, define $Y_i := \log X_i$. Then

$$\mu = E[Y_i] = \int_1^2 \log x\, dx = (x \log x)\Big|_1^2 - \int_1^2 1\, dx = 2\log 2 - 1$$

and

$$\sigma^2 = \mathrm{Var}(Y_i) = E[Y_i^2] - (E[Y_i])^2 = \int_1^2 (\log x)^2\, dx - (2\log 2 - 1)^2 = 1 - 2(\log 2)^2.$$

Then

$$\begin{aligned} P\left\{\prod_{i=1}^{100} X_i > a^{100}\right\} &= P\left\{\sum_{i=1}^{100} \log X_i > 100 \log a\right\} \\ &= P\left\{\frac{\sum_{i=1}^{100}(Y_i - \mu)}{10\sigma} > \frac{10(\log a - \mu)}{\sigma}\right\}. \end{aligned}$$

By the central limit theorem,

$$P\left\{\prod_{i=1}^{100} X_i > a^{100}\right\} \approx 1 - \Phi\left(\frac{10(\log a - \mu)}{\sigma}\right).$$

By $\Phi(-x) = 1 - \Phi(x)$,

$$\begin{aligned} P\left\{\prod_{i=1}^{100} X_i > a^{100}\right\} &\approx \begin{cases} 1 - \Phi\left(\frac{10(\log a - \mu)}{\sigma}\right) & \text{if } \log a \geq \mu, \\ \Phi\left(\frac{10(\mu - \log a)}{\sigma}\right) & \text{if } \log a < \mu. \end{cases} \\ &= \begin{cases} 1 - \Phi\left(\frac{10(\log a - 2\log 2 + 1)}{\sqrt{1 - 2(\log 2)^2}}\right) & \text{if } \log a \geq 2\log 2 - 1, \\ \Phi\left(\frac{10(2\log 2 - 1 - \log a)}{\sqrt{1 - 2(\log 2)^2}}\right) & \text{if } \log a < 2\log 2 - 1. \end{cases} \end{aligned}$$

$\square$

9 (10 pts)

Let X, Y be independent standard normal variables. Find the probability density function of X/Y.

Solution. Let f denote the probability density function of standard normal variable. Let F denote the cumulative distribution function of X/Y. Then for $t \in \mathbb{R}$,

$$\begin{aligned} F(t) &= P\{X/Y \le t\} \\ &= P\{X \le tY, Y \ge 0\} + P\{X \ge tY, Y < 0\} \\ &= \int_0^\infty \int_{-\infty}^{ty} f(x)f(y)\,dxdy + \int_{-\infty}^0 \int_{ty}^\infty f(x)f(y)\,dxdy. \end{aligned}$$

Then for $t \in \mathbb{R}$, the probability density function of X/Y is

$$\begin{aligned} g(t) &= \frac{dF(t)}{dt} \\ &= \int_0^\infty f(y)f(ty)y\,dy - \int_{-\infty}^0 f(y)f(ty)y\,dy \\ &= \int_0^\infty f(y)f(ty)y\,dy + \int_0^\infty f(-x)f(-tx)x\,dx && \text{by taking } x = -y \\ &= \int_0^\infty f(y)f(ty)y\,dy + \int_0^\infty f(x)f(tx)x\,dx && \text{by } f(x) = f(-x) \\ &= 2\int_0^\infty f(x)f(tx)x\,dx \\ &= \frac{1}{\pi}\int_0^\infty xe^{-\frac{(1+t^2)x^2}{2}}\,dx \\ &= \frac{1}{\pi(t^2+1)}\int_0^\infty ye^{-\frac{y^2}{2}}\,dy && \text{by taking } y = \sqrt{t^2+1}\,x \\ &= \frac{1}{\pi(t^2+1)}. \end{aligned}$$

$\square$

Alternatively, let $U = X, V = X/Y$. Then we can get the joint PDF of U, V and obtain the PDF of V by integration.

10 (10 pts)

Let $f(x)$ be a continuous function defined for $0 \leq x \leq 1$. Consider the functions

$$B_n(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}.$$

Let $X_1, X_2, \ldots,$ be independent Bernoulli random variables with mean x.

(i) Prove that $B_n(x) = E\left[f\left(\frac{X_1+\cdots+X_n}{n}\right)\right]$.

(ii) Give a probabilistic proof for the fact that $\lim_{n\to\infty} B_n(x) = f(x)$.

Solution.

(i) Let $Y_n = \sum_{i=1}^{n} X_i$. Then $Y_n \sim \text{Bin}(n, x)$. Then

$$\begin{aligned} B_n(x) &= E\left[f\left(\frac{X_1+\cdots+X_n}{n}\right)\right] = E\left[f\left(\frac{Y_n}{n}\right)\right] \\ &= \sum_{k=0}^{n} f\left(\frac{k}{n}\right) P\{Y_n = k\} \\ &= \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}. \end{aligned}$$

(ii) By the strong law of large numbers,

$$\frac{X_1+\cdots+X_n}{n} \xrightarrow{a.e.} x \qquad \text{as } n \to \infty.$$

Since f is continuous,

$$f\left(\frac{X_1+\cdots+X_n}{n}\right) \xrightarrow{a.e.} f(x) \qquad \text{as } n \to \infty.$$

Since f is continuous on $[0,1]$, there is some $M > 0$ such that

$$\left|f\left(\frac{X_1+\cdots+X_n}{n}\right)\right| \leq M.$$

By Lebesgue Dominated Convergence Theorem,

$$B_n(x) = E\left[f\left(\frac{X_1+\cdots+X_n}{n}\right)\right] \to E[f(x)] = f(x) \qquad \text{as } n \to \infty.$$

$\square$

Alternatively, we can follow the way in the homework: We first establish a convergent theorem based on the continuity of f and the convergence in measure. Then the weak law of large number will complete the proof.

— *THE END* —