LD := left divisible $\forall a, b \in G : \exists x : a \cdot x = b$
RD := right divisible $\forall a, b \in G : \exists x : x \cdot a = b$
LC := left cancellative
RC := right cancellative

Proposition: A semigroup G is a group iff it has a unit and is LC and RD.

Proof:
$\Rightarrow$ (G is a group $\to$ G has a unit and is LC and RD):
Since G is a group, then it is also divisible and cancellative, hence is LC and RD. In this direction we only need to prove the existence of a unit.

$$\begin{aligned}
&\forall g, h \in G : \exists l_g, r_g, l_h, r_h : l_g \cdot g = g \cdot r_g,\ l_h \cdot g = g \cdot r_h \text{ (RD, LD)}\\
&(g \cdot r_g) \cdot h = g \cdot h = g \cdot (l_h \cdot h) = (g \cdot l_h) \cdot h \ (associativity)\\
&(g \cdot r_g) \cdot h = (g \cdot l_h) \cdot h \Rightarrow g \cdot r_g = g \cdot l_h \ (RC)\\
&g \cdot r_g = g \cdot l_h \Rightarrow r_g = l_h \ (LC)\\
&\text{for } g = h \text{ we denote } l_g = r_g = u_g\\
&\text{but since } \forall g, h \in G : u_g \cdot g = g = g \cdot u_g,\ u_h \cdot g = g = g \cdot u_h \text{ and } u_g = u_h\\
&\text{then } u_g = u_h = u \text{ is a unit of } G
\end{aligned}$$

$\Leftarrow$ (S is a semigroup with a unit and is LC and RD $\to$ it is a group):
We only need to prove S is LD and RC for it to be a group.

LD:

$$\begin{aligned}
&\forall x \in S\ \exists x_l \in S : x_l \cdot x = u \text{ (unit of G, RD)}\\
&\text{let } q = x \cdot x_l\\
&q \cdot q = (x \cdot x_l) \cdot (x \cdot x_l) = x \cdot (x_l \cdot x) \cdot x_l = x \cdot (u \cdot x_l) = x \cdot x_l = q \text{ (associativity)}\\
&\text{similarly for q: } \exists q_l \in S : q_l \cdot q = u\\
&x \cdot x_l = q = u \cdot q = (q_l \cdot q) \cdot q = q_l \cdot (q \cdot q) = q_l \cdot q = u
\end{aligned}$$

We've shown, that for $x \in S$ if we start from the left inverse ($x_l \cdot x = u$), then it is also the right inverse ($x \cdot x_l = u$). $\forall x, b \in S\ \exists c :\ x \cdot c = b$, let $b = x_l \cdot b$, then $x \cdot (x_l \cdot b) = u \cdot b = b$. This means, that S is DL.

RC:

$$\begin{aligned}
&\forall g, a, b \in S, \text{ for which } a \cdot g = b \cdot g :\\
&\exists g' : gg' = u \text{ (LD)}\\
&(a \cdot g) \cdot g' = (b \cdot g) \cdot g'\\
&a \cdot (g \cdot g') = b \cdot (g \cdot g') \text{ (associativity)}\\
&a \cdot u = b \cdot u\\
&a = b
\end{aligned}$$

This showed, that $\forall g, a, b \in S : (a \cdot g = b \cdot g) \Rightarrow a = b$, which is the definition of RC.
Since S is also LD and RC, we can conclude, that it is a group. $\square$

Note:
The first implication would hold true even if we replaced RD with LD, but this wouldn't be true for the second one. We show this with a counterexample:

$$\begin{aligned}
&S = \{0, 1, 2\}\\
&\forall a, b \in S \backslash \{0\} : a \times b = b, 0 \times a = a \times 0 = a\\
&\quad \text{for } a, b \in \{1, 2\} :\ a \times (b \times c) = a \times c = c = b \times c = (a \times b) \times c\\
&\quad \text{for } a, b, \text{ or } c = 0 \text{ associativity also holds (intuitively or by an exhaustive proof)}\\
&0 \text{ is obviously the unit by definition}\\
&\forall a, b \in S : \exists x \in S : a \times x = b \text{ for } x = b \text{ (LD holds)}\\
&\forall x, a, b \in S : x \times a = x \times b \Rightarrow a = b \text{ because } x \times a = a \text{ and } x \times b = b \text{ (LC holds)}\\
&x \times 1 = 2, \text{ has no solutions, because } x \times 1 = 1 \text{ (is not RD)}
\end{aligned}$$

We constructed a semigroup with a unit and which is LD and LC, but which is not RD. $\square$