# Recall

Let $X, Y$ be two r.v.s and $\alpha, \beta \in \mathbb{R}$. Then $E[\alpha X + \beta Y] = \alpha E[X] + \beta E[Y]$ (the linearity of $E[\cdot]$) .

## Cumulative distribution function

The *cumulative distribution function* (CDF) of a r.v. $X$ is defined by

$$F(t) \coloneqq P(X \leq t), \ \forall\, t \in \mathbb{R}$$

which has the following properties:

· Non-decreasing. · Right-continuous. · $\lim_{t\to-\infty} F(t) = 0$ and $\lim_{t\to+\infty} F(t) = 1$.

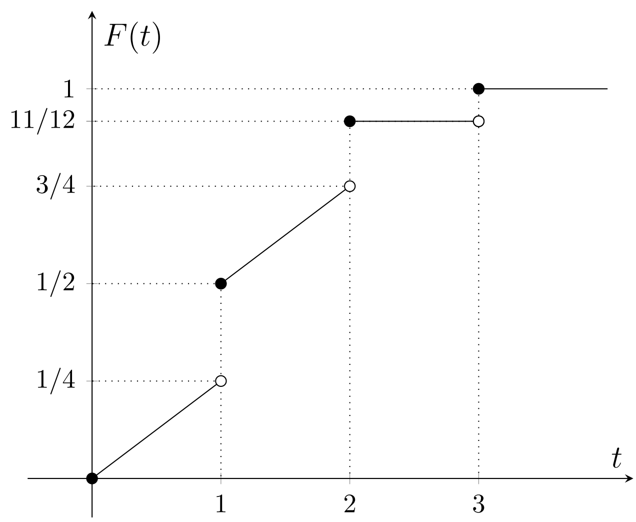
All probability questions about $X$ can be answered in terms of CDF. In particular, for $x \in \mathbb{R}$, $P(X < x) = \lim_{t\to x-} F(t)$.

**Example 1.** Suppose r.v. $X$ has CDF

$$F(t) = \begin{cases} 0 & t \in (-\infty, 0) \\ t/4 & t \in [0, 1) \\ 1/2 + (t-1)/4 & t \in [1, 2) \\ 11/12 & t \in [2, 3) \\ 1 & t \in [3, +\infty). \end{cases}$$

Find $P(X = i),\ i = 1, 2, 3$ and $P(1 \leq X < 3)$.

*Solution.* Below is the graph of $F(t)$.




Then

$$P(X = 1) = P(X \leq 1) - P(X < 1) = F(1) - \lim_{t\to1-} F(t) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4},$$
$$P(X = 2) = P(X \leq 2) - P(X < 2) = F(2) - \lim_{t\to2-} F(t) = \frac{11}{12} - \frac{3}{4} = \frac{1}{6},$$

$$P(X=3) = P(X \le 3) - P(X < 3) = F(3) - \lim_{t \to 3-} F(t) = 1 - \frac{11}{12} = \frac{1}{12}.$$

And

$$P(1 \le X < 3) = P(X < 3) - P(X < 1) = \lim_{t \to 3-} F(t) - \lim_{t \to 1-} F(t) = \frac{11}{12} - \frac{1}{4} = \frac{2}{3}.$$

$\square$

*Remark.* Since the CDF of discrete r.v.s should be of the shape of step functions, the r.v. $X$ in Example 1 is not discrete. Later we will learn that $X$ is not a *continuous* r.v. either because the CDF of continuous r.v. should be (absolutely) continuous (following Ross' definition of continuous r.v.).

## Common discrete random variables

Usually, there are several equivalent ways to characterize a common r.v. $X$. (1) The story/ backgroud/ definition of $X$. (2) The explicit PMF/ CDF/ (PDF) of $X$. (3) Express $X$ in terms of other r.v.s. The story shows the specialty of $X$ and hints us the type of examples that we can use $X$ to model. The PMF or CDF way is concise and allows us to do the computations.

*Bernoulli r.v..* with parameter $p \in [0, 1]$. Denote $X \sim Bern(p)$.

(1) $X$ is the outcome of a trial that succeeds with probability $p$ and fails with probability $1 - p$.

(2) The PMF of $X$ is $p(1) = P(X = 1) = p$ and $p(0) = P(X = 0) = 1 - p$.

Note $E[X] = p$ and $\mathrm{Var}(X) = p(1 - p)$.

*Binomial r.v..* with parameter $(n, p)$. Denote $X \sim Bin(n, p)$.

(1) $X$ is the number of successes that occur in the $n$ independent Bernoulli trials with parameter $p$.

(2) The PMF of $X$ is $p(k) = P(X = k) = \sum_{k=0}^{n} \binom{n}{k} p^k (1 - p)^{n-k}$ for $k = 0, \ldots, n$.

(3) Let $X_1, \ldots, X_n$ be independent Bernoulli r.v.s with parameter $p$. Then $X = \sum_{k=1}^{n} X_k$.

Note $E[X] = np$ and $\mathrm{Var}(X) = np(1 - p)$.

*Poisson r.v..* with parameter $\lambda > 0$. Denote $X \sim Poisson(\lambda)$.

(2) The PMF of $X$ is $p(k) = P(X = k) = \dfrac{\lambda^k e^{-\lambda}}{k!}$ for $k = 0, 1, 2, \ldots$

(1) Approximate binomial r.v. with parameter $(n, p)$ when $n$ large and $p$ small such that $np$ moderate. This story is not precise but flexible to model many examples.

Note $E[X] = \mathrm{Var}(X) = \lambda$.

# Geometric r.v. & computing examples

**Example 2** (Geometric r.v. with parameter $p$)**.** Denote $X \sim Geom(p)$.

(1) $X$ is the number of independent Bernoulli trials with parameter $p \in (0,1)$ such that first success occur.

(2) By independence, the PMF is $p(k) = (1-p)^{k-1}p$ for $k = 1, 2, 3, \ldots$

Then we show $E[X] = \dfrac{1}{p}$ and $\mathrm{Var}(X) = \dfrac{1-p}{p^2}$.

By definition, $E[X] = \sum_{k=1}^{\infty} k(1-p)^{k-1}p$. Note that for $x \in (-1,1)$, (by a result about the uniform convergence of power series) we have

$$\sum_{k=1}^{\infty} kx^{k-1} = \sum_{k=0}^{\infty}(x^k)' = (\sum_{k=0}^{\infty} x^k)' = (\frac{1}{1-x})' = \frac{1}{(1-x)^2}.$$

Then setting $x = 1-p$ leads to $E[X] = p\sum_{k=1}^{\infty} k(1-p)^{k-1} = p\dfrac{1}{[1-(1-p)]^2} = \dfrac{1}{p}$.

To obtain $\mathrm{Var}(X)$, it suffices to compute $E[X^2]$

$$\begin{aligned}
E[X^2] &= \sum_{k=1}^{\infty} k^2(1-p)^{k-1}p \\
&= \sum_{k=1}^{\infty}(k-1+1)^2(1-p)^{k-1}p \\
&= \sum_{k=1}^{\infty}(k-1)^2(1-p)^{k-1}p + \sum_{k=1}^{\infty} 2(k-1)(1-p)^{k-1}p + \sum_{k=1}^{\infty}(1-p)^{k-1}p \\
\text{(let } n = k-1) \qquad &= (1-p)\sum_{n=1}^{\infty} n^2(1-p)^{n-1}p + 2(1-p)\sum_{k=1}^{\infty} n(1-p)^{n-1}p + 1 \\
&= (1-p)E[X^2] + 2(1-p)E[X] + 1
\end{aligned}$$

Then substitute $E[X] = \dfrac{1}{p}$ and solve the equation to get $E[X^2] = \dfrac{2-p}{p^2}$.

Hence $\mathrm{Var}(X) = E[X^2] - (E[X])^2 = \dfrac{1-p}{p^2}$.

*Remark.* Observe that we have use two different ways to compute $E[X]$ and $\mathrm{Var}(X)$ in Example 2 both of which can be recursively extended to deal with series like $\sum_{k=1}^{\infty} k^p x^k$ with $p \in \mathbb{N}$.

**Example 3.** Let $X \sim Bin(n,p)$. Prove

$$E\left[\frac{1}{1+X}\right] = \frac{1-(1-p)^{n+1}}{(n+1)p}$$

*Proof.* Recall the PMF of $Bin(n,p)$ and by the formula of $E[g(X)]$,

$$E\left[\frac{1}{1+X}\right] = \sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k} p^k(1-p)^{n-k}$$

$$
\begin{aligned}
&= \frac{1}{(n+1)p}\sum_{k=0}^{n}\frac{1}{k+1}\frac{(n+1)!}{k![(n+1)-(k+1)]!}\cdot p^{k+1}(1-p)^{[(n+1)-(k+1)]} \\
&= \frac{1}{(n+1)p}\sum_{k=0}^{n}\binom{n+1}{k+1}p^{k+1}(1-p)^{[(n+1)-(k+1)]} \\
\text{(let } j=k+1) \quad &= \frac{1}{(n+1)p}\left[\sum_{j=0}^{n}\binom{n+1}{j}p^{j}(1-p)^{[(n+1)-j]}-(1-p)^{n+1}\right] \\
\text{( by Binomial Thm)} \quad &= \frac{1}{(n+1)p}[1-(1-p)^{n+1}].
\end{aligned}
$$

□

**Example 4.** Let $X$ be a r.v. with non-negative integral values. Prove

$$\sum_{k=0}^{\infty} kP(X\ge k)=\frac{1}{2}(E[X^2]+E[X]).$$

*Proof.*

$$
\begin{aligned}
\sum_{k=0}^{\infty} kP(X\ge k) &= \sum_{k=0}^{\infty}\sum_{i=k}^{\infty} kP(X=i) \\
&= \sum_{i=0}^{\infty}\sum_{k=0}^{i} kP(X=i) \\
&= \sum_{i=0}^{\infty}\frac{i(i+1)}{2}P(X=i) \\
&= \frac{1}{2}E[X^2]+\frac{1}{2}E[X],
\end{aligned}
$$

where in the second equality we have changed the order of summation (see e.g. [Tutorial 4, Example 2] for details). □

*Remark.* Recall for the r.v. $X$ in Example 4 we also have the layer-cake $E[X]=\sum_{k=0}^{\infty}P(X\ge k)$ (see e.g. [Tutorial 4, Example 2]). Together we can express $E[X^2]$ in terms of $P(X\ge k)$ if $E[X]<\infty$, thus $\mathrm{Var}(X)$. This process can recursively continue to express $E[X^p], p\in\mathbb{N}$ in terms of $P(X\ge k)$.