MSc in Computer Science
at University of Milan

Formal Methods
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1 Commutative sum in $\mathbb{N}$

Lemma 1. *$\forall\ n \in \mathbb{N}$ it holds that $n + 0 = n$.*

Proof. We can proof this by induction on n.

The base case is $n = 0$, and we have to show that $0 + 0 = 0$. This is trivially true.

The inductive step is to show that $\forall\ n \in \mathbb{N}$ it holds that $n+0 = n$. We can show that $(n+1)+0 = n+1$ by using the inductive hypothesis.

$$\begin{aligned}(n+1)+0 &= n + (1+0) \quad \text{by associativity} \\ &= n+1 \quad \text{by the inductive hypothesis}\end{aligned}$$

□

Lemma 2. *$\forall\ n, m \in \mathbb{N}$ it holds that $n + (m + 1) = (n + m) + 1$.*

Proof. We can proof this by induction on n.

The base case is $n = 0$, and we have to show that $\forall\ m \in \mathbb{N}$ it holds that $0 + (m + 1) = (0 + m) + 1$. This is trivially true.

The inductive step is to show that $\forall\ n \in \mathbb{N}$ it holds that $n + (m + 1) = (n + m) + 1$. We can show that $(n + 1) + (m + 1) = ((n + 1) + m) + 1$ by using the inductive hypothesis.

$$\begin{aligned}(n+1)+(m+1) &= (n + (m+1)) + 1 \quad \text{by associativity} \\ &= ((n+m)+1)+1 \quad \text{by the inductive hypothesis} \\ &= ((n+1)+m)+1 \quad \text{by associativity}\end{aligned}$$

□

Theorem 1. *$\forall\ n, m \in \mathbb{N}$ it holds that $m + n = n + m$.*

Proof. We can proof this by induction on n.

The base case is $n = 0$, and we have to show that $\forall\ m \in \mathbb{N}$ it holds that $m + 0 = 0 + m$.

$$m + 0 = m \quad \text{by Lemma 1}$$

The inductive step is to show that $\forall\ n \in \mathbb{N}$ it holds that $m+n = n+m$. We can show that $m+(n+1) = (n + 1) + m$ by using the inductive hypothesis.

$$\begin{aligned}m + (n+1) &= (m+n)+1 \quad \text{by Lemma 2} \\ &= (n+m)+1 \quad \text{by the inductive hypothesis} \\ &= (n+1)+m \quad \text{by Lemma 2}\end{aligned}$$

□