Unit 2 (Starting with Recitation 3)

Thursday, July 22, 2021 2:52 PM

Recitation 3:

$(a+bi)(c+di) = ac + adi + bci - bd$

$\Rightarrow ac - bd + (ad+bc)i$

$(1+\sqrt{3}i)(a+bi) = a - \sqrt{3}b + (b+\sqrt{3}a)i$

$(1+\sqrt{3}i) = 2e^{i\pi/3}$

$2e^{i\pi/3} \times re^{i\theta} = 2re^{i(\theta + \frac{\pi}{3})}$

$e^{it} = \cos t + i\sin t$

$e^{i4t} = \cos 4t + i\sin 4t = (\cos t + i\sin t)^4$

$= \cos^4 + 4i\cos^3\sin - 6\cos^2\sin^2 - 4i\cos\sin^3 + \sin^4$

$\sin 4t = 4\cos^3\sin - 4\cos\sin^3$

$1 = x^3 \Rightarrow x = e^{\frac{2\pi i}{3}} \quad x^2 = e^{\frac{4\pi i}{3}} \quad x^3 = 1 \ldots$

$i = e^{\frac{\pi i}{2}} = x^4 \Rightarrow x = e^{(\frac{\pi}{8} + \frac{2\pi k}{4})i}$

$x^6 - 2x^3 + 2 = 0 \qquad y = x^3 \rightarrow y^2 - 2y + 2 = 0$

$\sqrt{\Delta} = \sqrt{4-8} = \sqrt{-4} = 2i \Rightarrow \frac{+2 \pm 2i}{2} < \begin{matrix} 1-i \\ 1+i \end{matrix} = x^3$

$x^3 = \sqrt{2}e^{\frac{\pi}{4}i} = \sqrt[6]{2}\, e^{\frac{\pi}{12}i} \times e^{i\frac{2\pi k}{3}}$

$x^3 = \sqrt{2}e^{-\frac{\pi}{4}i} = \sqrt[6]{2}\, e^{-\frac{\pi}{12}i}$

Recitation 4.

$\cos 3t = \frac{e^{3it} - e^{-3it}}{2} \times e^{2t} \rightarrow$

$(2-3i)t \qquad (2+3i)t \quad (2-3i)t$

$\sin 3t = \frac{e}{2} \times$

$\frac{1}{2}\int (e^{(2+3i)t} - e^{(2-3i)t})\, dt = \frac{1}{2}\alpha\left(\frac{e^{(2+3i)t}}{2+3i} - \frac{e^{(2-3i)t}}{2-3i}\right)$

$\Rightarrow \frac{1}{2}\alpha\, \frac{(2-3i)e^{(2+3i)} - (2+3i)e^{(2-3i)}}{13} = \cdots$

$\ddot{x} + x = 0 \qquad x = e^{rt}$

$\Rightarrow r^2 e^{rt} + e^{rt} = 0 \Rightarrow r^2 + 1 = 0$

$\Rightarrow r = \pm i$

$5e^{-t} + 81e^{3t} + (-5e^{-3} + 27e^{3t})A + B(5e^{-t} + 9e^{3t}) = 0$

$\Rightarrow (5 - 5A + 5B)\, e^{-t} + (81 + 27A + 9B)\, e^{3t} = 0$

$\Rightarrow -5A + 5B = -5 \Rightarrow -A + B = -1$

$81 + 9B + 27A = 0 \qquad 9 + B + 3A = 0$

$\left.\begin{array}{l} A - B = 1 \\ B + 3A = -9 \end{array}\right\} \Rightarrow 4A = -8 \Rightarrow A = -2$

$\Rightarrow B = -3 \quad \Rightarrow \ddot{y} - 2\dot{y} - 3y = 0$

$x = e^{rt} \qquad r^2 + Ar + B = 0 \qquad x(0) = 0 \quad x'(0) = 0 \rightarrow x''(0) = 0$

Differentiating this infinite times, we realize $x = 0$

$r^2 + r - 3 \rightarrow \quad y'' + y' - 3y \rightarrow 2y'' + 2y' - 6y = 0$

Recitation 5.

$r^2 + br + 16 = 0 \qquad b^2 > 64 \Rightarrow b > 8$

$r = \frac{-b \pm \sqrt{b^2 - 64}}{2} \rightarrow e^{\frac{-b}{2}}\left(\overset{x_2}{e^{\frac{\sqrt{b^2-64}}{2}}} + \overset{x_1}{e^{\frac{-\sqrt{b^2-64}}{2}}}\right)$

$e^{\frac{1}{2}(-b + \sqrt{b^2-64})t} \qquad e^{\frac{1}{2}(-b - \sqrt{b^2-64})t}$

x

$$x(t) = A e^{\frac{1}{2}(-b + \sqrt{b^2-6a})t} + B e^{\frac{1}{2}(-b-\sqrt{b^2-6a})t}$$

$$x(0) = A + B = 1$$

$$x'(t) = A\alpha \frac{1}{2}(-b + \sqrt{b^2-6a}) + B\gamma \frac{1}{2}(-b - \sqrt{b^2-6a}) = -\frac{b}{2}$$

$$\Rightarrow A(-b + \sqrt{b^2-6a}) + B(-b - \sqrt{b^2-6a}) = -b$$

$$Ab + Bb = b$$

$$\Rightarrow A\sqrt{b^2-6a} - B\sqrt{b^2-6a} = 0$$

$$\Rightarrow A = B \Rightarrow A = B = \frac{1}{2}$$

$$x(t) = A e^{\frac{1}{2}(-b+\sqrt{b^2-6a})t} + B e^{\frac{1}{2}(-b-\sqrt{b^2-6a})t} = 0$$

We Know $\forall x \in \mathbb{R}$; $e^x > 0$ $\rightarrow$ Either always Positive / Passes once.

For $b < 8$ $\rightarrow$ Complex Solutions

$$r = \frac{-b \pm i\sqrt{6a-b^2}}{2} \Rightarrow e^{-\frac{b}{2}t}\left(\overset{A}{\cos} \frac{\sqrt{6a-b^2}}{2} t + \overset{B}{\sin} \frac{\sqrt{6a-b^2}}{2} t\right)$$

$$x(0) = A = 1$$

$$x(t) = -\frac{b}{2} e^{-\frac{b}{2}t}(A\cos \dots + B\sin \dots) + e^{-\frac{b}{2}t} \frac{\sqrt{6a-b^2}}{2}(-A\sin \dots + B\cos \dots)$$

$$= -\frac{b}{2}A + \frac{\sqrt{6a-b^2}}{2}B = -\frac{b}{2} \Rightarrow B = 0$$

$$\Rightarrow e^{-\frac{b}{2}} \cos \frac{\sqrt{6a-b^2}}{2} t$$

$$T\omega = \frac{\sqrt{6a-b^2}}{2} T = 2\pi \Rightarrow \frac{4\pi}{\sqrt{6a-b^2}} = T$$

Real Part of roots of $mr^2 + br + K = 0$

for $m > 0$, $b, K \geq 0$ is ≤ 0.

$$m\ddot{x} + b\dot{x} + Kx = 0 \Rightarrow m\ddot{x} = -b\dot{x}$$

$$y = \dot{x} \qquad \dot{y} = \frac{-b}{m} y$$

$$\to y = Ce^{\frac{-b}{m}t} \neq 0$$

$$\frac{dx}{dt} = Ce^{\frac{-b}{m}t} \to x = \frac{-m}{b} Ce^{\frac{-b}{m}t} + D$$

$$r^2 - 7r - 8 = 0 \qquad \sqrt{49+32} = \sqrt{81} = 9$$

$$\frac{+7 \pm 9}{2} < \begin{matrix} 8 \\ -1 \end{matrix} \qquad e^{-t} \quad e^{+8t}$$

$$(r+3-i)(r+3+i) = 0$$

oh yeah.

$$z^4 = 4 \longrightarrow z = \sqrt{2} e^{i\theta} \to \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

$$z^2 + 2z + 2 = 0 \Rightarrow \frac{-2 \pm \sqrt{4-8}}{2} < \begin{matrix} -1+i \\ -1-i \end{matrix}$$

$$\frac{3}{1-i} \cdot \frac{1+i}{1+i} = \frac{3+3i}{2} = \frac{3}{2}\sqrt{2}\, e^{i\frac{\pi}{4}}$$

$$\frac{2}{1-i} \cdot \frac{1+i}{1+i} = \frac{}{2} = \frac{2}{2}\sqrt{2}\, e^{\frac{\pi}{4}}$$

$$e^{1-\pi i} = e\, e^{-\pi i} = -e$$

$$e^{3}\, e^{\frac{\pi i}{2}} = e^{3} i$$

$Z = a + ib \rightarrow f(t) = Re(e^{Zt})$

$f(t) = \cos(2\pi t) \rightarrow a = 0 \quad b = 2\pi$

$f(t) = e^{-t} \rightarrow a = -1 \quad b = 0$

$f(t) = e^{-t}\cos(2\pi t) \rightarrow a = -1 \quad b = 2\pi$

$f(t) = 1 \rightarrow a = 0,\ b = 0$

$$Z(t) = \frac{e^{4it}}{1+i} \qquad |Z(t)| = \frac{1}{\sqrt{2}}$$

$$\text{Polar} \rightarrow \frac{e^{4it}}{\sqrt{2}\, e^{i\frac{\pi}{4}}}$$

$$z'(t) = \frac{4i\, e^{4it}}{1+i} = \frac{4ie^{4it}}{\sqrt{2}\, e^{i\frac{\pi}{4}}} = \frac{4i}{\sqrt{2}}\, e^{(4t-\frac{\pi}{4})i}$$

$$Im\,(z'(t)) = \frac{4}{\sqrt{2}}\cos\left(4t - \frac{\pi}{4}\right)$$

$$f(t) = Im\left(\frac{e^{4it}}{1+i}\right) = Im\left(\frac{1}{\sqrt{2}}\, e^{(4t-\frac{\pi}{4})i}\right) = \frac{1}{\sqrt{2}}\sin\left(4t - \frac{\pi}{4}\right)$$

$$f'(t) = \frac{4}{\sqrt{2}}\cos\left(4t - \frac{\pi}{4}\right)$$

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3. Mathlet - Swinging Door.

Char. Poly. $\rightarrow \frac{1}{2}r^2 + \frac{3}{2}r + \frac{5}{8} = 0$

$$\sqrt{\frac{9}{4} - \frac{5}{4}} = \quad \frac{-3/2 \pm 1}{1} \begin{cases} -5/2 \\ -1/2 \end{cases}$$

$$\rightarrow x(t) = Ae^{-\frac{5}{2}t} + Be^{-t/2}$$

$x(0) = \quad A + B = x_0$

$x'(0) = -\frac{5}{2}A - \frac{1}{2}B = V_0 \Rightarrow -5A - B = 2V_0$

$\Rightarrow -4A = x_0 + 2V_0 \Rightarrow A = \frac{x_0 + 2V_0}{-4}$

$\frac{x_0 + 2v_0}{-4} + B = \frac{-4x_0}{-4} \Rightarrow B = \frac{-5x_0 - 2V_0}{-4}$

$= B = \frac{5x_0 + 2v_0}{4}$

$$\Rightarrow 4x(t) = (-x_0 - 2v_0)e^{-\frac{5}{2}t} + (5x_0 + 2v_0)\,e^{-t/2}$$

if $(x_0 = -2v_0)$ OR $x_0 = -\frac{2}{5}V_0 \rightarrow$ Single exp

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$x(0) = 0.25 = A + B$, $\quad -\frac{5}{2}A - \frac{1}{2}B = V_0 < 0$

$t_1 > 0$

$x(t_1) = 0 \quad Ae^{-5/2 t_1} + Be^{-1/2 t_1} = 0 \Rightarrow Ae^{-2t_1} + B = 0$

$A(1 - e^{-2t_1}) = 0.25 \quad t_1 > 0 \rightarrow A > 0, B < 0$

$\frac{1}{4} = A + B \Rightarrow \frac{-5}{8} = -\frac{5}{2}A - \frac{5}{2}B = V_0 - 2B$

$\frac{1}{4} = A + B \Rightarrow \frac{-5}{8} = -\frac{1}{2}A - \frac{1}{2}B = V_0 - 2B$

Since $B < 0 \to -2B > 0 \Rightarrow -\frac{5}{8} = V_0 + C \Rightarrow V_0 < -\frac{5}{8}$

$$\frac{1}{2}r^2 + \frac{1}{4}r + \frac{51}{25} = 0 \qquad -\frac{1}{4} \pm 2i$$

$$\sqrt{\frac{1}{16} - \frac{4 \times 51}{50}} \simeq 2i$$

$$x(t) = e^{-\frac{1}{4}t}(A\cos 2t + B\sin 2t)$$

$$x(0) = A = 0$$

$$x'(0) = -\frac{1}{4}e^{-\frac{1}{4}}(A\cos 2t + B\sin 2t) + e^{-\frac{1}{4}t}(-2A\sin 2t + 2B\cos 2t)$$

$$2B = 1 \Rightarrow B = \frac{1}{2} \qquad 2.4 - 0.8 = 1.6$$

$$mr^2 + br + K = 0$$

$\frac{\sqrt{4mK - b^2}}{2m} \to$ Angular Frequency
