LD := left divisible $\forall a, b \in G : \exists x : a \cdot x = b$
RD := right divisible $\forall a, b \in G : \exists x : x \cdot a = b$
LC := left cancellative
RC := right cancellative

Proposition: A semigroup G is a group iff it has a unit and is LC and LD. In this case LD could be replaced with RD, as we will see in the proof.

Proof:
$\Rightarrow$ (G is a group $\rightarrow$ G has a unit and is LC and LD):
Since G is a group, then it is also divisible and cancellative, hence is LC and LD. In this direction we only need to prove the existence of a unit.

$$\begin{aligned}
&\forall g, h \in G : \exists l_g, r_g, l_h, r_h : l_g \cdot g = g \cdot r_g,\ l_h \cdot g = g \cdot r_h \text{ (RD, LD)}\\
&(g \cdot r_g) \cdot h = g \cdot h = g \cdot (l_h \cdot h) = (g \cdot l_h) \cdot h \ (associativity)\\
&(g \cdot r_g) \cdot h = (g \cdot l_h) \cdot h \Rightarrow g \cdot r_g = g \cdot l_h \ (RC)\\
&g \cdot r_g = g \cdot l_h \Rightarrow r_g = l_h \ (LC)\\
&\text{for } g = h \text{ we denote } l_g = r_g = u_g\\
&\text{but since } \forall g, h \in G : u_g \cdot g = g = g \cdot u_g,\ u_h \cdot g = g = g \cdot u_h \text{ and } u_g = u_h\\
&\text{then } u_g = u_h = u \text{ is a unit of } G
\end{aligned}$$

$\Leftarrow$ (S is a semigroup with a unit and is LC and LD $\rightarrow$ it is a group):
We only need to prove S is RD and RC for it to be a group.

RD:

$$\begin{aligned}
&\forall x \in S\ \exists x_r \in S : x \cdot x_r = u \text{ (unit of G, LD)}\\
&\text{let } q = x \cdot x_r\\
&q \cdot q = (x_r \cdot x) \cdot (x_r \cdot x) = x_r \cdot (x \cdot x_r) \cdot x = x_r \cdot (u \cdot x) = x_r \cdot x = q \text{ (associativity)}\\
&\text{similarly for q: } \exists q_r \in S : q \cdot q_r = u \text{ (LD)}\\
&x_r \cdot x = q = q \cdot u = q \cdot (q \cdot q_r) = (q \cdot q) \cdot q_r = q \cdot q_r = u
\end{aligned}$$

We've shown, that $\forall x \in S\ \exists x' : x' \cdot x = x \cdot x' = u$. This means, we've found the inverse. We can multiply the resulting equation from left by y and get the final form for RD. $\forall y, x \in S\ \exists x' : (y \cdot x') \cdot x = y \cdot u = y$, thus $\forall y, x \in S\ \exists t : t \cdot x = y$.

Note: If we started from RD, we could prove LD, since we used only the one sided divisibility and the existence of a unit. This means, that for each semigroup with a unit, LD is equivalent to RD. Proof follows easily from the previous one.

RC:

$$\begin{aligned}
&\forall g, a, b \in S, \text{ for which } a \cdot g = b \cdot g :\\
&\exists g' : g \cdot g' = u \text{ (LD)}\\
&(a \cdot g) \cdot g' = (b \cdot g) \cdot g'\\
&a \cdot (g \cdot g') = b \cdot (g \cdot g') \text{ (associativity)}\\
&a \cdot u = b \cdot u\\
&a = b
\end{aligned}$$

This showed, that $\forall g, a, b \in S : (a \cdot g = b \cdot g) \Rightarrow a = b$, which is the definition of RC.
Since S is also LD and RC, we can conclude, that it is a group. $\square$