5

= Seminar 5 =

OBS 1) DACĂ ADUNĂM 2 SERII CONV. ⇒ SERIE CONV.

2) " —— " — " 1 SERIE CONV. CU UNA DIV. ⇒ SERIE DIV.

3) " —— " — " 2 SERII DIV ⇒ PUTEM OBȚINE 1 CONV. SAU 1 DIV.

1) CONV. SERIEI $\sum_{n=1}^{\infty} \frac{a^n+n}{3^n+n^3}$; $a>0$

Fie $x_n = \frac{a^n+n}{3^n+n^3}$; $n \in \mathbb{N}^*$

0 (CR. RAPORT SAU L'H)

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{\cancel{3^n}\left(\left(\frac{a}{3}\right)^n + \left(\frac{n}{3^n}\right)\right)^n}{\cancel{3^n}\left(1+\frac{n^3}{3^n}\right)} = \begin{cases} 0 ; a \in (0;3) \\ 1 ; a = 3 \\ +\infty ; a \in (3;\infty) \end{cases}$$

I) $a \in [3;\infty)$

$\lim_{n\to\infty} x_n \neq 0 \Rightarrow \sum_{n=1}^{\infty} x_n$ = DIV.

II) $a \in (0,3)$

$$x_n = \frac{a^n+n}{3^n+n^3} \le \frac{a^n+n}{3^n} = \left(\frac{a}{3}\right)^n + \frac{n}{3^n}$$

$$\sum_{n=1}^{\infty}\left(\frac{a}{3}\right)^n = \text{CONV. (SERIE GEOMETRICĂ; } q = \frac{a}{3} \in (-1,1))$$

$$\sum_{n=1}^{\infty} \frac{n}{3^n} = \text{CONV (CRIT. RAPORT)}$$

DECI, $\sum_{n=1}^{\infty} \frac{a^n+n}{3^n} = \text{CONV}$

CONFORM CR. DE COMPARAȚIE CU INEGALITĂȚI, $\sum_{n=1}^{\infty} x_n = \text{CONV.}$

2) a) $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$

b) CONV. $\sum_{n=1}^{\infty}\left(1-\cos\frac{1}{n}\right)x^n;\ x>0$?

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a) $\lim_{x\to 0} \frac{1-\cos x}{x^2} \overset{l'H}{=} \lim_{x\to 0}\frac{1\cdot \sin x}{2\cdot x} = \frac{1}{2}$

b) Fie $x_n = \left(1-\cos\frac{1}{n}\right)x^n;\ (\forall)\, n\in\mathbb{N}^*$

$$\lim_{n\to\infty}\frac{x_{n+1}}{x_n} = \lim_{n\to\infty}\frac{x^{n+1}\left(1-\cos\frac{1}{n+1}\right)}{x^n\left(1-\cos\frac{1}{n}\right)} = \lim_{n\to\infty} x\cdot \frac{\overbrace{\frac{1-\cos\frac{1}{n+1}}{\frac{1}{(n+1)^2}}}^{\frac{1}{2}}\cdot\frac{1}{(n+1)^2}}{\underbrace{\frac{1-\cos\frac{1}{n}}{\frac{1}{n^2}}}_{\frac{1}{2}}\cdot\frac{1}{n^2}} = x$$

CR. RAPORT:
- $x\in(0,1) \to$ CONV
- $x\in(1,\infty) \to$ DIV
- $x=1$, CR. NU DECIDE $\Rightarrow \sum_{n\ge 1}\left(1-\cos\frac{1}{n}\right)$

Fie $a_n = 1-\cos\frac{1}{n};\ b_n = \frac{1}{n^2};\ (\forall)\, n\in\mathbb{N}^*$

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{1}{2}\in(0,\infty)$$

$$\sum_{n\ge 1} a_n \sim \sum_{n\ge 1} b_n = \sum_{n\ge 1}\frac{1}{n^2}$$ CONV. (SERIA ARITMETICĂ GENERALIZATĂ CU $\alpha = 2$)

DECI, $\sum_{n\geq 1} a_n$ CONV.

= SERII CU TERMENI OARECARE =

1) CONV.:

a) $\sum_{n\geq 1}^{\infty} \frac{\cos(nx)}{n^\lambda}$; $x\in\mathbb{R}$; $\lambda>0$

Fie $x_n = \frac{1}{n^\lambda}$; $y_n = \cos(nx)$, $(\forall) n\in\mathbb{N}^*$

$x_n \searrow 0$ (1)

? $(\exists) M>0$ a.î. $(\forall) n\in\mathbb{N}^*$, $|y_1+\ldots+y_n|\leq M$

$|y_1+\ldots+y_n| = |\cos x + \ldots \cos nx|$, $(\forall) n\in\mathbb{N}^*$

"M" NU POATE DEPINDE DE "n", DAR POATE DEPINDE DE "λ" și "x"

Fie $z = \cos x + i\cdot\sin x$

$z^2 = \cos 2x + i\cdot\sin 2x$

......

$z^n = \cos nx + i\cdot \sin nx$

PRESUPUN $z\neq 1$, i.e. $\cos x + i\cdot\sin x \neq 1$, i.e. $x\in\mathbb{R}-\{2k\pi / k\in\mathbb{Z}\}$

$z+\ldots+z^n = z\frac{z^n-1}{z-1} = \frac{z^{n+1}-z}{z-1} = \frac{\cos(n+1)x + i\cdot\sin(n+1)x - \cos x - i\cdot\sin x}{\cos x + i\cdot \sin x - 1} =$

$\sin 2\frac{x}{2} = 2\cdot\sin\frac{x}{2}\cdot\cos\frac{x}{2}$

$= \frac{(\cos(n+1)x - \cos x) + i(\sin(n+1)x - \sin x)}{(\cos x - 1) + i\cdot\sin x} =$

$= \frac{-2\cdot\sin\frac{n+2}{2}x\cdot\sin\frac{n}{2}x + 2i\cdot\cos\frac{n+2}{2}\cdot x\cdot\sin\frac{n}{2}\cdot x}{-2\sin^2\frac{x}{2} + 2i\sin\frac{x}{2}\cdot\cos\frac{x}{2}} =$

$= \frac{\sin\frac{n}{2}x}{\sin\frac{x}{2}}\cdot\frac{\cos\frac{n+2}{2}x + i\cdot\sin\frac{n+2}{2}x}{\cos\frac{x}{2} + i\cdot\sin\frac{x}{2}} =$

$= \frac{\sin\frac{n}{2}x}{\sin\frac{x}{2}}\cdot\frac{\left(\cos\frac{x}{2} + i\cdot\sin\frac{x}{2}\right)^{n+2}}{\cancel{\cos\frac{x}{2} + i\cdot\sin\frac{x}{2}}} = \frac{\sin\frac{n}{2}x}{\sin\frac{x}{2}}\left(\cos\frac{x}{2} + i\cdot\sin\frac{x}{2}\right)^{\cdot(n+1)}$

$|y_1+\ldots+y_n| = |Re(z+\ldots+z^n)| = \left|\frac{\sin\frac{n}{2}x}{\sin\frac{x}{2}}\cdot\cos\frac{n+1}{2}x\right| =$

$$= \frac{\left|\sin \frac{n}{2}x\right|}{\left|\sin \frac{x}{2}\right|} \cdot \underbrace{\left|\cos \frac{n+1}{2} x\right|}_{\leq 1} \leq \frac{1}{\left|\sin \frac{x}{2}\right|} \overset{not}{=} M \quad (2)$$

$(1) + (2) \Rightarrow$ (CR. ABEL $-$ D. (I)) $\sum\limits_{n=1}^{\infty} x_n \cdot y_n$ CONV

DACĂ $x \in \{2k\pi \mid k \in \mathbb{Z}\} \Rightarrow$ SERIA DEVINE $\sum\limits_{n=1}^{\infty} \frac{1}{n^\lambda}$ $\nearrow$ CONV; $\lambda \in (1;\infty)$ $\searrow$ DIV; $\lambda \in (0;1]$

b) $\sum\limits_{n=1}^{\infty} \frac{\cos \frac{1}{n} \cdot \cos n}{n} =$

Fie $x_n = \cos \frac{1}{n}$; $y = \frac{\cos n}{n}$; $(\forall) n \in \mathbb{N}^*$

$-1 \leq x_n \leq 1$, $(\forall) n \in \mathbb{N}^* \Rightarrow (x_n)_n$ MĂRGINIT

$$\left.\begin{array}{l} \underset{[0;\frac{\pi}{2}]}{x} \longrightarrow \underset{\mathbb{R}}{\cos x} \searrow \\ \left(\frac{1}{n}\right)_n \searrow \end{array}\right\} \Rightarrow (x_n)_n \searrow$$

DECI $(x_n)_n \rightarrow$ MONOTON + MĂRGINIT (1)

$\sum\limits_{n \geq 1} y_n : \sum\limits_{n \geq 1} \frac{\cos n}{n} \overset{= CONV}{\checkmark}$ (Vezi a) PT $x=1$ și $\lambda = 1$ (2)

$(1) + (2) \Rightarrow \sum\limits_{n \geq 1} x_n \cdot y_n$ CONV. (CONFORM A-D II)

c) $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

Fie $x_n = \frac{1}{\sqrt{n}}$; $(\forall) n \in \mathbb{N}^*$

$x_n \searrow 0$

CONFORM LEIBNIZ, $\sum\limits_{n=1}^{\infty} (-1)^n \cdot x_n$ CONV.

d) $\sum\limits_{n=1}^{\infty} \frac{(-1)^n \sqrt{n} + 1}{n} = \frac{(-1)^n \sqrt{n}}{n} + \frac{1}{n} = \frac{(-1)^n}{\sqrt{n}} + \frac{1}{n}$, $(\forall) n \in \mathbb{N}^*$

$\sum\limits_{n \geq 1} \frac{(-1)^n}{\sqrt{n}}$ CONV (VEZI c); $\sum\limits_{n \geq 1} \frac{1}{n}$ DIV (SERIE ARMONICĂ GENERALIZATA $\alpha = 1$)

DECI, $\sum\limits_{n \geq 1} \frac{(-1)^n \sqrt{n} + 1}{n}$ DIV

= Seminar 6 =

6

DEF: Fie (X,d) = SP. METRIC, $x \in X$ și $r>0$.

1) $B(x,r) = \{y \in X \mid d(y,x) < r\}$ (BILA DESCHISĂ DE CENTRU x ȘI RAZA r)

2) $B[x,r] = \{y \in X \mid d(y,x) \le r\}$ (BILA ÎNCHISĂ " ——— " ——— ")

PROP: (X,d) = SP. METRIC și $\tau_d = \{\emptyset\} \cup \{A \subseteq X \mid (\forall) x \in A, (\exists) r>0$ a.î. $B(x,r) \subseteq A\} \Rightarrow (X,\tau_d)$ = SP. TOPOLOGIC.

OBS: 1) DÂNDU-SE (X,d) = SP. METRIC, PUTEM DEFINI (X,τ_d) = SP. TOPOLOGIC. CA ATARE, PUTEM DEFINI NOȚIUNILE DE "MULȚIME DESCHISĂ", "M. ÎNCHISĂ", "VECINĂTATE"... ÎNTR-UN SP. METRIC, REFERINDU-NE LA (X,τ_d) = SP. TOPOLOGIC.

2) $(\mathbb{R},d)$ = SP. METRIC, UNDE $d: \mathbb{R}\times\mathbb{R} \to [0,\infty)$; $d(x,y) = |x-y|$.

3) CONSIDERĂM $(\mathbb{R},d)$ (SP. M. DE MAI SUS), $x \in \mathbb{R}$ și $r>0$:

i) $B(x,r) = (x-r, x+r)$

ii) $B[x,r] = [x-r, x+r]$

4) ÎN $(\mathbb{R},d)$ = SP. METRIC:

i) INTERVALELE DE FORMA $(-\infty,a)$; $(a,+\infty)$; (a,b) SUNT MULȚIMI DESCHISE, UNDE $a,b \in \mathbb{R}$ și $a \le b$.

ii) " ——— " ——— " $(-\infty,a]$; $[a,\infty)$; $[a,b]$ " — " — " ÎNCHISE " — " — ".

(CA ATARE, MULȚIMILE FORMATE DINTR-UN SINGUR EL. SUNT M. ÎNCHISE)

PROP: (ADAPTAREA DEF. DE LA CURS PT. SP. METRICE) (X,d) = SP.M și $\emptyset \ne A \subseteq X$:

1) $x \in \mathring{A}$ (MULȚIME PCT. INTERIOARE) $\Leftrightarrow (\exists) r>0$ a.î. $B(x,r) \subseteq A$

OBS: $\mathring{A}$ = CEA MAI MARE M. DESCHISĂ INCLUSĂ ÎN A (ÎN SENSUL INCLUZIUNII)

2) $x \in \bar{A} \Leftrightarrow (\forall) r>0$, AVEM $B(x,r) \cap A \ne \emptyset \Leftrightarrow (\exists) (x_n)_n \subseteq A$ a.î. $x_n \xrightarrow[n]{d} x$.

OBS: i) $\bar{A}$ = CEA MAI MICĂ M. ÎNCHISĂ CE INCLUDE A.

ii) $\overline{CA} = C\mathring{A}$

iii) $\mathring{\overline{CA}}$ ... $\overset{\circ}{CA} = C\bar{A}$

3) $x \in A' \Leftrightarrow (\forall) r>0$, AVEM $B(x,r) \cap (A \setminus \{x\}) \ne \emptyset \Leftrightarrow (\exists) (x_n)_n \subseteq A \setminus \{x\}$ a.î.

$x_n \xrightarrow[n]{d} x$.

OBS: i) $A' \subseteq \bar{A}$

ii) $\bar{A} = A \cup A'$

4) $Fr(A) = \bar{A} \setminus \mathring{A} = \bar{A} \cap \overline{CA}$

5) $Iz(A) = A \setminus A'$

OBS: 1) "x" DIN PROP. DE MAI SUS, "$\in X$".

2) $\mathring{A}$ = M. PCT. INTERIOARE (INTERIORUL LUI A)

$\bar{A}$ = ADERENȚA / ÎNCHIDEREA LUI A

A' = M. PCT. ACUMULARE A

$Fr(A)$ = FRONTIERA LUI A

$Iz(A)$ = M. PCT. IZOLATE A LUI A

① FACEȚI ANALIZA TOPOLOGICĂ A MULȚIMII $A \subseteq \mathbb{R}$, UNDE:

$(\mathring{A}, \bar{A}, A', Fr(A), Iz(A))$

a) $A = \mathbb{Q}$

SOL: 1) $\mathring{A} = ?$

$x \in \mathring{A} \Leftrightarrow (\exists)\, r>0$ a.î. $\underbrace{B(X,r)}_{(X-r,X+r)} \subseteq \underbrace{A}_{\mathbb{Q}}$

$(X-r, X+r) \subseteq \mathbb{Q} \Rightarrow$ ↯ PT. CĂ ÎNTRE ORICE 2 NR. $\mathbb{R}$, $(\exists)$ O INFINITATE DE NR. $\in \mathbb{Q}$ ȘI O INFINITATE DE NR. IRAȚIONALE.

DECI $\Rightarrow \mathring{A} = \emptyset$

2) $\bar{A} = ?$

$x \in \bar{A} \Leftrightarrow (\forall)\, r>0$, AVEM $\underbrace{B(X,r)}_{(X-r,X+r)} \cap \underbrace{A}_{\mathbb{Q}} \neq \emptyset$

$(X-r, X+r) \cap \mathbb{Q} \neq \emptyset$

$\bar{A} \subseteq \mathbb{R}$ (DIN DEFINIȚIE)

FIE $x \in \mathbb{R}$ ȘI $r>0$. AVEM $(X-r, X+r) \cap \mathbb{Q} \neq \emptyset$, PT. CĂ " —— " —— ",

DECI $x \in \bar{A}$, i.e. $\mathbb{R} \subseteq \bar{A}$. PRIN URMARE, $\bar{A} = \mathbb{R}$.

3) $A' = ?$

$x\in A' \Leftrightarrow (\forall)\, r>0$, AVEM $(x-r,x+r)\cap(\mathbb{Q}-\{x\})\neq\emptyset \Rightarrow A'=\mathbb{R}$

4) $Fr(A)=\bar{A}-\mathring{A}=\mathbb{R}-\emptyset=\mathbb{R}$

5) $Iz(A)=A-A'=\mathbb{Q}-\mathbb{R}=\emptyset$

b) $A=[0,2)\cup\{3,4\}$

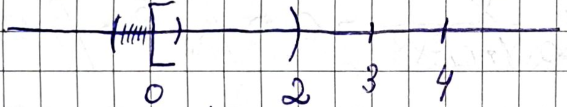
1) $\mathring{A}=?$

$x\in\mathring{A} \Leftrightarrow (\exists)\, r>0$ a.î. $(x-r,x+r)\subseteq A$

$$\left.\begin{array}{l}\mathring{A}\subseteq A\\ \left.\begin{array}{l}(0,2)\text{ DESCHISĂ}\\ (0,2)\subseteq A\end{array}\right\}\Rightarrow (0,2)\subseteq\mathring{A}\end{array}\right\}\Rightarrow (0,2)\subseteq\mathring{A}\subseteq[0,2)\cup\{3,4\}$$

STUDIEM DACĂ $\{0,3,4\}\in\mathring{A}$

$0\in\mathring{A}$? : $0\in\mathring{A}\Leftrightarrow(\exists)\, r>0$ a.î. $(-r,r)\subseteq A \Rightarrow 0\notin\mathring{A}$



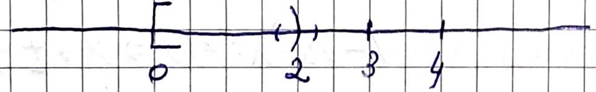
AȘADAR, $\mathring{A}=(0,2)$.

2) $\bar{A}=?$

$x\in\bar{A}\Leftrightarrow(\forall)\, r>0$, AVEM $(x-r,x+r)\cap A\neq\emptyset$

$$\left.\begin{array}{l}A\subseteq\bar{A}\\ \left.\begin{array}{l}[0,2)\cup\{3,4\}\text{ ÎNCHISĂ}\\ A\subseteq[0,2]\cup\{3,4\}\end{array}\right\}\Rightarrow \bar{A}\subseteq[0,2]\cup\{3,4\}\end{array}\right\}\Rightarrow [0,2)\cup\{3,4\}\subset[0,2]\cup\{3,4\}$$

$2\in\bar{A}$? : $2\in\bar{A}\Leftrightarrow(\forall)\, r>0,\ (2-r,2+r)\cap A\neq\emptyset$



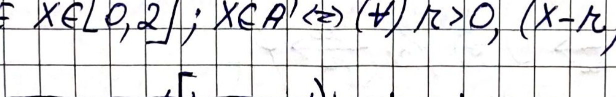
$\Rightarrow 2\in\bar{A}\Rightarrow\bar{A}=[0,2]\cup\{3,4\}$

3) $A'=?$

$x\in A'\Leftrightarrow(\forall)\, r>0,\ (x-r,x+r)\cap(A\setminus\{x\})\neq\emptyset$

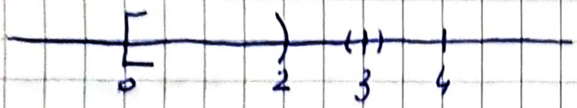
$A'\subseteq\bar{A}=[0,2]\cup\{3,4\}$

FIE $x\in[0,2]$; $x\in A'\Leftrightarrow(\forall)\, r>0,\ (x-r,x+r)\cap(A\setminus\{x\})\neq\emptyset$



DECI, $x\in A'$

$3 \in A'$: $3 \in A' \Leftrightarrow (\forall) r>0,\ (3-r, 3+r) \cap (A \setminus \{3\}) \neq \emptyset$



DECI, $3 \notin A'$

ANALOG, $4 \notin A'$

PRIN URMARE, $A' = [0, 2]$

4) $Fr(A) = \{0, 2, 3, 4\}$

5) $Iz(A) = A \setminus A' = \{3, 4\}$

② STUDIAȚI CONV. SIMPLĂ ȘI UNIFORMĂ PT. ȘIRURILE DE FUNCȚII:

a) $f_n : [-1, 1] \to \mathbb{R}$; $f_n(x) = \frac{x}{1+n^2x^2}$, $(\forall) n \in \mathbb{N}^*$

Sol: STUDIEM CONV. SIMPLĂ

Fie $x \in [-1, 1]$. $\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} \frac{x}{1+n^2x^2} = 0 \Rightarrow f_n \xrightarrow{S} f$, UNDE

$f : [-1, 1] \to \mathbb{R}$; $f(x) = 0$

STUDIEM CONV. UNIFORMĂ

<u>VARIANTA 1</u>: $\sup\limits_{x\in[-1,1]} |f_n(x) - f(x)| \to ?$

$|f_n(x) - f(x)| = \left|\frac{x}{1+n^2x^2} - 0\right| = \left|\frac{x}{1+n^2x^2}\right|$

Fie $f_n : [-1; 1] \to \mathbb{R}$; $f_n = \frac{x}{1+n^2x^2}$

$f_n'(x) = \frac{1+n^2x^2 - 2n^2x^2}{(1+n^2x^2)^2} \Rightarrow f_n'(x) = 0 \Leftrightarrow 1+n^2x^2 - 2n^2x^2 = 0$

$1+n^2x^2 = 2n^2x^2$

$1 - n^2x^2 = 0 \Rightarrow x = \pm\frac{1}{n}$

x	-1		$-\frac{1}{n}$		$\frac{1}{n}$		1
$f_n'(x)$		$- - -$	0	$+++$	0	$- -$	
$f_n(x)$	$-\frac{1}{1+n^2}$	$\searrow$	$-\frac{1}{2n}$	$\nearrow$	$\frac{1}{2n}$	$\searrow$	$\frac{1}{1+n^2}$

DECI $\sup\limits_{[-1,1]} |f_n(x) - f(x)| = \frac{1}{2n} \xrightarrow{n} 0$

AȘADAR $f_n \xrightarrow{u} f$

<u>VARIANTA 2</u>:

$\sup\limits_{x\in[-1,1]} |f_n(x) - f(x)|$ $\begin{cases} \leq a_n \xrightarrow{n} 0 \Rightarrow f_n \xrightarrow[n]{u} f \\ \lim > 0 \text{ și } \lim \xrightarrow{n} 0 \end{cases}$

7

= Seminar 7 =

1) STUDIAȚI CONVERGENȚA SIMPLĂ ȘI UNIFORMĂ A ȘIRURILOR DE FUNCȚII

$f_n:[0,\infty)\to\mathbb{R}$; $f_n(x)=\sqrt{x^2+\frac{1}{n}}$, $(\forall)\, n\in\mathbb{N}^*$

CONVERGENȚĂ SIMPLĂ: FIE $x\in[0,\infty)$

$$\lim_{n\to\infty} f_n(x)=\lim_{n\to\infty}\sqrt{x^2+\underbrace{\frac{1}{n}}_{\to 0}}=\sqrt{x^2}=|x|=x \Rightarrow$$

$\Rightarrow f_n \xrightarrow[n]{s} f$ UNDE $f:[0,\infty)\to\mathbb{R}$; $f(x)=x$

CONV. UNIFORMĂ: $\sup\limits_{x\in[0,\infty)}|f_n(x)-f(x)|=\sup\limits_{x\in[0,\infty)}\left|\sqrt{x^2+\frac{1}{n}}-x\right|=$

$$=\sup_{x\in[0,\infty)}\frac{\cancel{x^2}+\frac{1}{n}-\cancel{x^2}}{\sqrt{x^2+\frac{1}{n}}+x}=\sup_{x\in[0,\infty)}\frac{\frac{1}{n}}{\sqrt{x^2+\frac{1}{n}}+x}\leq\frac{\frac{1}{n}}{\sqrt{0^2+\frac{1}{n}}+0}=\frac{1}{\sqrt{n}}\xrightarrow[n]{}0\Rightarrow$$

$\Rightarrow f_n \xrightarrow[n]{u} f$

= FUNCȚII CU MAI MULTE VARIABILE $\mathbb{R}$ =

1) a) CONTINUITATEA LUI f?

b) DETERMINAȚI $\frac{\delta f}{\delta x}$; $\frac{\delta f}{\delta y}$ = ?

c) DERIVABILITATEA LUI "f"?

i) $f:\mathbb{R}^2\to\mathbb{R}$; $f(x,y)=\begin{cases}\frac{xy}{\sqrt{x^2+y^2}} ; & (x,y)\neq(0,0)\\ 0 ; & (x,y)=(0,0)\end{cases}$

a) "f" CONT. PE $\mathbb{R}-\{(0,0)\}$. STUDIEM CONT. LUI "f" ÎN $(0,0)$

FIE $(x,y)\neq(0,0)$

$$|f(x,y)-f(0,0)|=\left|\frac{xy}{\sqrt{x^2+y^2}}-0\right|=\frac{|xy|}{\sqrt{x^2+y^2}}=|x|\cdot\underbrace{\frac{|y|}{\sqrt{x^2+y^2}}}_{\leq 1\ (\sqrt{x^2+y^2}\geq\sqrt{y^2}=|y|)}\leq$$

$\leq |x| \xrightarrow[(x,y)\to(0,0)]{} 0 \Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) \Rightarrow$ "f" E CONT. ÎN $(0,0)$

b) FIE $(x,y) \neq (0,0)$

$$\frac{\delta f}{\delta x}(x,y) = \frac{y\sqrt{x^2+y^2} - xy \cdot \frac{2x}{2\sqrt{x^2+y^2}}}{x^2+y^2}$$

NU COINCID

$$\frac{\delta f}{\delta y}(x,y) = \frac{x\sqrt{x^2+y^2} - xy \cdot \frac{2y}{2\sqrt{x^2+y^2}}}{x^2+y^2}$$

NU COINCID

AU LEGĂTURĂ

$$\frac{\delta f}{\delta x}(0,0) = \lim_{t\to 0} \frac{f((0,0)+t(e_1)) - f(0,0)}{t} =$$

$$= \lim_{t\to 0} \frac{f((0,0)+t(1,0)) - f(0,0)}{t} = \lim_{t\to 0} \frac{f(t,0) - f(0,0)}{t} =$$

$$= \lim_{t\to 0} \frac{\frac{t\cdot 0}{\sqrt{t^2+0^2}} - 0}{t} = \lim_{t\to 0} \frac{0-0}{t} = 0$$

$$\frac{\delta f}{\delta y}(0,0) = \lim_{t\to 0} \frac{f((0,0)+t(e_2)) - f(0,0)}{t} = \lim_{t\to 0} \frac{f((0,0)+t(0,1)) - f(0,0)}{t} =$$

$$= \lim_{t\to 0} \frac{f(0,t) - f(0,0)}{t} = \lim_{t\to 0} \frac{\frac{0\cdot t}{\sqrt{0^2+t^2}} - 0}{t} = \lim_{t\to 0} \frac{0-0}{t} = 0$$

DECI:

$$\frac{\delta f}{\delta x}(x,y) = \begin{cases} \frac{y\sqrt{x^2+y^2} - xy \cdot \frac{x}{\sqrt{x^2+y^2}}}{x^2+y^2} ; & (x,y) \neq (0,0) \\ 0 ; & (x,y) = (0,0) \end{cases}$$

$$\frac{\delta f}{\delta y}(x,y) = \begin{cases} \frac{x\sqrt{x^2+y^2} - xy \cdot \frac{y}{\sqrt{x^2+y^2}}}{x^2+y^2} ; & (x,y) \neq (0,0) \\ 0 ; & (x,y) = (0,0) \end{cases}$$

c) $\frac{\delta f}{\delta x}; \frac{\delta f}{\delta y}$ (EXISTĂ ȘI) SUNT CONT. PE $\mathbb{R}^2 \setminus \{(0,0)\}$ } f DERIVATĂ PE $\mathbb{R}^2 - \{(0,0)\}$

$\mathbb{R}^2 \setminus \{(0,0)\}$ DESCHISĂ

STUDIEM DERIVABILITATEA ÎN $(0,0)$

DEF: DACĂ "f" AR FI DERIVABILĂ ÎN $(0,0)$ ATUNCI $\overbrace{f'(0,0)}^{T} : \mathbb{R}^2 \to \mathbb{R}$;

$$\underbrace{f'(0,0)}_{T}(u,v) \overset{t}{=} \left[\left(\underset{0}{\frac{\delta f}{\delta x}(0,0)} \quad \underset{0}{\frac{\delta f}{\delta y}(0,0)} \right) \cdot \begin{pmatrix} u \\ v \end{pmatrix} \right] = 0$$

TRANSPUS MATRICI

$$l = \lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - f'(0,0)((x,y)-(0,0))}{\|(x,y)-(0,0)\|}$$

$= 0 \Rightarrow$ "f" DERIV. ÎN $(0,0)$

$\neq 0 \Rightarrow$ "f" NU " — " — "

$$= \lim_{(x,y)\to(0,0)} \frac{\frac{xy}{\sqrt{x^2+y^2}} - 0 - 0}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

FIE $(x_n, y_n) = \left(\frac{1}{n}, \frac{1}{n}\right)$, $(\forall)\, n \in \mathbb{N}^*$

AVEM $\lim_{n\to\infty} (x_n, y_n) = (0,0)$ ŞI $\lim_{n\to\infty} \frac{x_n y_n}{x_n^2 + y_n^2} = \lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \frac{1}{2} \neq 0$

DECI $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} \neq 0$, i.e. $l \neq 0$, i.e. "f" NU E DERIV. ÎN $(0,0)$

ii) $f: \mathbb{R}^2 \to \mathbb{R}$; $f(x,y) = \begin{cases} \frac{x^3y^4}{x^6+y^6}; & (x,y) \neq (0,0) \\ 0; & (x,y) = (0,0) \end{cases}$

a) "f" CONT. PE $\mathbb{R}^2 - \{(0,0)\}$. STUDIEM CONT. ÎN $(0,0)$

V1 (MAJORĂRI AD-HOC): FIE $(x,y) \neq (0,0)$

$$|f(x,y) - f(0,0)| = \left|\frac{x^3y^4}{x^6+y^6} - 0\right| = \frac{|x^3y^4|}{x^6+y^6} = |y| \cdot \frac{|x^3y^3|}{x^6+y^6} \leq \frac{1}{2}|y| \xrightarrow[(x,y)\to(0,0)]{} 0 \Rightarrow$$

$\leq \frac{1}{2}$ $\left(\frac{x^6+y^6}{2} \geq \sqrt{x^6y^6} = |x^3y^3|\right)$

$\Rightarrow$ "f" CONT. ÎN $(0,0)$

V2 (ALGORITM): $\to$ CÂND JOS E PUTERE PARĂ

FIE $(x,y) \neq (0,0)$

$$|f(x,y) - f(0,0)| = \left|\frac{x^3y^4}{x^6+y^6} - 0\right| = \frac{|x^3y^4|}{x^6+y^6} = \frac{|x|^3 \cdot |y|^4}{x^6+y^6} = \left(\frac{|x|^6}{x^6+y^6}\right)^{\frac{3}{6}} \cdot \left(\frac{|y|^6}{x^6+y^6}\right)^{\frac{4}{6}} \cdot (x^6+y^6)^{\frac{3}{6}+\frac{4}{6}-1} =$$

$$= \underbrace{\left(\frac{x^6}{x^6+y^6}\right)^{\frac{3}{6}}}_{\leq 1} \cdot \underbrace{\left(\frac{y^6}{x^6+y^6}\right)^{\frac{4}{6}}}_{\leq 1} \cdot (x^6+y^6)^{\frac{1}{6}} \leq (x^6+y^6)^{\frac{1}{6}} \xrightarrow[(x,y)\to(0,0)]{} 0 \Rightarrow f \text{ CONT. ÎN } (0,0)$$

b) FIE $(x,y) \neq (0,0)$

$$\frac{\partial f}{\partial x}(x,y) = \frac{3x^2y^4(x^6+y^6) - x^3y^4 \cdot 6x^5}{(x^6+y^6)^2}$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{4y^3x^3(x^6+y^6) - x^3y^4\, 6y^5}{(x^6+y^6)^2}$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{t\to0} \frac{f((0,0)+t(e_1)) - f(0,0)}{t} = \lim_{t\to0} \frac{f(t,0) - f(0,0)}{t} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{t\to0} \frac{f((0,0)+t(e_2)) - f(0,0)}{t} = \lim_{t\to0} \frac{f(0,t) - f(0,0)}{t} = 0$$

c) $\left.\begin{array}{l} \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \text{ CONT. PE } \mathbb{R}^2 - \{(0,0)\} \\ \mathbb{R}^2 - \{(0,0)\} \text{ DESCHISĂ} \end{array}\right\} \Rightarrow$ "f" DERIV. PE $\mathbb{R}^2 \setminus \{(0,0)\}$

STUDIEM DERIV. ÎN $(0,0)$

DACĂ "f" AR FI DERIV. ÎN (0,0) ATUNCI: $f'(0,0): \mathbb{R}^2 \to \mathbb{R}$; $f'(0,0)(u,v) \overset{(*)}{=}$

$\overset{(*)}{=} \left[\left(\frac{\partial f}{\partial x}(0,0) \quad \frac{\partial f}{\partial y}(0,0) \right) \cdot \begin{pmatrix} u \\ v \end{pmatrix} \right] = 0$

$$l = \lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - f'(0,0)((x,y)-(0,0))}{\|(x,y)-(0,0)\|} = \lim_{(x,y)\to(0,0)} \frac{\frac{x^3y^4}{x^6+y^6} - 0 - 0}{\|(x,y)\| = \sqrt{x^2+y^2}} =$$

$$= \lim_{(x,y)\to(0,0)} \frac{x^3y^4}{(x^6+y^6)\sqrt{x^2+y^2}}$$

FIE $(x_n, y_n) = \left(\frac{1}{n}, \frac{1}{n}\right)$, $(\forall)\, n \in \mathbb{N}^*$

AVEM $\lim_{n\to\infty} (x_n, y_n) = (0,0)$ ȘI $\lim_{n\to\infty} \frac{x_n^3 y_n^4}{(x_n^6 + y_n^6)\sqrt{x_n^2+y_n^2}} = \lim_{n\to\infty} \frac{\frac{1}{\cancel{n^7}}}{\frac{2}{\cancel{n^6}} \cdot \frac{\sqrt{2}}{\cancel{n}}} = \frac{1}{2\sqrt{2}} \neq 0$

$l \neq 0 \Rightarrow f$ NU E DERIV. ÎN (0,0)

PCT. EXTREM LOCAL PT. FUNCȚII CU MAI MULTE VARIABILE

8

T. FERMAT, CAZUL MULTIDIMENSIONAL:

Fie $f: \Delta \subseteq \mathbb{R}^p \to \mathbb{R}$; $p \geq 1$ și $c \in \Delta$ a.î.:

1) $c \in \mathring{\Delta}$

2) c = PCT. EXTREM LOCAL "f"

3) "f" DERIV. ÎN "c"

ATUNCI $f'(c) \equiv 0$ (i.e. "c" = PCT. CRITIC "f")

1) PCT. EXTREM LOCAL, SPECIFICÂND NAT. LOR PT.:

a) $f: \mathbb{R}^2 \to \mathbb{R};\ f(x,y) = x^3 + 8y^3 - 2xy$

$\left.\begin{array}{l} \mathbb{R}^2 \to \text{DESCHISĂ} \\ \left.\begin{array}{l} \dfrac{\delta f}{\delta x}(x,y) = 3x^2 - 2y \\ \dfrac{\delta f}{\delta y}(x,y) = 24y^2 - 2x \end{array}\right\} \text{CONTINUE PE } \mathbb{R}^2 \end{array}\right\} \Rightarrow f \text{ DERIV PE } \mathbb{R}^2$

$\begin{cases} \dfrac{\delta f}{\delta y}(x,y) = 0 \\ \dfrac{\delta f}{\delta x} = 0 \end{cases} \Leftrightarrow \begin{cases} 3x^2 - 2y = 0 \\ 24y^2 - 2x = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$ sau $\begin{cases} x = \frac{1}{3} \\ y = \frac{1}{6} \end{cases}$

PCT. CRITICE "f": $(0,0)$ și $\left(\frac{1}{3}; \frac{1}{6}\right)$

DERIVATE PARȚIALE DE ORDIN 2:

$\dfrac{\delta^2 f}{\delta x^2}(x,y) = \dfrac{\delta}{\delta x}\left(\dfrac{\delta f}{\delta x}\right)(x,y) = 6x$

$\dfrac{\delta^2 f}{\delta y^2}(x,y) = 48y$

$\dfrac{\delta^2 f}{\delta x \delta y}(x,y) = \dfrac{\delta}{\delta x}\left(\dfrac{\delta f}{\delta y}\right)(x,y) = -2$

$\dfrac{\delta^2 f}{\delta y \delta x}(x,y) = \dfrac{\delta}{\delta y}\left(\dfrac{\delta f}{\delta x}\right) = -2$

TOATE DERIV. PARȚIALE DE ORDIN 2 SUNT CONT.

$f''(x,y) = \begin{pmatrix} \dfrac{\delta^2 f}{\delta x^2}(x,y) & \dfrac{\delta^2 f}{\delta x \delta y}(x,y) \\ \dfrac{\delta^2 f}{\delta y \delta x}(x,y) & \dfrac{\delta^2 f}{\delta y^2}(x,y) \end{pmatrix} = \begin{pmatrix} 6x & -2 \\ -2 & 48y \end{pmatrix}$

$f''(0,0) = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}$

$\left.\begin{array}{l} \Delta_1 = 0 \\ \Delta_2 = \begin{vmatrix} 0 & -2 \\ -2 & 0 \end{vmatrix} = -4 < 0 \end{array}\right\} \Rightarrow (0,0)$ pct. șa a lui f

$f''\left(\frac{1}{3}, \frac{1}{6}\right) = \begin{pmatrix} 2 & -2 \\ -2 & 8 \end{pmatrix}$

$\left.\begin{array}{l} \Delta_1 = 2 > 0 \\ \Delta_2 = 12 > 0 \end{array}\right\} \Rightarrow \left(\frac{1}{3}; \frac{1}{6}\right)$ pct. minim local "f"

b) $f:\mathbb{R}^2\to\mathbb{R}$; $f(x,y)=x^3+3xy^2-15x-12y$

$\mathbb{R}^2$ deschisă

$$\left.\begin{aligned}&\frac{\partial f}{\partial x}(x,y)=3x^2+3y^2-15\\&\frac{\partial f}{\partial y}(x,y)=6xy-12\end{aligned}\right\}\text{ sunt continue pe } \mathbb{R}^2 \Rightarrow f \text{ DERIVABILĂ PE } \mathbb{R}^2$$

$$\begin{cases}\frac{\partial f}{\partial x}(x,y)=0\\ \frac{\partial f}{\partial y}(x,y)=0\end{cases} \Leftrightarrow \begin{cases}3x^2+3y^2-15=0\\ 6xy-12=0\end{cases} \Leftrightarrow$$

$$\begin{cases}x=1\\ y=2\end{cases} \quad \begin{cases}-1\\ -2\end{cases} \quad \begin{cases}2\\ 1\end{cases} \quad \begin{cases}-2\\ 1\end{cases}$$

$$\frac{\partial^2 f}{\partial x^2}(x,y)=6x$$

$$\frac{\partial^2 f}{\partial y^2}(x,y)=6x$$

$$\frac{\partial^2 f}{\partial x\partial y}=6y$$

$$\frac{\partial^2 f}{\partial y\partial x}(x,y)=6y$$

lim sup
inf

Serii
topologie
Deriv. continuitate...

TOATE DERIV. PARȚIALE-S CONT..

$$\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 f}{\partial y\partial x}$$

$$f''(x,y)=\begin{pmatrix}6x & 6y\\ 6y & 6x\end{pmatrix}$$

$$f''(1,2)=\begin{pmatrix}6 & 12\\ 12 & 6\end{pmatrix} \quad \left.\begin{aligned}&\Delta_1=6>0\\&\Delta_2=\begin{vmatrix}6 & 12\\ 12 & 6\end{vmatrix}=-108<0\end{aligned}\right\}\Rightarrow (1\ 2) \text{ punct șa}$$

>0 ; >0 .. $\Rightarrow$ minim

<0, >0 ; <0 ; >0 ; <0 $\Rightarrow$ maxim

primul ; altfel nu

$=0$; $=0$ $\Rightarrow$ criteriul nu decide

>0 ; <0
<0 ; <0 } punct șa