Advanced Macroeconomics I/Economic Dynamics

Clemens Knoppe

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Contents

This script accompanies the lecture Economic Dynamics (Advanced Macroeconomic I). Mainly, it provides a brief recap of the theoretical contents of the tutorials. There is no guarantee of completeness or even correctness at all times.[1] The main reference to the course is the textbook by Gandolfo (2009), which most of the materials are actually taken from. However, as a brief overview and recap of the most important solution methods, this will hopefully be useful to some of you. As it is meant to be a brief overview of the solution techniques, there are no general proofs, but only the "cookbook" style recipes on how to solve certain classes of problems.

1 Linear difference equations

1.1 General form of linear difference equations

Difference equations describe how a variable "updates" its own state depending on its own past and, possibly, other factors.[2] These updates happen in *discrete* time, i.e. time progresses in steps from e.g. $t-1$ to t and then to $t+1$ etc. Essentially, these equations describe autoregressive (AR) processes of the state variable. This section explains the solution methods of linear difference equations.
The general form of such equations, where we set the leading coefficient $b_0 = 1$,[3] is:

$$\begin{aligned} y_t + b_1 y_{t-1} + \cdots + b_{n-1} y_{t-n+1} + b_n y_{t-n} &= g(t) \\ \text{or equivalently: } y_{t+n} + b_1 y_{t+n-1} + \cdots + b_{n-1} y_{t+1} + b_n y_t &= g(t) \end{aligned} \tag{1.1}$$

The highest lag in the equation determines the *order* of the equation. I.e. if y_t depends on its own past values as far back as $t-n$, it is an n-th order difference equation.
In this form, we have all the parts that include the state variable at any point in time, i.e. the endogenous parts, on the left-hand side (LHS) of the equation. On the right-hand side (RHS), we summarised everything else, i.e. the exogenous parts in the function $g(t)$.[4] It can be a constant, a function of time (e.g. economic growth), any exogenous sequence, e.g. a stochastic process or any other imaginable exogenous factor. If $g(t) = 0$, i.e. there are only endogenous factors, the equation is called homogeneous; it is called heterogeneous if $g(t) \neq 0$. Generally, the homogeneous part of the equation determines the dynamics, while the exogenous factors are relevant for the steady state (equilibrium) of the system.
To solve linear difference equations, we can apply a general scheme, that follows three main steps: (1) solve the homogeneous part of the equation to analyse dynamics and stability conditions; (2) determine the particular solution to find the steady state; and (3) combine both, apply boundary conditions, and determine the general solution.

1.2 Homogeneous solution

1.2.1 General form

Even if the equation is heterogeneous, we determine the dynamics using the homogeneous part, by setting $g(t) = 0$. It is the homogeneous part that defines the dynamic behaviour of the model, so we disregard the exogenous factors in the first step.
The final form of the homogeneous solution that we aim for is

$$\boxed{y_t = A_1 \lambda_1^t + A_2 \lambda_2^t + \cdots + A_n \lambda_n^t} \tag{1.2}$$

where λ_i are the roots of the characteristic equation (see below; also referred to as eigenvalues), and A_i are arbitrary constants that we keep undefined for now.
As a solution approach, insert $y_t \sim \lambda^t$ into the homogeneous equation (and $y_{t-1} \sim \lambda^{t-1}$, etc.):

$$\begin{aligned} &\lambda^t + b_1 \lambda^{t-1} + \cdots + b_{n-1} \lambda^{t-n+1} + b_n \lambda^{t-n} = 0 \\ \Leftrightarrow\ &\lambda^{t-n} \left(\lambda^n + b_1 \lambda^{n-1} + \cdots + b_{n-1} \lambda^1 + b_n \right) = 0 \end{aligned} \tag{1.3}$$

To determine the values of $\lambda_{1,\ldots,n}$, we find the roots of the characteristic equation, i.e. the values of λ for which the equation

$$\lambda^n + b_1 \lambda^{n-1} + \cdots + b_{n-1} \lambda^1 + b_n = 0 \tag{1.4}$$

holds. For an nth order equation, we expect to find n roots, which we can insert into the general form of our homogeneous solution (equation 1.2).

[1] If you find errors, please let us know, so we can correct them.
[2] Another way to express the same concept is that the *change* in one variable depends on its current state and, possibly, past states.
[3] In case the equation is not in this form yet, we can divide both sides of the equation by the coefficient to achieve this form. Due to the linearity, it does not change the general form of the equation
[4] In this general form, we indicate it as a function of time, to account for the possibility that these do not have to be constants in all cases.

1.2.2 Complex roots

The roots can be complex numbers. Complex roots always appear as pairs of complex conjugates, i.e. $\lambda = \alpha \pm \theta i$, where $i = \sqrt{-1}$ is the imaginary unit. In that case, the basic form of the general solution of a second-order system with complex conjugate roots of the characteristic equation is

$$y_t = r^t \left[B_1 \cos(\omega t) + B_2 \sin(\omega t)\right] \tag{1.5}$$

where r is the modulus, or absolute value of the number (can also be interpreted as the length of the vector in the two-dimensional complex plane, that λ represents), ω is the angle between λ, represented as a vector, and the x-axis (abscissa). See Figure (1) for a visual explanation of the concepts.

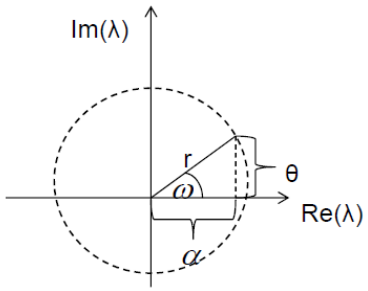


Figure 1: Representing complex eigenvalues with trigonometric functions. α is the magnitude of the real part of the eigenvalue, θ is the imaginary part. r is the modulus (absolute value, or length, if we interpret λ as a vector) of λ and ω is the angle between the real axis and λ.

r is calculated with the Pythagorean theorem:

$$r = \sqrt{\alpha^2 + \theta^2} \tag{1.6}$$

And we can find the angle ω by the following trigonometric identities:

$$r\ \cos\omega = \alpha \qquad\qquad r\ \sin\omega = \theta \tag{1.7}$$

Due to the trigonometric functions in the solution, the dynamics will display cyclical behaviour. I.e. the time series of y will look like a sinusoidal "wave".

1.2.3 Stability

The system is called "stable" if it converges to a finite value as $t \to \infty$. The conditions for stability are quite simple: clearly $A_i \lambda_i^t$ can only converge to a finite value, if $|\lambda_i| < 1\ \forall i$. Hence, the stability condition for system with real roots is that all roots are smaller than 1 in absolute value. In the case of complex roots, they must lie within the unit circle, such that $r < 1$.
Simplified stability conditions based on the coefficients $\{b_i\}_{i=1}^n$:

$n = 1 \Rightarrow |-b_1| < 1$
$n = 2 \Rightarrow 1 + b_1 + b_2 > 0 \quad \wedge \quad 1 - b_2 > 0 \quad \wedge \quad 1 - b_1 + b_2 > 0$
$n = 3 \Rightarrow 1 + b_1 + b_2 + b_3 > 0 \quad \wedge \quad 1 - b_1 + b_2 - b_3 > 0 \quad \wedge \quad 1 - b_2 + b_1 b_3 - b_3^2 > 0 \quad \wedge \quad b_2 < 3$
$n > 3 \Rightarrow$ see the main reference Gandolfo (2009).

Why imaginary eigenvalues relate to cyclical behaviour

Consider the first-order system of difference equations of two state variables (i.e. a 2-dimensional system):

$$\mathbf{y}_{t+1} = \mathbf{A}\mathbf{y}_t$$
$$\Leftrightarrow \begin{pmatrix} y_{1,t+1} \\ y_{2,t+1} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix}$$

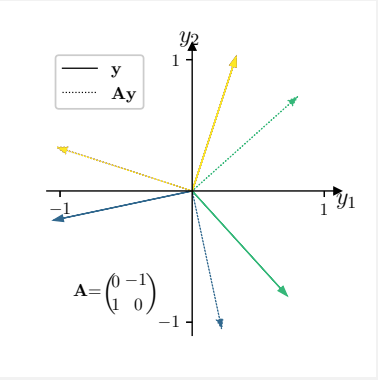
The matrix essentially acts as a function, which takes the vector $\mathbf{y}_t$ as an input/argument, and returns an updated state vector $\mathbf{y}_{t+1}$.
The definition of eigenvectors and eigenvalues is stated by the equation

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$$

i.e. an eigenvector $\mathbf{v}$ of a matrix $\mathbf{A}$ is one that does not change direction upon being multiplied with the matrix, but it is only scaled by a factor λ. To put it into geometric terms: you can multiply any vector, here 2-d, with the matrix and obtain a different vector. If you represent vectors as arrows in your coordinate system, almost all of them will change direction as a result. Those that remain on the same span, i.e. they are on the same line, only stretched, squeezed, or reversed, are eigenvectors. The factor by which they are scaled are the corresponding eigenvalues.
In some cases, the matrix rotates everything in space though. Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

as one example of such a rotational matrix. Any vector multiplied with that matrix will be rotated counter-clockwise by 90°, as depicted in the figure to the right. You can furthermore verify this by applying the matrix to any other vector. Therefore, there are no real-valued vectors that remain on the same span. Eigenvectors can only be found in the imaginary dimensions instead, and they will be scaled along those dimensions as well. Recall that the imaginary dimensions are orthogonal to the real dimensions. This is why we can find stable axes, i.e. axes along which the matrix only scales the space, but does not rotate, in those imaginary dimensions when the matrix defines a rotation in the real dimensions.



If you track any point in the phase diagram of such a system (i.e. starting in some initial condition, moving forward in time by using the system matrix to update the state), it will circle around the fixed point, either converging towards it, diverging away from it, or staying in a stable orbit around it, depending on the real part of the eigenvalue. If you track the behaviour over time in one of the two dimensions, i.e. of one of the state variables, those circles now relate to a cyclical movement around the fixed point.

1.3 Particular solution

1.3.1 Approach

The particular solution determines the steady state of the system. The steady state, also called a fixed point, or fix point, is defined as a state, where the state variables do not change anymore: $y_{t+1} = y_t = \bar{y}$, and is usually interpreted as the equilibrium.
The general solution approach is to try inserting something of a similar functional form as the function $g(t)$ and solve for the unknown parameters, e.g.

- $g(t) = const. \rightarrow$ try: $\bar{y} = \mu$.
- $g(t) = a_0 + a_1 t$ (linear function of time) $\rightarrow$ try: $\bar{y} = \alpha + \beta t$, where α and β are model parameters that we need to determine.
- $g(t) = Bd^t$ (exponential function of time) $\rightarrow$ try: $\bar{y} = Cd^t$.

You can think of exogenous sequences that are functions of time as a type of "trend", while the dynamics define fluctuations around that trend. In an economic context that could be a long-run trend growth of GDP, while the fluctuations are business cycles. In the case that $g(t) = x_t$, where x_t is any *exogenous* variable that also changes over time, we will have to apply lag operators to solve behaviour of x over time and determine the steady state in dependence of that value.

1.3.2 $g(t)$ is constant

The simplest case is that $g(t) = \text{const.}$, so we insert $\bar{y} \sim \mu$ for all realisations of y:

$$\mu + b_1\mu + \cdots + b_n\mu = g(t) \tag{1.8}$$

$$\bar{y} = \mu = \frac{g(t)}{1 + \sum_{i=1}^{n} b_i} \tag{1.9}$$

1.3.3 $g(t)$ is linear in t

If the particular part of the equation is a linear function of time, i.e. of the general form $g(t) = a_0 + a_1 t$, try inserting $\bar{y}_t \sim \alpha + \beta t$,[5] where α, β are undefined constants:

$$\alpha + \beta t + b_1(\alpha + \beta(t-1)) + \cdots + b_n(\alpha + \beta(t-n)) = a_0 + a_1 t \tag{1.10}$$

In the case of a second-order equation, by taking all parts that depend on t on one side and the rest on the other side, we obtain

$$\alpha(1 + b_1) - a_0 - b_1\beta = t(a_1 - (1 + b_1)\beta) \tag{1.11}$$

There are only constant (parameters) on the LHS, while the RHS depends on t. Therefore, equality can only be achieved, if both sides equal 0. by rearranging, we can find the model parameters α and β with the equations

$$\alpha(1 + b_1) - a_0 - b_1\beta = 0 \tag{1.12}$$

$$a_1 - (1 + b_1)\beta = 0 \tag{1.13}$$

1.3.4 $g(t)$ is exponential in t

If the particular part of the equation is an exponential function of time, i.e. it has the general form $g(t) = Bd^t$, try inserting $\bar{y}_t \sim Cd^t$, where we leave C as an undefined constant for now.
In the case of a first-order equation, we then obtain

$$\begin{aligned} Cd^t + b_1 Cd^{t-1} &= Bd^t \\ \Leftrightarrow C &= \frac{Bd}{d + b_1} \\ \Rightarrow \bar{y}_t &= \frac{Bd}{d + b_1} d^t \end{aligned} \tag{1.14}$$

1.3.5 $g(t)$ is an exogenous, time-variant sequence

Relevant for PhD students only.
This case treats the cases where $g(t)$ is a variable, say x_t, that changes over time and has an effect on y, but does not depend on it, i.e. it is purely exogenous. In the context of economic models, this could be e.g. a policy variable. We cannot treat it as a constant, so we need to solve the behaviour of the variable over time in order to find the steady state.
In this case, we apply lag operators, which shift variables in time, i.e. $Ly_t = y_{t-1}$. We can calculate with those operators in the usual way, so that $L^2 y_t = LLy_t = Ly_{t-1} = y_{t-2}$, and $L^{-1} y_t = y_{t+1}$.
In the case of a first order equation of the form

$$y_t + b_1 y_{t-1} = x_t \tag{1.15}$$

with $\lambda = -b_1$, we can express it as

$$y_t(1 - \lambda L) = x_t \tag{1.16}$$

$$y_t = (1 - \lambda L)^{-1} x_t \tag{1.17}$$

The term $(1 - \lambda L)^{-1}$ can be expanded as an infinite sequence, but the direction depends on convergence. If the system is stable, i.e. $|\lambda| < 1$, the system will eventually converge from any value y_{t-i} to the steady state. Hence, we apply a *backward solution*. In the case that $|\lambda| > 1$, the system explodes to $\pm\infty$. However, in the opposite direction, coming from any future value y_{t+i}, the system will converge to the steady state. Hence, we can apply the so-called *forward solution*.

[5] Note that the steady state $\bar{y}_t$ is indexed with the time subscript t in this case because the steady state now changes over time.

$$(1-\lambda L)^{-1} = \begin{cases} \sum_{i=0}^{\infty} \lambda^i L^i & \text{, if } |\lambda| < 1 \text{ (Backward solution)} \\ -\sum_{i=1}^{\infty} \left(\frac{1}{\lambda}\right)^i L^{-i} & \text{, if } |\lambda| > 1 \text{ (Forward solution)} \end{cases}$$

$$\implies \bar{y}_t = \frac{x_t}{1-\lambda L} = \begin{cases} \sum_{i=0}^{\infty} \lambda^i L^i x_t = \sum_{i=0}^{\infty} \lambda^i x_{t-i} & \text{, if } |\lambda| < 1 \text{ (Backward solution)} \\ -\sum_{i=1}^{\infty} \left(\frac{1}{\lambda}\right)^i L^{-i} x_t = -\sum_{i=1}^{\infty} \left(\frac{1}{\lambda}\right)^i x_{t+i} & \text{, if } |\lambda| > 1 \text{ (Forward solution)} \end{cases}$$

Note the difference in the exponent of L: the backward solution considers an infinite sum of past values of x (positive exponents on the lag operator) to determine the steady state, while the forward solution is found by taking an infinite sum of future values of x (negative exponent on the lag operator)

1.4 General solution

Finally, we find the general solution by adding the dynamics (homogeneous solution) and the steady state (particular solution):

$$\boxed{y_t = \sum_{i=1}^{n} A_i \lambda_i^t + \bar{y}} \tag{1.18}$$

We can then solve for the constants A_i by using n boundary conditions, i.e. inserting known combinations of values of y and t. E.g. $(t^*, y^*) \to (0, y_0)$ etc. For a first-order equation, we find

$$y_0 = A\lambda^0 + \bar{y} \Leftrightarrow A = y_0 - \bar{y} \tag{1.19}$$

2 Systems of difference equations

2.1 General form

Typically, economic models have more than one endogenous variable. E.g. the baseline New Keynesian macroeconomic model:[6]

$$x_t = E_t x_{t+1} - \frac{1}{\sigma}(i_t - E_t \pi_{t+1}) + u_t \tag{2.1}$$
$$\pi_t = \beta E_t \pi_{t+1} + \kappa x_t + \nu_t \tag{2.2}$$
$$i_t = \delta_\pi \pi_t + \delta_x x_t + k_t \tag{2.3}$$

where eq. 2.1 is an IS curve, defined by its a negative relationship between the output gap x_t and nominal interest rates i_t adjusted for expected future inflation π_{t+1}; eq. 2.2 is the Phillip's curve, with the standard positive relationship between inflation π_t and the current output gap; and eq. 2.3 is a Taylor rule that defines interest rate setting by the central bank based on the deviations of inflation and output from their targets. u_t, ν_t, k_t are exogenous (stochastic, zero mean) shock processes, and σ, β, κ, $\delta_{\pi,x}$ are model parameters. We apply an expectations operator (E_t) on variable values from the future to indicate that due to the stochastic factors they are not known yet in the present, but that agents need to form an expectation about it. Expectations are usually assumed to be rational, hence the use of the mathematical (objective) expectations operator, but other types of expectation formation can be studied in this framework too.
In this model, we are dealing with several interdependent equations that describe a system in which several variables update their state depending on the current, last, or expected future state of a number of other model variables. As the equations are interdependent, they need to be solved as such, rather than independently. This is less complicated than it first seems, if we use some simple linear algebra.
To generalise, assume the following form for a system of first-order difference equations:

$$\begin{aligned} y_{1,t} &= a_{11} y_{1,t-1} + a_{12} y_{2,t-1} + \cdots + a_{1n} y_{n,t-1} + g_1(t) \\ y_{2,t} &= a_{21} y_{1,t-1} + a_{22} y_{2,t-1} + \cdots + a_{2n} y_{n,t-1} + g_2(t) \\ &\vdots \\ y_{n,t} &= a_{n1} y_{1,t-1} + a_{n2} y_{2,t-1} + \cdots + a_{nn} y_{n,t-1} + g_n(t) \end{aligned} \tag{2.4}$$

[6] See Advanced Macroeconomics II.

This is clearly equivalent to[7]

$$\begin{pmatrix} y_{1,t} \\ y_{2,t} \\ \vdots \\ y_{n,t} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} y_{1,t-1} \\ y_{2,t-1} \\ \vdots \\ y_{n,t-1} \end{pmatrix} + \begin{pmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{pmatrix} \tag{2.5}$$

and can be expressed much more compactly using matrix notation:

$$\mathbf{y}_t = \mathbf{A}\mathbf{y}_{t-1} + \mathbf{g}(\mathbf{t}) \tag{2.6}$$

where bold font indicates vectors and matrices, and $\mathbf{A}$ is the coefficient matrix.
We can also transform an n-th order difference equation into a system of n first-order difference equations. This is useful when n is large, which would make the solution of the difference equation complicated. It also makes implementations in the computer quite efficient, as matrix operations can easily be parallelised.
Consider the equation

$$y_t = c_1 y_{t-1} + c_2 y_{t-2} + \dots + c_n y_{t-n} = g(t) \tag{2.7}$$

This is equivalent to

$$\begin{pmatrix} y_t \\ y_{t-1} \\ y_{t-2} \\ \vdots \\ y_{t-n+2} \\ y_{t-n+1} \end{pmatrix} = \begin{pmatrix} c_1 & c_2 & c_3 & \dots & c_{n-1} & c_n \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix} \begin{pmatrix} y_{t-1} \\ y_{t-2} \\ y_{t-3} \\ \vdots \\ y_{t-n+1} \\ y_{t-n} \end{pmatrix} + g(t) \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \tag{2.8}$$

The first row of the coefficient matrix contains all the coefficients from the original difference equation. The other rows of the matrix encode the relationship between lags of different orders. Often we would introduce auxiliary variables to denote the lags of the original state variable, e.g. $z_t = y_{t-1}$ etc. Ultimately, it allows us to express the equation in the more compact form

$$\Leftrightarrow \mathbf{y}_t = \mathbf{C}\mathbf{y}_{t-1} + \mathbf{g}(\mathbf{t}) \tag{2.9}$$

The solution method of systems of difference equations is very similar to that of difference equations, and we can follow the same scheme: first, solve the homogeneous part of the equations to determine the model dynamics and stability; then calculate the particular solution to determine the steady state; and finally, put everything together and apply boundary conditions to find the general solution.

2.2 Homogeneous solution

To find the homogeneous solution, we now set the *vector* of constants to be a zero-vector: $\mathbf{g}(\mathbf{t}) = \mathbf{0}$. This is equivalent to setting the constant in a difference equation to zero. The remaining homogeneous equation describes the endogenous dynamics of the model.
The homogeneous solution has the final form

$$\boxed{\boldsymbol{y_t} = A_1 \boldsymbol{v^{(1)}} \lambda_1^t + A_2 \boldsymbol{v^{(2)}} \lambda_2^t + \dots + A_n \boldsymbol{v^{(n)}} \lambda_n^t} \tag{2.10}$$

This solution can be derived using a simple eigendecomposition of the coefficient matrix $\boldsymbol{A}$.[8] Consider the eigenvector and eigenvalue definition $\boldsymbol{A}\boldsymbol{v} = \lambda \boldsymbol{v}$. We can state this relationship for all eigenvector-eigenvalue pairs at once using a matrix $\boldsymbol{V}$, where each column represents an eigenvector, and the diagonal matrix of eigenvalues $\boldsymbol{\Lambda}$:

$$\boldsymbol{V} = \underbrace{\begin{pmatrix} \boldsymbol{v}^{(1)} & \boldsymbol{v}^{(2)} & \dots & \boldsymbol{v^{(n)}} \end{pmatrix}}_{\text{column vector } \boldsymbol{v^{(i)}} \text{ from } (\boldsymbol{A}-\lambda_i \boldsymbol{I})\boldsymbol{v^{(i)}}=\mathbf{0}} = \begin{pmatrix} v_1^{(1)} & v_1^{(2)} & \dots & v_1^{(n)} \\ v_2^{(1)} & v_2^{(2)} & \dots & v_2^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ v_n^{(1)} & v_n^{(2)} & \dots & v_n^{(n)} \end{pmatrix} \tag{2.11}$$

$$\boldsymbol{\Lambda} = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix} \tag{2.12}$$

[7] In case it is not immediately clear, pause for a moment and try to understand on both technical and intuitive levels, why they are equivalent.

[8] Provided that there are n distinct eigenvalues λ_i with $i \in \{1, 2, \dots, n\}$ for the $n \times n$ matrix $\boldsymbol{A}$. Check the slides or the main reference (Gandolfo, 2009) for solutions to coefficient matrices with repeated eigenvalues.

Now the eigenvector equation becomes

$$\begin{aligned} \boldsymbol{AV} &= \boldsymbol{V\Lambda} \\ \Leftrightarrow \boldsymbol{A} &= \boldsymbol{V\Lambda V^{-1}} \end{aligned} \tag{2.13}$$

Hence $\boldsymbol{y}_1 = \boldsymbol{V\Lambda V^{-1}y}_0$, and $\boldsymbol{y}_2 = \left(\boldsymbol{V\lambda V^{-1}}\right)\boldsymbol{y}_1 = \left(\boldsymbol{V\lambda V^{-1}}\right)^2\boldsymbol{y}_0$, etc. Exploiting the fact that $\left(\boldsymbol{V\Lambda V^{-1}}\right)^t = \boldsymbol{V\Lambda^t V^{-1}}$, we can re-write:

$$\boldsymbol{y}_t = \boldsymbol{V\Lambda^t V^{-1}y_0} = \boldsymbol{V\Lambda^t a} \tag{2.14}$$

with $\boldsymbol{V^{-1}y_0} = \boldsymbol{a}$ and

$$\boldsymbol{\Lambda^t} = \begin{pmatrix} \lambda_1^t & 0 & \dots & 0 \\ 0 & \lambda_2^t & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n^t \end{pmatrix}, \qquad \boldsymbol{a} = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{pmatrix} \tag{2.15}$$

Expressed as a system of equations:

$$\boldsymbol{y_t = V\Lambda^t a} \Longrightarrow \begin{cases} y_{1,t} &= A_1 v_1^{(1)}\lambda_1^t + A_2 v_1^{(2)}\lambda_2^t + \dots + A_n v_1^{(n)}\lambda_n^t \\ y_{2,t} &= A_1 v_2^{(1)}\lambda_1^t + A_2 v_2^{(2)}\lambda_2^t + \dots + A_n v_2^{(n)}\lambda_n^t \\ & \vdots \\ y_{n,t} &= A_1 v_n^{(1)}\lambda_1^t + A_2 v_n^{(2)}\lambda_2^t + \dots + A_n v_n^{(n)}\lambda_n^t \end{cases}, \tag{2.16}$$

Hence, by solving for eigenvalues and eigenvectors of the coefficient matrix, we can determine the dynamic behaviour of the system, depending on initial conditions.
The final solution approach is similar to the approach for single linear difference equations. We replace the vector $\mathbf{y}_t \sim \boldsymbol{v}\lambda^t$, to obtain

$$\boldsymbol{v}\lambda^t = \boldsymbol{Av}\lambda^{t-1} \Longleftrightarrow \lambda^{t-1}\left(\boldsymbol{Av} - \lambda\boldsymbol{v}\right) = 0 \Longleftrightarrow \lambda^{t-1}\left(\boldsymbol{Av} - \lambda\boldsymbol{Iv}\right) = \mathbf{0} \tag{2.17}$$

$$\Longleftrightarrow \lambda^{t-1}\left(\boldsymbol{A} - \lambda\boldsymbol{I}\right)\boldsymbol{v} = \mathbf{0} \xrightarrow{\lambda \neq 0} \left(\boldsymbol{A} - \lambda\boldsymbol{I}\right)\boldsymbol{v} = \mathbf{0}$$

$$\xrightarrow{\boldsymbol{v} \neq \mathbf{0}} \det\left(\boldsymbol{A} - \lambda\boldsymbol{I}\right) = \mathbf{0} \tag{2.18}$$

We find that the scalar λ and vector $\mathbf{v}$ that solve the system are an eigenvalue-eigenvector pair of the coefficient matrix $\mathbf{A}$. $\det\left(\boldsymbol{A} - \lambda\boldsymbol{I}\right) = \mathbf{0}$ then yields the characteristic equation:

$$\det\left[\begin{pmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} - \lambda \end{pmatrix}\right] = 0 \tag{2.19}$$

$$\Longleftrightarrow \lambda^n + b_1\lambda^{n-1} + \dots + b_{n-1}\lambda + b_n = 0 \tag{2.20}$$

with coefficient: $b_i = (-1)^i \times$ **sum of all *i*th order principal minors of $\boldsymbol{A}$**

$$\xrightarrow{\text{in particular:}} b_1 = -\text{tr}(\boldsymbol{A}), \quad b_n = (-1)^n \det(\boldsymbol{A})$$

The eigendecomposition illustrates the stability conditions again: we raise the eigenvalues to the power of t, hence if $|\lambda_i| < 1$, $\lim_{t\to\infty} \lambda_i^t = 0$. If this condition holds for all eigenvalues, the vector that describes the homogeneous solution eventually collapses to the zero vector $\mathbf{0}$ and the system converges to its steady state (particular solution). Alternatively, the same simplified stability conditions based on coefficients $\{b_i\}_{i=1}^n$ as for linear difference equations can be applied if the system is sufficiently small.

2.3 Particular solution

The particular solution determines the steady state of the system. In the case of a system of equations, this implies finding a vector with a value for each state variable, at which none of those variables will diverge from its stable trend path/steady state value anymore. The solution method is equivalent to difference equations. Insert a functional form similar to $\mathbf{g}(\mathbf{t})$ as a solution for $\bar{\mathbf{y}}$.

E.g. if $\boldsymbol{g(t) = k} = const.vector \Longrightarrow$ try $\bar{\boldsymbol{y}} = \boldsymbol{\mu} = const.vector$
$\Longrightarrow \boldsymbol{\mu = A\mu + k} \qquad \Longrightarrow \boldsymbol{\mu} = (\boldsymbol{I - A})^{-1}\boldsymbol{k}$

For other types of non-homogeneous parts $\mathbf{g(t)}$, the same principles apply. See also the main reference Gandolfo (2009).

2.4 General solution

The general solution is the sum of the homogeneous solution and the general solution:

$$\boxed{\boldsymbol{y_t} = A_1\boldsymbol{v^{(1)}}\lambda_1^t + A_2\boldsymbol{v^{(2)}}\lambda_2^t + \cdots + A_n\boldsymbol{v^{(n)}}\lambda_n^t + \bar{\boldsymbol{y}}} \tag{2.21}$$

Use n boundary conditions (e.g. initial conditions) (e.g. $(t_1^*, \bar{\boldsymbol{y}}_{\boldsymbol{1}}^*), (t_2^*, \bar{\boldsymbol{y}}_{\boldsymbol{2}}^*), ..., (t_n^*, \bar{\boldsymbol{y}}_{\boldsymbol{n}}^*)$) to fix the n arbitrary constants $A_1, A_2, \ldots, A_n$

3 Linear differential equations

3.1 General form

Differential equations, as opposed to *difference* equations, model processes in continuous time. Hence, instead of the difference of a variable from one period to the next, depending on its current value, the *rate of change*, i.e. the derivative, now depends on the current value of this variable, as time flows continuously, rather than moving forward in discrete steps.
Recall the standard form of a first-order homogeneous difference equation $dy_t \equiv y_{t+dt} - y_t = ay_t dt \Leftrightarrow y_{t+dt} = (1+a)y_t dt$. In general, we choose the unit of time, such that $dt = 1$, leading to the equation $y_{t+1} = by_t$, with $b = 1 + a$. Most importantly, we choose $dt > 0$, i.e. there is a discrete jump in time, from one moment to another. Differential equations, on the other hand, model systems in continuous time, i.e. $dt \to 0$. In that case, we obtain $dy_t = ay_t dt \Leftrightarrow \frac{dy}{dt} = y' = ay_t$. Instead of summing over differences in order to obtain the value of y in time t, we now have to integrate the rate of change over time, up to point t, to calculate y_t, i.e. $y(t) = y(0) + \int_0^t y'(\tau)d\tau$. In the case of linear equations we will not have to calculate integrals explicitly though, as we have a standard solution approach.
In the most general form, we can have an arbitrary order of differentiation:

$$y^{(n)} + b_1 y^{(n-1)} + \cdots + b_{n-2}y'' + b_{n-1}y' + b_n y = g(t) \tag{3.1}$$

where we set the leading coefficient $b_0 = 1$, and $y^{(n)}$ represents the n-th derivative of y. $g(t)$ is an exogenous process (possible a function of time, but could be a constant too). If $g(t) = 0$, we call it a homogeneous equation, and otherwise it is a heterogeneous equation.
It should come to no surprise that these equations are analysed and solved in a very similar way to difference equations. After all, they are merely the limiting case that we obtain by letting the difference in time from one period to the next approach zero asymptotically.

3.2 Homogeneous solution

The final solution of the homogeneous part of the equation has the form

$$\boxed{y_t = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} + \cdots + A_{n-1}e^{\lambda_{n-1} t} + A_n e^{\lambda_n t}} \tag{3.2}$$

where λ_i is the ith eigenvalue of the equation, i.e. root of the characteristic equation.
The way we approach the equation to obtain the final solution is similar to what we did with difference equations. Only now, we insert $y_t \sim e^{\lambda t}$ into the homogeneous equation ($g(t) = 0$):

$$\lambda^n e^{\lambda t} + b_1\lambda^{n-1}e^{\lambda t} + \cdots + b_{n-1}\lambda e^{\lambda t} + b_n e^{\lambda t} = 0 \tag{3.3}$$

We can factor out $e^{\lambda t}$:

$$\Leftrightarrow e^{\lambda t}\left(\lambda^n + b_1\lambda^{n-1} + \cdots + b_{n-1}\lambda + b_n\right) = 0 \tag{3.4}$$

And we obtain the characteristic equation:

$$\begin{aligned} &\lambda^n + b_1\lambda^{n-1} + \cdots + b_{n-1}\lambda + b_n = 0 \\ \Longleftrightarrow\ &(\lambda - \lambda_1)(\lambda - \lambda_2)\ldots(\lambda - \lambda_n) = 0, \qquad \lambda_i \in \mathbb{C} \end{aligned} \tag{3.5}$$

Exponential functions as solutions to linear differential equations

In this box, two explanations for the emergence of exponential functions in the solutions to linear differential equations are provided. The first one comes from compounding differences over smaller and smaller time increments, taking the size of the increments to the limit of $dt \to 0$. The second is the formal solution that we obtain by simply integrating a first-order differential equation.

Compounding changes as $dt \to 0$ Differential equations are the continuous-time equivalent of difference equations, which follow the scheme $y_{t+1} = (1+a)y_t$. We obtained this form by defining the difference in a variable of interest, i.e. the change of the state variable from one period to the next, as a function of the current state of that variable. Now consider taking a smaller time increment, e.g. cut it in half and apply the change at intermediate steps: $y_{t+1} = \left(1+\frac{a}{2}\right)^2 y_t$. More generally, for a division into any number m intermediate intervals and applying any number of time unit steps, we can express this as: $y_t = \left(1+\frac{a}{m}\right)^{mt} y_0$. By taking the limit of infinitely many infinitesimally small time increments, this yields $\lim_{m\to\infty}\left(1+\frac{a}{m}\right)^{mt} = \sum_{n=0}^{\infty}\frac{(at)^n}{n!} = e^{at}$. Note how the change from one period to the next is not the same, even though our parameter a is unchanged. That is because the rate of change, i.e. the derivative also changes continuously as the state variable changes its state. I.e. after one infinitesimally small time increment, the state variable has changed its value by a an infinitesimal amount too, meaning the rate of change adjusts as well.

Integrating differential equations The standard procedure to solve differential equations is to integrate them over time. Since we have a function for the rate of change of a state variable y over time, the integral of this function yields the time paths of the variable: $y(t) = y(0) + \int_0^t \frac{\partial y}{\partial \tau} d\tau$. In this class, you usually do not have to integrate linear differential equations by hand to derive the solution. The general approach we teach will always suffice for tutorial and exam questions. However, for the sake of completeness, the full solution of a first-order homogeneous equations would be done as follows:
We start with the equation

$$y' = \frac{dy}{dt} = ay \Leftrightarrow \frac{dy}{y} = adt$$

The value of $y(t)$, for any given time t can now be derived by simple integration:

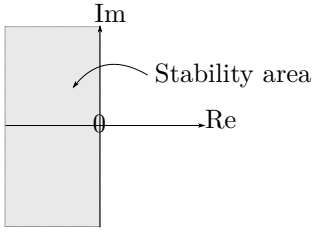
$$\int \frac{dy}{y} dt = a \int dt$$
$$\Leftrightarrow \ln(y) = at + c$$

where c is an arbitrary constant of integration. Finally, we apply the inverse of the logarithm, i.e. the exponential function:

$$y(t) = e^{at+c} = e^c e^{at} = Ae^{at}$$

with $A = e^c$, the undetermined constant, which is to be determined by use of boundary conditions, e.g. initial values (since we do not account for $y(0)$ in the integration step). Moreover, it is straightforward to verify that this solution satisfies our problem. If $y(t) = Ae^{at}$, then clearly $y'(t) = aAe^{at} = ay(t)$.

Stability conditions are that the real parts of all eigenvalues are negative: $Re(\lambda_i) < 0$. If that is fulfilled, $\lim_{t\to\infty} A_i e^{\lambda_i t} = 0$. Hence, the system will converge towards the steady state (particular solution): $\lim_{t\to\infty} y_t = \bar{y}$.



If the order of the equation is not too high and we are only interested in stability, the following conditions based on the coefficients $\{b_i\}_{i=1}^n$ of the characteristic equation can be useful:

$n = 1 \Rightarrow b_1 > 0$
$n = 2 \Rightarrow b_1 > 0 \quad \wedge \quad b_2 > 0$
$n = 3 \Rightarrow b_1 > 0 \quad \wedge \quad b_1 b_2 - b_0 b_3 > 0 \quad \wedge \quad b_3 > 0$
$n > 3 \Rightarrow$ see the main reference Gandolfo (2009).

Complex exponents, DeMoivre's Theorem, and Euler's formula

When dealing with continuous-time systems, we do not need to transform complex numbers with trigonometric functions as we did to solve discrete-time systems. We can simply raise e to the power of a complex number, and it generates cyclical behaviour. We can show however, that trigonometric functions arise naturally in this case.

Recall the Taylor series of the exponential function e:[†]

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

Since we can separate the real and the imaginary part of an exponent by setting $e^{a+ix} = e^a e^{ix}$, we now focus only on the purely imaginary exponential:

$$\begin{aligned} e^{ix} &= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \dots \\ &= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} - i\frac{x^7}{7!} + \dots \\ &= \underbrace{\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right)}_{\cos(x)} + i\underbrace{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right)}_{\sin(x)} \\ &= \cos(x) + i\sin(x) \end{aligned}$$

This identity can furthermore be applied to derive DeMoivre's Theorem:

$$\begin{aligned} (\cos(x) + i\sin(x))^n &= \left(e^{ix}\right)^n \\ &= e^{inx} \\ &= \cos(nx) + i\sin(nx) \end{aligned}$$

And, on a side note, if we set $x = \pi$, we retrieve Euler's formula:

$$\begin{aligned} e^{i\pi} &= \underbrace{\cos(\pi)}_{=-1} + i\underbrace{\sin(\pi)}_{=0} \\ \Leftrightarrow e^{i\pi} + 1 &= 0 \end{aligned}$$

[†]Note how this definition naturally yields the property $\frac{\partial}{\partial x}e^x = e^x$.

3.3 Particular solution

In order to determine the steady state, try a functional form similar to $g(t)$ as solution.

3.3.1 $g(t)$ is a constant

If the heterogeneous part is a constant, we insert a constant value $\mu \sim y(t)$. $\mu' = 0$, so we obtain

$$b_n\mu = k \Longleftrightarrow \bar{y} = \mu = \frac{k}{b_n} \tag{3.6}$$

3.3.2 $g\,(t)$ is an exponential function

If $g(t)$ is an exponential function, i.e. of form $g\,(t) = Be^{at}$, try $\bar{y}\,(t) = Ce^{at}$, where C is a yet undetermined constant. $\frac{d}{dt}Ce^{at} = aCe^{at}$, hence

$$\begin{aligned} y' + b_1 y &= Be^{at} \\ \Longleftrightarrow aCe^{at} + b_1 Ce^{at} &= Be^{at} \\ \Longleftrightarrow C &= \frac{B}{a + b_1} \end{aligned} \tag{3.7}$$

3.3.3 $g(t)$ is a polynomial function

If $g(t)$ is a polynomial function of degree m, try a polynomial function of the same degree and determine the coefficients. E.g. a first-degree polynomial: $g(t) = c_0 + c_1 t \Rightarrow$ try $\bar{y}(t) = \alpha + \beta t$

$$\begin{aligned} y' + b_1 y &= c_0 + c_1 t \\ \Leftrightarrow \beta + b_1(\alpha + \beta t) &= c_0 + c_1 t \\ \Leftrightarrow t(b_1\beta - c_1) + b_1\alpha + \beta - c_0 &= 0 \end{aligned} \tag{3.8}$$

which holds $\forall t$ if and only if $b_1\beta - c_1 = 0$ and $\beta + b_1\alpha - c_0 = 0$, since all coefficients are constants, while t varies.

$$\Rightarrow \begin{cases} \beta &= \frac{c_1}{b_1} \\ \alpha &= \frac{1}{b_1}\left(c_0 - \frac{c_1}{b_1}\right) \end{cases} \tag{3.9}$$

For higher-order polynomials, the method needs to be adjusted accordingly. E.g. try $\bar{y} = \alpha + \beta t + \gamma t^2$ if $g(t) = c_0 + c_1 t + c_2 t^2$ etc. Note how the constant function $g(t)$ is a special case of this general class of functions.

See the main reference Gandolfo (2009) for an overview of more types of functional forms, and combinations of several forms.

3.4 General Solution

For the general solution, simply add up the homogeneous and particular solutions:

$$\boxed{y(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} + \cdots + A_{n-1} e^{\lambda_{n-1} t} + A_n e^{\lambda_n t} + \bar{y}(t)} \tag{3.10}$$

Finally, use n initial (boundary) conditions $(t_1^*, y_1^*), \ldots, (t_n^*, y_n^*)$ to fix the n arbitrary constants $A_1, A_2, \ldots, A_n$

4 Systems of linear differential equations

4.1 General form

The step from single linear differential equations to systems of equations is entirely analogous to the same step in discrete-time systems. Instead of a single state variable and its derivatives, we model several state variables, of which the derivatives depend on the values of all others:

$$\begin{aligned} y_1' &= a_{11} y_1 + a_{12} y_2 + \cdots + a_{1n} y_n + g_1(t) \\ y_2' &= a_{21} y_1 + a_{22} y_2 + \cdots + a_{2n} y_n + g_2(t) \\ &\vdots \\ y_n' &= a_{n1} y_1 + a_{n2} y_2 + \cdots + a_{nn} y_n + g_n(t) \end{aligned} \tag{4.1}$$

which can be expressed in vector notation:

$$\begin{pmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} + \begin{pmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{pmatrix} \tag{4.2}$$

and can be written more compactly using matrix notation:

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g}(\mathbf{t}) \tag{4.3}$$

where bold font indicates vectors and matrices, and $\mathbf{A}$ is the coefficient matrix. As with difference equations, this allows us to rewrite higher order equations and systems of higher-order equations as larger systems of first-order differential equations. I.e. we have a choice between larger dimensions of the system, or higher order of the equations. Consider for example the n-th order differential equation

$$y^{(n)} + b_1 y^{(n-1)} + \cdots + b_{n-2} y'' + b_{n-1} y' + b_n y = g(t) \tag{4.4}$$

Let $z_1 \equiv y'$, $z_2 \equiv z_1' = y''$ etc, i.e. z_i represents the ith time-derivative of y. Then the equation above can be re-written as the following system of first-order differential equations:

$$\begin{pmatrix} y' \\ z_1' \\ \vdots \\ z_{n-2}' \\ z_{n-1}' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -b_n & -b_{n-1} & -b_{n-2} & \dots & -b_1 \end{pmatrix} \begin{pmatrix} y \\ z_1 \\ \vdots \\ z_{n-2} \\ z_{n-1} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ g(t) \end{pmatrix} \tag{4.5}$$
$$\Leftrightarrow \mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g}(\mathbf{t})$$

This method can simplify the analysis considerably, and is particularly useful for the simulation of dynamic systems in continuous time.

4.2 Homogeneous solution

As for discrete-time systems, we simply extend the single-equation approach to vectors and matrices. The general solution approach for the homogeneous part of the equation (dynamics) is to insert $\boldsymbol{y_t} \sim \boldsymbol{v}e^{\lambda t}$:

$$\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y} \tag{4.6}$$
$$\Leftrightarrow \boldsymbol{v}\lambda e^{\lambda t} = \boldsymbol{A}\boldsymbol{v}e^{\lambda t}$$
$$\Leftrightarrow e^{\lambda t}\left(\boldsymbol{A} - \lambda \boldsymbol{I}\right)\boldsymbol{v} = 0 \qquad ; e^{\lambda t} \neq 0;\ \boldsymbol{v} \neq \boldsymbol{0}$$
$$\Rightarrow \det\left(\boldsymbol{A} - \lambda \boldsymbol{I}\right) = 0 \tag{4.7}$$

Which leads is the characteristic polynomial ($\chi(\lambda)$) and characteristic equation:

$$\det\left[\begin{pmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} - \lambda \end{pmatrix}\right] = 0 \tag{4.8}$$

$$\Longrightarrow \lambda^n + b_1\lambda^{n-1} + \dots + b_{n-1}\lambda + b_n = 0 \text{ (characteristic equation)}$$

$$\text{with } b_i = (-1)^i \times \text{sum of all } i\text{th order principal minors of } \boldsymbol{A}$$

$$\xrightarrow{\text{in particular:}} b_1 = -\mathrm{tr}(\boldsymbol{A}), \quad b_n = (-1)^n \det(\boldsymbol{A})$$

The final solution of the homogeneous part of the equation is as follows:

$$\boldsymbol{y} = \boldsymbol{V}e^{\boldsymbol{\Lambda} t}\boldsymbol{a} \tag{4.9}$$

with

$$\boldsymbol{V} = \underbrace{\begin{pmatrix} \boldsymbol{v}^{(1)} & \boldsymbol{v}^{(2)} & \dots & \boldsymbol{v}^{(\boldsymbol{n})} \end{pmatrix}}_{\boldsymbol{v}^{(\boldsymbol{i})} \text{ from } (\boldsymbol{A} - \lambda_i \boldsymbol{I})\boldsymbol{v}^{(\boldsymbol{i})} = 0} = \begin{pmatrix} v_1^{(1)} & v_1^{(2)} & \dots & v_1^{(n)} \\ v_2^{(1)} & v_2^{(2)} & \dots & v_2^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ v_n^{(1)} & v_n^{(2)} & \dots & v_n^{(n)} \end{pmatrix}, \qquad \boldsymbol{\Lambda} = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}, \qquad \boldsymbol{a} = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{pmatrix}$$

Hence

$$\boldsymbol{y} = \begin{pmatrix} y_{1,t} \\ y_{2,t} \\ \vdots \\ y_{n,t} \end{pmatrix} = \begin{pmatrix} A_1 v_1^{(1)} e^{\lambda_1 t} + A_2 v_1^{(2)} e^{\lambda_2 t} + \dots + A_n v_1^{(n)} e^{\lambda_n t} \\ A_1 v_2^{(1)} e^{\lambda_1 t} + A_2 v_2^{(2)} e^{\lambda_2 t} + \dots + A_n v_2^{(n)} e^{\lambda_n t} \\ \vdots \\ A_1 v_n^{(1)} e^{\lambda_1 t} + A_2 v_n^{(2)} e^{\lambda_2 t} + \dots + A_n v_n^{(n)} e^{\lambda_n t} \end{pmatrix} \tag{4.10}$$

Or, more simply:

$$\boxed{\boldsymbol{y} = A_1 \boldsymbol{v}^{(1)} e^{\lambda_1 t} + A_2 \boldsymbol{v}^{(2)} e^{\lambda_2 t} + \dots + A_n \boldsymbol{v}^{(n)} e^{\lambda_n t}} \tag{4.11}$$

The dynamics converge, i.e. the system is stable if $\lim_{t \to \infty} e^{\lambda_i t} = 0 \, \forall i$, which holds if $Re(\lambda_i) < 0$. This is equivalent to the stability conditions for single differential equations of arbitrary order. For simplified stability conditions based on $\{b_i\}_{i=1}^n$: see section 3.2.

4.3 Particular solution

The particular solution, i.e. the steady state vector $\bar{y}$ is found by trying a similar functional form as the non-homogeneous part $\boldsymbol{g(t)}$:

$\Rightarrow$ e.g. if $\boldsymbol{g(t)} = \boldsymbol{k} = const.vector \Rightarrow$ try $\bar{\boldsymbol{y}} = \boldsymbol{\mu} = const.vector \Rightarrow \boldsymbol{\mu} = (-\boldsymbol{A})^{-1}\boldsymbol{k}$

For other functional forms, check the main reference (Gandolfo, 2009).

4.4 General Solution

The general solution is the sum of the homogeneous and particular solutions, where the former describes the dynamics off of the steady state, and the latter defines the steady state of the system:

$$\boxed{\boldsymbol{y} = \sum_i^n A_i \boldsymbol{v}^{(i)} e^{\lambda_i t} + \bar{\boldsymbol{y}}} \tag{4.12}$$

The constants $\{A_i\}_{i=1}^n$ are determined by n boundary conditions, e.g. initial conditions.

4.5 A special case of stability: saddle-path dynamics

A special type of dynamic, with great importance especially in macroeconomic models with rational expectations, is the saddle path. This refers to a combination of stable and unstable eigenvalues. One example, represented in its phase space in Figure 2, is the Dornbusch model from tutorial IV. Note how almost any initial condition would lead the model to diverge further and further away from its equilibrium.

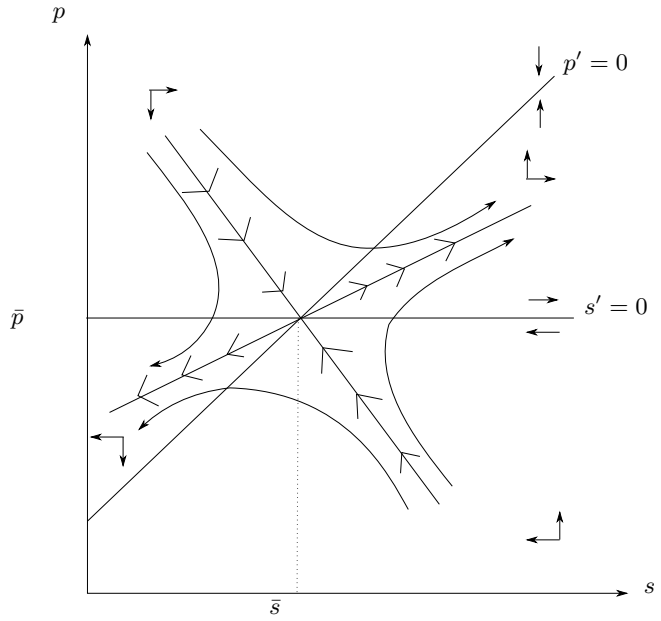


Figure 2: Example for saddle-path dynamics: phase diagram of a version of the famous Dornbusch overshooting model (Figure from tutorial IV). There is exactly one convergent trajectory, defined by the eigenvector corresponding to the stable eigenvalue. Any point that does not lie exactly on the stable saddle path would lead to divergent dynamics, i.e. explosion to $\pm\infty$.

A further inspection of the general solution of a two-dimensional model can be helpful here. Assume $\lambda_1 < 0$ and $\lambda_2 > 0$. Then, as $t \to \infty$:

$$\boldsymbol{y} = A_1 \boldsymbol{v}^{(1)} \underbrace{e^{\lambda_1 t}}_{\to 0} + A_2 \boldsymbol{v}^{(2)} \underbrace{e^{\lambda_2 t}}_{\to \infty} + \bar{\boldsymbol{y}}$$

The stable saddle path is defined by the first term. The second (divergent) term can be eliminated if $A_2 = 0$. However, these constants depend on initial conditions. The "trick" by which the stable arm of the saddle path can be reached is to assume rational expectations and that one of the variables is a so-called jump-variable. It is not pre-determined, but depends on agents' expectations of the future. If something happens that removes the economy from its steady state, the agents receive new information, hence update their expectation about the future, and are able to choose a new value for that jump variable.

A typical example in discrete time is the consumption equation in classical (RBC-type) models: agents maximise discounted future utility from consumption and leisure (max $E_t \sum_{k=0}^{\infty} \beta^k U(C_{t+k}, N_{t+k})$, subject to some budget constraint. That means, agents always try to optimise their behaviour such that the present

and future values of consumption are optimised, while the past values do not matter. The final consumption equation then usually has the form $C_t^{-\sigma} = \beta(1 + r_t)E_t C_{t+1}^{-\sigma}$, with the discount factor β, real interest rate r_t, and an elasticity parameter σ. The equation shows that current consumption depends on expected future consumption, but not on its own past. If the agents receive new information, e.g. through a policy change announcement, their expectation of the future changes, and hence the chosen value of consumption can be changed discontinuously ("jump"), unrestricted by its previous value.
It is furthermore assumed that the agents will always choose a value of the jump variable such that they will end on the stable saddle path that takes them back to the steady state (sometimes due to an optimality condition). In modelling practice, it is hence important to clarify which variables are pre-determined, i.e. their trajectory depend on the system's current and past states, and which variables can behave discontinuously, depending on expectations of the future. The Blanchard-Khan conditions then state that the number of jump variables has to equal the number of unstable eigenvalues, so that there is exactly one solution that ensures a trajectory back to the equilibrium.

5 Non-linear dynamics

5.1 General form

Recall the general formulation of dynamic models, where the vector of changes in state variables relates to the current values of the state variables through some functional form $F : \mathbb{R}^n \to \mathbb{R}^n$:[9]

$$\boldsymbol{y}_{t+1} = F(\boldsymbol{y}) \qquad \text{(discrete time)} \tag{5.1}$$

$$\boldsymbol{y}' = F(\boldsymbol{y}) \qquad \text{(continuous time)} \tag{5.2}$$

with the state vector

$$\boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \tag{5.3}$$

The considerations below are valid for systems in discrete and continuous time. For simplicity, only continuous systems will be treated explicitly in this section.
So far we have treated only linear systems, i.e. systems where the state variables only appear at power of one (y^1). That includes combinations of state variables, i.e. they can only be added, but there are no products of several state variables in F. This allowed us to express F as a linear operator, i.e. a matrix $\boldsymbol{A}$, so that $\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y}$. The matrix contains only the coefficients of the individual terms, but is independent of the state of the system. Ultimately, we could use simple linear algebra to solve the model and describe the system's behaviour over time.
If we allow for non-linear functions, this is not possible anymore. Consider the following 2-equation model:

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} a_{11}y_1^2 + a_{12}y_1y_2 \\ a_{21}y_1y_2 + a_{22}\sqrt{y_2} \end{pmatrix} \tag{5.4}$$

If we tried to express it using a matrix, that matrix would have to contain the state variables in it:

$$\boldsymbol{A} = \begin{pmatrix} a_{11}y_1 & a_{21}y_1 \\ a_{21}y_2 & \frac{a_{22}}{\sqrt{y_2}} \end{pmatrix} \tag{5.5}$$

Hence, the matrix that defines the dynamic behaviour changes with the values of the state variables.[10] While the rates of change are always a function of the current state in differential equations (by definition), those functions are globally defined exactly the same way in the case of linear systems. When you are dealing with non-linear systems, that is not true anymore, the dynamics can be completely different in different parts of the state space. Moreover, there can be several equilibria, which can each be stable or unstable, independently of each other.[11] Due to these differences in local dynamics, non-linear models are typically analysed locally, i.e. we analyse the dynamics within a relatively small range of values around points of interest (i.e. steady states).

[9] This notation formalises the same idea: F is a function that that maps a real-valued ($\mathbb{R}$) object in n dimensions, i.e. a vector of that length, to another real-valued object in n dimensions.

[10] Note that it might not even be possible to construct such a matrix for all types of systems.

[11] This can of course be problematic when you try to extrapolate the behaviour of real-world systems to domains that have never been empirically observed. Often, the "true" model of the world is unknown and we can only have models that approximate its behaviour relatively well. Without data to inform these approximations, the dynamics outside the range of observed values hence becomes difficult, if not impossible, to predict.

5.2 Local vs global analysis, linearisation

Even locally, non-linearities would often complicate the analysis too much to be manageable. Hence, we use linear approximations of the model, which describe it accurately enough within a small range of values.
In order to approximate the dynamics in the vicinity of a steady state, we apply a Taylor series expansion. Consider as an example the first-order different equation

$$y'(t) = f(y(t)) \tag{5.6}$$

where f can be non-linear. Then, a small distance from the steady state, the dynamics can be described by

$$f(\bar{y}+\epsilon) = \underbrace{f(\bar{y})}_{=\bar{y}'=0} + \epsilon f'(\bar{y}) + \underbrace{\frac{\epsilon^2}{2} f''(\bar{y}) + \dots}_{\text{higher-order terms}}$$

The first term drops out because the function that describes the rate of change of the state variable evaluates to 0, by virtue of the definition of the steady state. As ϵ is considered small, the higher-order terms can be neglected in most cases (unless e.g. $f'(\bar{y}) = 0$).
To analyse the stability of a steady state $\bar{y}$, we would hence use as a coefficient $f'(\bar{y})$. For systems of several dimensions, we introduce the Jacobian, i.e. the matrix of first derivatives:

$$\boldsymbol{J} = \begin{pmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} & \cdots & \frac{\partial f_1}{\partial y_n} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} & \cdots & \frac{\partial f_2}{\partial y_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial y_1} & \frac{\partial f_n}{\partial y_2} & \cdots & \frac{\partial f_n}{\partial y_n} \end{pmatrix} \tag{5.7}$$

And we can use this equation, evaluated at the steady state (first derivatives will typically involve the state variables) to analyse the local stability around that steady state:

$$J|_{\boldsymbol{y}=\bar{\boldsymbol{y}}} = \begin{pmatrix} \frac{\partial f_1}{\partial y_1}\Big|_{\boldsymbol{y}=\bar{\boldsymbol{y}}} & \frac{\partial f_1}{\partial y_2}\Big|_{\boldsymbol{y}=\bar{\boldsymbol{y}}} & \cdots & \frac{\partial f_1}{\partial y_n}\Big|_{\boldsymbol{y}=\bar{\boldsymbol{y}}} \\ \frac{\partial f_2}{\partial y_1}\Big|_{\boldsymbol{y}=\bar{\boldsymbol{y}}} & \frac{\partial f_2}{\partial y_2}\Big|_{\boldsymbol{y}=\bar{\boldsymbol{y}}} & \cdots & \frac{\partial f_2}{\partial y_n}\Big|_{\boldsymbol{y}=\bar{\boldsymbol{y}}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial y_1}\Big|_{\boldsymbol{y}=\bar{\boldsymbol{y}}} & \frac{\partial f_n}{\partial y_2}\Big|_{\boldsymbol{y}=\bar{\boldsymbol{y}}} & \cdots & \frac{\partial f_n}{\partial y_n}\Big|_{\boldsymbol{y}=\bar{\boldsymbol{y}}} \end{pmatrix} \tag{5.8}$$

The stability conditions are identical to those for linear systems. The key difference is that local stability does not inform about global stability.
Note how the coefficient matrix of a linear system is at the same time the Jacobian matrix. Because it is linear already, this "approximation" is now exact. Moreover, the Jacobian is identical at all possible states, since the derivatives do not include the state variables. Hence, stability analysis can be performed globally for linear systems, while it typically differs in different regions of the state space of non-linear systems, so that only local stability analysis is possible.

5.3 Bifurcation theory

Bifurcation theory treats another type of stability, i.e. *structural* stability. All stability analyses above consider quantitative changes of the system. E.g. we can analyse whether a system returns to a its original steady state, if it was slightly perturbed. Structural stability on the other hand refers to qualitative behaviour of the model, e.g. number and stability of equilibria, in dependence on the model parameters.
Typically, we are interested in finding the bifurcation points, i.e. the values of a model parameter at which the qualitative model behaviour changes. Take as an example the saddle-node bifurcation and its prototype equation

$$y' = f(y, \alpha) = y^2 - \alpha \tag{5.9}$$

The steady state of this model can be expressed as a function of the parameter α, i.e. $\bar{y} = \bar{y}(\alpha)$. Depending on the value of that parameter, there can be different numbers of equilibria:

$$y' = 0 \Leftrightarrow y^2 - \alpha = 0 \tag{5.10}$$

$$\Leftrightarrow \bar{y} = \pm\sqrt{\alpha} \;\Rightarrow\; \begin{cases} \alpha < 0: \text{ no real-valued solutions} \\ \alpha = 0: \text{ 1 solution} \\ \alpha > 0: \text{ 2 solutions} \end{cases} \tag{5.11}$$

Similar problems occur in the cases of the transcritical and pitchfork bifurcations. The prototypical equations are

$$y' = \alpha y - y^2 \qquad \text{(transcritical bifurcation)} \tag{5.12}$$
$$y' = \alpha y - y^3 \qquad \text{(pitchfork bifurcation)} \tag{5.13}$$

Finding the bifurcation points involves finding the roots of the steady state equation $\bar{y} = \bar{y}(\alpha)$.
The final bifurcation that is relevant in this class is the Hopf bifurcation. It requires a system of at least two state variables and two equations, i.e.

$$\begin{aligned} y_1' &= f_1(y_1, y_2, \alpha) \\ y_2' &= f_2(y_1, y_2, \alpha) \end{aligned} \tag{5.14}$$

If the eigenvalues of the Jacobian are complex, then there might be a value of α at which the real part of both eigenvalues vanishes, i.e. $Re(\lambda_{1,2}(\alpha_0)) = 0$. In that case, there is a Hopf bifurcation at α_0. To each side of this critical parameter, there might appear either stable of unstable cycles, while it displays a stable limit cycle behaviour at α_0.

6 Dynamic optimisation

The last chapter of this class treats a related, but different problem: the optimisation of dynamic problems. These kinds of problems are ubiquitous in economics, as individual behaviour is mostly defined as some type of optimisation problem, e.g. utility maximisation or cost minimisation. You should be familiar with the Lagrange method for static optimisation problems from your previous studies. Now we have to address dynamic problems as well, for which we require a few additional techniques.

6.1 Economic interpretation of the Hamiltonian and its FOCs

6.1.1 Setting up the problem

The explanations in this script follow mainly Dorfman (1969), which provides an intuitive approach to understand the use of the Hamiltonian equation and the first-order conditions we derive from it. The explanation below is essentially a condensed version of the elaborations in the paper.
Optimisation techniques are easiest to understand with an example, so we will consider a profit maximisation problem here. This has the simple advantage that we can give names to our variables, which should be helpful with intuition-building. In order to make the insights easily transferable to other problems, we will keep the functional forms abstract enough to be able to generalise though.
Consider the immediate profit u at time t of a firm to be a function of the amount of capital that is employed in production k, as well as some choice variable x, investments (which represent a cost at one point in time, but increase future capital):

$$u = u(k, x, t) \tag{6.1}$$

Instead of maximising profit in one moment of time (static optimisation), we now consider profit over a specific period of time between $t = 0$ and $t = T$. Hence we need to integrate the profit function over time:

$$W(k_0, x) = \int_0^T u(k, x, t)dt \tag{6.2}$$

which is a function of the initial amount of capital k_0, as well as the time path x. The critical part is that capital changes as a consequence of our choice of the control variable x, leading to the state equation:

$$k' = f(k, x, t) \tag{6.3}$$

6.1.2 FOC 1

The first "trick" we apply, is to split the time period into two parts: one very small amount of time in the beginning, ranging from t to $t + \delta$, and the remainder between $t + \delta$ and T. The initial time period is chosen so small that there is no scope to alter the control variable in the meantime (this will be ensured rigorously by letting $\delta \to 0$ later on). Then our profit integral function can be expressed as the sum of profits in both subperiods:

$$\begin{aligned} W(k_t, x, t) &= \delta u(k, x, t) + \int_{t+\delta}^{T} u(k, x, t)dt \\ &= \delta u(k, x, t) + W(t_{k+\delta}, x, t + \delta) \end{aligned} \tag{6.4}$$

And we define the optimal choice, i.e. value of the maximum profit, given by the best possible choice of the time path of the control variable as

$$V^*(k_t, t) = \max_{\{x\}} W(k_t, x, t) \tag{6.5}$$

It is not a function of the control, as x is now fixed at the optimal value. We furthermore define the marginal value of capital in this optimum as

$$\mu(t) = \frac{\partial}{\partial k_t} V^*(k, t) \tag{6.6}$$

This is the dynamic Lagrangian multiplier, or co-state variable, known as the "shadow price" in economic applications. In a static environment (solved by the Lagrangian), it denotes how much additional value a marginal increase in the available budget (constraint) could generate. In the dynamic setting, the interpretation is identical, but takes into account the point in time when this marginal increase is applied, as the effect on the integral over the entire time period would be affected by that change.
Now let's apply the split into two time periods to the maximum value function. We only use the maximum value for the second subperiod, and define the control variable x to be free in the initial moment between t and $t + \delta$, so it can become subject to optimisation.

$$V(k_t, x_t, t) = \delta u(k_t, x_t, t) + V^*(k_{t+\delta}, t + \delta) \tag{6.7}$$

Note how this creates a smaller optimisation problem, where we assume the future to be optimal already, and we only need to optimise the first subperiod. We can do this by simply setting the derivative with respect to x equal to 0:

$$\frac{\partial V}{\partial x} = \delta \frac{\partial}{\partial x} u(k, x, t) + \frac{\partial}{\partial x} V^*(k_{t+\delta}, t + \delta) \stackrel{!}{=} 0 \tag{6.8}$$

For the second term, we have to apply the chain rule of differentiation:

$$\frac{\partial}{\partial x} V^*(k_{t+\delta}, t + \delta) = \frac{\partial V^*}{\partial k_{t+\delta}} \frac{\partial k_{t+\delta}}{\partial x_t} \tag{6.9}$$

And we apply a first-order Taylor expansion (Euler step) around k_t as an approximation to calculate the amount of capital after a very short amount of time δ[12]:

$$k_{t+\delta} = k_t + \delta k' = k_t + \delta f(k_t, x_t, t) \tag{6.10}$$

Only the second term depends on x (the initial value of capital is given!), so:

$$\frac{\partial k_{t+\delta}}{\partial x_t} = \delta \frac{\partial f}{\partial x_t} \tag{6.11}$$

Inserting these, as well as equation 6.6, we obtain

$$\delta \frac{\partial u}{\partial x} + \delta \mu_{t+\delta} \frac{\partial f}{\partial x} = 0 \tag{6.12}$$

where δ cancels out.
Applying a Taylor expansion/Euler step to estimate the shadow price after the time increment ($\mu_{t+\delta} = \mu_t + \delta \mu'$), we finally obtain

$$\frac{\partial u}{\partial x} + \mu_t \frac{\partial f}{\partial x} + \delta \mu' \frac{\partial f}{\partial x} = 0 \tag{6.13}$$

And, since the problem is defined in continuous time, we let the time increment $\delta \to 0$, so the last term on the left vanishes, yielding the first-order condition

$$\boxed{\frac{\partial u}{\partial x} + \mu_t \frac{\partial f}{\partial x} = 0} \tag{6.14}$$

This equation has an intuitive economic interpretation: the immediate marginal utility derived from a change in x (first term) has to exactly balance out the increase of capital as a result of a marginal change to x, valued at the marginal value of capital μ (second term).

[12] This creates a little error term for finitely small values of the time increment, but vanishes as $\delta \to 0$

6.1.3 FOC 2

Now assume we know the optimal time path for the control variable x. Then, clearly $V(k_t, x_t, t) = V^*(k_t, t)$, and hence

$$V^*(k_t, t) = \delta u(k_t, x_t, t) + V^*(k_{t+\delta}, t + \delta) \tag{6.15}$$

Differentiating with respect to the amount of capital yields

$$\begin{aligned} \frac{\partial V^*}{\partial k} = \mu_t &= \delta \frac{\partial u}{\partial k} + \frac{\partial V^*(k_{t+\delta}, t+\delta)}{\partial k} && (6.16) \\ &= \delta \frac{\partial u}{\partial k} + \frac{\partial k_{t+\delta}}{\partial k} \mu_{t+\delta} \\ &= \delta \frac{\partial u}{\partial k} + \left(1 + \delta \frac{\partial f}{\partial k}\right)(\mu + \delta \mu') \\ \Leftrightarrow \mu_t &= \delta \frac{\partial u}{\partial k} + \mu_t + \delta \mu \frac{\partial f}{\partial k} + \delta \mu' + \delta^2 \mu' \frac{\partial f}{\partial k} && (6.17) \end{aligned}$$

where we can subtract μ_t on both sides. Furthermore, δ^2 vanishes faster than δ as $\delta \to 0$, so we can neglect the last term. We are ultimately left with the equation

$$0 = \delta \left(\frac{\partial u}{\partial k} + \mu \frac{\partial f}{\partial k} + \mu' \right) \tag{6.18}$$

which finally yields the second FOC:

$$\boxed{-\mu' = \tfrac{\partial u}{\partial k} + \mu \tfrac{\partial f}{\partial k}} \tag{6.19}$$

This equation also has a clear economic interpretation: μ' is the rate of change of the marginal value of capital, i.e. its appreciation. Its negative, the depreciation, has to equal the sum of capital's marginal utility in the current moment and to future capital, valued at its current marginal value μ.

6.1.4 The Hamiltonian

It is not necessary to derive the third FOC, as it is given by the state equation. Given the optimality conditions derived above, it should be easier to understand the Hamiltonian equation (eq. 6.20) now: it sums the immediate utility, i.e. the utility derived from capital in the current moment, and the value derived from increasing future capital. Using the first-order conditions hence solves the problem of maximising utility not only at one point in time, but also for the future, in situations where choices in the present have an effect on the future, as specified by the state equation.

$$H = u(k, x, t) + \mu f(k, x, t) \tag{6.20}$$

And, most importantly, it naturally yields the FOCs from above:

$$\begin{aligned} \frac{\partial H}{\partial x} &\overset{!}{=} 0 && \Leftrightarrow 0 = \frac{\partial u}{\partial x} + \mu_t \frac{\partial f}{\partial x} && (6.21) \\ \frac{\partial H}{\partial k} &\overset{!}{=} -\mu' && \Leftrightarrow -\mu' = \frac{\partial u}{\partial k} + \mu \frac{\partial f}{\partial k} && (6.22) \\ \frac{\partial H}{\partial \mu} &\overset{!}{=} k' && \Leftrightarrow k' = f(k, x, t) && (6.23) \end{aligned}$$

We can then derive the optimal time paths for all variables from these conditions. Note that the Hamiltonian is defined at points in time, rather than over the entire time period. A good rule of thumb is hence that you will at some point have to perform an integration step.

6.2 Boundary conditions

To fully define the time paths, we must apply boundary conditions. These serve to fix the time paths in a specific position, restricting the degrees of freedom such that only one solution is possible. Without the boundary conditions we can derive the differential equations that govern optimal choices and time paths of all three variables. However, as in the case of solving differential equations without the additional difficulty of choosing the optimal one, the exact path is only defined if we know where we start from. Hence we always require the initial value of the state variable, here k:

$$k(0) = k_0 \tag{6.24}$$

However, we are dealing with two main variables here, as we need to account for the control variable too.[13] Typically we do not know its initial value, as the optimisation problem starts by choosing the best value from the first point in time. Instead, as a second boundary condition, we apply a terminal condition on the state variable. There are two main cases: either it is given, i.e. there is a desired value for the state variable at time T, or it is not given. In the first case, the boundary condition is straighforward:

$$k(T) = k_T \tag{6.25}$$

If it is not given, we apply the transversality condition. Either the state variable has to be exhausted fully ($k(T) = 0$), or the shadow price must equal zero ($\mu(T) = 0$), as there cannot be any possible improvement of optimal value by marginally changing capital anymore. That is to say that since the period $T + 1$ is not considered in the optimisation problem, adding capital into that period does not add value. Hence

$$k(T)\mu(T) = 0 \quad \Leftrightarrow \quad k(T) = 0 \vee \mu(T) = 0 \tag{6.26}$$

In the case of an infinite-horizon problem, i.e. one where a function is optimised into the infinite future, this becomes

$$\lim_{T\to\infty} k(T)\mu(T) = 0 \tag{6.27}$$

References

Dorfman, R. (1969). An economic interpretation of optimal control theory. *The American Economic Review*, *59*(5), 817–831.

Gandolfo, G. (2009). *Economic dynamics*. Springer, Berlin.

[13] The shadow price/Lagrangian multiplier is implicitly given by the other equations.