a) Since the COM of link i is located at P_i, they could be treated as discrete masses in space.

$$\Rightarrow \quad \bar{x} = \sum_{i=1}^{4} m_i x_i \Big/ \sum_{i=1}^{4} m_i = \frac{x_1 + x_2 + x_3 + x_4}{4}$$

$$\bar{y} = \sum_{i=1}^{4} m_i y_i \Big/ \sum_{i=1}^{4} m_i = \frac{y_1 + y_2 + y_3 + y_4}{4}$$

$$\Rightarrow \quad \boxed{P_{cm} = (\bar{x}, \bar{y}) = \frac{1}{4} \sum_{i=1}^{4} P_i}$$

b) $$\frac{\partial P_4}{\partial \theta} = J(\theta)$$

$$\cancel{\Rightarrow \frac{\partial P_4}{\partial t} \cdot \frac{\partial t}{\partial \theta} = J \Rightarrow \dot{P}_4 = J \cdot \dot{\theta}}$$

For revolut joint, $J_{Pi} = Z_{i-1} \times (P - P_{i-1})$

$$\boxed{\begin{aligned} \frac{\partial P_4}{\partial \theta} = J(\theta) &= (J_{P_1}, J_{P_2}, J_{P_3}, J_{P_4}) \\ &= (Z_0 \times (P_4 - P_0), Z_1 \times (P_4 - P_1), Z_2 \times (P_4 - P_2), \\ &\quad Z_3 \times (P_4 - P_3)) \end{aligned}}$$

where Z_i is the joint axis of joint i,
P_i is the vector from the origin of the world coordinate system to the origin of the ith link coordinate system.

c) $P_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $P_1 = P_0 + \begin{pmatrix} l_1 \cdot \cos(\theta_1) \\ l_2 \cdot \sin(\theta_1) \\ 0 \end{pmatrix} = \begin{pmatrix} l_1 \cdot \cos(\theta_1) \\ l_2 \cdot \sin(\theta_1) \\ 0 \end{pmatrix}$

$$P_2 = P_1 + \begin{pmatrix} l_2 \cdot \cos(\theta_1 + \theta_2) \\ l_2 \cdot \sin(\theta_1 + \theta_2) \\ 0 \end{pmatrix} = \begin{pmatrix} l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2) \\ l_1 \cdot \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2) \\ 0 \end{pmatrix}$$

$$P_3 = P_2 + \begin{pmatrix} l_3 \cos(\theta_1 + \theta_2 + \theta_3) \\ l_3 \sin(\theta_1 + \theta_2 + \theta_3) \\ 0 \end{pmatrix} = \begin{pmatrix} l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2) + l_3 \cos(\theta_1 + \theta_2 + \theta_3) \\ l_1 \cdot \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2) + l_3 \sin(\theta_1 + \theta_2 + \theta_3) \\ 0 \end{pmatrix}$$

$$P_4 = \begin{pmatrix} l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2) + l_3 \cos(\theta_1 + \theta_2 + \theta_3) + l_4 \cos(\theta_1 + \theta_2 + \theta_3 + \theta_4) \\ l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2) + l_3 \sin(\theta_1 + \theta_2 + \theta_3) + l_4 \sin(\theta_1 + \theta_2 + \theta_3 + \theta_4) \\ 0 \end{pmatrix}$$

$$z_i = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{for } i = 0 \dots 4$$

$$\Rightarrow \frac{\partial P_4}{\partial \theta} = (J_{P_1}, J_{P_2}, J_{P_3}, J_{P_4})$$

$$J_{P_1} = \begin{pmatrix} -l_1 S_1 - l_2 S_{12} - l_3 S_{123} - l_4 S_{1234} \\ l_1 C_1 + l_2 C_{12} + l_3 C_{123} + l_4 C_{1234} \end{pmatrix}$$

$$J_{P_2} = \begin{pmatrix} -l_2 S_{12} - l_3 S_{123} - l_4 S_{1234} \\ l_2 C_{12} + l_3 C_{123} + l_4 C_{1234} \end{pmatrix}$$

$$J_{P_3} = \begin{pmatrix} -l_3 S_{123} - l_4 S_{1234} \\ l_3 C_{123} + l_4 C_{1234} \end{pmatrix}$$

$$J_{P_4} = \begin{pmatrix} -l_4 S_{1234} \\ l_4 C_{1234} \end{pmatrix}$$

Here, S_{ij} means $\sin(\theta_i + \dots + \theta_j)$, C_{ij} means $\cos(\theta_i + \dots + \theta_j)$

d) $$\frac{\partial P_3}{\partial \theta} = \begin{bmatrix} -l_1S_1 - l_2S_{12} - l_3S_{123} & -l_2S_{12} - l_3S_{123} & -l_3S_{123} & 0 \\ l_1C_1 + l_2C_{12} + l_3C_{123} & l_2C_{12} + l_3C_{123} & l_3C_{123} & 0 \end{bmatrix}$$

$$\frac{\partial P_2}{\partial \theta} = \begin{bmatrix} -l_1S_1 - l_2S_{12} & -l_2S_{12} & 0 & 0 \\ l_1C_1 + l_2C_{12} & l_2C_{12} & 0 & 0 \end{bmatrix}$$

$$\frac{\partial P_1}{\partial \theta} = \begin{bmatrix} -l_1S_1 & 0 & 0 & 0 \\ l_1C_1 & 0 & 0 & 0 \end{bmatrix}$$

e) $$\frac{\partial P_{cm}}{\partial \theta} = \frac{\partial \frac{1}{4}\sum_{i=1}^{4} P_i}{\partial \theta}.$$

$$= \frac{1}{4}\sum_{i=1}^{4} \frac{\partial P_i}{\partial \theta}$$

$\Rightarrow$

$$\frac{\partial P_{cm}}{\partial \theta} = \frac{1}{4} \cdot \begin{bmatrix} -4l_1S_1 - 3l_2S_{12} - 2l_3S_{123} - l_4S_{1234} & -3l_2S_{12} - 2l_3S_{123} - l_4S_{1234} & -2l_3S_{123} - l_4S_{1234} & -l_4S_{1234} \\ 4l_1C_1 + 3l_2C_{12} + 2l_3C_{123} + l_4C_{1234} & 3l_2C_{12} + 2l_3C_{123} + l_4C_{1234} & 2l_3C_{123} + l_4C_{1234} & l_4C_{1234} \end{bmatrix}$$

g) Jacobian Transpose derivation:

Minimize cost fuction:

$$F = \frac{1}{2}(x_{target} - x)^T (x_{target} - x)$$
$$= \frac{1}{2}(x_{target} - f(\theta))^T (x_{target} - f(\theta))$$
$$\Rightarrow \Delta\theta = -\alpha \left(\frac{\partial F}{\partial \theta}\right)^T$$
$$= \alpha\left((x_{target} - x)^T \frac{\partial f(\theta)}{\partial \theta}\right)^T$$
$$= \alpha J^T(\theta) \Delta x$$

$$\Rightarrow \boxed{\Delta\theta = \alpha J^T(\theta) \Delta x}$$ ⇒ Jacobian transpose for inverse kinematics

h) The formula of the Pseudo Inverse

$$\boxed{\begin{aligned}\Delta\theta &= \alpha J^T(\theta)(J(\theta)J^T(\theta))^{-1} \Delta x \\ &= J^\# \Delta x\end{aligned}}$$

i) The formula of Pseud Inverse with Null-space optimization for inverse kinemcitics:

$$\boxed{\begin{aligned}&\Delta\theta = \alpha J^\# \Delta x + (I - J^\# J)(\theta_0 - \theta) \\ &\text{where } J^\# = J^T(\theta)(J(\theta)J^T(\theta))^{-1}\end{aligned}}$$

j) Derivation:

For a small step Δx, minimize the cost function:

$$F = \frac{1}{2}\Delta\theta^T W \Delta\theta + \lambda^T(\Delta x - J\Delta\theta)$$

$\Rightarrow$

(1) $\frac{\partial F}{\partial \lambda} = 0 \Rightarrow \Delta x = J\Delta\theta$

(2) $\frac{\partial F}{\partial \Delta\theta} = 0 \Rightarrow \Delta\theta^T W - \lambda^T J = 0$

$\Rightarrow \Delta\theta = (W^{-1})^T J^T \lambda$

Since W is a diagonal matrix, $(W^{-1})^T = W^{-1}$

$\Rightarrow \Delta\theta = W^{-1} J^T \lambda$

$J\Delta\theta = J W^{-1} J^T \lambda$

$\lambda = (J W^{-1} J^T)^{-1} J\Delta\theta$

By (1): $\Delta x = J\Delta\theta$

$\Rightarrow \lambda = (J W^{-1} J^T)^{-1} \Delta x$

Plug this into $\Delta\theta = W^{-1} J^T \lambda$

$$\boxed{\Delta\theta = W^{-1} J^T (J W^{-1} J^T)^{-1} \Delta x}$$

k) Formula:

$$\Delta\theta = \alpha J_W^{\#} \Delta x + (I - J_W^{\#} J)(\theta_0 - \theta)$$

where $J_W^{\#} = W^{-1} J^T (J W^{-1} J^T)^{-1}$