THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH3280 Introductory Probability 2021-22 Term 1
Solution to Homework 2

1. If two fair dice are rolled, what is the conditional probability that the first one lands on 6 given that the sum of the dice is i? Compute for all values of i between 2 and 12.

   Solution. Denote the events

$$\begin{aligned} E &= \{ \text{ the first die lands on 6 } \}, \\ F_i &= \{ \text{ the sum of dice is } i \} \text{ for } i = 2, \dots, 12. \end{aligned}$$

   Then

$$P(E \cap F_i) = \begin{cases} 0 & i = 2, \dots, 6 \\ 1/36 & i = 7, \dots, 12. \end{cases}$$

   And

$$P(F_i) = \begin{cases} 6/36 & i = 7 \\ 5/36 & i = 8 \\ 4/36 & i = 9 \\ 3/36 & i = 10 \\ 2/36 & i = 11 \\ 1/36 & i = 12. \end{cases}$$

   Hence

$$P(E|F_i) = \frac{P(E \cap F_i)}{P(F_i)} = \begin{cases} 0 & i = 2, \dots, 6 \\ 1/6 & i = 7 \\ 1/5 & i = 8 \\ 1/4 & i = 9 \\ 1/3 & i = 10 \\ 1/2 & i = 11 \\ 1 & i = 12. \end{cases}$$

   □

2. What is the probability that at least one of a pair of fair dice lands on 6, given that the sum of the dice is i, $i = 2, 3, \cdots, 12$?

   Solution. Denote the event $\widetilde{E} = \{$ at least one of the dice lands on 6 $\}$. In the same notation of the solution to Question 1,

$$P(\widetilde{E} \cap F_i) = \begin{cases} 0 & i = 2, \dots, 6 \\ 2/36 & i = 7, \dots, 11 \\ 1/36 & i = 12. \end{cases}$$

Hence

$$P(\widetilde{E}|F_i) = \frac{P(\widetilde{E} \cap F_i)}{P(F_i)} = \begin{cases} 0 & i = 2, \ldots, 6 \\ 1/3 & i = 7 \\ 2/5 & i = 8 \\ 1/2 & i = 9 \\ 2/3 & i = 10 \\ 1 & i = 11, 12. \end{cases}$$

□

3. Two cards are randomly chosen without replacement from an ordinary deck of 52 cards. Let B be the event that both cards are aces, let A_h be the event that the ace of hearts is chosen, and let A be the event that at least one ace is chosen. Find

   (a) $P(B|A_h)$.

   (b) $P(B|A)$.

Solution. (a) By combination arguments on the desired outcomes of the the other card without considering the order,

$$P(B \cap A_h) = \frac{\binom{3}{1}}{\binom{52}{2}} \text{ and } P(A_h) = \frac{\binom{51}{1}}{\binom{52}{2}},$$

then

$$P(B|A_h) = \frac{\binom{3}{1}}{\binom{51}{1}} = \frac{1}{17}.$$

Alternatively, by condition on the first card,

$$P(B \cap A_h) = \frac{1}{52} \times \frac{3}{52} + \frac{3}{52} \times \frac{1}{52} \text{ and } P(A_h) = \frac{1}{52} \times 1 + \frac{51}{52} \times \frac{1}{51}$$

then

$$P(B|A_h) = \frac{\frac{6}{52 \times 51}}{\frac{2}{52}} = \frac{1}{17}.$$

(b) Notice $B \cap A = B$ and $P(A) = 1 - P(A^c)$. Then

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{P(B)}{1 - P(A^c)}.$$

By combination,

$$P(B|A) = \frac{\binom{4}{2}/\binom{52}{2}}{1 - \binom{48}{2}/\binom{52}{2}} = \frac{1}{33}.$$

Alternatively, by permutation,

$$P(B|A) = \frac{\frac{4}{54} \times \frac{3}{51}}{1 - \frac{48}{52} \times \frac{47}{51}} = \frac{1}{33}.$$

Since the sample spaces consist of equally-likely outcomes, it is also acceptable to compute the conditional probability by counting the corresponding outcomes. □

4. Urn I contains 2 blue and 4 white balls, whereas urn II contains 1 blue and 1 white ball. A ball is randomly chosen from urn I and put into urn II, and a ball is then randomly selected from urn II. What is

   (a) the probability that the ball selected from urn II is blue?

   (b) the conditional probability that the transferred ball was blue given that a blue ball is selected from urn II?

Solution. Denote the events

$$B = \{ \text{ the ball selected from urn II is blue } \},$$
$$W = \{ \text{ the transferred ball was white } \}.$$

(a) By law of total probability,

$$P(B) = P(W)P(B|W) + P(W^c)P(B|W^c) = \frac{4}{6} \times \frac{1}{3} + \frac{2}{6} \times \frac{2}{3} = \frac{4}{9}.$$

(b) By Bayes' formula

$$P(W^c|B) = \frac{P(W^c)P(B|W^c)}{P(B)} = \frac{\frac{2}{6} \times \frac{2}{3}}{\frac{4}{9}} = \frac{1}{2}.$$

□

5. E and F play a series of games. Each game is independently won by E with probability p and by F with probability $1 - p$. They stop when the total number of wins of one of the players is two greater than that of the other player. The player with the greater number of total wins is declared the winner of the series.

   (a) Find the probability that a total of 4 games are played.

   (b) Find the probability that E is the winner of the series.

Solution. Let N denote the total number of games which could be infinite. For any $n \le N$, denote f_n the number of games that F wins and e_n the number of games that E wins in the first n games. Notice that $f_n + e_n = n$ and $f_n, e_n \in \mathbb{Z}_{\ge 0}$. There are two observations.

If $N < \infty$, then by the winning criterion, $2f_N + 2 = N$ or $2e_N + 2 = N$, thus N must be even.

If $n = 2k < N$, then $f_n = e_n$. Since, otherwise, $f_n = e_n + 1$ or $e_n = f_n + 1$ will imply n is odd, which is a contradiction.

For each game, write the result by 1 if E wins and by 0 if E loses. Hence for any $2k < N$, the results of $(2k - 1)$-th and $(2k)$-th game must be (10) or (01). If $2k = N$, then the results of $(2k - 1)$-th and $2k$-th game must be (11) or (00).

(a) According to the above reasoning, the results of the first two games should be (10) or (01) and that of the last two games should be (11) or (00).

By the independence of the games,

$$\begin{aligned} P\{ \text{ a total 4 games are played } \} &= [p(1-p) + (1-p)p] \cdot [p^2 + (1-p)^2] \\ &= 2p(1-p)(2p^2 - 2p + 1). \end{aligned}$$

(b) For $k \in \mathbb{N}$, let E_{2k} denote the event that E becomes the winner after a total $2k$ games. According to the above reasoning, when E wins right after $2k$ games ($N = 2k$), for any $i < k, i \in \mathbb{N}$, the results of $(2i-1)$-th and $(2i)$-th game must be (10) or (01). The results of $(2k-1)$-th and $2k$-th game must be (11).

By the countable additivity of probability

$$\begin{aligned} P\{E \text{ is the winner of the series }\} &= \sum_{k=1}^{\infty} P(E_{2k}) \\ &= \sum_{k=1}^{\infty} [2p(1-p)]^{k-1}p^2 \\ &= \frac{p^2}{1-2p(1-p)} = \frac{p^2}{2p^2-2p+1} = \frac{p^2}{p^2+(1-p)^2} \end{aligned}$$

where $p \in (0,1)$.

Alternatively, denote the events

$$\begin{aligned} A &= \{ \ E \text{ wins the series } \}, \\ B &= \{ \ E \text{ wins the first two games } \}, \\ C &= \{ \ E \text{ wins a game and loses a game in the first two games } \}, \\ D &= \{ \ E \text{ loses the first two games } \}, \end{aligned}$$

Notice that $P(A|B) = 1$ and $P(A|D) = 0$. It follows from the independence that $P(A|C) = P(A)$. By law of total probability,

$$\begin{aligned} P(A) &= P(B)P(A|B) + P(C)P(A|C) + P(D)P(A|D) \\ &= p^2 + 2p(1-p)P(A) + 0, \end{aligned}$$

from which we solve that $P(A) = \dfrac{p^2}{1-2p(1-p)}$.

□

6. Let $A \subset B$. Express the following probabilities as simply as possible:

(a) $P(A|B)$;

(b) $P(A|B^c)$;

(c) $P(B|A)$;

(d) $P(B|A^c)$.

Solution. We express the desired probabilities in terms of $P(A)$ and $P(B)$.

(a) By $A \cap B = A$, $P(A|B) = \dfrac{P(A)}{P(B)}$.

(b) By $A \cap B^c = \emptyset$, $P(A|B^c) = 0$.

(c) By $B \cap A = A$, $P(B|A) = 1$.

(d) Since $P(B \cap A^c) = P(B) - P(B \cap A) = P(B) - P(A)$ and $P(A^c) = 1 - P(A)$, we have $P(B|A^c) = \dfrac{P(B) - P(A)}{1 - P(A)}$

□

7. Independent trials that result in a success with probability q and a failure with probability $1-q$ are called Bernoulli trials. Let Q_n denote the probability that n Bernoulli trials result in an even number of successes (0 being considered an even number). Show that

$$Q_n = q(1-Q_{n-1}) + (1-q)Q_{n-1},\ n \geq 1$$

and use this formula to prove (by induction) that

$$Q_n = \frac{1+(1-2q)^n}{2}. \tag{1}$$

Solution. Let E_n denote the event that the n-th trial is success and F_n denote the event that there is an even number of successes in n trials. Then

$$P(F_n|F_{n-1}) = P(E_n^c) \text{ and } P(F_n|F_{n-1}^c) = P(E_n).$$

By law of total probability and the independence of trials

$$\begin{aligned}
Q_n = P(F_n) &= P(F_{n-1})P(F_n|F_{n-1}) + P(F_{n-1}^c)P(F_n|F_{n-1}^c) \\
&= P(F_{n-1})P(E_n^c) + (1-P(F_{n-1}))P(E_n) \\
&= (1-q)Q_{n-1} + q(1-Q_{n-1}).
\end{aligned}$$

Next we prove (1) by induction. It is readily checked that $Q_1 = 1-q$ since 0 is even. Suppose (1) holds for Q_{n-1}. We check that

$$\begin{aligned}
Q_n &= q(1-Q_{n-1}) + (1-q)Q_{n-1} \\
&= q\left(1 - \frac{1+(1-2q)^{n-1}}{2}\right) + (1-q)\frac{1+(1-2q)^{n-1}}{2} \\
&= \frac{q - q(1-2q)^{n-1} + 1 - q + (1-q)(1-2q)^{n-1}}{2} = \frac{1+(1-2q)^n}{2}.
\end{aligned}$$

□

8. Let C_n denote the probability that no run of 3 consecutive heads appears in n tosses of a fair coin. Show that

$$C_n = \frac{1}{2}C_{n-1} + \frac{1}{4}C_{n-2} + \frac{1}{8}C_{n-3}$$
$$C_0 = C_1 = C_2 = 1$$

Find C_8. (*Hint:* Condition on the first trial)

Solution. Denote the events

$$\begin{aligned}
T &= \{ \text{ 1st trial is tail } \}, \\
HT &= \{ \text{ 1st trial is head, 2nd trial is tail } \}, \\
HHT &= \{ \text{ 1st and 2nd trials are head, 3rd trial is tail } \}, \\
HHH &= \{ \text{ the first three tirals are all head } \}, \\
E_n &= \{ \text{ no run of 3 consecutive heads appears } \} \text{ for } n \in \mathbb{Z}_{\geq 3}.
\end{aligned}$$

Then $P(E_n|HHH) = 0$. By law of total probability and the independence of trials

$$\begin{aligned}
C_n = P(E_n) &= P(T)P(E_n|T) + P(HT)P(E_n|HT) + P(HHT)P(E_n|HHT) + 0 \\
&= P(T)P(E_{n-1}) + P(HT)P(E_{n-2}) + P(HHT)P(E_{n-3}) \\
&= \frac{1}{2}C_{n-1} + \frac{1}{4}C_{n-2} + \frac{1}{8}C_{n-3}.
\end{aligned}$$

By iteration we have the first several values of $C_n, n = 1, \ldots, 9$

$$\begin{bmatrix} 1 & 1 & 1 & \frac{7}{8} & \frac{13}{16} & \frac{3}{4} & \frac{11}{16} & \frac{81}{128} & \frac{149}{256} & \frac{137}{256} \end{bmatrix}$$

Hence $C_8 = 149/256$. $\square$

9. For a fixed event B, show that the collection $P(A|B)$, defined for all events A, satisfies the three conditions for a probability. Conclude from this that

$$P(A|B) = P(A|B \cap C)P(C|B) + P(A|B \cap C^c)P(C^c|B) \tag{2}$$

Then directly verify the preceding equation.

Solution. Fix a event B with $P(B) > 0$. Then

(i) Since $0 \le P(A \cap B) \le P(B)$, $0 \le P(A|B) \le 1$.

(ii) Since $B \subset S$, $P(S|B) = \dfrac{P(S \cap B)}{P(B)} = \dfrac{P(B)}{P(B)} = 1$.

(iii) For disjoint events $(E_n)_{n=1}^{\infty}$, then $(E_n \bigcap B)_{n=1}^{\infty}$ disjoint,

$$P(\bigcup_{n=1}^{\infty} E_n|B) = \frac{P((\bigcup_{n=1}^{\infty} E_n) \cap B)}{P(B)} = \frac{P(\bigcup_{n=1}^{\infty} (E_n \cap B))}{P(B)} = \frac{\sum_{n=1}^{\infty} P(E_n \cap B)}{P(B)} = \sum_{n=1}^{\infty} P(E_n|B).$$

where the third equality is by the countable additivity of $P(\cdot)$. Hence $P(\cdot|B)$ satisfies the axioms of probability. It follows from law of total probability that (2) holds.

Direct check by the definition of conditional probability and the finite additivity of $P(\cdot)$:

$$\begin{aligned}
P(A|BC)P(C|B) + P(A|BC^c)P(C^c|B) &= \frac{P(ABC)}{P(BC)}\frac{P(BC)}{P(B)} + \frac{P(ABC^c)}{P(BC^c)}\frac{P(BC^c)}{P(B)} \\
&= \frac{P(ABC) + P(ABC^c)}{P(B)} \\
&= \frac{P(AB)}{P(B)} \\
&= P(A|B).
\end{aligned}$$

$\square$

— *THE END* —