



MSc in Computer Science at University of Milan

Formal Methods
course held by **Momigliano Alberto Davide Adolfo**

Email:
federico.bruzzone@studenti.unimi.it

Created by:
Federico Bruzzone

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1 Commutative sum in \mathbb{N}

Lemma 1. $\forall n \in \mathbb{N}$ it holds that $n + 0 = n$.

Proof. We can proof this by induction on n .

The base case is $n = 0$, and we have to show that $0 + 0 = 0$. This is trivially true.

The inductive step is to show that $\forall n \in \mathbb{N}$ it holds that $n + 0 = n$. We can show that $(n + 1) + 0 = n + 1$ by using the inductive hypothesis.

$$\begin{aligned}(n + 1) + 0 &= n + (1 + 0) \quad \text{by associativity} \\ &= n + 1 \quad \text{by the inductive hypothesis}\end{aligned}$$

□

Lemma 2. $\forall n, m \in \mathbb{N}$ it holds that $n + (m + 1) = (n + m) + 1$.

Proof. We can proof this by induction on n .

The base case is $n = 0$, and we have to show that $\forall m \in \mathbb{N}$ it holds that $0 + (m + 1) = (0 + m) + 1$. This is trivially true.

The inductive step is to show that $\forall n \in \mathbb{N}$ it holds that $n + (m + 1) = (n + m) + 1$. We can show that $(n + 1) + (m + 1) = ((n + 1) + m) + 1$ by using the inductive hypothesis.

$$\begin{aligned}(n + 1) + (m + 1) &= (n + (m + 1)) + 1 \quad \text{by associativity} \\ &= ((n + m) + 1) + 1 \quad \text{by the inductive hypothesis} \\ &= ((n + 1) + m) + 1 \quad \text{by associativity}\end{aligned}$$

□

Theorem 1. $\forall n, m \in \mathbb{N}$ it holds that $m + n = n + m$.

Proof. We can proof this by induction on n .

The base case is $n = 0$, and we have to show that $\forall m \in \mathbb{N}$ it holds that $m + 0 = 0 + m$.

$$m + 0 = m \quad \text{by Lemma 1}$$

The inductive step is to show that $\forall n \in \mathbb{N}$ it holds that $m + n = n + m$. We can show that $m + (n + 1) = (n + 1) + m$ by using the inductive hypothesis.

$$(n + 1) + m = (n + m) + 1$$

$$\begin{aligned}m + (n + 1) &= (m + n) + 1 \quad \text{by Lemma 2} \\ &= (n + m) + 1 \quad \text{by the inductive hypothesis}\end{aligned}$$

