Since there is no lecture last Thursday, we pick some examples based on previous knowledge.

Example 1. Let $U, V \overset{i.i.d.}{\sim} U(0,1)$, i.e., U, V are independent random variables with common distribution $U(0,1)$ (the standard uniform distribution). Define

$$X \coloneqq \min(U,V) \quad \text{and} \quad Y \coloneqq \max(U,V).$$

Find a PDF f_X of X and a joint PDF $f_{X,Y}$ of X, Y.

Solution. Let F_X denote the CDF of X. Then for $t \in (0,1)$,

$$\begin{aligned} F_X(t) &= P(X \le t) \\ \text{(by def. of } X\text{)} \qquad &= P(U \le t \text{ or } V \le t) \\ \text{(De Morgan's Law)} \qquad &= 1 - P(U > t \text{ and } V > t) \\ \text{(by independence)} \qquad &= 1 - P(U > t)P(V > t) \\ \text{(by def. of } U, V\text{)} \qquad &= 1 - (1-t)^2 = 2t - t^2. \end{aligned}$$

Notice that $F_X(t) = 0$ if $t < 0$ and $F_X(t) = 1$ if $t > 1$. By differentiation, we have

$$f_X(x) = \begin{cases} 2 - 2x & x \in (0,1) \\ 0 & \text{otherwise.} \end{cases}$$

Next we will compute the joint PDF of X, Y. Fix $0 < x < y < 1$. Let $\varepsilon > 0$ be small. Then by applying the finite additivity with respect to the partition $\{U < V\} \sqcup \{U > V\} \sqcup \{U = V\}$,

$$\begin{aligned} & P\{X \in [x - \tfrac{\varepsilon}{2}, x + \tfrac{\varepsilon}{2}],\ Y \in [y - \tfrac{\varepsilon}{2}, y + \tfrac{\varepsilon}{2}]\} \\ &= P\Big\{\{X \in [x - \frac{\varepsilon}{2}, x + \frac{\varepsilon}{2}],\ Y \in [y - \frac{\varepsilon}{2}, y + \frac{\varepsilon}{2}]\} \cap \{U < V\}\Big\} \\ &\quad + P\Big\{\{X \in [x - \frac{\varepsilon}{2}, x + \frac{\varepsilon}{2}],\ Y \in [y - \frac{\varepsilon}{2}, y + \frac{\varepsilon}{2}]\} \cap \{U > V\}\Big\} + 0 \\ &= P\Big\{U \in [x - \frac{\varepsilon}{2}, x + \frac{\varepsilon}{2}],\ V \in [y - \frac{\varepsilon}{2}, y + \frac{\varepsilon}{2}]\Big\} + P\Big\{V \in [x - \frac{\varepsilon}{2}, x + \frac{\varepsilon}{2}],\ U \in [y - \frac{\varepsilon}{2}, y + \frac{\varepsilon}{2}]\Big\} \\ &= \varepsilon \times \varepsilon + \varepsilon \times \varepsilon \\ &= 2\varepsilon^2. \end{aligned}$$

Then

$$f_{X,Y}(x,y) = \lim_{\varepsilon \to 0} \frac{P\{X \in [x - \frac{\varepsilon}{2}, x + \frac{\varepsilon}{2}],\ Y \in [y - \frac{\varepsilon}{2}, y + \frac{\varepsilon}{2}]\}}{\varepsilon^2} = \lim_{\varepsilon \to 0} \frac{2\varepsilon^2}{\varepsilon^2} = 2. \tag{1}$$

Hence the joint PDF of X, Y is

$$f_{X,Y}(x,y) = \begin{cases} 2 & 0 < x < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

□

Remark. In (1), we have used the local characterization of PDF at most points which can be formally remembered as (density "=" mass / volume).

Example 2 (A flavor of symmetry arguments)**.** Let $X_1, X_2, \ldots$ be a sequence of i.i.d. continuous random variables. Let $N \geq 2$ be such that

$$X_1 \geq X_2 \geq \cdots \geq X_{N-1} \text{ and } X_{N-1} < X_N, \tag{2}$$

i.e., N is the random variable representing the first position that the sequence $X_1, X_2, \ldots$ stops decreasing. Show that $E[N] = e$.

Proof. Let $n \in \mathbb{Z}_{\geq 2}$. We first determine $P(N = n)$. Since $(X_i)_{i=1}^n$ are i.i.d., for any permutation $(i_1, \ldots, i_n)$ of $(1, \ldots, n)$, we have $P(X_{i_1} \geq \cdots \geq X_{i_n}) = P(X_1 \geq \cdots \geq X_n)$. Then by the finite additivity,

$$n! \times P(X_{i_1} \geq \cdots \geq X_{i_n}) = \sum_{\substack{(j_1,\ldots,j_n) \\ \text{permutation of } (1,\ldots,n)}} P(X_{j_1} \geq \cdots \geq X_{j_n}) = 1,$$

where in the last equality we have used $P(X_i = X_j) = 0\ , \forall 1 \leq i < j \leq n$ since $(X_i)_{i=1}^n$ are continuous. This implies $P(X_{i_1} \geq \cdots \geq X_{i_n}) = \frac{1}{n!}$ for every permutation $(i_1, \ldots, i_n)$ of $(1, \ldots, n)$.

To interpret (2), we can reason by first fixing the positions of $X_1, \ldots, X_{n-1}$ and then placing X_n on the positions just before $X_k, 1 \leq k \leq n-1$, i.e., in the order of decreasing from left to right

$$\bigcirc X_1 \bigcirc X_2 \bigcirc \cdots \bigcirc X_{n-1},$$

where $\bigcirc$ are positions that X_n can take. Hence

$$P(N = n) = \frac{n-1}{n!},$$

thus

$$E[N] = \sum_{n=2}^{\infty} nP(N = n) = \sum_{n=2}^{\infty} n\frac{n-1}{n!} = \sum_{n=2}^{\infty} \frac{1}{(n-2)!} = e.$$

Alternatively, by law of total probability,

$$\begin{aligned}
P(N = n) &= P(X_1 \geq \cdots \geq X_{n-1},\ X_{n-1} < X_n) \\
&= P(X_1 \geq \cdots \geq X_{n-1})P(X_1 \geq \cdots \geq X_{n-1},\ X_{n-1} < X_n \mid X_1 \geq \cdots \geq X_{n-1}) + 0 \\
&= P(X_1 \geq \cdots \geq X_{n-1})P(X_{n-1} < X_n \mid X_1 \geq \cdots \geq X_{n-1}) \\
&= P(X_1 \geq \cdots \geq X_{n-1})[1 - P(X_{n-1} \geq X_n \mid X_1 \geq \cdots \geq X_{n-1})] \\
&= \frac{1}{(n-1)!}(1 - \frac{1/n!}{1/(n-1)!}) \\
&= \frac{n-1}{n!}.
\end{aligned}$$

$\square$

Example 3 (Basic order statistics)**.** For $n \geq 2$, let $X_1, \ldots, X_n$ be i.i.d. continuous random variables with common PDF f and CDF F. For $i \in \{1, \ldots, n\}$, define

$$X_{(i)} := \text{ the } i\text{-th smallest item of } \{X_1, \ldots, X_n\}.$$

Find the joint PDF $f_{X_{(1)},\ldots,X_{(n)}}$ of $X_{(1)}, \ldots, X_{(n)}$.

Proof. Let $x_1 < \cdots < x_n \in \mathbb{R}$. Then similar to (1), we have

$$\begin{aligned}
& f_{X_{(1)},\ldots,X_{(n)}}(x_1, \ldots, x_n) \\
&= \lim_{\varepsilon \to 0} \frac{P\left\{x_1 - \frac{\varepsilon}{2} < X_{(1)} < x_1 + \frac{\varepsilon}{2}, \ldots,\ x_n - \frac{\varepsilon}{2} < X_{(n)} < x_n + \frac{\varepsilon}{2}\right\}}{\varepsilon^n} \\
\text{(by def. of } X_{(i)}) \quad &= \lim_{\varepsilon \to 0} \frac{P\left\{\bigsqcup_{\substack{(i_1,\ldots,i_n) \\ \text{permutation of } (1,\ldots,n)}} \bigcap_{k=1}^n \{x_k - \frac{\varepsilon}{2} < X_{i_k} < x_k + \frac{\varepsilon}{2}\}\right\}}{\varepsilon^n} \\
\text{(by finite additivity)} \quad &= \lim_{\varepsilon \to 0} \frac{\sum_{\substack{(i_1,\ldots,i_n) \\ \text{permutation of } (1,\ldots,n)}} P\left\{\bigcap_{k=1}^n \{x_k - \frac{\varepsilon}{2} < X_{i_k} < x_k + \frac{\varepsilon}{2}\}\right\}}{\varepsilon^n} \\
\text{(by i.i.d.)} \quad &= \lim_{\varepsilon \to 0} \frac{\sum_{\substack{(i_1,\ldots,i_n) \\ \text{permutation of } (1,\ldots,n)}} \prod_{k=1}^n [F(x_k + \frac{\varepsilon}{2}) - F(x_k - \frac{\varepsilon}{2})]}{\varepsilon^n} \\
&= n! \lim_{\varepsilon \to 0} \prod_{k=1}^n \frac{F(x_k + \frac{\varepsilon}{2}) - F(x_k - \frac{\varepsilon}{2})}{\varepsilon} \\
&= n! \prod_{k=1}^n f(x_k).
\end{aligned}$$

Hence the joint PDF of $X_{(1)}, \ldots, X_{(n)}$ is

$$f_{X_{(1)},\ldots,X_{(n)}}(x_1, \ldots, x_n) = \begin{cases} n! \prod_{k=1}^n f(x_k) & x_1 < \cdots < x_n \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

$\square$

Remark. Recall U, V from Example 1. Let $X_1 = U$ and $X_2 = V$ in Example 3. Then $X_{(1)} = \min(U, V)$ and $X_{(2)} = \max(U, V)$. It follows from (3) that

$$\begin{aligned}
f_{X_{(1)},X_{(2)}}(x, y) &= \begin{cases} 2\chi_{[0,1]}(x)\chi_{[0,1]}(y) & x < y \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} 2 & 0 < x < y < 1 \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}$$

which recovers the result in Example 1.