

Natural Deduction for Propositional Logic

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September 9, 2019

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Natural Deduction

- In our examples, we (informally) infer new sentences.
- In natural deduction, we have a collection of proof rules.
 - These proof rules allow us to infer new sentences logically followed from existing ones.
- Suppose we have a set of sentences: $\phi_1, \phi_2, \dots, \phi_n$ (called premises), and another sentence ψ (called a conclusion).
- The notation

$$\phi_1, \phi_2, \dots, \phi_n \vdash \psi$$

is called a sequent.

- A sequent is valid if a proof (built by the proof rules) can be found.
- We will try to build a proof for our examples. Namely,

$$p \wedge \neg q \implies r, \neg r, p \vdash q.$$

Proof Rules for Natural Deduction – Conjunction

- Suppose we want to prove a conclusion $\phi \wedge \psi$. What do we do?
 - Of course, we need to prove both ϕ and ψ so that we can conclude $\phi \wedge \psi$.
- Hence the proof rule for conjunction is

$$\frac{\phi \quad \psi}{\phi \wedge \psi} \wedge i$$

- Note that premises are shown above the line and the conclusion is below. Also, $\wedge i$ is the name of the proof rule.
- This proof rule is called “conjunction-introduction” since we introduce a conjunction (\wedge) in the conclusion.

Proof Rules for Natural Deduction – Conjunction

- For each connective, we have introduction proof rule(s) and also elimination proof rule(s).
- Suppose we want to prove a conclusion ϕ from the premise $\phi \wedge \psi$. What do we do?
 - We don't do any thing since we know ϕ already!
- Here are the elimination proof rules:

$$\frac{\phi \wedge \psi}{\phi} \wedge e_1$$

$$\frac{\phi \wedge \psi}{\psi} \wedge e_2$$

- The rule $\wedge e_1$ says: if you have a proof for $\phi \wedge \psi$, then you have a proof for ϕ by applying this proof rule.
- Why do we need two rules?
 - Because we want to manipulate syntax only.

Examples

Example

Prove $p \wedge q, r \vdash q \wedge r$.

Proof.

We are looking for a proof of the form:

$$\begin{array}{c} p \wedge q \quad r \\ \vdots \\ q \wedge r \end{array}$$



Examples

Example

Prove $p \wedge q, r \vdash q \wedge r$.

Proof.

We are looking for a proof of the form:

$$\frac{\frac{p \wedge q}{q} \wedge e_2 \quad r}{q \wedge r} \wedge i$$

We will write proofs in lines:

1	$p \wedge q$	premise
2	r	premise
3	q	$\wedge e_2$ 1
4	$q \wedge r$	$\wedge i$ 3, 2



Proof Rules for Natural Deduction – Double Negation

- Suppose we want to prove ϕ from a proof for $\neg\neg\phi$. What do we do?
 - There is no difference between ϕ and $\neg\neg\phi$. The same proof suffices!
- Hence we have the following proof rules:

$$\frac{\phi}{\neg\neg\phi} \neg\neg i$$

$$\frac{\neg\neg\phi}{\phi} \neg\neg e$$

Examples

Example

Prove $p, \neg\neg(q \wedge r) \vdash \neg\neg p \wedge r$.

Proof.

We are looking for a proof like:

$$\begin{array}{c} p \quad \neg\neg(q \wedge r) \\ \vdots \\ \neg\neg p \wedge r \end{array}$$



Examples

Example

Prove $p, \neg\neg(q \wedge r) \vdash \neg\neg p \wedge r$.

Proof.

We are looking for a proof like:

$$\frac{\frac{p}{\neg\neg p} \quad \neg\neg i \quad \frac{\frac{\neg\neg(q \wedge r)}{q \wedge r} \quad \neg\neg e \quad \frac{q \wedge r}{r} \quad \wedge e_2}{\neg\neg p \wedge r} \quad \wedge i$$



Examples

Example

Prove $p, \neg\neg(q \wedge r) \vdash \neg\neg p \wedge r$.

Proof.

We are looking for a proof like:

1	p	premise
2	$\neg\neg(q \wedge r)$	premise
3	$\neg\neg p$	$\neg\neg i$ 1
4	$q \wedge r$	$\neg\neg e$ 2
5	r	$\wedge e_2$ 4
6	$\neg\neg p \wedge r$	$\wedge i$ 3, 5



Proof Rules for Natural Deduction – Implication

- Suppose we want to prove ψ from proofs for ϕ and $\phi \implies \psi$. What do we do?
 - We just put the two proofs for ϕ and $\phi \implies \psi$ together.
- Here is the proof rule:

$$\frac{\phi \quad \phi \implies \psi}{\psi} \implies e$$

- This proof rule is also called *modus ponens*.
- Here is another proof rule related to implication:

$$\frac{\phi \implies \psi \quad \neg\psi}{\neg\phi} MT$$

- This proof rule is called *modus tollens*.

Example

Example

Prove $p \implies (q \implies r), p, \neg r \vdash \neg q$.

Proof.

1	$p \implies (q \implies r)$	premise
2	p	premise
3	$\neg r$	premise
4	$q \implies r$	\implies e 2, 1
5	$\neg q$	MT 4, 3



Proof Rules for Natural Deduction – Implication

- Suppose we want to prove $\phi \implies \psi$. What do we do?
 - ▶ We assume ϕ to prove ψ . If succeed, we conclude $\phi \implies \psi$ without any assumption.
 - ▶ Note that ϕ is added as an assumption and then removed so that $\phi \implies \psi$ does not depend on ϕ .
- We use “box” to simulate this strategy.
- Here is the proof rule:

$$\frac{\boxed{\begin{array}{c} \phi \\ \vdots \\ \psi \end{array}}}{\phi \implies \psi} \implies i$$

- At any point in a box, you can only use a sentence ϕ before that point. Moreover, no box enclosing the occurrence of ϕ has been closed.

Example

Example

Prove $\neg q \implies \neg p \vdash p \implies \neg\neg q$.

Proof.

$$\frac{\neg q \implies \neg p \quad \frac{p}{\neg\neg p} \quad \neg\neg i}{\neg\neg q} MT$$
$$\frac{p \implies \neg\neg q}{p \implies \neg\neg q \implies i}$$

- | | | |
|---|------------------------------------|----------------|
| 1 | $\neg q \implies \neg p$ | premise |
| 2 | p | assumption |
| 3 | $\neg\neg p$ | $\neg\neg i$ 2 |
| 4 | $\neg\neg q$ | MT 1, 3 |
| 5 | $p \implies \neg\neg q \implies i$ | 2-4 |



Theorems

Example

Prove $\vdash p \implies p$.

Proof.

1	<table border="1"><tr><td>p</td><td>assumption</td></tr></table>	p	assumption
p	assumption		
2	$p \implies p \implies i\ 1 - 1$		



In the box, we have $\phi \equiv \psi \equiv p$.

Definition

A sentence ϕ such that $\vdash \phi$ is called a theorem.

Examples

Example

Prove $p \wedge q \implies r \vdash p \implies (q \implies r)$.

Proof.

1	$p \wedge q \implies r$	premise	
2	p	assumption]
3	q	assumption	
4	$p \wedge q$	$\wedge i$ 2, 3	
5	r	$\implies e$ 4, 1]
6	$q \implies r$	$\implies i$ 3-5	
7	$p \implies (q \implies r)$	$\implies i$ 2-6	



Proof Rules for Natural Deduction – Disjunction

- Suppose we want to prove $\phi \vee \psi$. What do we do?
 - We can either prove ϕ or ψ .
- Here are the proof rules:

$$\frac{\phi}{\phi \vee \psi} \vee i_1$$

$$\frac{\psi}{\phi \vee \psi} \vee i_2$$

- Note the symmetry with $\wedge e_1$ and $\wedge e_2$.

$$\frac{\phi \wedge \psi}{\phi} \wedge e_1$$

$$\frac{\phi \wedge \psi}{\psi} \wedge e_2$$

- Can we have a corresponding symmetric elimination rule for disjunction? Recall

$$\frac{\phi \quad \psi}{\phi \wedge \psi} \wedge i$$

Proof Rules for Natural Deduction – Disjunction

- Suppose we want to prove χ from $\phi \vee \psi$. What do we do?
 - ▶ We assume ϕ to prove χ and then assume ψ to prove χ .
 - ▶ If both succeed, χ is proved from $\phi \vee \psi$ without assuming ϕ and ψ .
- Here is the proof rule:

$$\frac{\phi \vee \psi \quad \boxed{\begin{array}{c} \phi \\ \vdots \\ \chi \end{array}} \quad \boxed{\begin{array}{c} \psi \\ \vdots \\ \chi \end{array}}}{\chi} \vee e$$

- In addition to nested boxes, we may have parallel boxes in our proofs.

Example

Recall that our syntax does not admit commutativity.

Example

Prove $p \vee q \vdash q \vee p$.

Proof.

$$\frac{p \vee q \quad \boxed{\frac{p}{q \vee p} \vee i_2} \quad \boxed{\frac{q}{q \vee p} \vee i_1}}{q \vee p} \vee e$$

- | | | | |
|---|------------|----------------------|---|
| 1 | $p \vee q$ | premise | |
| 2 | p | assumption |] |
| 3 | $q \vee p$ | $\vee i_2$ 2 | |
| 4 | q | assumption |] |
| 5 | $q \vee p$ | $\vee i_1$ 4 | |
| 6 | $q \vee p$ | $\vee e$ 1, 2-3, 4-5 | |



Example

Example

Prove $q \implies r \vdash p \vee q \implies p \vee r$.

Proof.

1	$q \implies r$	premise		
2	$p \vee q$	assumption]	
3	p	assumption]	
4	$p \vee r$	$\vee i_1$ 3]	
5	q	assumption]	
6	r	\implies e 5, 1		
7	$p \vee r$	$\vee i_2$ 6]	
8	$p \vee r$	\vee e 2, 3-4, 5-7]	
9	$p \vee q \implies p \vee r$	\implies i 2-8		



Example

Example

Prove $p \wedge (q \vee r) \vdash (p \wedge q) \vee (p \wedge r)$.

Proof.

1	$p \wedge (q \vee r)$	premise	
2	p	$\wedge e_1$ 1	
3	$q \vee r$	$\wedge e_2$ 1	
4	q	assumption]
5	$p \wedge q$	$\wedge i$ 2, 4	
6	$(p \wedge q) \vee (p \wedge r)$	$\vee i_1$ 5]
7	r	assumption]
8	$p \wedge r$	$\wedge i$ 2, 7	
9	$(p \wedge q) \vee (p \wedge r)$	$\vee i_2$ 8]
10	$(p \wedge q) \vee (p \wedge r)$	$\vee e$ 3, 4-6, 7-9	



Example

Example

Prove $(p \wedge q) \vee (p \wedge r) \vdash p \wedge (q \vee r)$.

Proof.

1	$(p \wedge q) \vee (p \wedge r)$	premise	
2	$p \wedge q$	assumption]
3	p	$\wedge e_1$ 2	
4	q	$\wedge e_2$ 2	
5	$q \vee r$	$\vee i_1$ 4	
6	$p \wedge (q \vee r)$	$\wedge i$ 3, 5	
7	$p \wedge r$	assumption]
8	p	$\wedge e_1$ 7	
9	r	$\wedge e_2$ 7	
10	$q \vee r$	$\vee i_2$ 9	
11	$p \wedge (q \vee r)$	$\wedge i$ 8, 10	
12	$p \wedge (q \vee r)$	$\vee e$ 1, 2-6, 7-11	

Contradiction

Definition

Contradictions are sentences of the form $\phi \wedge \neg\phi$ or $\neg\phi \wedge \phi$.

- Examples:
 - ▶ $p \wedge \neg p, \neg(p \vee q \implies r) \wedge (p \vee q \implies r)$.
- Logically, any sentence can be proved from a contradiction.
 - ▶ If $0 = 1$, then $100 \neq 100$.
- Particularly, if ϕ and ψ are contradictions, we have $\phi \dashv\vdash \psi$.
 - ▶ $\phi \dashv\vdash \psi$ means $\phi \vdash \psi$ and $\psi \vdash \phi$ (called provably equivalent).
- Since all contradictions are equivalent, we will use the symbol \perp (called “bottom”) for them.
- We are now ready to discuss proof rules for negation.

Proof Rules for Natural Deduction – Negation

- Since any sentence can be proved from a contradiction, we have

$$\frac{\perp}{\phi} \perp e$$

- When both ϕ and $\neg\phi$ are proved, we have a contradiction.

$$\frac{\phi \quad \neg\phi}{\perp} \neg e$$

- ▶ The proof rule could be called $\perp i$. We use $\neg e$ because it eliminates a negation.

Example

Example

Prove $\neg p \vee q \vdash p \implies q$.

Proof.

1	$\neg p \vee q$	premise		
2	$\neg p$	assumption]
3	p	assumption]	
4	\perp	$\neg e$ 3, 2		
5	q	$\perp e$ 4]	
6	$p \implies q$	$\implies i$ 3-5]	
7	q	assumption]
8	p	assumption]	
9	q	copy 7]	
10	$p \implies q$	$\implies i$ 8-9]	
11	$p \implies q$	$\vee e$ 1, 2-6, 7-10		

Proof Rules for Natural Deduction – Negation

- Suppose we want to prove $\neg\phi$. What do we do?
 - ▶ We assume ϕ and try to prove a contradiction. If succeed, we prove $\neg\phi$.
- Here is the proof rule:

$$\frac{\boxed{\begin{array}{c} \phi \\ \vdots \\ \bot \end{array}}}{\neg\phi} \neg i$$

Example

Example

Prove $p \implies q, p \implies \neg q \vdash \neg p$.

Proof.

1	$p \implies q$	premise	
2	$p \implies \neg q$	premise	
3	p	assumption]
4	q	\implies e 3, 1	
5	$\neg q$	\implies e 3, 2	
6	\perp	\neg e 4, 5	
7	$\neg p$	\neg i 3-6	



Example

Example

Prove $p \wedge \neg q \implies r, \neg r, p \vdash q$.

Proof.

1	$p \wedge \neg q \implies r$	premise	
2	$\neg r$	premise	
3	p	premise	
4	$\neg q$	assumption]
5	$p \wedge \neg q$	$\wedge i$ 3, 4	
6	r	$\implies e$ 5, 1	
7	\perp	$\neg e$ 6, 2]
8	$\neg\neg q$	$\neg i$ 4-7	
9	q	$\neg\neg e$ 8	



Derived Rules

- Some rules can actually be derived from others.

Examples

Prove $p \implies q, \neg q \vdash \neg p$ (modus tollens).

Proof.

1	$p \implies q$	premise	
2	$\neg q$	premise	
3	p	assumption	
4	q	$\implies e$ 3, 1	
5	\perp	$\neg e$ 4, 2	
6	$\neg p$	$\neg i$ 3-5	



Derived Rules

Examples

Prove $p \vdash \neg\neg p$ ($\neg\neg i$)

Proof.

1	p	premise	
2	$\neg p$	assumption]
3	\perp	$\neg e$ 1, 2	
4	$\neg\neg p$	$\neg i$ 2-3	



- These rules can be replaced by their proofs and are not necessary.
 - They are just macros to help us write shorter proofs.

Reductio ad absurdum (RAA)

Example

Prove $\neg p \implies \perp \vdash p$ (RAA).

Proof.

1	$\neg p \implies \perp$	premise	
2	$\neg p$	assumption]
3	\perp	$\implies e$ 2, 1	
4	$\neg\neg p$	$\neg i$ 2-3	
5	p	$\neg\neg e$ 4	



Tertium non datur, Law of the Excluded Middle (LEM)

Example

Prove $\vdash p \vee \neg p$.

Proof.

1	$\neg(p \vee \neg p)$	assumption]	
2	p	assumption]	
3	$p \vee \neg p$	$\vee i_1$ 2		
4	\perp	$\neg e$ 3, 1]	
5	$\neg p$	$\neg i$ 2-4		
6	$p \vee \neg p$	$\vee i_2$ 5		
7	\perp	$\neg e$ 6, 1]	
8	$\neg\neg(p \vee \neg p)$	$\neg i$ 1-7		
9	$p \vee \neg p$	$\neg\neg e$ 8		



Proof Rules for Natural Deduction (Summary)

Conjunction (\wedge)

$$\frac{\phi \quad \psi}{\phi \wedge \psi} \wedge i$$

$$\frac{\phi \wedge \psi}{\phi} \wedge e_1 \quad \frac{\phi \wedge \psi}{\psi} \wedge e_2$$

Disjunction (\vee)

$$\frac{\phi}{\phi \vee \psi} \vee i_1 \quad \frac{\psi}{\phi \vee \psi} \vee i_2$$

$$\frac{\phi \vee \psi \quad \boxed{\begin{array}{c} \phi \\ \vdots \\ \chi \end{array}} \quad \boxed{\begin{array}{c} \psi \\ \vdots \\ \chi \end{array}}}{\chi} \vee e$$

Implication (\implies)

$$\frac{\boxed{\begin{array}{c} \phi \\ \vdots \\ \psi \end{array}}}{\phi \implies \psi} \implies i$$

$$\frac{\phi \quad \phi \implies \psi}{\psi} \implies e$$

Proof Rules for Natural Deduction (Summary)

Negation (\neg)

$$\frac{\boxed{\begin{array}{c} \phi \\ \vdots \\ \perp \end{array}}}{\neg\phi} \neg i$$

$$\frac{\phi \quad \neg\phi}{\perp} \neg e$$

Contradiction (\perp)

(no introduction rule)

$$\frac{\perp}{\phi} \perp e$$

Double negation ($\neg\neg$)

(no introduction rule)

$$\frac{\neg\neg\phi}{\phi} \neg\neg e$$

Useful Derived Proof Rules

$$\frac{\phi \implies \psi \quad \neg \psi}{\neg \phi} \text{ MT}$$
$$\boxed{\begin{array}{c} \neg \phi \\ \vdots \\ \bot \end{array}} \text{ RAA}$$
$$\frac{}{\phi}$$

$$\frac{\phi}{\neg \neg \phi} \neg \neg i$$

$$\frac{}{\phi \vee \neg \phi} \text{ LEM}$$

Provable Equivalence

- Recall $p \dashv\vdash q$ means $p \vdash q$ and $q \vdash p$.
- Here are some provably equivalent sentences:

$$\neg(p \wedge q) \dashv\vdash \neg q \vee \neg p$$

$$\neg(p \vee q) \dashv\vdash \neg p \wedge \neg q$$

$$p \implies q \dashv\vdash \neg q \implies \neg p$$

$$p \implies q \dashv\vdash \neg p \vee q$$

$$p \wedge q \implies p \dashv\vdash r \vee \neg r$$

$$p \wedge q \implies r \dashv\vdash p \implies (q \implies r)$$

- Try to prove them.

Proof by Contradiction

- Although it is very useful, the proof rule RAA is a bit puzzling.

$$\frac{\boxed{\begin{array}{c} \neg\phi \\ \vdots \\ \perp \end{array}}}{\phi} \text{RAA}$$

- Instead of proving ϕ directly, the proof rule allows indirect proofs.
 - If $\neg\phi$ leads to a contradiction, then ϕ must hold.
- Note that indirect proofs are not “constructive.”
 - We do not show why ϕ holds; we only know $\neg\phi$ is impossible.
- In early 20th century, some logicians and mathematicians chose not to prove indirectly. They are intuitionistic logicians or mathematicians.
- For the same reason, intuitionists also reject

$$\frac{}{\phi \vee \neg\phi} \text{LEM}$$

$$\frac{\neg\neg\phi}{\phi} \neg\neg e$$

Proof by Contradiction

Theorem

There are $a, b \in \mathbb{R} \setminus \mathbb{Q}$ such that $a^b \in \mathbb{Q}$.

Proof.

Let $b = \sqrt{2}$. There are two cases:

- If $b^b \in \mathbb{Q}$, we are done since $\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$.
- If $b^b \notin \mathbb{Q}$, choose $a = b^b = \sqrt{2}^{\sqrt{2}}$. Then $a^b = \sqrt{2}^{\sqrt{2} \cdot \sqrt{2}} = \sqrt{2}^2 = 2$.
Since $\sqrt{2}^{\sqrt{2}}, \sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$, we are done.



- An intuitionist would criticize the proof since it does not tell us what a, b give $a^b \in \mathbb{Q}$.
 - We know (a, b) is either $(\sqrt{2}, \sqrt{2})$ or $(\sqrt{2}^{\sqrt{2}}, \sqrt{2})$.

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Well-Formedness

Definition

A well-formed formula is constructed by applying the following rules finitely many times:

- atom: Every propositional atom p, q, r, \dots is a well-formed formula;
 - \neg : If ϕ is a well-formed formula, so is $(\neg\phi)$;
 - \wedge : If ϕ and ψ are well-formed formulae, so is $(\phi \wedge \psi)$;
 - \vee : If ϕ and ψ are well-formed formulae, so is $(\phi \vee \psi)$;
 - \implies : If ϕ and ψ are well-formed formulae, so is $(\phi \implies \psi)$.
-
- More compactly, well-formed formulae are defined by the following grammar in Backus Naur form (BNF):

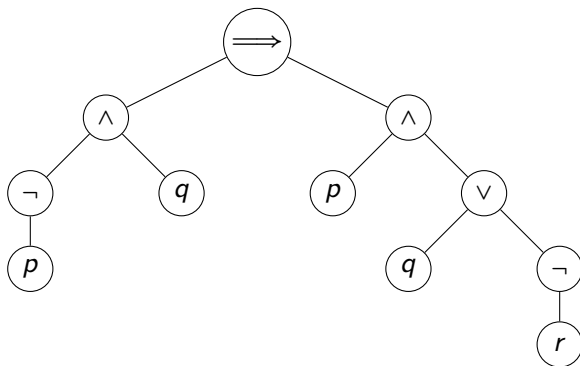
$$\phi ::= p \mid (\neg\phi) \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (\phi \implies \phi)$$

Inversion Principle

- How do we check if $((\neg p) \wedge q) \implies (p \wedge (q \vee (\neg r)))$ is well-formed?
- Although a well-formed formula needs five grammar rules to construct, the construction process can always be inverted.
 - This is called inversion principle.
- To show $((\neg p) \wedge q) \implies (p \wedge (q \vee (\neg r)))$ is well-formed, we need to show both $((\neg p) \wedge q)$ and $(p \wedge (q \vee (\neg r)))$ are well-formed.
- To show $((\neg p) \wedge q)$ is well-formed, we need to show both $(\neg p)$ and q are well-formed.
 - q is well-formed since it is an atom.
- To show $(\neg p)$ is well-formed, we need to show p is well-formed.
 - p is well-formed since it is an atom.
- Similarly, we can show $(p \wedge (q \vee (\neg r)))$ is well-formed.

Parse Tree

- The easiest way to decide whether a formula is well-formed is perhaps by drawing its parse tree.



Subformulae

- Given a well-formed formula, its subformulae are the well-formed formulae corresponding to its parse tree.
- For instance, the subformulae of the well-formed formulae $((\neg p) \wedge q) \implies (p \wedge (q \vee (\neg r)))$ are

p

q

r

$(\neg p)$

$(\neg r)$

$((\neg p) \wedge q)$

$(q \vee (\neg r))$

$(p \wedge (q \vee (\neg r)))$

$((\neg p) \wedge q) \implies (p \wedge (q \vee (\neg r)))$

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From \vdash to \models

- We have developed a calculus to determine whether $\phi_1, \phi_2, \dots, \phi_n \vdash \psi$ is valid.
 - ▶ That is, from the premises $\phi_1, \phi_2, \dots, \phi_n$, we can conclude ψ .
 - ▶ Our calculus is syntactic. It depends on the syntactic structures of $\phi_1, \phi_2, \dots, \phi_n$, and ψ .
- We will introduce another relation between premises $\phi_1, \phi_2, \dots, \phi_n$ and a conclusion ψ .

$$\phi_1, \phi_2, \dots, \phi_n \models \psi.$$

- ▶ The new relation is defined by 'truth values' of atomic formulae and the semantics of logical connectives.

Truth Values and Models

Definition

The set of truth values is $\{F, T\}$ where F represents 'false' and T represents 'true.'

Definition

A valuation or model of a formula ϕ is an assignment from each proposition atom in ϕ to a truth value.

Truth Values of Formulae

Definition

Given a valuation of a formula ϕ , the truth value of ϕ is defined inductively by the following truth tables:

ϕ	ψ	$\phi \wedge \psi$	ϕ	ψ	$\phi \vee \psi$
F	F	F	F	F	F
F	T	F	F	T	T
T	F	F	T	F	T
T	T	T	T	T	T

ϕ	ψ	$\phi \implies \psi$	ϕ	$\neg\phi$	\top	\perp
F	F	T	F	T	T	F
F	T	T	T	F		
T	F	F				
T	T	T				

Example

- $\phi \wedge \psi$ is T when ϕ and ψ are T.
- $\phi \vee \psi$ is F when ϕ or ψ is T.
- \perp is always F; \top is always T.
- $\phi \implies \psi$ is T when ϕ “implies” ψ .

Example

Consider the valuation $\{q \mapsto \text{T}, p \mapsto \text{F}, r \mapsto \text{F}\}$ of $(q \wedge p) \implies r$. What is the truth value of $(q \wedge p) \implies r$?

Proof.

Since the truth values of q and p are T and F respectively, the truth value of $q \wedge p$ is F. Moreover, the truth value of r is F. The truth value of $(q \wedge p) \implies r$ is T. □

Truth Tables for Formulae

- Given a formula ϕ with propositional atoms p_1, p_2, \dots, p_n , we can construct a truth table for ϕ by listing 2^n valuations of ϕ .

Example

Find the truth table for $(p \implies \neg q) \implies (q \vee \neg p)$.

Proof.

p	q	$\neg p$	$\neg q$	$p \implies \neg q$	$q \vee \neg p$	$(p \implies \neg q) \implies (q \vee \neg p)$
F	F	T	T	T	T	T
F	T	T	F	T	T	T
T	F	F	T	T	F	F
T	T	F	F	F	T	T



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Validity of Sequent Revisited

- Informally $\phi_1, \phi_2, \dots, \phi_n \vdash \psi$ is valid if we can derive ψ with assumptions $\phi_1, \phi_2, \dots, \phi_n$.
 - We have formalized “deriving ψ with assumptions $\phi_1, \phi_2, \dots, \phi_n$ ” by “constructing a proof in a formal calculus.”
- We can give another interpretation by valuations and truth values.
- Consider a valuation ν over all propositional atoms in $\phi_1, \phi_2, \dots, \phi_n, \psi$.
 - By “assumptions $\phi_1, \phi_2, \dots, \phi_n$,” we mean “ $\phi_1, \phi_2, \dots, \phi_n$ are T under the valuation ν .”
 - By “deriving ψ ,” we mean ψ is also T under the valuation ν .
- Hence, “we can derive ψ with assumptions $\phi_1, \phi_2, \dots, \phi_n$ ” actually means “if $\phi_1, \phi_2, \dots, \phi_n$ are T under a valuation, then ψ must be T under the same valuation.”

Semantic Entailment

Definition

We say

$$\phi_1, \phi_2, \dots, \phi_n \models \psi$$

holds if for every valuations where $\phi_1, \phi_2, \dots, \phi_n$ are T, ψ is also T. In this case, we also say $\phi_1, \phi_2, \dots, \phi_n$ semantically entail ψ .

• Examples

- ▶ $p \wedge q \models p$. For every valuation where $p \wedge q$ is T, p must be T. Hence $p \wedge q \models p$.
- ▶ $p \vee q \not\models q$. Consider the valuation $\{p \mapsto T, q \mapsto F\}$. We have $p \vee q$ is T but q is F. Hence $p \vee q \not\models q$.
- ▶ $\neg p, p \vee q \models q$. Consider any valuation where $\neg p$ and $p \vee q$ are T. Since $\neg p$ is T, p must be F under the valuation. Since p is F and $p \vee q$ is T, q must be T under the valuation. Hence $\neg p, p \vee q \models q$.

- The validity of $\phi_1, \phi_2, \dots, \phi_n \vdash \psi$ is defined by syntactic calculus. $\phi_1, \phi_2, \dots, \phi_n \models \psi$ is defined by truth tables. Do these two relations coincide?

Soundness Theorem for Propositional Logic

Theorem (Soundness)

Let $\phi_1, \phi_2, \dots, \phi_n$ and ψ be propositional logic formulae. If $\phi_1, \phi_2, \dots, \phi_n \vdash \psi$ is valid, then $\phi_1, \phi_2, \dots, \phi_n \models \psi$ holds.

Proof.

Consider the assertion $M(k)$:

“For all sequents $\phi_1, \phi_2, \dots, \phi_n \vdash \psi$ ($n \geq 0$) that have a proof of length k , then $\phi_1, \phi_2, \dots, \phi_n \models \psi$ holds.”

$k = 1$. The only possible proof is of the form

1 ϕ premise

This is the proof of $\phi \vdash \phi$. For every valuation such that ϕ is T, ϕ must be T. That is, $\phi \models \phi$.

Soundness Theorem for Propositional Logic

Proof (cont'd).

Assume $M(i)$ for $i < k$. Consider a proof of the form

1	ϕ_1	premise
2	ϕ_2	premise
	\vdots	
n	ϕ_n	premise
	\vdots	
k	ψ	justification

We have the following possible cases for justification:

- $\wedge i$. Then ψ is $\psi_1 \wedge \psi_2$. In order to apply $\wedge i$, ψ_1 and ψ_2 must appear in the proof. That is, we have $\phi_1, \phi_2, \dots, \phi_n \vdash \psi_1$ and $\phi_1, \phi_2, \dots, \phi_n \vdash \psi_2$. By inductive hypothesis, $\phi_1, \phi_2, \dots, \phi_n \models \psi_1$ and $\phi_1, \phi_2, \dots, \phi_n \models \psi_2$. Hence $\phi_1, \phi_2, \dots, \phi_n \models \psi_1 \wedge \psi_2$ (Why?).

Soundness Theorem for Propositional Logic

Proof (cont'd).

ii \vee e. Recall the proof rule for \vee e:

$$\frac{\eta_1 \vee \eta_2 \quad \boxed{\begin{array}{c} \eta_1 \\ \vdots \\ \psi \end{array}} \quad \boxed{\begin{array}{c} \eta_2 \\ \vdots \\ \psi \end{array}}}{\psi} \vee e$$

In order to apply \vee e, $\eta_1 \vee \eta_2$ must appear in the proof. We have $\phi_1, \phi_2, \dots, \phi_n \vdash \eta_1 \vee \eta_2$. By turning “assumptions” η_1 and η_2 to “premises,” we obtain proofs for $\phi_1, \phi_2, \dots, \phi_n, \eta_1 \vdash \psi$ and $\phi_1, \phi_2, \dots, \phi_n, \eta_2 \vdash \psi$. By inductive hypothesis, $\phi_1, \phi_2, \dots, \phi_n \models \eta_1 \vee \eta_2$, $\phi_1, \phi_2, \dots, \phi_n, \eta_1 \models \psi$, and $\phi_1, \phi_2, \dots, \phi_n, \eta_2 \models \psi$. Consider any valuation such that $\phi_1, \phi_2, \dots, \phi_n$ evaluates to T. $\eta_1 \vee \eta_2$ must be T. If η_1 is T under the valuation, ψ is also T (Why?). Similarly for η_2 is T. Thus $\phi_1, \phi_2, \dots, \phi_n \models \psi$.

Soundness Theorem for Propositional Logic

Proof (cont'd).

- iii Other cases are similar. Prove the case of \implies e to see if you understand the proof.



- The soundness theorem shows that our calculus does not go wrong.
- If there is a proof of a sequent, then the conclusion must be true for all valuations where all premises are true.
- The theorem also allows us to show the non-existence of proofs.
- Given a sequent $\phi_1, \phi_2, \dots, \phi_n \vdash \psi$, how do we prove there is no proof for the sequent?
 - Try to find a valuation where $\phi_1, \phi_2, \dots, \phi_n$ are T but ψ is F.

- 1 Natural Deduction
- 2 Propositional logic as a formal language
- 3 Semantics of propositional logic
 - The meaning of logical connectives
 - Soundness of Propositional Logic
 - Completeness of Propositional Logic

Completeness Theorem for Propositional Logic

- “ $\phi_1, \phi_2, \dots, \phi_n \vdash \psi$ is valid” and “ $\phi_1, \phi_2, \dots, \phi_n \models \psi$ holds” are very different.
 - ▶ “ $\phi_1, \phi_2, \dots, \phi_n \vdash \psi$ is valid” requires proof search (syntax);
 - ▶ “ $\phi_1, \phi_2, \dots, \phi_n \models \psi$ holds” requires a truth table (semantics).
- If “ $\phi_1, \phi_2, \dots, \phi_n \models \psi$ holds” implies “ $\phi_1, \phi_2, \dots, \phi_n \vdash \psi$ is valid,” then our natural deduction proof system is complete.
- The natural deduction proof system is both sound and complete.
That is
 $\phi_1, \phi_2, \dots, \phi_n \vdash \psi$ is valid iff $\phi_1, \phi_2, \dots, \phi_n \models \psi$ holds.

Completeness Theorem for Propositional Logic

- We will show the natural deduction proof system is complete.
- That is, if $\phi_1, \phi_2, \dots, \phi_n \models \psi$ holds, then there is a natural deduction proof for the sequent $\phi_1, \phi_2, \dots, \phi_n \vdash \psi$.
- Assume $\phi_1, \phi_2, \dots, \phi_n \models \psi$. We proceed in three steps:
 - ① $\models \phi_1 \implies (\phi_2 \implies (\dots (\phi_n \implies \psi)))$ holds;
 - ② $\vdash \phi_1 \implies (\phi_2 \implies (\dots (\phi_n \implies \psi)))$ is valid;
 - ③ $\phi_1, \phi_2, \dots, \phi_n \vdash \psi$ is valid.

Completeness Theorem for Propositional Logic (Step 1)

Lemma

If $\phi_1, \phi_2, \dots, \phi_n \models \psi$ holds, then $\models \phi_1 \implies (\phi_2 \implies (\dots (\phi_n \implies \psi)))$ holds.

Proof.

Suppose $\models \phi_1 \implies (\phi_2 \implies (\dots (\phi_n \implies \psi)))$ does not hold. Then there is valuation where $\phi_1, \phi_2, \dots, \phi_n$ is T but ψ is F. A contradiction to $\phi_1, \phi_2, \dots, \phi_n \models \psi$. □

Definition

Let ϕ be a propositional logic formula. We say ϕ is a tautology if $\models \phi$.

- A tautology is a propositional logic formula that evaluates to T for all of its valuations.

Completeness Theorem for Propositional Logic (Step 2)

- Our goal is to show the following theorem:

Theorem

If $\models \eta$ holds, then $\vdash \eta$ is valid.

- Similar to tautologies, we introduce the following definition:

Definition

Let ϕ be a propositional logic formula. We say ϕ is a theorem if $\vdash \phi$.

- Two types of theorems:
 - ▶ If $\vdash \phi$, ϕ is a theorem proved by the natural deduction proof system.
 - ▶ The soundness theorem for propositional logic is another type of theorem proved by mathematical reasoning (less formally).

Completeness Theorem for Propositional Logic (Step 2)

Proposition

Let ϕ be a formula with propositional atoms p_1, p_2, \dots, p_n . Let I be a line in ϕ 's truth table. For all $1 \leq i \leq n$, let \hat{p}_i be p_i if p_i is T in I ; otherwise \hat{p}_i is $\neg p_i$. Then

- ① $\hat{p}_1, \hat{p}_2, \dots, \hat{p}_n \vdash \phi$ is valid if the entry for ϕ at I is T;
- ② $\hat{p}_1, \hat{p}_2, \dots, \hat{p}_n \vdash \neg\phi$ is valid if the entry for ϕ at I is F.

Proof.

We prove by induction on the height of the parse tree of ϕ .

- ϕ is a propositional atom p . Then $p \vdash p$ or $\neg p \vdash \neg p$ have one-line proof.
- ϕ is $\neg\phi_1$.
 - ▶ If ϕ is T at I . Then ϕ_1 is F. By IH, $\hat{p}_1, \hat{p}_2, \dots, \hat{p}_n \vdash \neg\phi_1 (\equiv \phi)$.
 - ▶ If ϕ is F at I . Then ϕ_1 is T. By IH, $\hat{p}_1, \hat{p}_2, \dots, \hat{p}_n \vdash \phi_1$. Using $\neg\text{-}i$, we have $\hat{p}_1, \hat{p}_2, \dots, \hat{p}_n \vdash \neg\neg\phi_1 (\equiv \neg\phi)$.

Completeness Theorem for Propositional Logic (Step 2)

Proof (cont'd).

- ϕ is $\phi_1 \implies \phi_2$.
 - ▶ If ϕ is F at I , then ϕ_1 is T and ϕ_2 is F at I . By IH, $\hat{p}_1, \hat{p}_2, \dots, \hat{p}_n \vdash \phi_1$ and $\hat{p}_1, \hat{p}_2, \dots, \hat{p}_n \vdash \neg\phi_2$. Consider

1	$\phi_1 \implies \phi_2$	assumption]
\vdots			
i	ϕ_1	IH	
i + 1	ϕ_2	\implies e i, 1	
\vdots			
j	$\neg\phi_2$	IH	
j + 1	\perp	\neg e i+1, j	
j + 2	$\neg(\phi_1 \implies \phi_2)$	\neg i 1-(j+1)	

Completeness Theorem for Propositional Logic (Step 2)

Proof (cont'd).

- ϕ is $\phi_1 \implies \phi_2$.
 - ▶ If ϕ is T at I , we have three subcases. Consider the case where ϕ_1 and ϕ_2 are F at I . Then

1	ϕ_1	assumption]
	\vdots		
i	$\neg\phi_1$	IH	
i + 1	\perp	\neg e 1, i	
i + 2	ϕ_2	\perp e (i+1)	
i + 3	$\phi_1 \implies \phi_2$	\implies i 1-(i+2)	

The other two subcases are simple exercises.

Completeness Theorem for Propositional Logic (Step 2)

Proof (cont'd).

- ϕ is $\phi_1 \wedge \phi_2$.
 - ▶ If ϕ is T at I , then ϕ_1 and ϕ_2 are T at I . By IH, we have $\hat{p}_1, \hat{p}_2, \dots, \hat{p}_n \vdash \phi_1$ and $\hat{p}_1, \hat{p}_2, \dots, \hat{p}_n \vdash \phi_2$. Using \wedge i, we have $\hat{p}_1, \hat{p}_2, \dots, \hat{p}_n \vdash \phi_1 \wedge \phi_2$.
 - ▶ If ϕ is F at I , there are three subcases. Consider the subcase where ϕ_1 and ϕ_2 are F at I . Then

1	$\phi_1 \wedge \phi_2$	assumption]
2	ϕ_1	\wedge e 1	
	\vdots		
i	$\neg\phi_1$	IH	
i + 1	\perp	\neg e 2, i	
i + 2	$\neg(\phi_1 \wedge \phi_2)$	\neg i 1-(i+1)	

The other two subcases are simple exercises.

Completeness Theorem for Propositional Logic (Step 2)

Proof.

- ϕ is $\phi_1 \vee \phi_2$.

- ▶ If ϕ is F at I , then ϕ_1 and ϕ_2 are F at I . Then

1	$\phi_1 \vee \phi_2$	assumption]]
2	ϕ_1	assumption		
\vdots				
i	$\neg\phi_1$	IH]]
$i + 1$	\perp	\neg e 2, i		
$i + 2$	ϕ_2	assumption		
\vdots]]
j	$\neg\phi_2$	IH		
$j + 1$	\perp	\neg e $i+2$, j		
$j + 2$	\perp	\vee e 2-($i+1$), ($i+2$)-($j+1$)]]
$j + 3$	$\neg(\phi_1 \vee \phi_2)$	\neg i 1-($j+2$)		

- ▶ If ϕ is T at I , there are three subcases. All of them are simple exercises.



Completeness Theorem for Propositional Logic (Step 2)

Theorem

If ϕ is a tautology, then ϕ is a theorem.

Proof.

Let ϕ have propositional atoms p_1, p_2, \dots, p_n . Since ϕ is a tautology, each line in ϕ 's truth table is T. By the above proposition, we have the following 2^n proofs for ϕ :

$$\begin{array}{rcl} \neg p_1, \neg p_2, \dots, \neg p_n & \vdash & \phi \\ p_1, \neg p_2, \dots, \neg p_n & \vdash & \phi \\ \neg p_1, p_2, \dots, \neg p_n & \vdash & \phi \\ & \vdots & \\ p_1, p_2, \dots, p_n & \vdash & \phi \end{array}$$

We apply the rule LEM and the \vee rule to obtain a proof for $\vdash \phi$. (See the following example.) □

Completeness Theorem for Propositional Logic (Step 2)

Example

Observe that $\models p \implies (q \implies p)$. Prove $\vdash p \implies (q \implies p)$.

Proof.

1	$p \vee \neg p$	LEM	
2	p	assumption	
3	$q \vee \neg q$	LEM	
4	q	assumption	
	\vdots		
i	$p \implies (q \implies p)$	$p, q \vdash p \implies (q \implies p)$	
i + 1	$\neg q$	assumption	
	\vdots		
j	$p \implies (q \implies p)$	$p, \neg q \vdash p \implies (q \implies p)$	
j + 1	$p \implies (q \implies p)$	\vee 3, 4-i, (i+1)-j	
j + 2	$\neg p$	assumption	
j + 3	$q \vee \neg q$	LEM	
j + 4	q	assumption	
	\vdots		
k	$p \implies (q \implies p)$	$\neg p, q \vdash p \implies (q \implies p)$	
k + 1	$\neg q$	assumption	
	\vdots		
l	$p \implies (q \implies p)$	$\neg p, \neg q \vdash p \implies (q \implies p)$	
l + 1	$p \implies (q \implies p)$	\vee (j+3), (j+4)-k, (k+1)-l	
l + 2	$p \implies (q \implies p)$	\vee 1, 2-(j+1), (j+2)-(l+1)	

Completeness Theorem for Propositional Logic (Step 3)

Lemma

If $\phi_1 \implies (\phi_2 \implies (\dots(\phi_n \implies \psi)))$ is a theorem, then $\phi_1, \phi_2, \dots, \phi_n \vdash \psi$ is valid.

Proof.

Consider

1	ϕ_1	premise
2	ϕ_2	premise
	\vdots	
n	ϕ_n	premise
	\vdots	
i	$\phi_1 \implies (\phi_2 \implies (\dots(\phi_n \implies \psi)))$	theorem
i + 1	$\phi_2 \implies (\dots(\phi_n \implies \psi))$	\implies e 1, i
i + 2	$\phi_3 \implies (\dots(\phi_n \implies \psi))$	\implies e 2, (i+1)
	\vdots	
i + n - 1	$\phi_n \implies \psi$	\implies e (n-1), (i+n-2)
i + n	ψ	\implies e n, (i+n-1)

