Cvičení 11

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1

Nejprve rozvážíme, že limita bude 0 a to dokážeme:

chceme: $\lim\limits_{x\to\infty} \big[\sin(\sqrt{n+1}) - \sin(\sqrt{n})\big] \Leftrightarrow$
$\forall\epsilon > 0 \;\; \exists n_0 : \forall n : n > n_0 \Rightarrow |\sin(\sqrt{n+1}) - \sin(\sqrt{n})| < \epsilon \Leftrightarrow$
$\forall\epsilon > 0 \;\; \exists n_0 : \forall n : n > n_0 \Rightarrow ||\sin(\sqrt{n+1}) - \sin(\sqrt{n})|| < \epsilon \Leftrightarrow$
$\lim\limits_{x\to\infty} \big[|\sin(\sqrt{n+1}) - \sin(\sqrt{n})|\big]$
použijeme dva policajty: $0 \le |\sin(\sqrt{n+1}) - \sin(\sqrt{n})|$ a druhý:
využijeme $\sin(x+y) = \sin(x)\cos(y) + \sin(y)\cos(x)$
$|\sin(\sqrt{n+1}) - \sin(\sqrt{n})| = |\sin(\sqrt{n+1} + \sqrt{n} - \sqrt{n}) - \sin(\sqrt{n})| =$
$|\sin(\sqrt{n})\cos(\sqrt{n+1} - \sqrt{n}) + \sin(\sqrt{n+1})\cos(\sqrt{n})| =$
$|\sin(\sqrt{n})\big(\cos(\sqrt{n+1} - \sqrt{n}) - 1\big) + \sin(\sqrt{n+1} - \sqrt{n})\cos(\sqrt{n})| \le$
$|\sin(\sqrt{n})\big(\cos(\sqrt{n+1} - \sqrt{n}) - 1\big)| + |\sin(\sqrt{n+1} - \sqrt{n})\cos(\sqrt{n})| \le$
$|\cos(\sqrt{n+1} - \sqrt{n}) - 1| + |\sin(\sqrt{n+1} - \sqrt{n})| = \qquad (\sin(\sqrt{n}) \in [-1,1], \cos(\sqrt{n}) \in [-1,1])$
$\left|\cos\left(\frac{1}{\sqrt{n+1}+\sqrt{n}}\right) - 1\right| + \left|\sin\left(\frac{1}{\sqrt{n+1}+\sqrt{n}}\right)\right|$ (rozšíření) $\Rightarrow$
$0 \le |\sin(\sqrt{n+1}) - \sin(\sqrt{n})| \le \left|\cos\left(\frac{1}{\sqrt{n+1}+\sqrt{n}}\right) - 1\right| + \left|\sin\left(\frac{1}{\sqrt{n+1}+\sqrt{n}}\right)\right|$
$\lim\limits_{x\to\infty} \left|\cos\left(\frac{1}{\sqrt{n+1}+\sqrt{n}}\right) - 1\right| + \left|\sin\left(\frac{1}{\sqrt{n+1}+\sqrt{n}}\right)\right| =$ (abs, cos, sin jsou spojité, lim vnitřní fce)
$\left|\cos\left(\lim\limits_{x\to\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}}\right) - 1\right| + \left|\sin\left(\lim\limits_{x\to\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}}\right)\right| = \qquad (\lim\limits_{x\to\infty} -1 = -1)$
$\left(0 \le \frac{1}{\sqrt{n+1}+\sqrt{n}} \le \frac{1}{2\sqrt{n}}, \text{ale} \lim\limits_{x\to\infty}\frac{1}{2\sqrt{n}} = 0 \Rightarrow \lim\limits_{x\to\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}} = 0\right)$
$= |\cos(0) - 1| + |\sin(0)| = 0 \Rightarrow \lim\limits_{x\to\infty}\left|\cos\left(\frac{1}{\sqrt{n+1}+\sqrt{n}}\right) - 1\right| + \left|\sin\left(\frac{1}{\sqrt{n+1}+\sqrt{n}}\right)\right| = 0 \Rightarrow$
$\lim\limits_{x\to\infty} |\sin(\sqrt{n+1}) - \sin(\sqrt{n})| = 0 \Rightarrow \lim\limits_{x\to\infty} \sin(\sqrt{n+1}) - \sin(\sqrt{n}) = 0$ (z úvodní ekvivalence)

2

$$\left(\frac{x^2+1}{x^2-2}\right)^{x^2} = \exp\left(x^2 \cdot ln\left(\frac{x^2+1}{x^2-2}\right) \cdot \frac{\frac{x^2+1}{x^2-2} - 1}{\frac{x^2+1}{x^2-2} - 1}\right) = \exp\left(\frac{3x^2}{x^2-2} \cdot \frac{ln\left(\frac{x^2+1}{x^2-2}\right)}{\frac{x^2+1}{x^2-2} - 1}\right) \Rightarrow$$

$$\lim_{x\to\infty}\left(\frac{x^2+1}{x^2-2}\right)^{x^2} = \lim_{x\to\infty}\exp\left(\frac{3x^2}{x^2-2} \cdot \frac{ln\left(\frac{x^2+1}{x^2-2}\right)}{\frac{x^2+1}{x^2-2} - 1}\right) = \exp\left(\lim_{x\to\infty}\left[\frac{3x^2}{x^2-2}\right] \cdot \lim_{x\to\infty}\left[\frac{ln\left(\frac{x^2+1}{x^2-2}\right)}{\frac{x^2+1}{x^2-2} - 1}\right]\right)$$

(exp je všude spojitá, můžeme tedy limitovat vnitřní funkci; aritmetika limit)

$$= \exp\left(\frac{\lim_{x\to\infty} 3}{\lim_{x\to\infty} 1 - \frac{2}{x^2}} \cdot \lim_{\frac{x^2+1}{x^2-2}\to 1}\frac{ln\left(\frac{x^2+1}{x^2-2}\right)}{\frac{x^2+1}{x^2-2} - 1}\right) \text{ (neboť } \lim_{x\to\infty}\frac{x^2+1}{x^2-2} = 1) = \exp(3 \cdot 1) = e^3$$

3

$$\lim_{n\to\infty}(exp(1/n)-4/n)^n = \lim_{n\to\infty} exp(n\cdot ln(exp(1/n)-4/n)) =$$

$$\lim_{n\to\infty} exp\left(n\cdot ln(exp(1/n)-4/n)\cdot\frac{exp(1/n)-4/n-1}{exp(1/n)-4/n-1}\right) = \text{ (exp je všude spojitá, proto lim vnitřní funkce; AL)}$$

$$exp\left(\lim_{n\to\infty} n\cdot(exp(1/n)-4/n-1)\cdot\lim_{n\to\infty}\frac{ln(exp(1/n)-4/n)}{exp(1/n)-4/n-1}\right)$$

$$1: \lim_{n\to\infty}\frac{ln(exp(1/n)-4/n)}{exp(1/n)-4/n-1} = 1,\ \text{neboť}\ \lim_{n\to\infty} exp(1/n)-4/n = 0,\ \text{tedy}\ \lim_{y\to 1}\frac{ln(y)}{y-1} = 1 \text{ ze známé limity (substituce)}$$

$$2: \lim_{n\to\infty} n\cdot(exp(1/n)-4/n-1) = \lim_{xn\to\infty}\left[\frac{exp(1/n)-1}{1/n}\right] - \lim_{n\to\infty} 4 = -3$$

$$\text{neboť}\ \lim_{n\to\infty}\frac{1}{n} = 0,\ \text{tedy}\ \lim_{1/n\to 0}\left[\frac{exp(1/n)-1}{1/n}\right] - 4 = -3$$