Cvičení 9

Vilém Zouhar

L1

$$\lim_{x\to b} f^2(x) = \lim_{x\to b} f(x)\cdot f(x) = \lim_{x\to b} f(x) \cdot \lim_{x\to b} f(x) \text{ (pokud je výraz definovaný)} = \big(\lim_{x\to b} f(x)\big)^2$$
$$\Rightarrow \lim_{x\to b} f(x) = \sqrt{\lim_{x\to b} f^2(x)}, \Rightarrow \lim_{x\to b} \sqrt{f(x)} = \sqrt{\lim_{x\to b} f(x)}$$

1

$$\lim_{x\to\infty}[\sqrt{(x+a)(x+b)} - x] = \lim_{x\to\infty} \frac{x^2+(a+b)x+ab-x^2}{\sqrt{(x+a)(x+b)}+x} = \lim_{x\to\infty} \frac{x((a+b)+\frac{ab}{x})}{x(sgn(x)\cdot\sqrt{1+\frac{a+b}{x}+\frac{ab}{x^2}}+1)} =$$
$$\text{(z aritmetiky limit)} \quad \frac{\lim_{x\to\infty}[(a+b)+\frac{ab}{x}]}{\lim_{x\to\infty}\left[sgn(x)\cdot\sqrt{1+\frac{a+b}{x}+\frac{ab}{x^2}}+1\right]} =$$
$$\frac{a+b}{\lim_{x\to\infty} sgn(x)\cdot \lim_{x\to\infty}\sqrt{1+\frac{a+b}{x}+\frac{ab}{x^2}} + \lim_{x\to\infty} 1} = \frac{a+b}{\lim_{x\to\infty}\sqrt{1+\frac{a+b}{x}+\frac{ab}{x^2}}+1} =$$
$$\text{(z L1)} \quad \frac{a+b}{\sqrt{\lim_{x\to\infty}\left[1+\frac{a+b}{x}+\frac{ab}{x^2}\right]}+1} = \frac{a+b}{2}$$

2

$$\lim_{x\to\frac{\pi}{6}} \frac{2sin^2(x)+sin(x)-1}{2sin^2(x)-3sin(x)+1} = \lim_{x\to\frac{\pi}{6}} \frac{2(sin(x)+1)(sin(x)-\frac{1}{2})}{2(sin(x)-1)(sin(x)-\frac{1}{2})} =$$
$$\lim_{x\to\frac{\pi}{6}} \frac{sin(x)+1}{sin(x)-1} = -3$$

L2

Neumím formálně ukázat, že $\lim_{x\to 0}\frac{a^x-1}{x} = ln(a)$, ale náznak:

$$y = a^x - 1, x = log_a(y+1) \quad \Rightarrow \quad \lim_{x\to 0}\frac{a^x-1}{x} = \lim_{y\to 0}\frac{y}{log_a(y+1)} \quad (y\to 0 \Leftrightarrow x\to 0)$$
$$\lim_{y\to 0}\frac{y}{log_a(y+1)} = \lim_{y\to 0}\frac{ln(a)}{\frac{1}{y}ln(y+1)} = \lim_{y\to 0}\frac{ln(a)}{ln\big((y+1)^{\frac{1}{y}}\big)} = \frac{ln(a)}{\lim_{y\to 0} ln\big((y+1)^{\frac{1}{y}}\big)} =$$
$$= \frac{ln(a)}{ln\big(\lim_{y\to 0}(y+1)^{\frac{1}{y}}\big)} = \frac{ln(a)}{ln\big(\lim_{y\to 0} e^{\frac{1}{y}ln(y+1)}\big)} = \frac{ln(a)}{ln\big(e^{\lim_{y\to 0}\frac{ln(y+1)}{y}}\big)} = \text{ (ze slíbené limity)}$$
$$= \frac{ln(a)}{ln(e^1)} = ln(a)$$

L3

Podobně neumím ukázat, že $\lim_{x\to b} a^{f(x)} = a^{\lim_{x\to b} f(x)}$, obecně mi chybí znalost o limitě složené funkce, za pomocí které by se ukázala jak L2, tak L3.

3

$$\lim_{x\to 0}\left(\frac{2^x+8^x}{2}\right)^{\frac{1}{x}} = \lim_{x\to 0} e^{\frac{1}{x}\cdot ln\left(\frac{2^x+8^x}{2}\right)} = \lim_{x\to 0} e^{\frac{1}{x}\cdot\frac{ln\left(\frac{2^x+8^x}{2}\right)\cdot\left(\frac{2^x+8^x}{2}-1\right)}{\frac{2^x+8^x}{2}-1}} = \text{(z L3, nedokázané formálně)}$$

$$= e^{\lim_{x\to 0}\left[\frac{\left(\frac{2^x+8^x}{2}-1\right)}{x}\cdot\frac{ln\left(\frac{2^x+8^x}{2}\right)}{\frac{2^x+8^x}{2}-1}\right]} = e^{\lim_{x\to 0}\left[\frac{\left(\frac{2^x+8^x}{2}-1\right)}{x}\right]\cdot\lim_{x\to 0}\left[\frac{ln\left(\frac{2^x+8^x}{2}\right)}{\frac{2^x+8^x}{2}-1}\right]} = \text{(ze slíbené limity)}$$

$$= e^{\lim_{x\to 0}\left[\frac{\left(\frac{2^x+8^x}{2}-1\right)}{x}\right]\cdot 1} = e^{\lim_{x\to 0}\frac{2^x+8^x-2}{2x}} = e^{\frac{1}{2}\left(\lim_{x\to 0}\frac{2^x-1}{x}+\lim_{x\to 0}\frac{8^x-1}{x}\right)} = \text{(z L2, nedokázané formálně)}$$

$$= e^{\frac{1}{2}\left(ln(2)+ln(8)\right)} = \left(e^{ln(2)}\cdot e^{ln(8)}\right)^{\frac{1}{2}} = \sqrt{16} = 4$$

Stejným způsobem lze mimo jiné ukázat, že: $\lim_{x\to 0}\left(\frac{a^x+b^x}{2}\right)^{\frac{1}{x}} = \sqrt{ab}$