Answer to recitation problems

Friday, July 9, 2021 4:53 PM

$\frac{dx}{dt} = Kx - a$ Modeling the ...

$\dot{x} - Kx = -a$

input: $-a$, System Response: x

impulse Response?

Solution: $x = \frac{a}{K} + Ce^{Kt}$

$a = 0 \rightarrow x = Ce^{Kt}$

$\rightarrow$ doubling time $= \frac{\ln 2}{K}$

Multiplication factor $= e$

General Solution: $x = \frac{a}{K} + Ce^{Kt}$

$x = \frac{a}{K} + \left(x_0 - \frac{a}{K}\right) e^{Kt}$

$x = \frac{\alpha}{K} + (x_0 - \ldots$

$x = \frac{\alpha}{K} + Ce^{Kt}$ if $c =$

$x = \frac{\alpha}{K}$

$x = \frac{\alpha}{K}$

$\alpha = \frac{oxy\alpha}{year}$ $K = \frac{1}{year}$

$\frac{\alpha}{K} = \frac{\frac{oxy\alpha}{year}}{\frac{1}{year}} = oxy\alpha$

$K = 2/10$ $\alpha = 20$

$\frac{20}{2/10} = 100$

$\rightarrow x = 100 + (x_0 - 100)e^{t/5}$

$t = \ldots \ln(100 - x_0))$

$$t_e = 5(\ln 100 - \ln(100 - x_0))$$

Standard form: $y' + P(x)y = q(x)$

$x^2(y' - xe^x) + (\ln x)y = 0$

$x^2y' - x^3e^x + y\ln x = 0$

$x^2y' = x^3e^x - y\ln x$

↳ $y' = xe^x - \dfrac{y\ln x}{x^2}$

$P(x) = \dfrac{\ln x}{x^2}$

$q(x) = xe^x$

Homogenous Solution concept check.

$x_h(a) = Ce^{-P(a)}$ < 0 → this part is always +

→ $C < 0$ → x_h is always negative

SoLP Large norm → x is constant

$\dfrac{dy}{dt} + ky = kx = c$

→ $x = c$. ⇒ $+$ $y(0) = 100$

$\rightarrow x = 0. \Rightarrow /dt$ $\quad y(0) = 100$

$\Rightarrow y = c e^{-Kt}$

$\Rightarrow y = 100 e^{-Kt}$

$y' + Ky = Kx \qquad \frac{dy}{dt} = -Ky$

$y' + Ky = 0 \qquad \frac{dy}{y} = -K dt$

$\longrightarrow \ln y = -Kt$

$\Rightarrow y_h = e^{-Kt}$

$\Rightarrow y_I = u(t)\, e^{-Kt}$

$\frac{dy}{dt} = u'(t) e^{-Kt} + \underline{-K.u(t) e^{-Kt} + K u(t)} e^{-Kt}$

$$\Rightarrow u'(t)y_h = q \rightarrow u'(t)e^{-Kt} = Kt$$

$$\Rightarrow u'(t) = \frac{Kt}{e^{-Kt}} = \frac{du}{dt}$$

$$\Rightarrow u = \int \frac{Kt}{e^{-Kt}}\,dt$$

$z = Kt \quad dz = K\,dt$

$$u = \int Kt\,e^{Kt}\,dt = \int z\,e^{z}\frac{dz}{K} = \frac{1}{K}\int \ldots$$

$$= \frac{1}{K}\left(ze^{z} - e^{z}\right) \qquad z = Kt$$

$$\Longrightarrow \frac{1}{K}\left(Kt\,e^{Kt} - e^{Kt}\right)$$

$$= t\,e^{Kt} - \frac{e^{Kt}}{K}$$

$\rightsquigarrow \ln y = -Kt$

$$\Rightarrow y_h = e^{-Kt}$$

approx

n is constant q

$$y_I = u(t)\,e^{-Kt}$$

$$u'(t)y_h = q \Rightarrow u'(t)e^{-Kt} = K\bar{T}$$

$$\int du = \int K\bar{T}\,e^{Kt}\,dt$$

$$\Rightarrow u = \bar{T}e^{Kt} + C$$

$$\Rightarrow y_I = \left(\bar{T}e^{Kt} + c\right)e^{-Kt}$$

$$= \bar{T} + Ce^{-Kt}$$

$$y_I(0) = g_0 \Rightarrow \bar{T} + C = g_0$$

$$\Rightarrow y_I = \bar{T} + (g_0 - \bar{T})\,e^{-Kt}$$

$\dot{x} + 2x = 0 \qquad x = Ce^{-2t}$

$\dot{x} + 2x = 1 \qquad x_p = \frac{1}{2}$

$\dot{x} + 2x = t \qquad x_p = \frac{t}{2} - \frac{1}{4}$

$\dot{x} + 2x = e^{-2t} \qquad x_p = te^{-2t}$

$\dot{x} + 2x = 5 \,, \quad \dot{x} + 2x = 6t \qquad \dot{x} + 2x = -7e^{-2t}$

$x = 5/2 \qquad x = 3t - 3/2 \qquad x = -7te^{-2t}$

$$\Longrightarrow x = 5/2 + 3t - 3/2 - 7te^{-2t}$$

$$= 3t + \frac{1}{2} - 7te^{-2t} + Ce^{-2t}$$

e.g.: $\quad \dot{x} + P(x)\, x^2 = 0$

$$\dot{x}_1 + \dot{x}_2 + P(x)\left(x_1^2 + x_2^2 + 2x_1 x_2\right) = q(t)$$

$$\dot{x}_1 + P(x)\, x_1^2 = 0 \qquad \dot{x}_2 + P(x)\, x_2^2 = 0$$

$$\Longrightarrow 2P(x)\, x_1 x_2 = q(t)$$

$T_e = 25$ $\quad T(7) = 34$ $\quad T(8) = 31$

$$T(t) = T_e + (T_0 - T_e)\, e^{-kt}$$

$$34 = 25 + 12\, e^{-kt_1} \qquad t_2 = t_1 + 1$$

$$31 = 25 + 12\, e^{-kt_2}$$

$$\frac{3}{4} = e^{-kt_1} \Longrightarrow \frac{3}{2} = e^{k} \Longrightarrow k = \ln\frac{3}{2} \simeq 0.4$$

$$\frac{1}{2} = e^{-kt_2}$$

$$\Longrightarrow \frac{3}{4} = e^{-\frac{2t_1}{5}}$$

$$\Longrightarrow \ln\frac{3}{4} = -\frac{2t_1}{5} \simeq -\frac{3}{10}$$

$$\Longrightarrow t_1 = \frac{3}{4}$$

$$\frac{dy}{dx} - y\tan x = 1$$ linear and not Separable.

$$y' = y\tan x$$

$$\Rightarrow \int \frac{dy}{y} = \int \tan x\, dx$$

$u = \cos x \rightarrow dx = \frac{-du}{\sin x}$

$du = -\sin x\, dx$

$$\Rightarrow \ln y = \int \frac{\sin x}{\cos x}\, dx$$

$$\Rightarrow \ln y = \int \frac{-du}{u} \qquad \ln y = -\ln u + C$$

$$y = C\frac{1}{u} = \frac{C}{\cos x}$$

Sanity Check. $y' = C(\cos x^{-1})' = C \times \frac{+\sin x}{\cos^2 x}$

$$\frac{C\sin x}{\cos^2 x} - \frac{\sin}{\cos}\frac{C}{\cos} = 0 \quad \checkmark$$

$$y = \frac{c}{\cos x} \qquad y_p = \frac{u(x)}{\cos x}$$

$$y_p' \to \;= \frac{u'(x)}{\cos x} + u\frac{\sin x}{\cos^2 x}$$

$$\frac{u'}{\cos} + \overbrace{u\frac{\sin}{\cos^2} - \frac{\sin}{\cos}\frac{u}{\cos}}^{0} = 1$$

$$\Rightarrow \frac{u'}{\cos} = 1 \Rightarrow u' = \cos$$

$$\Rightarrow u = \sin x$$

$$\Rightarrow y_p = \frac{\sin x}{\cos x} = \tan x$$

$$y_p' = \frac{\cos^2+\sin^2}{\cos^2} = \frac{1}{\cos^2} - \tan^2 = 1 \quad ?$$

$$\checkmark \qquad \frac{1}{\cos^2} - \frac{\sin^2}{\cos^2} = \frac{\sin^2+\cos^2-\sin^2}{\cos^2}$$

$$y' - xy = 1 \qquad u = e^{-\int x\,dx} = e^{-\frac{x^2}{2}}$$

$$\Rightarrow e^{-x^2/2}y' - xye^{-x^2/2} = e^{-x^2/2}$$

$$(e^{-x^2/2}y)' = y'e^{-x^2/2} - xe^{-\frac{x^2}{2}}y = e^{-\frac{x^2}{2}}$$

$$e^{-x^2/2}y = \int e^{-x^2/2}dx = \sqrt{2\pi}$$

$$\int e^{-a(x+b)^2}dx = \sqrt{\frac{\pi}{a}} \qquad \Rightarrow y = (\sqrt{2\pi} + c)\, e^{x^2/2}$$

$$y' = (\sqrt{2\pi}+c)\,x\,e^{x^2/2} - (\sqrt{2\pi}+c)\,x\,e^{x^2/2} = c$$

Unsolvable

Part A HW 1 ?

$y' = 7y \rightarrow y = Ce^{7t}$

$C=0$ is a solution.

$te^{7t} \rightarrow e^{7t} + \cancel{7te^{7t} = 7te^{7t}}$ Nope

Infinitely Many Solutions / [illegible]

Nice!

order is 6

$kg \times \frac{m}{s^2} = -kg \times \frac{m}{s^2} - b \times \frac{m}{s} \rightarrow b = kg/s$

$ty = y'\cos t + \sin t$

$\Rightarrow \frac{t}{\cos t} y = y' + \tan t \Rightarrow y' - \frac{t}{\cos t} y = -\tan t$ ✓

$y''' = yy'' + 2e^t y$ NOPE

$t^2 y'' = (y' - 1)\log t \qquad u = y'$

$\Rightarrow u' = (u-1)\frac{\log t}{t^2} \Rightarrow u' - \frac{\log t}{t^2} u = -\frac{\log t}{t}$

Lecture 2 $\quad y' = \frac{-\pi}{2} t (1+y^2)$

$y'(1) = 0$

$y(0) = 1$

Separation of Variables

$\Rightarrow \int \frac{dy}{1+y^2} = \int \frac{-\pi}{2} t \, dt$

$\int \frac{dy}{1+y^2} = \frac{-\pi t^2}{4} \Rightarrow \tan^{-1} y = \frac{-\pi t^2}{4}$

$\Rightarrow \int \frac{dy}{1+y^2}$

$\Rightarrow y = \tan \frac{-\pi t^2}{4} + C$

$y(0) = 0 + C \Rightarrow C = 1$

$y(1) = \tan \frac{-\pi}{4} + 1 = 0$

$t, \; t^2+1$

$3t^2+4t+5 = 3(t^2+1)+4t+2$ ✗

$t^2+2t+1 = t^2+1+2t$ ✓

$0 \times (t^2+1) + t = t$ ✓

$t^2+a-2t \qquad t^2+1-2t-5$ ✗

0 ✓ $\quad 1$ ✗

$y(-1)=?$ $\qquad \dot{y}+5y = 5t^9+9t^8 \qquad y(0)=2$

$(y'+5y)e^{5t} = 5t^9e^{5t}+9t^8e^{5t}$

$\Rightarrow ye^{5t} = t^9e^{5t}+C$

$\Rightarrow y = t^9+Ce^{-5t}$

$y(0)=C \Rightarrow y = t^9+2ce^{-5t}$

$t, \; t^2+1$

$3t^2+4t+5 = 3(t^2+1)+4t+2$ ✗

$t^2+2t+1 = t^2+1+2t$ ✓

$0 \times (t^2+1) + t = t$ ✓

$t^2+a-2t \qquad t^2+1-2t-5$ ✗

0 ✓ $\quad 1$ ✗

$y(-1)=?$ $\qquad \dot{y}+5y = 5t^9+9t^8 \qquad y(0)=2$

$(y'+5y)e^{5t} = 5t^9e^{5t}+9t^8e^{5t}$

$\Rightarrow ye^{5t} = t^9e^{5t}+C$

$\Rightarrow y = t^9+Ce^{-5t}$

$y(0)=C \Rightarrow y = t^9+2e^{-5t}$

$y(-1) = -1+2e^5$

Part B HW 1.

$y = a\sin(2t) \quad y' = 2a\cos(2t)$

$2a\cos 2t + P(t)\, a\sin(2t) = 0$

$\Rightarrow 2\cos(2t) = -P(t)\sin(2t)$

$\Rightarrow -2\cot(2t) = P(t)$

$m\ddot{x} = -(k_1 + k_2)x$

$T(t) = T_e + (T_0 - T_e)e^{-kt}$

$55 = 40 + (70 - 40)e^{-\frac{1}{20}k}$

$= e^{-\frac{1}{20}k} \Rightarrow k = -\frac{1}{20}\ln\frac{1}{2}$

$\dot{T} = K(T_e - T)$

$T(\infty) \to ?$ $\quad \dot{T} = K(A_0 - T) + q$

$\dot{T} + KT = A_0 K + q$

$v = e^{Kt}$

$(uf) \Rightarrow Te^{Kt} = \int A_0 K e^{Kt} + q e^{Kt}\, dt$

$= A_0 e^{Kt} + \frac{q}{K} e^{Kt} + C$

$\Rightarrow T = A_0 + \frac{q}{K} + Ce^{-Kt}$

$\Rightarrow A_0 + \frac{q}{K}$

as $t \to \infty$

as $t \to \infty$, $T \Rightarrow A_0 + \frac{q}{K}$

for $T = 70 \Rightarrow q = 30K$

for $T = 55 \Rightarrow q = 15K$

$\Rightarrow \frac{q_{70}}{q_{55}} = 2$

$x(t+1) = (1+I)\,x(t) - q(t)$

$$x(t+1) = (1+\bar{I})x(t) - y(t)$$

$$\underbrace{\frac{x(t+1)-x(t)}{1}}_{\approx} \simeq \dot{x}(t) = \bar{I}x - y$$

$$\dot{x} = \bar{I}x - 1440t \Rightarrow \dot{x} - \bar{I}x = -1440t$$

$$\Rightarrow u = e^{-\bar{I}t} \Rightarrow x e^{-\bar{I}t} = -1440 \int t e^{-\bar{I}t}$$

$$\Rightarrow x e^{-\bar{I}t} = +1440 \times \frac{e^{-\bar{I}t}(\bar{I}t+1)}{\bar{I}^2} + C$$

$$\Rightarrow x = 1440\,\frac{\bar{I}t+1}{\bar{I}^2} + Ce^{+\bar{I}t}$$

$\bar{I} = 0.01$

$x(18) = 0$

$$\frac{1440\left(\frac{t}{100}+1\right)}{\frac{1}{100\times 100}} + Ce^{\frac{t}{100}}$$

$$\Rightarrow 1440000 \times 118 + Ce^{\frac{18}{100}} = 0$$

$$\Rightarrow C = -141{,}92{,}911.5$$

$$C = 207{,}088.5$$

$$x(0) = 1440000 + C = 2070088.5$$

$$207,10$$