Unit 4

Monday, August 2, 2021 1:31 AM

$P(D)e^{rt} = P(r)e^{rt}$

$\Rightarrow \frac{1}{P(r)}e^{rt}$ is a Particular Solution to $P(D)y = e^{rt}$

Example: Gives Particular Solution

$\ddot{x}+x = e^{2it} \to (D^2+1)x = e^{2it}$

$$\Rightarrow \frac{1}{(2i)^2+1}e^{2it} = \frac{1}{-3}e^{2it}$$

$$\Rightarrow (D^2+1)y = e^{-t} \Rightarrow \frac{1}{(-1)^2+1}e^{-t} = \frac{1}{2}e^{-t}$$

$(D^2+1) = e^{-t} \Rightarrow \frac{1}{2}e^{-t}$

$(D^2+1) = -3e^{2it} \Rightarrow \frac{(D^2+1)}{-3} = e^{2it}$

$$\Rightarrow \frac{1}{\frac{(2i)^2+1}{-3}}e^{2it} = \frac{-3}{-4+1}e^{2it} = e^{2it}$$

$\Rightarrow y_p = e^{2it} + \frac{1}{2}e^{-t}$

$r^2+1=0 \Rightarrow r = i, -i$

Existence and uniqueness theorem guarantees a [illegible]

$P(r) = 0 \Rightarrow$

Let's see what happens around r_0 where $P(r_0) = 0$

By differentiating w.r.t. r and setting $r = r_0$

$$\frac{\partial}{\partial r}(P(D)e^{rt}) = \frac{\partial}{\partial r}(P(r)e^{rt}) = P'(r)e^{rt} + P(r)te^{rt}$$

RHS: Using continuity of 2nd derivatives:

$$\frac{\partial}{\partial r}D = D\frac{\partial}{\partial r}$$

$$\Rightarrow P(D)\left(\frac{\partial}{\partial r}e^{rt}\right) = P(D)(te^{rt})$$

$\Rightarrow P(D)(te^{rt}) = P'(r)e^{rt} + tP(r)e^{rt}$

$P(D)(te^{rt}) = P'(r)e^{rt} + tP(r)e^{rt}$

$P(r_0) = 0 \Rightarrow P(D)(te^{r_0 t}) = P'(r_0)e^{r_0 t}$

$\Rightarrow P(D)\left(\frac{te^{r_0 t}}{P'(r_0)}\right) = e^{r_0 t}$

The term goes in by linearity. Importante!

$\Rightarrow y_p = \frac{te^{r_0 t}}{P'(r_0)}$ is a Particular Solution to $P(D)y = e^{r_0 t}$

$\Rightarrow$ Formally:

Suppose P is a Polynomial and $P(r_0) = 0$ but $P'(r_0) \neq 0$ for some number r_0. Then:

$x_p = \frac{t}{P'(r_0)} e^{r_0 t}$ is a Particular Solution to $P(D)x = e^{r_0 t}$

$\Rightarrow$ Generalized exponential response formula:

If P is a Polynomial and r_0 is a number such that

$P(r_0) = P'(r_0) = \cdots = P^{(m-1)}(r_0) = 0$; $P^{(m)}(r_0) \neq 0$

then $P(D)(t^m e^{r_0 t}) = P^{(m)}(r_0)e^{r_0 t}$

and $y_p = \frac{t^m}{P^{(m)}(r_0)} e^{r_0 t}$ is a Particular Solution to $P(D)y = e^{r_0 t}$

$y''' = 0 \Rightarrow y'' = C_1 \quad y' = C_1 x + C_2$

$y = C_1 x^2 + C_2 x + C_3$

well, yeah. $r = 0$!

$y = e^{rt}, te^{rt}, t^2 e^{rt}$

$\Rightarrow y = C_1 e^{0t} + C_2 te^{0t} + C_3 t^2 e^{0t}$

$\Rightarrow y = C_1 + C_2 t + C_3 t^2$

Now it makes sense.

I hate math.

Why these are the solutions!!

Lets take $(D-5)^3 y = 0$

Ch. Poly: $(r-5)^3 = 0$

We know $(D-5)e^{5t} = 0$, So what could it Do to $u(t)e^{5t}$

$\Rightarrow (D-5)ue^{5t} = (u'e^{5t} + 5ue^{5t}) - 5ue^{5t}$

$= u'e^{5t}$

$\Rightarrow (D-5)^2 ue^{5t} = u''e^{5t}$

$(D-5)^3 ue^{5t} = u'''e^{5t}$ (*)

$\Rightarrow$ In order for ue^{5t} to be a Solution to $(D-5)^3 y = 0$

u''' must be $0 \Rightarrow u = a + bt + ct^2$

$\Rightarrow ue^{5t} = ae^{5t} + bte^{5t} + ct^2e^{5t}$

Finally.

$(2D^2 + D + 1) = 1 + 2e^t$

Basis

$(2D^2 + D + 1) = e^{0t} \Rightarrow \frac{1}{1} = y \Rightarrow y = 1$

$\frac{(2D^2 + D + 1)}{2} = e^t \Rightarrow \frac{2}{2+1+1} = \frac{1}{2}e^t$

$\Rightarrow y_p = C_0 + \frac{C_1}{2}e^t$

y_g ?: $(2r^2 + r + 1) = 0$

$1 - 8 = \sqrt{7}i$

$\frac{-1 \pm \sqrt{7}i}{4} = e^{(-1-\sqrt{7}i)t}$

$\Rightarrow e^{\frac{-t}{4}}(e^{\frac{-\sqrt{7}it}{4}})$

$D^5 - 1 = e^{at}$

$D^5 - 1 = e^{at}$ $\quad \frac{1}{a^5 - 1} \neq 0 \quad a = 1$

$a =$

Cs & Cassher

$3D^2 - 1 \quad (3 \times 1 - 1) = 2$

$3D^2 - 1 \Rightarrow$ Keep

$\frac{t}{-1} e^{0t} = -t \quad (t)^{4n} = 0$

$\frac{D^2 + 9}{9} = 1 \Rightarrow \frac{1}{9} e^{at} \Rightarrow 1$

$\Rightarrow D^2 + 9 \quad \frac{1}{16} e^{3t}$

1/6 * e^(3*3)

$D^4 - 1 = e^{-2t} \quad \frac{1}{15} e^{2t}$

$10D^2 + 5 = 0 \quad \frac{1}{0}$

$e^{\frac{i}{\sqrt{2}}t} \quad 10r^2 + 5 = 0 \quad r = \frac{i}{\sqrt{2}}$

$\Rightarrow \quad e^{\frac{i}{\sqrt{2}}t}$

$10r^2 + 2r + 5 = 0 \quad \frac{r_0^{10}}{10}$

$4 - 5 \times 4 \times 10 = 4 - 200 = -196$

$14i \quad -2 \quad \frac{1}{2}$

$\frac{-2 \pm 14i}{20} \quad \frac{7}{10}$

$P(D)x = \cos wt \Rightarrow P(D)z = e^{iwt}$

$P(D)x = \cos\omega t \Rightarrow P(D)z = e^{i\omega t}$

$\Rightarrow z_p = \dfrac{e^{i\omega t}}{P(i\omega)}$

$x_p = Re(z_p)$

$P(D)z = 1 \qquad P(D)\,y = A\cos\omega t$

$\rightarrow \dfrac{P(D)}{A} y = \cos\omega t \Rightarrow$

$\dfrac{P(D)}{A} z = e^{i\omega t} \longrightarrow z_p = \dfrac{A}{1} e^{i\omega t}$

Ex!, $\dot{x} + Kx = Ky$

$\dot{x} + Kx = KA\cos(\omega t)$

$\dfrac{(D+K)}{KA} z = e^{i\omega t}$

$\dfrac{(D^2 + bD + 2)x}{2} = e^{it}$

$\dfrac{2}{-1+ib+2} = \dfrac{2}{ib+1} e^{it}$

$\dfrac{2}{\sqrt{2}\, e^{i\tan^{-1} b}}$

$P(D)x = \cos\omega t$

$P(D)x = \sin\omega t$

$P(D)x = e^{i\omega t}$

$Re\left(\dfrac{e^{i\omega t}}{P(i\omega)}\right) = A\cos(\omega t - \phi)$

$Im\left(\dfrac{e^{i\omega t}}{P(i\omega)}\right) = A\sin(\omega t - \phi)$

$r^3 + r^2 + r - 3 = 0$

$r = 1$

$r(r^2 + r + 1) = 3$

not Stable

Recitation 16 $\quad r^2 + 2r + 2 = e^{2it}$

$$\mathrm{Re}\,\frac{e^{2it}}{(2i)^2 + 2\,2i + 2} = \frac{e^{2it}}{-4 + 4i + 2} = \mathrm{Re}\,\frac{e^{2it}}{-2 + 4i}$$

$$\frac{1}{2} \times \frac{e^{2it}}{-1 + 2i} \times \frac{-1-2i}{1-2i} = \frac{1}{10} \times {-1-2i}\; e^{2it}$$

$$= -\frac{1}{10} e^{2it} - \frac{i}{5} e^{2it}$$

$$= -\frac{1}{10} (\cos 2t) - \frac{i}{5} (\cos 2t + i \sin 2t)$$

$$-\frac{1}{10} \cos 2t + \frac{1}{5} \sin 2t$$

$x(t) =$ ...

$x(t) = A e^{(-1+i)t} + \left(\frac{-1}{10}\cos 2t + \frac{1}{5}\sin 2t\right)$

$e^{-t}(A\cos t + B\sin t) + \frac{-1}{10}\cos 2t + \frac{1}{5}\sin 2t$

$x(0) = A - \frac{1}{10} = 0 \Rightarrow A = \frac{1}{10}$

$x'(0) = -e^{-t}(A\cos t + B\sin t) + e^{-t}(-A\sin t + B\cos t)$

$+ \frac{2}{10}\sin 2t + \frac{2}{5}\cos 2t \qquad A = \frac{1}{10}$

$= -A + B + \frac{4}{10} = 0$

$\Rightarrow B + \frac{3}{10} = 0 \Rightarrow B = -0.3$

$r^2 + 2r + 2 = e^{-t}e^{2it} = e^{(2i-1)t}$

$\frac{1}{(2i-1)^2 + 2(2i-1) + 2} e^{(2i-1)t}$

$(2i-1)t$

$$\overline{(2i-1)^2 + 2(2i-1) + 2} \qquad e^{(2i-1)t}$$

$$= \frac{1}{-4+1-\cancel{4i}+\cancel{4i}-\cancel{2}+2} = \frac{e}{-3}$$

$$= -\frac{1}{3} e^{-t} \cos(2t)$$

$$r^2 + 4 = 2e^{it} + 3e^{2it} + 4e^{3it}$$

(above the terms: cos, cos, sin)

$$r^2 + 4 = 2e^{it} \to \frac{2e^{it}}{3} = \frac{2}{3}\cos t$$

$$2 \to \frac{3e^{2it}}{4i} t = 3 \frac{\cos 2t + i\sin 2t}{4i} t$$

$$= 3\frac{\sin 2t}{4} t = \boxed{\frac{3t}{4} \sin 2t}$$

$$3 \to \frac{4e^{3it}}{-9+4} = \frac{4}{-5} e^{3it} = \boxed{2\sin 3t}$$

$$r^2 + \omega_n^2 = Ae^{i\omega t}$$

$$r^2 + \omega_n^2 = Ae^{i\omega t}$$

$$\frac{Ae^{i\omega t}}{-\omega^2 + \omega_n^2} = \frac{A\cos\omega t + Ai\sin\omega t}{-\omega^2 + \omega_n^2}$$

$$\frac{i^2 + i}{i^4 + 2i^2 + 5} = \frac{-1 + i}{1 - 2 + 5} = \frac{-1 + i}{4}$$

$$\frac{(ai)^2 + ai}{(ai)^4 + 2(ai)^2 + 5} = \frac{-a^2 + ai}{a^4 - 2a^2 + 5}$$

$$u^2 + 2u + 5 = 0 \rightarrow \frac{-2 \pm \sqrt{-16}}{2}$$

$$r^2 = -1 \pm 2i \overset{+}{=} \sqrt{5}\, e^{\arctan(-2)i} = r_1$$

$$\Rightarrow e^{\sqrt{5}t} e^{\arctan(-2)it} \rightarrow \text{has a Positive } e^{rt}$$

$$\mathrm{Re}\ \frac{e^{i\omega t}}{9 - \omega^2} = \frac{\cos\omega t}{9 - \omega^2}$$

$9-\omega^2 \qquad 9-\omega^2$

$$r^2+4r=2e^{2it}$$

$$\frac{2te^{2it}}{4i}=\frac{t}{2}\sin(2t)$$

$$\frac{-2p\pm\sqrt{4p^2-4\omega_n^2}}{2}=-p\pm\sqrt{p^2-\omega_n^2}=r$$

$$\frac{Q(i\omega)}{P(i\omega)}=\frac{K}{-\omega^2+bi\omega+K}=\frac{1}{-\omega^2+\frac{3}{2}i\omega+1}$$

Recitation 11.

$$mg=K(L-x_0)$$

$$m\ddot{x}=-K(x_0+x-L)-mg$$

$$m\ddot{x}=-K(x_0+x-L)-K(L-x_0)$$

$$= -Kx_0 - Kx + KL - KL + Kx_0$$

$$m\ddot{x} = -Kx$$

$$\frac{bi\omega + K}{-m\omega^2 + ib\omega + K} \cdot \frac{i\omega + 2}{-\omega^2 + i\omega + 2} \leq 1$$

$$\frac{1}{-\omega^2 + \alpha} \to \frac{1}{-\omega^2 + ib\omega + 2}$$

$$\frac{-\omega^2 + 2 - ib\omega}{(2-\omega^2)^2 - b^2\omega^2} \quad Re$$

$$A e^{\omega t} \to \omega^2 + b\omega + K = 0$$

$$b^2 - 4K < 0$$

$$\frac{-b}{2} > 0$$

$b^2 < 4K$

$r^3 + 5r^2 + 8r + 6 = 0 \qquad 3$

$r = -3,\ -1-i,\ -1+i$

$e^{-t} e^{-it} = e^{-t}(-\cos t - i\sin t)$

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(1) $(r\omega)^3 = \operatorname{Im}(e^{it})$

(2) $(r\omega)^3 = e^{-t}$

(1) $\Rightarrow \dfrac{1}{(i+1)^3} e^{it} = \dfrac{\cos t + i\sin t}{-2+2i}$

$\operatorname{Im}(\ldots) = \dfrac{i\sin t}{-2} + \dfrac{i\cos t}{-2} = \dfrac{(\sin t + \cos t)}{-2}$

$r = -3+2i,\ -3-2i,\ 2,\ 2$

$r^4 + a_3 r^3 + a_2 r^2 + a_1 r + a_0 = 0$

$(r-2),\ ,\ r^3 + 3a_3 r^2 + 2a_2 r + a_1 = \frac{1}{1}$

$(r-2)$

$r^3 + 3a_3 r^2 + 2a_2 r + a_1 = \frac{1}{4}$

$8 + 12a_3 + 4a_2 + a_1 = \frac{1}{4}$

$\rightarrow 48a_3 + 16a_2 + 4a_1 = -31$

$\rightarrow 9a_3 + 2a_2 + a_1 = -8$

$-16 + 8a_3 + 4a_2 + 2a_1 = 0$

$(9a_0 + 5a_1 + a_2 + a_3) = 112$

$(-12a_1 + 46a_0) = 118$

$(23a_0 - 6a_1) = 59$

We know that $t = 2$ gives zero.

$$\boxed{r^4 + a_3 r^3 + a_2 r^2 + a_1 r_1 + a_0 = 0}$$

(1) $16 + 8r_3 + 4r^2 + 2r_1 + a_0 = 0$

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r' $4r^3 + 3a_2 r^2 + 2a_2 r + a_1 = 1$

$p' \to 4r^3 + 3a_3 r^2 + 2a_2 r + a_1 = \frac{1}{4}$

② $32 + 12a_3 + 4a_2 + a_1 = 0.25$

③ $9a_0 + 5a_1 + a_2 + a_3 = 122$

④ $46a_0 - 12a_1 = 118$

Last Effort. $(4+e^{2t})\,(e^{-3t}\sin(2t))$

$(4+e^{2t})^{(4)} = 4(16+e^{2t} + 32e^{2t})$

$(\quad " \quad)^{(3)} = 4(8+e^{2t} + 12e^{2t})$

$'' = 4(4+e^{2t} + 4e^{2t})$

$' = 4(2+e^{2t} + e^{2t})$

$^{\circ} = 4+e^{2t}$

Div by $4e^{2t}$

$\ldots\ 8t a_0 + 12a_1 + 4t a_2 + 4a_3 +$

$2 + a_1 \in a_1$

$16t + 32 + 8ta_3 + 12a_3 + 4ta_2 + 4a_2 +$

$+ a_2 t = 0$

$r^2 + 4r = e^{ut}$

$u^2 + 4 = 0$

$a^n u = 0$

$e^{i\omega t}$

$-m\omega^2 + ib\omega + K$

$\longrightarrow -2\omega m + ib = 0 \Rightarrow \omega = -$

$-m\alpha \; \frac{-b^2}{\;} + \frac{-b^2}{\;} + K$

$2 + a_1 \ e \ a_1$

$$\frac{ib}{2m}$$

$$-m\alpha \frac{-b^2}{4m^2} + \frac{-b}{2m} + K$$

$$x_p = A\sin t + B\cos t$$

$$\sin(A-2B) + \cos(2A+$$

$$\left.\begin{aligned} A - 2B &= 0 \\ 4A + 2B &= 2 \end{aligned}\right\} \Rightarrow \begin{aligned} &5A = \\ &A = \end{aligned}$$

$$x_p = A\sin(2t) + B($$

$$\downarrow$$

$$(D^2 + 2D - 2)\,(A\sin(2t)$$

$B)$ Blet

$2 \Rightarrow A = \frac{2}{5}$

$2B \Rightarrow B = \frac{1}{5}$

$$\begin{array}{ccc} \sin & \to & \cos \\ \uparrow & & \downarrow \\ -\cos & \leftarrow & -\sin \end{array}$$

$\cos(2A)$

$+ B\cos(2t))$

$$(D^2 + 2D + 2)(A \sin(2t)$$

$$= -4A\sin(2t) - 4B\cos(2t)$$

$$-4B\sin(2t) + 12A\sin($$

$$= \sin(2t)$$

$$(-4B + 4A + 2B) = 0$$

$$(-4A - 4B + 2A) = 1$$

$$\Rightarrow B = -\frac{1}{5}$$

$$A = \frac{1}{10}$$

0.1*sin(

$+ 15\cos(2t))$

$+ 4A\cos(2t)$

$2t) + 2B\cos(2t)$

$= (2A - B) = 0$

$(-2A - 4B) = 1$

$-5B = 1$

$2A + \frac{1}{5} = 0$

$2A - B = 0$

$$y_p = At^2 + Bt + C$$

$$(D^2 + 2D + 2)(At^2 + B$$

$$2A + 4At + 2B + 2$$

$$(A + B + C) = 0$$

$$2A = 1 \Rightarrow A = \frac{1}{2}$$

$$B = $$

$$1 \qquad \frac{k - m\omega^2 + ib\omega}{}$$

$t + C)$

$2At^2 + 2Bt + 2C = t^2$

$(4A + 2B) = 0$

$B = 1$

$\frac{1}{2}$

$\frac{}{k - m\omega^2 + ib\omega}$

$$\frac{1}{K - m\omega^2 + ib\omega} \propto \frac{K - m\omega^2 + ib\omega}{(K - m\omega^2) - ib\omega}$$

Denom $= K^2 + m^2\omega^4 - 2Km\omega^2$

Min this $\quad m^2 u^2 + (b^2 - 2Km$

$\frac{d}{d\omega} \quad 2m^2 u + b^2$

$\omega^2 =$

$$\frac{4 - \Delta}{\quad}$$

$$= \frac{K - m\omega + ib\omega}{(K - m\omega^2)^2 + b^2\omega^2}$$

$$+ b^2\omega^2$$

$$)u + K^2 = 0$$

$$- 2Km = 0$$

$$\frac{2Km - b^2}{2m^2} < 0$$

$$< 0$$

$$a \cos\theta + b \sin\theta$$

$$A = \sqrt{a^2 + b^2} \qquad \phi$$

$$\frac{b}{a} = \sqrt{3} \longrightarrow \qquad b^{\,}$$

$$A = \sqrt{2a^2 + a^2} =$$

$$\frac{1}{2}\cos 2t + \frac{3}{4}$$

$$c = a - bi = \frac{1}{2}$$

$= \ln(2 + e^{\pi/3})$

$= \frac{\pi}{3} = \tan^{-1}\left(\frac{b}{a}\right)$

$^2 = 3a^2$

$2a = 1 \Rightarrow a = \frac{1}{2}$

$b = \frac{3}{4}$

$\sin 2\phi \qquad \phi = \frac{\pi}{3}$

$- \frac{3}{4} i$

$$e^{-i\frac{\pi}{3}}$$

$$e^{-\frac{\pi}{3}i} \quad e^{(2i-1)t}$$

$$\frac{1}{(2i-1)^2 + 2(2i-1) + 2}$$

$$\frac{e^{-\frac{\pi}{3}i}\, e^{(2i-1)t}}{-3}$$

$$e^{2it} = e^{\left(2t - \frac{\pi}{3}\right)i}$$

$$D^2 + 7D + 2$$

$$= \overline{\qquad\qquad}$$

$$-4 + 1 - 4i + 4i - 2 + 2$$

$$\overline{\qquad\qquad}$$

$$-3$$

$$e^{-t}(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})x$$

$$= e^{-t}\cos\frac{\pi}{3}\cos 2t -$$

$$\frac{e^{-2t}}{4 - \cos 2} = \frac{1 - e^{-2t}}{2}$$

$$\frac{1}{\quad}$$

$(\cos 2t + i \sin 2t)$

$e^{t} \sin\frac{\pi}{3} \sin 2t$

$e^{2} e^{2} e^{2}$

$= 5e^{-2t} \quad e^{-2t}$

$e^{t} e^{it} = e^{(i+1)t}$

1

$9n e 2$ =

$$\frac{t e^{(i+1)t}}{2i} \longrightarrow \mathcal{F}\, t$$

5*e^(2*t)

$12L = 60\%$ $9L$

$9L = \frac{90}{12}\% = 7.$

$12R = 10\%$ 7

$5A = 10$ $A = 2$

$2i - 2e \cdot 2$

$$\frac{e^{-t} \sin t}{2}$$

$= \frac{10}{12}\%$

5% Lecture

% Recitation

6% A

$5A = 10 \quad A = 2$

$4B = 25 \qquad 9$

$B = 6.25$

Before Exam : 35.

35.75%

Fall on Exam :

6 %

A

-e 6.25 = 15.25 B

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75 % from 55 %

Rаен = 45 %

So 75 %

10Σ!

Kull ldam Ba

83 16.6

0.3

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by !