Hazard-based parametric regression models

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Lecture Aims

- To discuss the importance of the hazard function in the analysis of survival data.
- To introduce a general hazard regression model.
- To discuss a real data example and available software.

Survival analysis

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- The **survival function** and the **hazard function** represent two quantities of interest in this area.
- The survival function provides information about the probability that an individual or population will survive beyond a certain time point: $S(t) = P(T > t)$.

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where $S_T(t) = P(T > t)$, and $f_T(t)$ is the probability density function of T.

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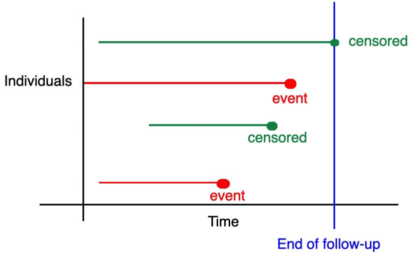

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- In some cases, we may know some additional characteristics about the individuals, meaning we have access to **covariates** $\mathbf{x}_i = (x_{i1}, \dots, x_ip)^\top$, (age, sex, ...).
- How do we incorporate information on covariates to the hazard function?

General/Extended hazard (GH) structure

- Etezadi-Amoli and Ciampi [1987] and Chen and Jewell [2001] proposed a very natural unifying hazard structure. The corresponding hazard and cumulative hazard functions are:

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h(t; \mathbf{x}, \boldsymbol{\alpha}, \boldsymbol{\beta}) &= h_0\left(t \exp\left\{\tilde{\mathbf{x}}^\top \boldsymbol{\alpha}\right\}\right) \exp\left\{\mathbf{x}^\top \boldsymbol{\beta}\right\}, \\
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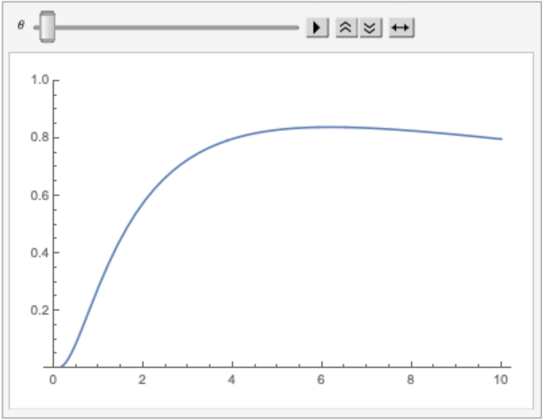
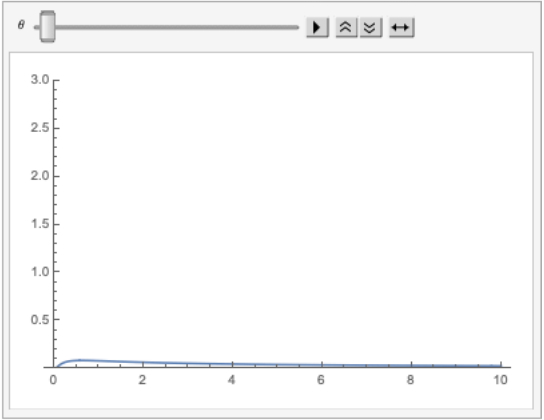
- When $\boldsymbol{\alpha} = \mathbf{0}$, we recover the PH model. For $\boldsymbol{\beta} = \mathbf{0}$ we obtain the AH model. The AFT model is obtained for $\boldsymbol{\alpha} = \boldsymbol{\beta}$.
- The GH model is **identifiable** provided that the baseline hazard is not a member of the Weibull family of distributions (when PH = AFT = AH). [Chen and Jewell, 2001]

- The GH structure also includes time-level effects through α ($\tilde{\mathbf{x}}$).



- The GH structure also includes hazard-level effects through $\beta(\mathbf{x})$.



Parametric GH Models and Inference

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- Once we choose a parametric baseline hazard (e.g. LogNormal, LogLogistic, Gamma, Generalised Gamma, Power Generalised Weibull, ...), we can estimate the parameters using maximum likelihood inference. (`HazReg` R and Julia packages)

Brief catalogue of parametric distributions

- [Gamma].
- [Weibull].
- [Lognormal].
- [Loglogistic].
- [Generalised Gamma].
- Among many many others. [PGW], [EW].

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By selecting a parametric from the catalogue of distributions, we are making assumptions about the possible hazard rates of the true distribution. Selecting the best model using formal tools is usually recommended (AIC, BIC).

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- In this example, we analyse the LeukSurv data set from the R package spBayesSurv. This data set contains information about the survival of acute myeloid leukemia in 1,043 patients.

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- We will fit several models (GH, PH, AFT, AH) with different baseline hazards, and select the best model using AIC and BIC.
- We summarise the best selected model with the available tools in the `HazReg` R package.

[HazReg]

Y.Q. Chen and N.P. Jewell. On a general class of semiparametric hazards regression models. *Biometrika*, 88(3):687–702, 2001.

J. Etezadi-Amoli and A. Ciampi. Extended hazard regression for censored survival data with covariates: a spline approximation for the baseline hazard function. *Biometrics*, 43:181–192, 1987.

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