Ex 8.1

1) $\pi(x) = 0{,}2\,\delta_1(x) + 0{,}6\,\delta_2(x) + 0{,}2\,\delta_3(x)$

$\pi(y) = 0{,}3\,\delta_1(y) + 0{,}5\,\delta_2(y) + 0{,}2\,\delta_3(y)$

2)

$\pi(y\mid x)$	$x=1$	$x=2$	$x=3$
$y=1$	0,5	1/3	0
$y=2$	0,5	0,5	0,5
$y=3$	0	1/6	0,5

$\pi(x\mid y)$	$x=1$	$x=2$	$x=3$
$y=1$	1/3	2/3	0
$y=2$	1/5	3/5	1/5
$y=3$	0	1/2	1/2

Ex 8.2

1) $bel(x_{k+1}) = \sum_{i \in \{1,2\}} \pi(x_{k+1} | x_k = i) \times bel(x_k = i)$

$\pi(x_{k+1} = j) = \sum_i \pi(x_{k+1} = j | x_k = i)\, \pi(x_k = i)$

$$\underbrace{\begin{pmatrix} P_{k+1,1} \\ P_{k+1,2} \end{pmatrix}}_{P_{k+1}} = \underbrace{\begin{pmatrix} 0{,}9 & 0{,}5 \\ 0{,}1 & 0{,}5 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} P_{k,1} \\ P_{k,2} \end{pmatrix}}_{P_k}$$

2) $P_k = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad P_{k+l} = A^l \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

3) $P_\infty = A P_\infty$ donc $A P_\infty = 1 P_\infty$
on cherche le vecteur propre de A associé à la valeur propre 1.
On prend bien celui qui a une forme de vecteur stochastique.

$$P_\infty = \begin{pmatrix} 0{,}833 \\ 0{,}167 \end{pmatrix}$$

Ex 8.3

1)

$$\text{pred}(x_1) = \pi(x_1 | x_0 = 0, u_0 = 1)\, \text{bel}(x_0 = 0) + \pi(x_1 | x_0 = 1, u_0 = 1)\, \text{bel}(x_0 = 1)$$

$$= (0{,}2\, \delta_0 + 0{,}8\, \delta_1) \times 0{,}5 + (0\, \delta_0 + 1\, \delta_1) \times 0{,}5$$

$$\text{pred}(x_1) = 0{,}1\, \delta_0 + 0{,}9\, \delta_1$$

2) $y_1 = 1$

$$\text{bel}(x_1) = \frac{\pi(y_1 = 1 | x_1)\, \text{pred}(x_1)}{\int \pi(y_1 = 1 | x_1)\, \text{pred}(x_1)\, dx_1}$$

$$\propto \pi(y_1 = 1 | x_1)\, \text{pred}(x_1)$$

$$\propto (0{,}2\, \delta_0(x_1) + 0{,}6\, \delta_1(x_1))\,(0{,}1\, \delta_0(x_1) + 0{,}9\, \delta_1(x_1))$$

$$\text{bel}(x_1) \propto 0{,}02\, \delta_0(x_1) + 0{,}54\, \delta_1(x_1)$$

$$\text{bel}(x_1) = \frac{0{,}02\, \delta_0(x_1) + 0{,}54\, \delta_1(x_1)}{0{,}56}$$

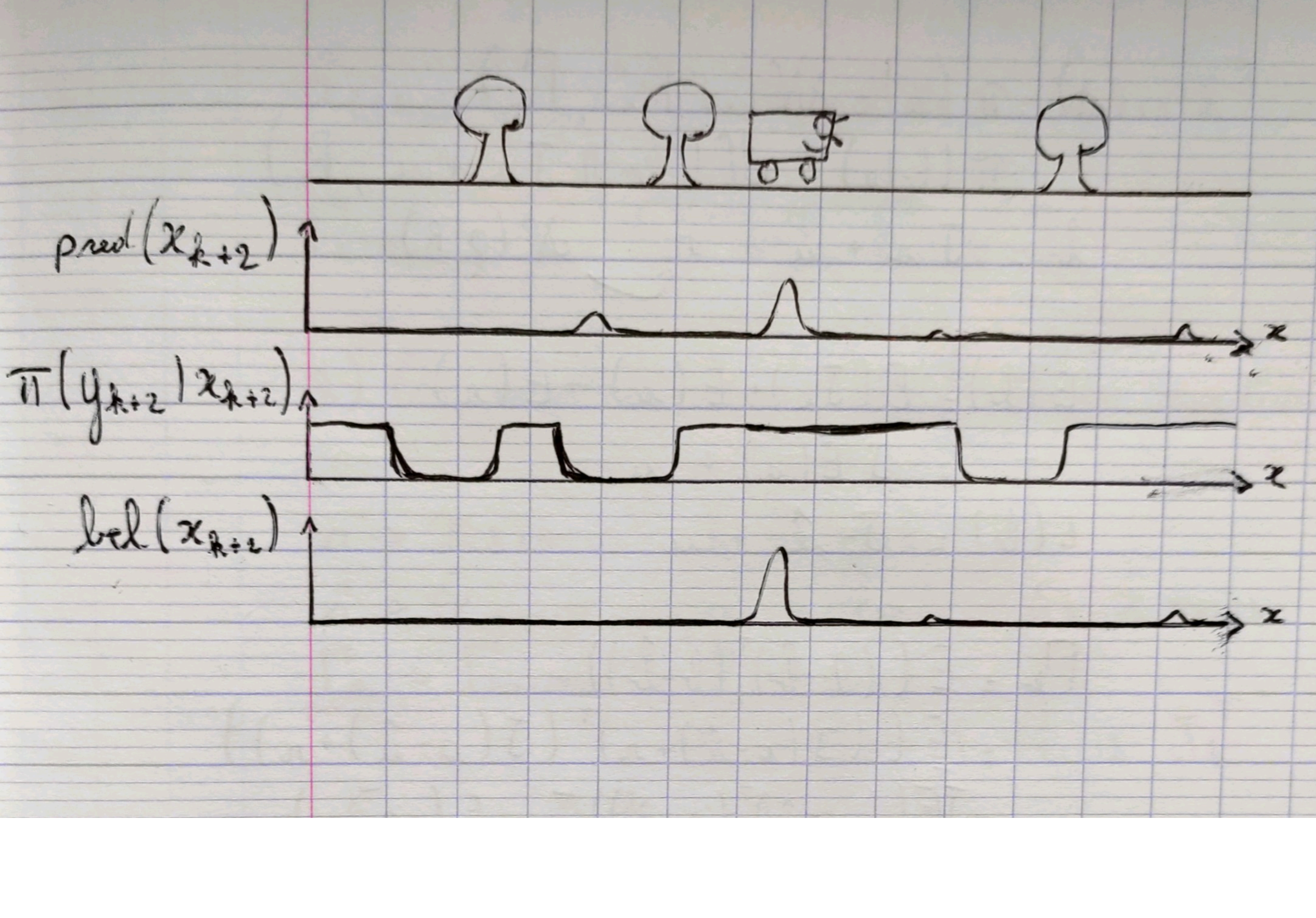
3)

Prédiction $P_{k+1}^{pred} = \begin{pmatrix} 1 & 0,8 \\ 0 & 0,2 \end{pmatrix} \delta_{-1}(u_k)\, P_k^{bel} + \delta_0(u_k) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} P_k^{bel} + \delta_1(u_k) \begin{pmatrix} 0,2 & 0 \\ 0,8 & 1 \end{pmatrix} P_k^{bel}$

Correction $P_k^{bel} \propto \left.\begin{array}{l} \delta_0(y_k) \begin{pmatrix} 0,8 & 0 \\ 0 & 0,4 \end{pmatrix} P_k^{pred} + \\ \delta_1(y_k) \begin{pmatrix} 0,2 & 0 \\ 0 & 0,6 \end{pmatrix} P_k^{pred} \end{array}\right\} q_k$

$$P_k^{bel} = \frac{q_k}{q_{k,1} + q_{k,2}}$$

Ex 8.4



Ex 8.5

1) $\pi(a) = \mathcal{N}(a \| \hat{a}, \Gamma_a)$

$\pi(b|a) = \mathcal{N}(b \| Ja + u, R)$

$b = Ja + u + r \quad \mathcal{N}(0, R)$

$E(b) = E(Ja) + E(u) + E(r)$

$= J E(a) + u$

$E(b) = J \hat{a} + u$

$\Gamma_b = E((b-\hat{b})^T (b-\hat{b}))$

$= E((J(a-\hat{a})+r)^T (J(a-\hat{a})+r))$

$= J^T E((a-\hat{a})^T(a-\hat{a}))J + E(r^T r)$

$\Gamma_b = J^T \Gamma_a J + R$

2) $\pi(x_k | x_{k+1}, y_{0:N}) = \pi(x_k | x_{k+1}, y_{0:k})$

Toutes mesures futures ne serviront pas pour trouver x_k.

$= \hat{x}_{k|k} + J_k (x_{k+1} - \hat{x}_{k+1|k})$

$\hat{x}_{k|x_{k+1}, y_{0 \to k}} = \hat{x}_{k|k} + J_k \tilde{z}_k$

$\Gamma_{k|x_{k+1}, y_{0 \to k}} = (I - J_k A_k)\Gamma_{k|k}$

$\tilde{z}_k = (x_{k+1} - u_k) - A_k \hat{x}_{k|k}$

$S_k = A_k \Gamma_{k|k} A_k^T + \Gamma_\alpha \to S_k = \Gamma_{k+1|k}$

$J_k = \Gamma_{k|k} A_k^T S_k^{-1}$

$\to \Gamma_{k+1|k}^{-1}$

$$\hat{x}_{k|x_{k+1}, y_{0\to k}} = \hat{x}_{k|k} + J_k (x_{k+1} - \hat{x}_{k+1|k})$$

$$\Gamma_{k|x_{k+1}, y_{0\to k}} = \Gamma_{k|k} - J_k \Gamma_{k+1|k} J_k^T$$

3) On remplace les a et b de la question 1

$a \to x_{k+1}$ $\quad$ $b \to x_k$

$\hat{a} \to \hat{x}_{k+1|N}$ $\quad$ $u \to \hat{x}_{k|k} - J_k \hat{x}_{k+1|k}$

$\Gamma_a \to \Gamma_{k+1|N}$ $\quad$ $J \to J_k$

$R \to \Gamma_{k|k} - J_k \Gamma_{k+1|k} J_k^T$

et on obtient

$\pi(x_k | y_{0\to N}) = \mathcal{N}(x_k ||$

$$\hat{x}_{k|N} \to x_{k|k} + J_k (\hat{x}_{k+1|N} - \hat{x}_{k+1|k}),$$

$$\Gamma_{k|N} \to \Gamma_{k|k} - J_k (\Gamma_{k+1|k} - \Gamma_{k+1|N}) J_k^T)$$

Ex 8.6

1) $x \sim \mathcal{N}(1,1)$ En utilisant la partie
$b \sim \mathcal{N}(0,1)$ prédiction du KF on a :

$$\bar{y} = 1 \cdot \bar{x} + \bar{b} = 1$$

$$\sigma_y^2 = 1 \cdot \sigma_x^2 \cdot 1 + \sigma_b^2 = 1^2 + 1^2 = 2$$

$$\pi_y(y) = \frac{1}{\sqrt{4\pi}} ex\left(-\frac{1}{4}(y-1)^2\right)$$

2) $x \to x_{k|k-1}$ prédiction
$v \to x_{k|k}$ correction $y=3$

$$\bar{v} = \bar{x} + K\tilde{y} = 1 + 1/2 \times 2 = 2$$

$$\sigma_v^2 = \sigma_x^2 - K\sigma_x^2 = 1 - 1/2 = 1/2$$

$$\tilde{y} = y - \bar{x} = 3 - 1 = 2$$

$$S = \sigma_x^2 + \sigma_b^2 = 2$$

$$K = \sigma_x^2 S^{-1} = 1/2$$

$$\pi_v(v) = \frac{1}{\sqrt{2\pi \, 1/2}} \exp\left(-\frac{1}{2}(v-2)\, 2\, (v-2)\right)$$

$$\pi_v(v) = \frac{1}{\sqrt{\pi}} \exp\left(-(v-2)^2\right)$$

3) $$\pi(x|y) = \frac{\pi(y|x)\,\pi(x)}{\pi(y)}$$

Dans le cas gaussien on peut faire le calcul en manipulant des moyennes et covariances. C'est ce que fait le filtre de Kalman.