Recall

For $n \in \mathbb{N}$, let $X, \widetilde{X}, Y$ and X_i, $i = 1, \ldots, n$ be random variables.

Properties of expectation

- (LOTUS) Let $g\colon \mathbb{R}^2 \to \mathbb{R}$.
  - If X, Y discrete with joint PMF $p(x, y)$, then $E[g(X, Y)] = \sum_x \sum_y g(x, y)p(x, y)$.
  - If X, Y joint continuous with joint PDF $f(x, y)$, then $E[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y)f(x, y)dxdy$.
- (linear) $\forall\, \alpha, \beta \in \mathbb{R}$, $E[\alpha X + \beta Y] = \alpha E[X] + \beta E[Y]$. By induction, $E[\sum_{i=1}^n X_i] = \sum_{i=1}^n E[X_i]$.

  In general, to justify $E[\sum_{i=1}^\infty X_i] = \sum_{i=1}^\infty E[X_i]$ we need some additional conditions and two important ones are: (1) $X_i \geq 0$ for all $i \in \mathbb{N}$. **Or** (2) $\sum_{i=1}^\infty E[|X_i|] < \infty$.
- (monotone) If $X \leq Y$, then $E[X] \leq E[Y]$. In particular, $|E[X]| \leq E[|X|]$.

Covariance

- $\mathrm{Cov}(X, Y) := E\left[(X - E[X])(Y - E[Y])\right] = E[XY] - E[X]E[Y]$.
- X, Y independent $\implies$ $E[XY] = E[X]E[Y]$ $\iff$ $\mathrm{Cov}(X, Y) = 0$.
- Properties of $\mathrm{Cov}(\cdot, \cdot)$:

  (1) $\mathrm{Cov}(X, X) = \mathrm{Var}(X) \geq 0$.

  (2) $\mathrm{Cov}(X, Y) = \mathrm{Cov}(Y, X)$.

  (3) $\mathrm{Cov}(\alpha X + \widetilde{X}, Y) = \alpha\,\mathrm{Cov}(X, Y) + \mathrm{Cov}(\widetilde{X}, Y)$ for $\alpha \in \mathbb{R}$.

  It follows from (2) and (3) that $\mathrm{Cov}(\cdot, \cdot)$ is bilinear. Note that $\mathrm{Cov}(X, X) = 0$ only implies $P(X = E[X]) = 1$ (see e.g., Chebyshev's inequality in later lectures).
- $\mathrm{Var}(\sum_{i=1}^n X_i) = \sum_{i=1}^n \mathrm{Var}(X_i) + 2\sum_{1\leq i<j\leq n} \mathrm{Cov}(X_i, X_j)$. In particular, $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\,\mathrm{Cov}(X, Y)$.

  Moreover, if $X_1, \ldots, X_n$ are pairwise independent, then $\mathrm{Var}(\sum_{i=1}^n X_i) = \sum_{i=1}^n \mathrm{Var}(X_i)$.

Examples

Example 1 (yet another example for $\mathrm{Cov}(X, Y) = 0 \not\Rightarrow X, Y$ independent)**.** Let X, Y be random variables with the joint PDF:

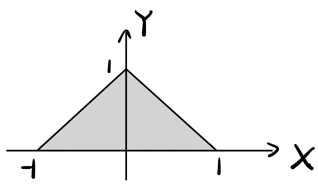

Figure 1: $f(x, y) = 1$ on the shadowed region

By symmetry, $E[XY] = 0$ and $E[X] = 0$, hence $\mathrm{Cov}(X,Y) = 0$. However, X, Y are NOT independent from the "shape" of $f(x,y)$.

Example 2. For $L > 0$, let $X, Y \overset{i.i.d.}{\sim} U(0,L)$. Find $E[|X-Y|]$.

Solution. By independence, the joint PDF of X, Y is

$$f(x,y) = f_X(x)f_Y(y) = \frac{1}{L}\chi_{(0,L)}(x)\frac{1}{L}\chi_{(0,L)}(y) = \frac{1}{L^2}\chi_{(0,L)\times(0,L)}(x,y).$$

Applying the formula of $E[g(X,Y)]$, we have

$$\begin{aligned} E[|X-Y|] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|x-y|f(x,y)dxdy \\ &= \int_0^L\int_0^x (x-y)\frac{1}{L^2}dydx + \int_0^L\int_0^y (y-x)\frac{1}{L^2}dxdy \\ &= \frac{1}{L^2}\int_0^L x^2dx = \frac{L}{3}. \end{aligned}$$

Alternatively, note $|X-Y| = \max(X,Y) - \min(X,Y)$. Define $U = \min(X,Y), V = \max(X,Y)$. Then by e.g., [Tutorial 10, Example 3], the joint PDF of U, V is $f(u,v) = \begin{cases} 2\frac{1}{L^2} & 0 < u < v < L \\ 0 & \text{otherwise.} \end{cases}$ Hence

$$E[|X-Y|] = E[V-U] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(v-u)f(u,v)dudv = \int_0^L\int_0^v (v-u)\frac{2}{L^2}dudv = \frac{L}{3}.$$

□

Denote the sample space by Ω. Let A, B be events $\subset \Omega$. Let χ_A be the *indicator variable* with respect to A, i.e., for $\omega \in \Omega$,

$$\chi_A(\omega) := \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A. \end{cases}$$

Then we have some readily-checked convenient facts:

(a) $E[\chi_A] = P(A)$.

(b) $\chi_{A^c} = 1 - \chi_A$; $\chi_{A\cap B} = \chi_A\chi_B$, $\chi_A^2 = \chi_A$; $\chi_{A\cup B} = 1 - \chi_{A^c\cap B^c} = 1 - (1-\chi_A)(1-\chi_B)$.

Example 3. In the above notations, prove $\mathrm{Cov}(\chi_A, \chi_B) = P(B)(P(A|B) - P(A))$ if $P(B) > 0$.

Proof. By (a) and (b),

$$\begin{aligned} \mathrm{Cov}(\chi_A,\chi_B) &= E[\chi_A\chi_B] - E[\chi_A]E[\chi_B] \\ &= E[\chi_{A\cap B}] - P(A)P(B) \\ &= P(A\cap B) - P(A)P(B) \\ &= P(B)(P(A|B) - P(A)). \end{aligned}$$

□

Remark. By Example 3, χ_A, χ_B independent $\iff$ $\text{Cov}(\chi_A, \chi_B) = 0$.

Example 4. Let X, Y be random variables with joint PDF

$$f(x,y) = \begin{cases} \frac{1}{y} e^{-y-x/y} & x > 0, y > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Show that $\text{Cov}(X, Y) = 1$.

Solution. By change of variable, let $t = x/y$ in the inner integral (where y is fixed) when it appears during the computations below. Then $dx = ydt$. Applying the formula for $E[g(X,Y)]$, we have

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f(x,y) dx dy = \int_0^{\infty} \int_0^{\infty} \frac{x}{y} e^{-y-x/y} dx dy = \int_0^{\infty} \int_0^{\infty} t e^{-y-t} y dt dy \\ &= \int_0^{\infty} y e^{-y} dy \int_0^{\infty} t e^{-t} dt = 1 \times 1 = 1. \end{aligned}$$

And

$$\begin{aligned} E[Y] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y f(x,y) dx dy = \int_0^{\infty} \int_0^{\infty} e^{-y-x/y} dx dy = \int_0^{\infty} \int_0^{\infty} e^{-y-t} y dt dy \\ &= \int_0^{\infty} y e^{-y} dy \int_0^{\infty} e^{-t} dt = 1 \times 1 = 1. \end{aligned}$$

And

$$\begin{aligned} E[XY] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f(x,y) dx = \int_0^{\infty} \int_0^{\infty} x e^{-y-x/y} dx dy = \int_0^{\infty} \int_0^{\infty} y t e^{-y-t} y dt dy \\ &= \int_0^{\infty} y^2 e^{-y} dy \int_0^{\infty} t e^{-t} dt = 2 \times 1 = 2. \end{aligned}$$

Hence $\text{Cov}(X,Y) = E[XY] - E[X]E[Y] = 2 - 1 \times 1 = 1$. □

Remark. To calculate the last integrals of the computations in Example 4, we can apply integration by parts to get a recursive formula $\int_0^{\infty} x^n e^{-x} dx = n \int_0^{\infty} x^{n-1} e^{-x} dx$ for $n \in \mathbb{N}$. Hence by induction, $\int_0^{\infty} x^n e^{-x} dx = n!$ (, which is exactly the definition of gamma function $\Gamma(n+1) = n\Gamma(n) = n!$).