A METHODOLOGY FOR THE DETERMINATION OF GLOBAL ELECTROMECHANICAL QUANTITIES FROM A FINITE ELEMENT ANALYSIS AND ITS APPLICATION TO THE EVALUATION OF MAGNETIC FORCES, TORQUES AND STIFFNESS

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Abstract - The paper will present two general methods for the deduction of global information from the final result of the finite element computation of an electromagnetic device.

The first one, called the local jacobian derivative, may be used for evaluation of the derivative of any integral quantity versus the parameter of motion of a rigid body. Typically, this method when applied to electromagnetic systems, can be used for the computation of magnetic force or torque by virtual-work principle.Compared with the popular Maxwell's tensor method, this procedure is easier to implement in a finite element package especially for 3D problems.

The second method which is based on a stationary property of the field solution, allows the evaluation of a second order derivative of any integral quantity. For instance, computation of the stiffness of a magnetic system (derivative of a force or a torque) may be achieved as the second order derivative of the magnetic energy. It may be pointed out that this method requires the field computation once for a linear problem as well as for a non-linear one.

INTRODUCTION

In electrical engineering, the calculation of total magnetic forces or torques acting on movable parts is of a great interest. The concept of Maxwell's tensor in empty space is a very convenient way of calculating such quantities [1]. Unfortunately when the magnetic field comes from a finite element solution, the implementation of such an algorithm depends on:

1. The type of the finite element crossed by the surface of integration (triangle, quadrilateral, tetrahedron, hexahedron,...).

2. The path of this surface inside the element.

On the other hand, the virtual work principle gives a more general algorithm which is easier to implement. Its introduction in the finite element context is possible using the "local jacobian derivative method" due to Rafinéjad [2] and resumed by Coulomb, Meunier and Sabonnadière [3].

The prediction of force or torque stiffnesses is fundamental for some electrical or magnetic devices (magnetic coupling, magnetic bearing,...). Traditionally, when numerical methods are used for field computations, stiffnesses are obtained by finite differences starting from a set of forces or torques calculated at various positions of the movable body ($dF/dx \simeq (F1-F2)/(x1-x2)$). This method may introduce important numerical errors in final results due to the already imprecise knowledge of forces or torques. In fact, direct computation of stiffness may be achieved in finite element work using both the local jacobian derivative and a stationary property of the field solution. This other method requires the field computation once for a linear problem as well as for a non-linear one [3], [4].

The first part of this paper will be dedicated to a brief description of force and torque calculation by Maxwell's tensor. In the second part, we shall see the use of virtual-work principle to compute force and torque in finite element context. In the third part, we shall develop the direct stiffness approach. A short example will be given to illustrate these methods.

FORCES AND TORQUES - MAXWELL'S TENSOR

The global magnetic forces and torques acting on a rigid body may be obtained via Maxwell's tensor using the following steps [1]:

1. Choice of an arbitrary surface S in empty space (or equivalent medium) including all the movable body and only it (Fig. 1).

2. Integration of the force by:

$$F = \int_S [\mu_o.(H.n).H - \frac{\mu_o}{2}.H^2.n].dS \quad (1)$$

3. Integration of the torque by:

$$C = \int_S [\mu_o.(H.n).(r \times H) - \frac{\mu_o}{2} H^2.(r \times n)].dS \quad (2)$$

with n the normal vector to the surface S, r the distance of a point to the axis of rotation, H the magnetic field and μ_o the permeability of free space.

Similar formulae are available for forces and torques to be due to electric fields.

This method has the following advantages:

1. The field computation is required once for a linear problem as well as for a non-linear one.

2. Only one surface integral is needed.

3. The choice of the surface S is arbitrary, provided that no medium other that empty space is crossed.

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Obviously, when an approximate field is substituted in (1) or (2) to the true one, the independance of the result relative to the choice of S disappears. For instance, when first order triangular finite elements are used for the field computation [5], the best results are obtained when S crosses triangles at the middle of the edges.

In finite element context, the optimum choice of S depends on the type of element. For example, in 3D analysis with tetahedra the 2D triangular case [5] may be easely extended. The choice of S with an hexahedron curvilinear mesh is less obvious...

Another difficulty arises from the multiplicity of possible topological crossing in some elements (Fig. 2) which implies the multiplicity of algorithms.

In fact, these disadvantages are relatively minor, so Maxwell's tensor remains very popular (see [9] for a special application). In the next part, we shall see another method based on virtual-work principle. This approach owns all the advantages of Maxwell's tensor without its drawbacks and it is particulary well suited for finite element implementation.

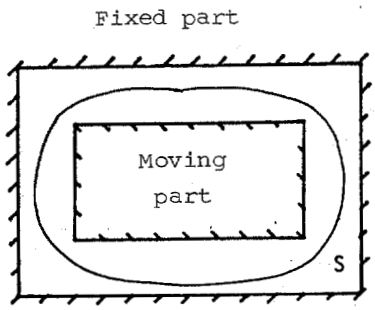

Fig. 1. Choice of an arbitrary surrounding surface for the integration of Maxwell's tensor.

FORCES AND TORQUES - VIRTUAL-WORK

The Maxwell's tensor and virtual work are closely related [5], so we can use the second method each time the first one applies.

Here, to illustrate the implementation of the virtual-work principle in a finite element context, we choose a simple 3D magnetostatic formulation based on a scalar magnetic potential $\Psi(x,y,z)$ such that:

$$H = - \text{grad } \psi \qquad (3)$$

Let a domain be discretized into finite elements. The unknown scalar potential is approximated by a linear combination of the shape functions weighted by nodal values:

$$\psi(x,y,z) = \sum \alpha_i(x,y,z) \cdot \psi_i \qquad (4)$$

If we consider the coenergy functional:

$$F(\psi(x,y,z)) = \int_\Omega \int_0^H B.dH \; d\Omega + \int_\Gamma B_n.\psi d\Gamma \qquad (5)$$

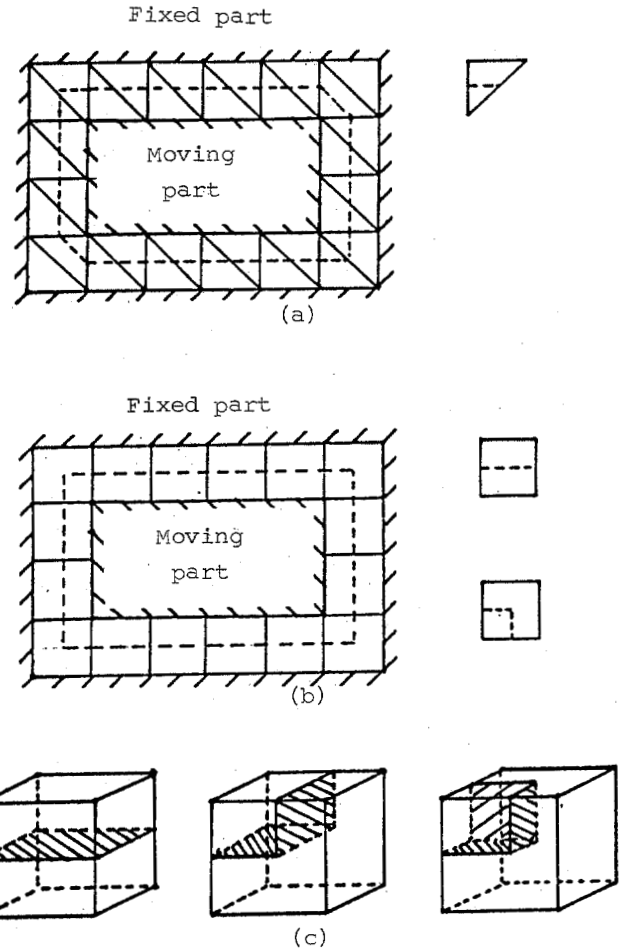

Fig. 2. Surfaces of integration for various meshes :
a) Triangles : one algorithm
b) Quadrilaterals : two algorithms.
c) Hexahedra : three algorithms.

the best set of nodal values satisfies the extremal conditions:

$$\frac{\partial F}{\partial \psi_i} = 0 \qquad (6) \text{ , for all degrees of freedom.}$$

These equations are linear or non linear, depending on the absence or the presence of saturable material. The optimal set $\Psi_{1opt}, \ldots, \Psi_{iopt}, \ldots$ allows the building of an approximate solution $\Psi_{opt}(x,y,z)$ and an approximate value of the magnetic coenergy W':

$$W' \simeq F_{opt} = F(\psi_{opt}(x,y,z)) \qquad (7)$$

We apply the virtual work principle at constant magnetic scalar potential. Force and torque acting on a rigid part of the domain may be approximated in some sense by:

$$F_s = \frac{\partial Fopt}{\partial s} \qquad C_\omega = \frac{\partial Fopt}{\partial \theta} \qquad (8) \; (9)$$

with Fs the force along the s direction; s the measure of the virtual moving of the movable part; C_ω the torque relative to the axis ω and θ the angular measure of the virtual rotation.

We remark that the movable part must be strictly surrounded with free space to allow virtual moving. This condition is quite similar to those imposed on the surface S including the body for Maxwell's tensor formulation.

Using approximation (4), the expression of the force is:

$$F_s = \frac{\partial Fopt}{\partial s} + \left[\frac{\partial Fopt}{\partial \psi}\right]^T . \left[\frac{\partial \psi}{\partial s}\right] \qquad (10)$$

where ∂ means either the derivative versus s at Ψ_i constant, or the derivative versus Ψ_i at s constant.

At the optimal solution, the vector $\left[\frac{\partial F_{opt}}{\partial \psi}^T\right]$ (derivatives of F_{opt} versus Ψ_i) vanishes (see equation (6)) and the expression becomes:

$$F_s = \frac{\partial F_{opt}}{\partial s} \qquad (11)$$

A similar expression arises for the torque:

$$C_\omega = \frac{\partial F_{opt}}{\partial \theta} \qquad (12)$$

Rafinéjad [2] uses (11) and (12) for direct evaluation of force and torque within finite element analysis. The elegance of his method lies in the formulation of the virtual distortion of the free space contained between the movable and the fixed parts. We call it the "local jacobian derivative" method.

LOCAL JACOBIAN DERIVATIVE

The domain virtually distorted during the virtual moving is broken up into a set of finite elements. On each of them, the integrals are expressed in terms of local coordinates (u,v,w) instead of global ones (x,y,z) (see [6] for explanations on isoparametric elements):

$$\iiint_{\Omega_d} q[x,y,z]\,dxdydz = \sum_e \iiint_{\Omega_e} q[x,y,z]dxdydz$$

$$= \sum_e \iiint_{\square e} q[x(u,v,w),y(u,v,w),z(u,v,w)].|G|dudvdw \qquad (13)$$

here q(x,y,z) is an arbitrary function ; the sum concerns all the finite elements ; |G| is the determinant of the jacobian matrix G:

$$\begin{vmatrix} x = x(u,v,w) \\ y = y(u,v,w) \\ z = z(u,v,w) \end{vmatrix} \qquad (14)$$

$$G = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix} \quad , \quad |G| = \det(G) \qquad (15)$$

We remark that the boundaries of every integral (13) in dudvdw are absolutely fixed. They only depend on the type of elements (noted $\square_e$). For instance, for all hexahedral finite elements, the u,v,w boundaries are -1 and +1, regardless of the true shape of each element [6]. This means that 1- the integral boundaries are independant of moving 2- to express any derivative versus any moving, we have just to take into account the derivative of integrand.

This technique may be applied each time we have to express any distortion of the domain: optimal shape design, virtual-work application, ...

VIRTUAL-WORK IMPLEMENTATION

Now, we continue the adaptation of the virtual-work principle to obtain force and torque within the finite element analysis.

Combining (11) and (5) we obtain:

$$F_s = \frac{\partial}{\partial s} \int_\Omega \int_0^H B.dH\ d\Omega + \frac{\partial}{\partial s} \int_\Gamma B_n.\psi\ d\Gamma \qquad (16)$$

The derivative of the surface integral on Γ vanishes because 1- the boundary conditions are kept constant during virtual work 2- the shape of Γ is unaffected by the moving of an interior sub-region.

We apply (13) to (16):

$$F_s = \frac{\partial}{\partial s} \sum_e \int_{\Omega_e} \int_0^H B.dH\ d\Omega$$

$$= \frac{\partial}{\partial s} \sum_e \int_{\square e} \int_0^H B.dH\ |G|\ dudvdw \qquad (17)$$

$$= \sum_e \int_{\square e} \left[\frac{\partial}{\partial s} \left[\int_0^H B.dH \right] . |G| + \int_0^H B.dH . \frac{\partial |G|}{\partial s} \right] dudvdw \qquad (18)$$

Without magnetostriction phenomena in the distorted area, the first derivative becomes

$$\frac{\partial}{\partial s} \int_0^H B.dH = B^T . \frac{\partial H}{\partial s} \qquad (19)$$

On each parametric element the potential is [6]:

$$\psi = \Sigma \alpha_i\ (u,v,w)\ .\ \psi_i = \psi(u,v,w) \qquad (20)$$

and the magnetic field is

$$H = -\ \mathrm{grad}\ \psi = -\ G^{-1} . \begin{vmatrix} \frac{\partial \psi}{\partial u} \\ \frac{\partial \psi}{\partial v} \\ \frac{\partial \psi}{\partial w} \end{vmatrix} \qquad (21)$$

where G^{-1} is the inverse of the jacobian matrix (15).

The derivative versus s at constant Y_i gives:

$$\frac{\partial H}{\partial s} = - \frac{\partial G^{-1}}{\partial s} \cdot \begin{vmatrix} \frac{\partial \psi}{\partial u} \\ \frac{\partial \psi}{\partial v} \\ \frac{\partial \psi}{\partial w} \end{vmatrix} \qquad (22)$$

$$\frac{\partial H}{\partial s} = G^{-1} \cdot \frac{\partial G}{\partial s} \cdot G^{-1} \cdot \begin{vmatrix} \frac{\partial \psi}{\partial u} \\ \frac{\partial \psi}{\partial v} \\ \frac{\partial \psi}{\partial w} \end{vmatrix} = - G^{-1} \cdot \frac{\partial G}{\partial s} \cdot H \qquad (23)$$

(using the derivative of identity $G.G^{-1} = I$).

Finally, the force expression is

$$F_s = \sum_e \int_{\Omega_e} \left[- B^T G^{-1} \cdot \frac{\partial G}{\partial s} \cdot H + \int_0^H B.dH. |G|^{-1} \cdot \frac{\partial |G|}{\partial s} \right] d\Omega \qquad (24)$$

We remark that the sum may be restricted to the sub-set of virtually distorted finite elements (area contained between movable and fixed parts). For others (entirely fixed or entirely moving elements) their jacobian matrices remain unaffected by virtual moving.

Torque is given by a similar formula:

$$C_\omega = \sum_e \int_{\Omega_e} \left[-B^T G^{-1} \cdot \frac{\partial G}{\partial \theta} \cdot H + \int_0^H B.dH. |G|^{-1} \cdot \frac{\partial |G|}{\partial \theta} \right] d\Omega \qquad (25)$$

JACOBIAN MATRIX AND DERIVATIVES

Regardless the type of finite element we have:

$$[x,y,z] = \sum_i \beta_i(u,v,w).[x_i,y_i,z_i] \qquad (26)$$

the sum affects the nodes i of the element. The β_i (u,v,w) functions are identical to the α_i(u,v,w) ones for isoparametric elements [6]. The node coordinates $(X_i Y_i Z_i)$ may depend on the virtual moving but the β functions are independant. Starting from (26), β_i(u,v,w), the node coordinates and their derivatives versus s or θ, the expression of G, G and their derivatives may be built easily. These expressions can be easely implemented in a finite element package :

$$G = \sum_i \begin{vmatrix} \frac{\partial \beta_i}{\partial u} \\ \frac{\partial \beta_i}{\partial v} \\ \frac{\partial \beta_i}{\partial w} \end{vmatrix} . [X_i, Y_i, Z_i] \qquad (27)$$

$$\frac{\partial G}{\partial s} = \sum_i \begin{vmatrix} \frac{\partial \beta_i}{\partial u} \\ \frac{\partial \beta_i}{\partial v} \\ \frac{\partial \beta_i}{\partial w} \end{vmatrix} . \left| \frac{\partial X_i}{\partial s}, \frac{\partial Y_i}{\partial s}, \frac{\partial Z_i}{\partial s} \right| \qquad (28)$$

and so on for the determinant and its derivative.

VIRTUAL-WORK SUMMARY

To compute force and torque acting on a rigid part, within a virtual-work finite element implementation:

1. Only one field computation is needed (linear or non linear problem).

2. Only one volume integral on the distorted area is needed.

3. The distorted area is arbitrary, provided that no medium other that free space is distorted and that this area includes all the body and only it. For instance, we can use the first finite element layer surrounding the body under study (Fig. 3).

For an user point of view, the numerical results of Maxwell's tensor and virtual-work are similar provided that a good surface S has been chosen for tensor integration.

For an implementation point of view this method is very efficient. We only have to introduce a volume integration on a layer of elements. It is easier to manage than a surface integration across a layer of elements, especially when a numerical quadrature for the volume integral is used [6].

For a theoretical point of view, the two methods are equivalent. Starting from a thin layer of distorted area for virtual work,

Rafinéjad shows that the virtual work expression (24) tends to the Maxwell's tensor expression (1) when the thickness tends towards zero (see [2] for demonstration).

In fact, the virtual-work method will be particulary well suited to any finite element formulation based on energy consideration. It has been successfully used in scalar or vectorial magnetostatic or magnetodynamic problems in bidimensional [7] and tridimensional approaches [4], [8].

STIFFNESS

Let p be an arbitrary direction of moving (not necessarily the same as s). An approximate value for the stiffness may be obtained by:

$$\frac{\partial}{\partial p}[F_s] = \frac{\partial}{\partial p}\left[\frac{\partial F_{opt}}{\partial s}\right] = \frac{\partial^2 F_{opt}}{\partial p \partial s} + \left[\frac{\partial^2 F_{opt}}{\partial \psi^T \partial s}\right] \cdot \left[\frac{\partial \psi}{\partial p}\right] \quad (29)$$

where ∂ means either derivative versus p at constant Ψ_i or derivative versus Ψ_i at constant p.

The scalar $\frac{\partial^2 F_{opt}}{\partial p \partial s}$ is evaluated using the local jacobian derivative method twice as for (24):

$$\frac{\partial^2 F_{opt}}{\partial p \partial s} = \sum_e \int_{\Omega_e} \Big[+ H^T \cdot \frac{\partial G^T}{\partial p} \cdot G^{-1T} \cdot \frac{\partial B^T}{\partial H} \cdot G^{-1} \cdot \frac{\partial G}{\partial s} \cdot H + B^T \cdot G^{-1} \cdot \frac{\partial G}{\partial p} \cdot G^{-1} \cdot \frac{\partial G}{\partial s} \cdot H - B^T \cdot G^{-1} \cdot \frac{\partial^2 G}{\partial p \partial s} \cdot H + B^T \cdot G^{-1} \cdot \frac{\partial G}{\partial s} \cdot G^{-1} \cdot \frac{\partial G}{\partial p} \cdot H - B^T \cdot G^{-1} \cdot \frac{\partial G}{\partial s} \cdot H \cdot |G|^{-1} \cdot \frac{\partial |G|}{\partial p} - B^T \cdot G^{-1} \cdot \frac{\partial G}{\partial p} \cdot H \cdot |G|^{-1} \cdot \frac{\partial |G|}{\partial s} + \int_0^H B \cdot dH \cdot |G|^{-1} \cdot \frac{\partial^2 |G|}{\partial p \partial s} \Big] \cdot d\Omega \quad (30)$$

The vector $\left[\frac{\partial^2 F_{opt}}{\partial \psi^T \partial s}\right]$ is the derivative of the force (24) versus nodal values:

$$\frac{\partial^2 F_{opt}}{\partial \psi_j \partial s} = \sum_e \int_{\Omega_e} \Big[\mathrm{grad}\alpha_j^T \cdot \frac{\partial B^T}{\partial H} \cdot G^{-1} \cdot \frac{\partial G}{\partial s} \cdot H + B^T \cdot G^{-1} \cdot \frac{\partial G}{\partial s} \cdot \mathrm{grad}\,\alpha_j - B^T \cdot \mathrm{grad}\,\alpha_j \cdot |G|^{-1} \cdot \frac{\partial |G|}{\partial s} \Big] \cdot d\Omega \quad (31)$$

The vector $\left[\frac{\partial \psi}{\partial p}\right]$ gives the effect of virtual moving on the nodal potential values. It cannot be obtained directly. Fortunately, we can state that whatever the position of the body along the p axis, the conditions (6) are always respected. This is the key of the method:

$$\frac{\partial}{\partial p}\left[\frac{\partial F_{opt}}{\partial \psi}\right] = \left[\frac{\partial^2 F_{opt}}{\partial p \partial \psi}\right] + \left[\frac{\partial^2 F_{opt}}{\partial \psi^T \partial \psi}\right] \cdot \left[\frac{\partial \psi}{\partial p}\right] = 0 \quad (32)$$

The vector $\left[\frac{\partial^2 F_{opt}}{\partial p \partial \psi}\right]$ is evaluated by the local jacobian derivative method once more:

$$\frac{\partial^2 F_{opt}}{\partial p \partial \psi_i} = \sum_e \int_{\Omega_e} \Big[\mathrm{grad}\,\alpha_i^T \cdot \frac{\partial G^T}{\partial p} \cdot G^{-1T} \cdot B + \mathrm{grad}\,\alpha_i^T \cdot \frac{\partial B}{\partial H^T} \cdot G^{-1} \cdot \frac{\partial G}{\partial p} \cdot H - \mathrm{grad}\,\alpha_i^T \cdot B \cdot |G|^{-1} \cdot \frac{\partial |G|}{\partial p} \Big] \cdot d\Omega \quad (33)$$

The matrix $\left[\frac{\partial^2 F_{opt}}{\partial \psi^T \partial \psi}\right]$ is the familiar one built during the computation of nodal values:

$$\frac{\partial^2 F_{opt}}{\partial \psi_j \partial \psi_i} = \sum_e \int_{\Omega} \mathrm{grad}\,\alpha_i^T \cdot \frac{\partial B}{\partial H^T} \cdot \mathrm{grad}\,\alpha_j \, d\Omega \quad (34)$$

The final stiffness expression is:

$$\frac{\partial}{\partial p}[F_s] = \frac{\partial^2 F_{opt}}{\partial p \partial s} - \left[\frac{\partial^2 F_{opt}}{\partial \psi^T \partial s}\right] \cdot \left[\frac{\partial^2 F_{opt}}{\partial \psi^T \partial \psi}\right]^{-1} \cdot \left[\frac{\partial^2 F_{opt}}{\partial p \partial \psi}\right] \quad (35)$$

A similar expression arises for the stiffness of a torque.

EXAMPLE

Figure 4 shows a simple bidimensional magnetostatic problem. It has been solved using 32 second order isoparametric quadrilaterals (8 nodes each) (4 layers along the r direction and 8 layers along the θ direction) The torque and its stiffness have been computed using virtual-work implementation. Figure 5 shows that analytical and computed results are in good agreement.

CONCLUSION

A method based on the virtual-work principle and allowing magnetic force and torque calculation has been exposed. This method is easy and natural to implement in a finite element package. It has been used in various situations and has been proved fast, accurate and reliable [7].

An original method of direct stiffness computation has also been described starting from the same basis. The first results shown here are fairly pleasing.

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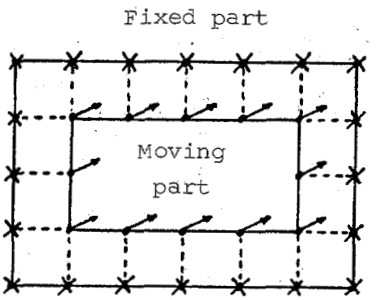

Fig. 3. Layer of distorted finite elements:
x Fixed nodes.
. Moving nodes.

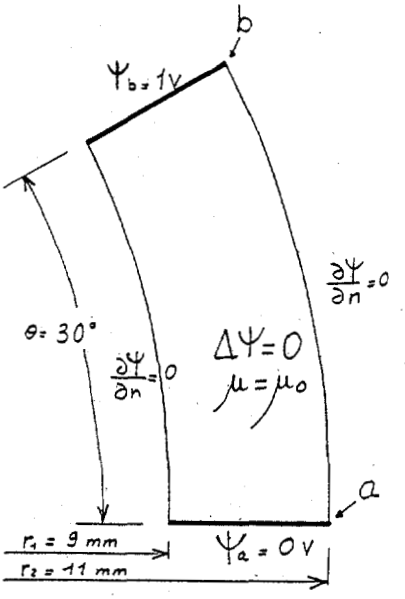

Fig. 4. Simple example. The torque and its stiffness between sides a and b are evaluated by virtual work finite elements.

	Analytical	Numerical
Coenergy **Joules x10^{12}**	K/θ 3.42237	3.42237
Torque **N.M x 10^{12}**	$-K/\theta^2$ - 6.53625	- 6.53625
Stiffness **SI x 10^{12}**	$2K/\theta^3$ 24.96663	24.96698

$$K = \frac{1}{2}\mu_0 (\Psi_b - \Psi_a)^2 \frac{L}{\ln \frac{r_2}{r_1}}$$

Fig. 5. Comparison between analytical and numerical results.