the thermometer shows us that we encounter other types of numbers besides the natural numbers, because there may be temperatures which may go below 0. Thus we encounter naturally what we shall call **negative integers** which we call **minus** 1, **minus** 2, **minus** 3, . . . , and which we write as

$$-1,\ -2,\ -3,\ -4, \ldots.$$

We represent the negative integers on a line as being on the other side of 0 from the positive integers, like this:

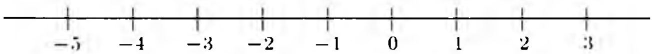

Fig. 1–2

The positive integers, negative integers, and zero all together are called the **integers.** Thus -9, 0, 10, -5 are all integers.

If we view the line as a thermometer, on which a unit of temperature has been selected, say the degree Fahrenheit, then each integer represents a certain temperature. The negative integers represent temperatures below zero.

Our discussion is already typical of many discussions which will occur in this course, concerning mathematical objects and their applicability to physical situations. In the present instance, we have the integers as mathematical objects, which are essentially abstract quantities. We also have different applications for them, for instance measuring distance or temperatures. These are of course not the only applications. Namely, we can use the integers to measure time. We take the origin 0 to represent the year of the birth of Christ. Then the positive integers represent years after the birth of Christ (called AD years), while the negative integers can be used to represent BC years. With this convention, we can say that the year -500 is the year 500 BC.

Adding a positive number, say 7, to another number, means that we must move 7 units to the right of the other number. For instance,

$$5 + 7 = 12.$$

Seven units to the right of 5 yields 12. On the thermometer, we would of course be moving upward instead of right. For instance, if the temperature at a given time is 5° and if it goes up by 7°, then the new temperature is 12°.

Observe the very simple rule for addition with 0, namely

N1. $$\boxed{0 + a = a + 0 = a}$$

for any integer a.

What about adding negative numbers? Look at the thermometer again. Suppose the temperature at a given time is $10°$, and the temperature drops by $15°$. The new temperature is then $-5°$, and we can write

$$10 - 15 = -5.$$

Thus -5 is the result of subtracting 15 from 10, or of adding -15 to 10.

In terms of points on a line, adding a negative number, say -3, to another number means that we must move 3 units to the left of this other number. For example,

$$5 + (-3) = 2$$

because starting with 5 and moving 3 units to the left yields 2. Similarly,

$$7 + (-3) = 4, \qquad \text{and} \qquad 3 + (-5) = -2.$$

Note that we have

$$3 + (-3) = 0 \qquad \text{or} \qquad 5 + (-5) = 0.$$

We can also write these equations in the form

$$(-3) + 3 = 0 \qquad \text{or} \qquad (-5) + 5 = 0.$$

For instance, if we start 3 units to the left of 0 and move 3 units to the right, we get 0. Thus, in general, we have the formulas (by assumption):

N2. $$\boxed{a + (-a) = 0 \qquad \text{and also} \qquad -a + a = 0.}$$

In the representation of integers on the line, this means that a and $-a$ lie on opposite sides of 0 on that line, as shown on the next picture:

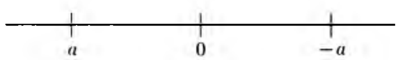

Fig. 1–3

Thus according to this representation we can now write

$$3 = -(-3) \qquad \text{or} \qquad 5 = -(-5).$$

In these special cases, the pictures are:

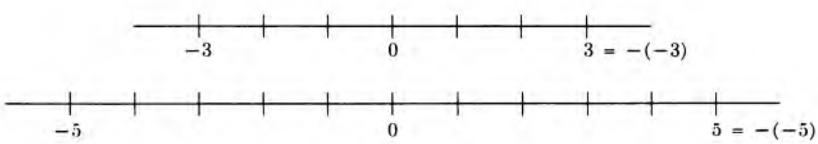

Fig. 1–4

Remark. We use the name

$$\textbf{minus } a \quad \text{for} \quad -a$$

rather than the words "negative a" which have found some currency recently. I find the words "negative a" confusing, because they suggest that $-a$ is a negative number. This is not true unless a itself is positive. For instance,

$$3 = -(-3)$$

is a positive number, but 3 is equal to $-a$, where $a = -3$, and a is a negative number.

Because of the property

$$a + (-a) = 0,$$

one also calls $-a$ the **additive inverse** of a.

The sum and product of integers are also integers, and the next sections are devoted to a description of the rules governing addition and multiplication.

§2. RULES FOR ADDITION

Integers follow very simple rules for addition. These are:

Commutativity. *If a, b are integers, then*

$$\boxed{a + b = b + a.}$$

For instance, we have

$$3 + 5 = 5 + 3 = 8,$$

or in an example with negative numbers, we have

$$-2 + 5 = 3 = 5 + (-2).$$

Associativity. *If a, b, c are integers, then*

$$\boxed{(a + b) + c = a + (b + c).}$$

In view of this, it is unnecessary to use parentheses in such a simple context, and we write simply

$$a + b + c.$$

For instance,

$$(3 + 5) + 9 = 8 + 9 = 17,$$
$$3 + (5 + 9) = 3 + 14 = 17.$$

We write simply

$$3 + 5 + 9 = 17.$$

Associativity also holds with negative numbers. For example,

$$(-2 + 5) + 4 = 3 + 4 = 7,$$
$$-2 + (5 + 4) = -2 + 9 = 7.$$

Also,

$$(2 + (-5)) + (-3) = -3 + (-3) = -6,$$
$$2 + (-5 + (-3)) = 2 + (-8) = -6.$$

The rules of addition mentioned above will not be proved, but we shall prove other rules from them.

To begin with, note that:

N3. $\boxed{\text{If } a + b = 0, \text{ then } b = -a \text{ and } a = -b.}$

To prove this, add $-a$ to both sides of the equation $a + b = 0$. We get

$$-a + a + b = -a + 0 = -a.$$

Since $-a + a + b = 0 + b = b$, we find

$$b = -a$$

as desired. Similarly, we find $a = -b$. We could also conclude that

$$-b = -(-a) = a.$$

As a matter of convention, we shall write

$$a - b$$

instead of

$$a + (-b).$$

Thus a sum involving three terms may be written in many ways, as follows:

$$\begin{aligned}(a - b) + c &= (a + (-b)) + c \\ &= a + (-b + c) && \text{by associativity} \\ &= a + (c - b) && \text{by commutativity} \\ &= (a + c) - b && \text{by associativity,}\end{aligned}$$

and we can also write this sum as

$$a - b + c = a + c - b,$$

omitting the parentheses. Generally, in taking the sum of integers, we can take the sum in any order by applying associativity and commutativity repeatedly.

As a special case of **N3**, for any integer a we have

N4. $$\boxed{a = -(-a).}$$

This is true because

$$a + (-a) = 0,$$

and we can apply **N3** with $b = -a$. Remark that this formula is true whether a is positive, negative, or 0. If a is positive, then $-a$ is negative. If a is negative, then $-a$ is positive. In the geometric representation of numbers on the line, a and $-a$ occur symmetrically on the line on opposite sides of 0. Of course, we can pile up minus signs and get other relationships, like

$$-3 = -(-(-3)),$$

or

$$3 = -(-3) = -(-(-(-3))).$$

Thus when we pile up the minus signs in front of a, we obtain a or $-a$ alternatively. For the general formula with the appropriate notation, cf. Exercises 5 and 6 of §4.

From our rules of operation we can now prove:

For any integers a, b we have

$$-(a + b) = -a + (-b)$$

or, in other words,

N5. $$\boxed{-(a+b) = -a-b.}$$

Proof. Remember that if x, y are integers, then $x = -y$ and $y = -x$ mean that $x + y = 0$. Thus to prove our assertion, we must show that

$$(a+b) + (-a-b) = 0.$$

But this comes out immediately, namely,

$$\begin{aligned}(a+b) + (-a-b) &= a+b-a-b && \text{by associativity}\\ &= a-a+b-b && \text{by commutativity}\\ &= 0+0\\ &= 0.\end{aligned}$$

This proves our formula.

Example. We have

$$\begin{aligned}-(3+5) &= -3-5 = -8,\\ -(-4+5) &= -(-4)-5 = 4-5 = -1,\\ -(3-7) &= -3-(-7) = -3+7 = 4.\end{aligned}$$

You should be very careful when you take the negative of a sum which involves itself in negative numbers, taking into account that

$$-(-a) = a.$$

The following rule concerning positive integers is so natural that you probably would not even think it worth while to take special notice of it. We still state it explicitly.

If a, b are positive integers, then $a + b$ is also a positive integer.

For instance, 17 and 45 are positive integers, and their sum, 62, is also a positive integer.

We assume this rule concerning positivity. We shall see later that it also applies to positive real numbers. From it we can prove:

If a, b are negative integers, then $a + b$ is negative.