0.1 Useful lemmas

Lemma—AJD. Suppose b is a bank of a pond. Then there is a $\Gamma \in A^*$ such that $b\Gamma$ belongs to a doubly infinite φ-orbit that meets X.

Proof. Induction on the width of the pond. Written in an email. □

Lemma. Suppose $\ell \in X\langle A\rangle \setminus Y\langle A\rangle$ and $r \in X\langle A\rangle \setminus Z\langle A\rangle$. Let Γ be such that $\ell\Gamma$ belongs to a doubly infinite φ-orbit. (Such a Γ exists by AJD's lemma.) Then:

1. If (ℓ, r) are banks of a pond, $\ell\Gamma$ and $r\Gamma$ share an orbit.
2. If $\ell\Gamma$ and $r\Gamma$ share an orbit, then there is at most one solution to the equation $\ell\varphi^k = r$.

Proof.
1. If $\ell\varphi^k = r$, then $\ell\Gamma\varphi^k = \ell\varphi^k\Gamma = r\Gamma$.
2. We will show that

$$\ell\Gamma\varphi^k = r\Gamma \implies (\text{If } \ell\varphi^m = r, \text{ then } k = m).$$

   This is a statement of the form $A \implies (B \implies C)$, which is equivalent to the statement $\neg(A \wedge B \wedge \neg C)$. We prove the second statement by showing that $A \wedge B \wedge \neg C$ causes a contradiction.

   Suppose that $\ell\Gamma\varphi^k = r\Gamma$, $\ell\varphi^m = r$, and that $k \neq m$. It follows that $\ell\Gamma\varphi^m = r\Gamma = \ell\Gamma\varphi^k$, meaning that $\ell\Gamma\varphi^{m-k} = \ell\Gamma$. Since $k \neq m$, this means that $\ell\Gamma$ belongs to a periodic φ-orbit. But this contradicts the fact that $\ell\Gamma$ is in a doubly-infinite orbit.

□

0.1.1 Test for banks

Now we outline a procedure to test if (ℓ, r) are banks of a pond.

1. Find a string Γ for which $\ell\Gamma$ is in a doubly infinite orbit.
2. Run the orbit sharing test on $\ell\Gamma$ and $r\Gamma$.
3. **If** the test fails, then (ℓ, r) are not the banks of a pond. [This is the contrapositive of part 1.]
4. **Else** we have k such that $\ell\Gamma\varphi^k = r\Gamma$. By part 2, (ℓ, r) are banks if and only if $\ell\varphi^k = r$. We can evaluate this equality.

0.2 Gathering pond data

1. Construct the quasi-normal basis X as before, and compute $X\langle A\rangle \setminus Y\langle A\rangle$ and $X\langle A\rangle \setminus Z\langle A\rangle$.
2. **For** each terminal element ℓ in $X\langle A\rangle \setminus Y\langle A\rangle$:
   (a) Let Γ iterate through A^*, starting with $\Gamma = 1$, then words of length $1, 2, \ldots$. **For** each Γ:
      i. Compute $w = \ell\Gamma\varphi$.

   ii. Test to see if w belongs to a doubly-infinite type C X-component.
   iii. **If** so, **break**.

   (b) At this stage, we have $\ell\Gamma$ which belongs to a doubly infinite orbit. Record the tuple (ℓ, Γ).

When I first wrote this, I used the proof of AJD's lemma to find Γ. This was probably more efficient but it was tricky to implement. I decided to keep things simple!.

3. Let R be a set. Initialise R with the contents of $X\langle A\rangle \setminus Z\langle A\rangle$.

4. Try to match ℓ to $r \in R$. **For** each $\ell \in X\langle A\rangle \setminus Y\langle A\rangle$:

   (a) **For** each $r \in R$:

      i. Use the previous proceedure and the tuples (ℓ, Γ) to see if (ℓ, r) are banks of a pond.
      ii. **If** so, record this fact, taking note of the value k for which $\ell\varphi^k = r$. Then remove r from R, **break** and **continue** to the next $\ell' \in L$.

   (b) If we exhaust all remaning initial words r, then ℓ is in a genuinely left semi-infinite φ-orbit.

5. Any elements remaining in R are in genuinely left semi-infinite φ-orbits.

0.3 Modifications to the orbit-sharing test

1. Let us skip forward to the point where we assume $u, v \in X\langle A\rangle$. If it is true that u and v share an orbit, then either they belong to the same X-component, or they are separated by a pond.

2. Look to see if u is in a pond-orbit. **For** each 'bank' word b:

   (a) Run the orbit-sharing test on b and u. Let S denote the solution set.
   (b) **If** the test returns $S = \emptyset$, **continue** to the next bank word.
   (c) Run the component-sharing test on (b, v) and (b', v), where b' is the other bank of the pond in question.
   (d) **If** one of these two tests returns a non-empty solution set, u and v share an orbit. Combine the two non-empty solution sets and **return**.
   (e) **Else**, **return** $\emptyset$.

   If we reach this point without returning, then u does not lie in a pond orbit. It may be the case that v *does* lie in a pond orbit; but then this certainly means that u and v do not share an X-component. Thus the rest of the tests in Lemma 4.24.2 should conclude that u and v do not share an orbit.

Remark This is pretty clunky and involves repeatedly invoking the orbit-sharing test. When I came to implement this, instead of the above I did the following. When we compute the list (7) of Lemma 4.24, (which I've started calling the *core* part of the component in question), I check to see if it contains a bank ℓ. If not, then proceed as normal. Otherwise, I check to see which r corresponds to ℓ, and add the 'core' part of the component containing r to the list (7).