1

$$\begin{gathered}\forall a,b \in G: \\ a^2b^2 = (ab)^2 \\ aabb = abab \\ a^{-1}aabbb^{-1} = a^{-1}abab^{-1} \\ ab = ba\end{gathered}$$

Thus we can conclude, that G is abelian.

2

Intersection

Identity

Since $e \in H$ and $e \in K$, then $e \in H \cap K$.

Inverse

$\forall x \in H \cap K : x \in H, x \in K \Rightarrow x^{-1} \in H, x^{-1} \in K \Rightarrow x^{-1} \in H \cap K$.

Closure under multiplication

$\forall x, y \in H \cap K : x, y \in H, x, y \in K \Rightarrow xy \in H, xy \in K \Rightarrow xy \in H \cap K$.
Thus we can conclude, that $H \cap K$ form a subgroup.

Union

Consider $A = \{id, (12)\}, B = \{id, (13)\} \subseteq S_3$. Then both A and B are subgroups, but $(12)(13) = (123) \notin A \cup B$. So it is not true, that $A \cup B$ must be a group.

3

$S_3 = \{id, (12), (13), (23), (123), (132)\}$

elements	normal
$\{id\}$	yes
$\{id, (12)\}$	no
$\{id, (13)\}$	no
$\{id, (23)\}$	no
$\{id, (123), (132)\}$	yes
$\{id, (12), (13), (23), (123), (132)\}$	yes

4

Subgroup

Identity

$\forall g \in G : ge = eg \Rightarrow e \in Z(G)$.

Closure under multiplication

$foralla, b \in Z(G) : \forall g \in G : ag = ga, ab = ba \Rightarrow g(ab) = abg = (ab)g \Rightarrow ab \in Z(G)$.

Inverse

$\forall x \in Z(G) : \forall g \in G : x^{-1}(gx)x^{-1} = x^{-1}(xg)x^{-1} \Rightarrow x^{-1}g = gx^{-1} \Rightarrow x^{-1} \in Z(G)$.

Normality

$\forall g \in G : g \cdot Z(G) = Z(G) \cdot g$, since $Z(G)$ commutes with every element of G, but this is also the definition of normality, because every left coset is also (exactly) a right coset.

5

Subgroup

Identity

$det(I_n) = 1 \Rightarrow I_n \in \mathrm{SL}_n(\mathbb{R}), \forall x \in \mathrm{SL}_n(\mathbb{R}) : xI_n = I_n x = x.$

Closure under multiplication

$foralla, b \in \mathrm{SL}_n(\mathbb{R}) : det(ab) = 1 \Rightarrow= ab \in \mathrm{SL}_n(\mathbb{R}).$

Inverse

$\forall x \in \mathrm{SL}_n(\mathbb{R}) : det(x^{-1}) = \frac{1}{det(x)} = 1 \Rightarrow x^{-1} \in \mathrm{SL}_n(\mathbb{R}).$

Normality

Using the alternative definition of normality:
$\forall g \in \mathrm{GL}_n(\mathbb{R}), \forall x \in \mathrm{SL}_n(\mathbb{R}) : det(gxg^{-1}) = \frac{det(g) \cdot 1}{det(g)} = 1 \Rightarrow gxg^{-1} \in \mathrm{SL}_n(\mathbb{R}).$

Quotient group

$\mathrm{GL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{R}) = \{\{x \in \mathrm{GL}_n(\mathbb{R}) : det(x) = r\}, r \in \mathbb{R}\backslash\{0\}\}$ (sets of matricies of given determinant)