Motion Planning for Autonomous Vehicles

Model Predictive Control (MPC)

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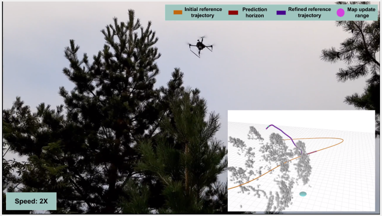
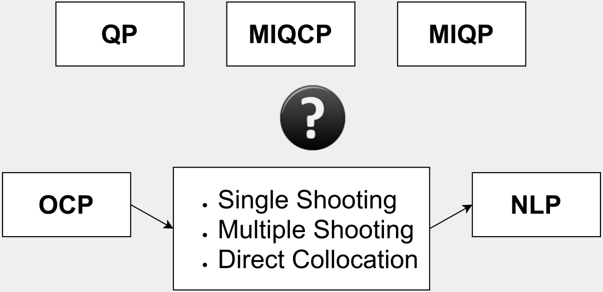


Model Predictive Control (MPC)

Contents

Ways to solve Optimal Control (OCP) Problems

An OCP problem can be transformed into an NLP problem. A problem is always solved better in a **nonlinear manner** as opposed to a **linearizing motion model** at every time since the motion model is nonlinear.



A **OCP** problem can **transform** into **NLP** in various ways, including MS (**Multiple-Shooting**) and DC (**Direct-Collocation**).

OCP Using Nonlinear Programming Problem (NLP)

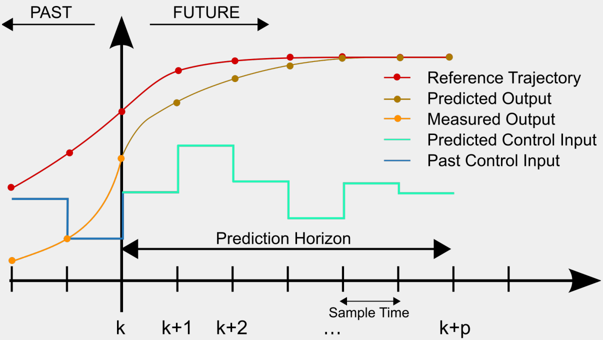
OCP

$$
\begin{aligned}
\min_{u} \quad & J_n(x_0, u) = \sum_{k=0}^{n-1} c(x(k), u(k)) \\
\text{s.t.} \quad & x_{k+1} = f(x(k), u(k)) \\
& x(0) = x_0 \\
& u(k) \in U \,\forall k \in [0, n-1],\ u \in U \subseteq \mathbf{R}^{n_u} \\
& x(k) \in X \,\forall k \in [0, n],\ x \in X \subseteq \mathbf{R}^{n_x}
\end{aligned}
$$

NLP

$$
\begin{aligned}
\min_{w} \quad & \varphi(F(w, x_0, t_k), w) \\
\text{s.t.} \quad & x_{k+1} = f(x(k), u(k)) \\
& g_1(F(w, x_0, t_k), w) \le 0 \\
& g_2(F(w, x_0, t_k), w) = 0
\end{aligned}
$$

In general, use the specified **model** to **predict** the motion of the system, **generate** a **locally optimal or feasible trajectory**, and **repeat** the procedure



Simon, D. (2014). Model Predictive Control in Flight Control Design: Stability and Reference Tracking (Doctoral dissertation, Linköping University Electronic Press).

Prediction Simulate states forward in time up to a defined horizon, **prediction horizon**, N_e from the **current state**

$$\underbrace{\mathbf{u}_k = \begin{bmatrix} u_{0|k} \\ u_{1|k} \\ \vdots \\ u_{N_e-1|k} \end{bmatrix}}_{\text{control inputs}} \xrightarrow{\text{estimate or calculate}} \underbrace{\mathbf{x}_k = \begin{bmatrix} x_{1|k} \\ x_{2|k} \\ \vdots \\ x_{N_e|k} \end{bmatrix}}_{\text{state vector}} \tag{1}$$

where $x_{i|k}$ denoted, current state x_k and $x_{i|k}$ denoted, i steps into the future, same for control as well

The **prediction** by **minimizing** a **stage cost**

$$J_{N_e}(x_k, \mathbf{u}_k) = \sum_{h=0}^{N_e} \left\| \mathbf{x}_{k+h} - \mathbf{x}_{k+h}^{ref} \right\|_Q^2 + \left\| \mathbf{u}_{k+h} - \mathbf{u}_{k+h}^{ref} \right\|_R^2$$

This can be solved numerically to estimate optimal $\mathbf{u}_k^*$

$$\mathbf{u}_k^* = \min_{\mathbf{u}} \quad J_{N_e}(x_k, \mathbf{u}_k), \quad Q \in \mathbb{R}^{n_x \times n_x} \succeq 0, \quad R \in \mathbb{R}^{n_u \times n_u} \succ 0$$

$$\begin{aligned} \text{s.t.} \quad & g_1(\mathbf{u}) = 0, \quad g_2(\mathbf{u}) \le 0 \\ & p^{lower} \le \mathbf{x}_{k+h} \le p^{upper} \quad \forall 0 \le h \le N_e \\ & u^{lower} \le \mathbf{u}_{k+h} \le u^{upper} \quad \forall 0 \le h \le N_e - 1, \end{aligned}$$

Apply the **first** element of $\mathbf{u}_k^*$ on the **system** and **repeat** the optimization

Model Predictive Control

- The **model** can be defined in **various ways**, multivariable, linear, or nonlinear, deterministic, stochastic or fuzzy
- Can handle **different types of constraints**, e.g., linear, quadratic, and nonlinear
- **Near-optimal** control inputs
- , however, requires **online optimization** that may be costly

Model Predictive Control: Prediction model

- If the plant **model** is linear, the model's **state depends** linearly on **control** inputs, i.e., $x_{k+1} = f(x_k, u_k)$. Hence, **cost**, in general, is quadratic in u_k subject to linear **constraints**. Such problems can be formulated as a convex quadratic program and **guaranteed** to have **global optimal solution** all the time.

$$\min_{\mathbf{u}} \mathbf{u}^\top R \mathbf{u} + 2r^\top \mathbf{u} \quad s.t\; A\mathbf{u} \leq b$$

$$\underset{x}{\text{minimize}} \quad f(x)$$

where $f(x) = \frac{1}{2}x^\top Q x + b^\top x + c$, where $c \in \mathbb{R}, x \in \mathbb{R}^2$, and Q is 2×2 matrix. First order necessary condition $\Delta f(x) = 0$

$$\begin{aligned} df &= \frac{1}{2}x^\top Q^\top dx + \frac{1}{2}x^\top Q dx + b^\top dx \\ &= \underbrace{\left(x^\top \frac{Q^\top + Q}{2} + b^\top\right)}_{d\dot{f}(x) = \Delta f(x)} dx \end{aligned} \tag{2}$$

Since $Q^\top = Q$, $\Delta f(x) = Qx + b$. Hence, the critical point: $Qx = -b$
Second order necessary condition $\Delta^2 f(x) = Q$. It can be either **minimum**, **maximum**, saddle point, or **singular** point, i.e., **at least one eigenvalue becomes zero**.

- If the plant **model** is nonlinear, the model's **state depends** non-linearly on **control** inputs, i.e., $x_{k+1} = Ax_k + Bu_k$. Hence, **cost**, in general, is nonconvex in u_k subject to convex and nonconvex **constraints**. Such problems are formulated as a nonlinear program and **does not guarantee** to have **global optimal solution** all the time. Therefore, the solution can have local minima, locally optimal, and may not be solved efficiently or reliably

$$\min_{\mathbf{u}} J(x_k, \mathbf{u}) \quad s.t\ g(x_k, \mathbf{u}) \leq 0$$

- **Discrete-time** necessary to have sampling interval δ, piecewise optimization is carried out
- **Continuous time** not necessary to have sampling interval δ, nor piece wise optimization is carried out. Can be linearized, good for nonlinear continuous-time systems

Model Predictive Control: Constraints

- **Hard** constraints are **satisfied** all the time, it is **not possible** to satisfy, the **problem** is **infeasible**
  - Box constraints

$$
\begin{aligned}
p^{lower} \leq \mathbf{x}_{k+h} \leq p^{upper} \quad & \forall 0 \leq h \leq N_e \\
u^{lower} \leq \mathbf{u}_{k+h} \leq u^{upper} \quad & \forall 0 \leq h \leq N_e - 1,
\end{aligned} \tag{3}
$$

  - System dynamics constraints

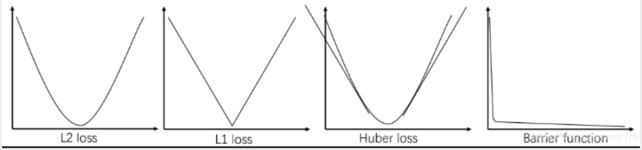
$$
g_1(\mathbf{w}) = \begin{bmatrix} \bar{\mathbf{x}}_k - \mathbf{x}_k \\ \vdots \end{bmatrix}. \tag{4}
$$

- **Soft** constraints may be **violated** to **avoid infeasibility**
  Consider the following **hard-constraints** optimization problem

$$\min_{x} \quad f(x)$$
$$\text{s.t.} \quad g_1(x) = c, \quad g_2(x) \leq d$$

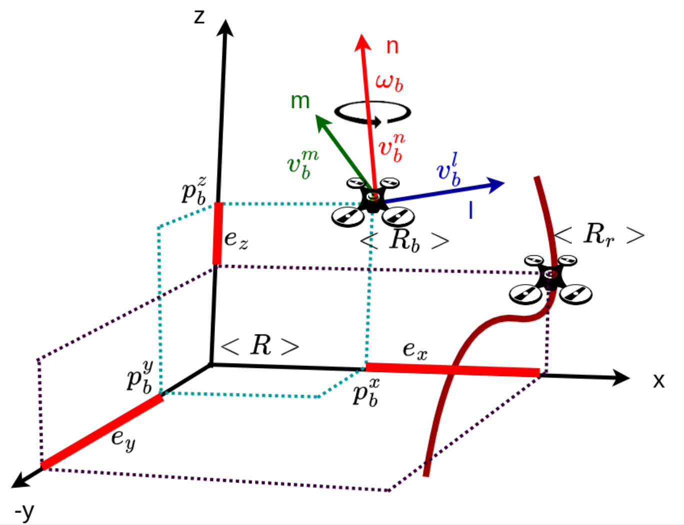
  Can be **converted** into a **soft constraints optimization problem**

$$\min_{x} \quad f(x) + \lambda_1 g_1(x) + \lambda_2 g_2(x)$$



  by using penalty terms or loss functions

Reference trajectory tracking



The system states $\mathbf{x}_k = [p_k^x, p_k^y, p_k^z, \alpha_k^z]^T \in \mathbb{R}^{n_x}$ and control inputs $\mathbf{u}_k = [v_k^x, v_k^y, v_k^z, \omega_k^z]^T \in \mathbb{R}^{n_u}$, where p_k^i and $v_k^i, i \in \{x, y, z\}$ denotes the quadrotor center position(m) and velocity (m/s) in each direction, i.e., x,y,z, at time t = k in the world coordinate frame; α_k^z and ω_k^z denote the yaw angle (rad) and yaw rate (rad/s) around the z-axis, respectively.
The simplified motion model is expressed by $\dot{\mathbf{x}}_k = \mathbf{f}_c(\mathbf{x}_k, \mathbf{u}_k)$

$$\dot{\mathbf{x}}_k = \begin{bmatrix} \dot{p}_k^x \\ \dot{p}_k^y \\ \dot{p}_k^z \\ \dot{\alpha}_k^z \end{bmatrix} = \begin{bmatrix} v_k^x cos(\alpha_k^z) - v_k^y sin(\alpha_k^z) \\ v_k^x sin(\alpha_k^z) + v_k^y cos(\alpha_k^z) \\ v_k^z \\ \omega_k^z \end{bmatrix}, \tag{5}$$

where $\mathbf{f}_c(\cdot) : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}$ and $n_x = n_u = 4$

Simplified Motion Model

Forward Euler discretization, $\mathbf{x}_{k+1} = \mathbf{f}_d(\mathbf{x}_k, \mathbf{u}_k)$ is introduced for a given sampling period in seconds, $\delta \in \mathbb{R} > 0$, e.g., $\delta = 0.1s$

$$\mathbf{x}_{k+1} = \begin{bmatrix} p_k^x \\ p_k^y \\ p_k^z \\ \alpha_k^z \end{bmatrix} + \delta \begin{bmatrix} v_k^x cos(\alpha_k^z) - v_k^y sin(\alpha_k^z) \\ v_k^x sin(\alpha_k^z) + v_k^y cos(\alpha_k^z) \\ v_k^z \\ \omega_k^z \end{bmatrix}, \tag{6}$$

where $\mathbf{f}_d(\cdot) : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \rightarrow \mathbb{R}^{n_x}$.

With Multiple Shooting

$$
\begin{aligned}
J_{N_e}(\mathbf{x}_k,\mathbf{u}_k) &= \sum_{h=0}^{N_e} \left\| \mathbf{x}_{k+h} - \mathbf{x}_{k+h}^{ref} \right\|_Q^2 + \left\| \mathbf{u}_{k+h} - \mathbf{u}_{k+h}^{ref} \right\|_R^2 \\
\min_{\mathbf{w}} \quad & J_{N_e}(\mathbf{x}_k,\mathbf{u}_k), \quad Q \in \mathbb{R}^{n_x \times n_x} \succeq 0, \quad R \in \mathbb{R}^{n_u \times n_u} \succ 0 \\
\text{s.t.} \quad & g_1(\mathbf{w}) = 0, \quad g_2(\mathbf{w}) \leq 0 \\
& p^{lower} \leq \mathbf{x}_{k+h} \leq p^{upper} \quad \forall 0 \leq h \leq N_e \\
& u^{lower} \leq \mathbf{u}_{k+h} \leq u^{upper} \quad \forall 0 \leq h \leq N_e - 1,
\end{aligned}
\tag{7}
$$

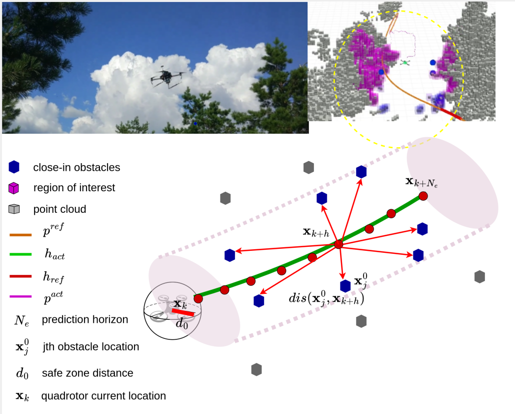
where $\mathbf{w} = [\mathbf{u}_k, \ldots, \mathbf{u}_{k+N_e-1}, \mathbf{x}_k, \ldots, \mathbf{x}_{k+N_e}]$ denotes the decision variables set to be minimized.

Notations u^{lower} and u^{upper} define the minimum and maximum linear and angular velocities allowed

Term $g_1(\mathbf{w})$ depicts the constraints that system dynamics imposes as follows:

$$g_1(\mathbf{w}) = \begin{bmatrix} \bar{\mathbf{x}}_k - \mathbf{x}_k \\ \vdots \\ f_d(\mathbf{x}_{k+h}, \mathbf{u}_{k+h}) - \mathbf{x}_{k+h+1} \\ \vdots \\ f_d(\mathbf{x}_{k+N_e-1}, \mathbf{u}_{k+N_e-1}) - \mathbf{x}_{k+N_e} \end{bmatrix}. \tag{8}$$

Reconstructing obstacle constraints in each iteration is necessary to **incorporate the dynamic environment changes** into the trajectory tracker



Term $g_2(\mathbf{w})$ describes the constraints imposed by obstacles.

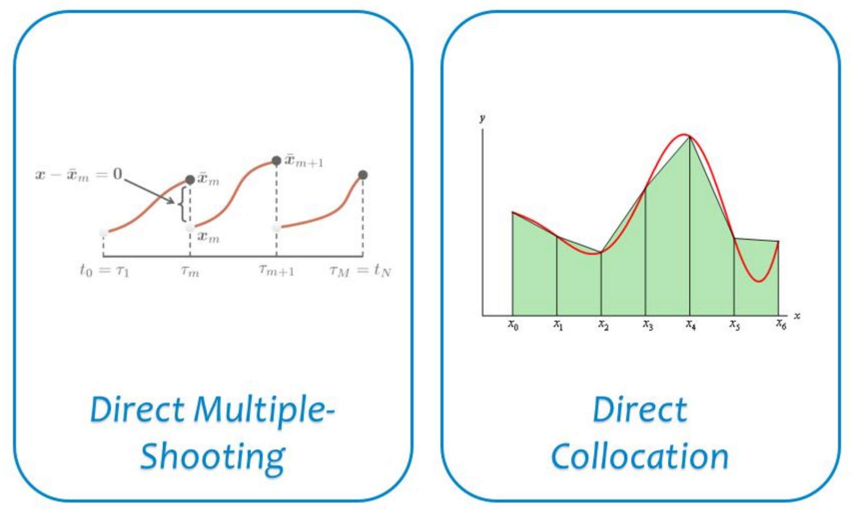
$$g_2(\mathbf{w}) = \begin{bmatrix} dis(\mathbf{x}_j^o, \mathbf{x}_k) \\ \vdots \\ dis(\mathbf{x}_j^o, \mathbf{x}_{k+h}) \\ \vdots \\ dis(\mathbf{x}_j^o, \mathbf{x}_{k+N_e}) \end{bmatrix}, \; j = 1, \ldots, N_o, \tag{9}$$

where $\bar{\mathbf{x}}_k$ is the initial position and N_o is the number of obstacles, and $dis(\mathbf{x}_j^o, \mathbf{x}_{k+h})$ can be calculated as follows:

$$-\sqrt{(x_j^o - x_{k+h})^2 + (y_j^o - y_{k+h})^2 + (z_j^o - z_{k+h})^2} + d^o$$

where d^o is the safe zone distance between the robot and close-in obstacles

Multiple shooting vs Direct Collocation

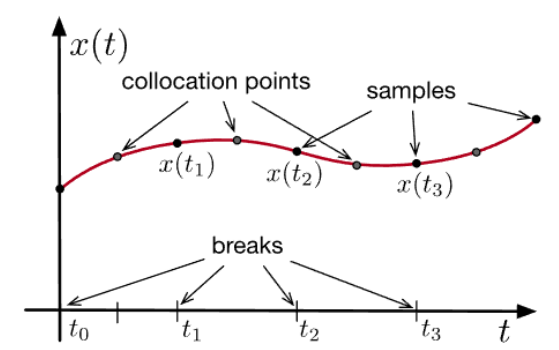


http://www.ee.ic.ac.uk/ICLOCS/Overview.html

With Direct Collocation (DC)

- **Multiple shooting**: **nonlinearity** with a sparsity structure to reduce the nonlinearity
- **Direct collocation**: add more degrees of freedom. Thus, exploits even more, but computation power increases dramatically
- **Collocation points** with respect to a **chosen polynomial**: Lagrangian 3rd order (N_d) polynomial, B-spline or Bézier
- **Fixed time interval** in multiple-shooting, but in DC, it has more **freedom** to determine how should **define points between two consecutive time interval**

- Kept the same discretization as in the multiple-shooting, i.e., $u(t) = u_k$, for $t \in [t_k, t_{k+1}], \quad k = 0, .., N_e - 1$, where N_e is the prediction horizon length

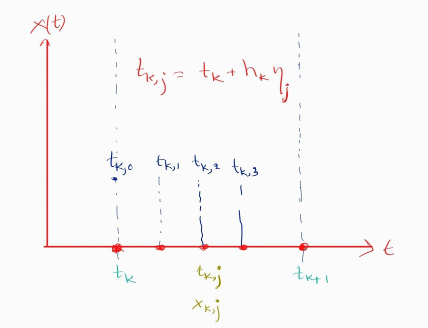


https://underactuated.csail.mit.edu/trajopt.html

- Consecutive time interval (t_k and t_{k+1}) is divided into small sub-intervals

$$t_{k,j} := t_k + h_k \eta_j, \quad k = 0, \ldots, N_e - 1,\ j = 0, \ldots, N_d$$

where Legendre points of order $N_d = 3$
$\eta = [0, 0.112, 0.500, 0.888]$ and $h_k = t_{k+1} - t_k$ and $x_{k,j}$ denote the states at $t_{k,j}$



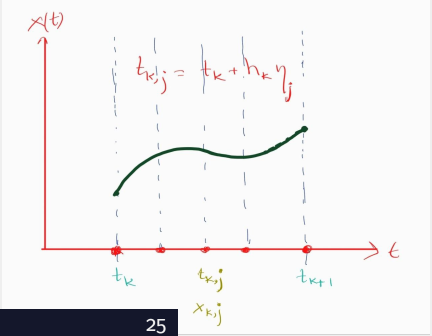
With Direct Collocation (DC)

- In each control interval, the Langrangian polynomial is defined as

$$L_j(\eta) = \prod_{r=0, r\neq j}^{N_d} \frac{\eta - \eta_r}{\eta_j - \eta_r}$$

with property

$$L_j(\eta) = \begin{cases} 1, & \text{if j =r} \\ 0, & \text{otherwise} \end{cases}$$



- State trajectory can be approximated using these basis functions

$$\bar{x}_k(t) = \Sigma_{r=0}^{N_d} L_r\left(\frac{t - t_k}{h_k}\right) x_{k,r}$$

Also, state at the end of the control interval

$$\bar{x}_{k+1}(t) = \Sigma_{r=0}^{N_d} L_r\left(1\right) x_{k,r}$$

And state time derivative at each collocation point except η_0

$$\dot{\bar{x}}_k(t) = \frac{1}{h_k} \Sigma_{r=0}^{N_d} \dot{L}_r\left(\eta_j\right) x_{k,r} := \frac{1}{h_k} \Sigma_{r=0}^{N_d} C_{r,j} x_{k,r}$$

With Direct Collocation (DC)

- Hence, these collocation equations that necessary to satisfy every state at every collocation point

$$h_k f_c(x_{k,j}, u_k) - \Sigma_{r=0}^{N_d} C_{r,j} x_{k,r} = 0, \quad k = 0, \ldots, N_e - 1, \quad j = 0, \ldots, N_d$$

  And the approximation of the end state

$$x_{k+1}(t) - \Sigma_{r=0}^{N_d} L_r(1) x_{k,r} = 0 \quad k = 0, \ldots, N_e - 1$$

Continuous nonlinear system linearization

- Consider a continuous-time nonlinear system

$$\dot{\mathbf{x}}(t) = \boldsymbol{f}_c(\mathbf{x}(t), \mathbf{u}(t)),$$

where $\boldsymbol{f}_c : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$, and reference trajectory

$$\dot{\mathbf{x}}^r(t) = \boldsymbol{f}_c(\mathbf{x}^r(t), \mathbf{u}^r(t)).$$

- Consider a continuous-time nonlinear system

$$\dot{\mathbf{x}}(t) = \boldsymbol{f}_c(\mathbf{x}(t), \mathbf{u}(t)),$$

where $\boldsymbol{f}_c : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$, and reference trajectory

$$\dot{\mathbf{x}}^r(t) = \boldsymbol{f}_c(\mathbf{x}^r(t), \mathbf{u}^r(t)).$$

- If $\mathbf{x}^r(t)$ is “close to” $\mathbf{x}(t)$,

$$\boldsymbol{f}(\mathbf{x},\mathbf{u}) \approx \boldsymbol{f}(\mathbf{x}^r, \mathbf{u}^r) + \frac{\partial}{\partial \mathbf{x}} \boldsymbol{f}(\mathbf{x}^r, \mathbf{u}^r)(\mathbf{x} - \mathbf{x}^r) + \frac{\partial}{\partial \mathbf{u}} \boldsymbol{f}(\mathbf{x}^r, \mathbf{u}^r)(\mathbf{u} - \mathbf{u}^r),$$

- Consider a continuous-time nonlinear system

$$\dot{\mathbf{x}}(t) = \boldsymbol{f}_c(\mathbf{x}(t), \mathbf{u}(t)),$$

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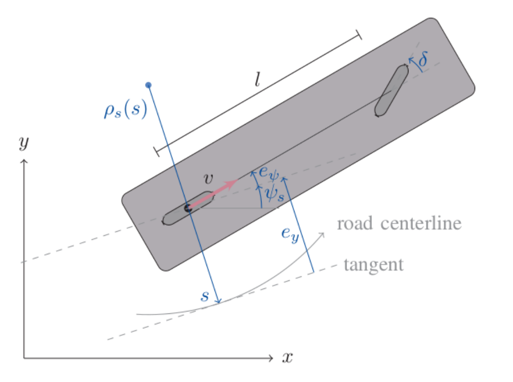
- If $\mathbf{x}^r(t)$ is “close to” $\mathbf{x}(t)$,

$$\boldsymbol{f}(\mathbf{x},\mathbf{u}) \approx \boldsymbol{f}(\mathbf{x}^r,\mathbf{u}^r) + \frac{\partial}{\partial \mathbf{x}}\boldsymbol{f}(\mathbf{x}^r,\mathbf{u}^r)(\mathbf{x}-\mathbf{x}^r) + \frac{\partial}{\partial \mathbf{u}}\boldsymbol{f}(\mathbf{x}^r,\mathbf{u}^r)(\mathbf{u}-\mathbf{u}^r),$$

- Hence,

$$\dot{\mathbf{x}} - \dot{\mathbf{x}}^r = \boldsymbol{f}(\mathbf{x},\mathbf{u}) - \boldsymbol{f}(\mathbf{x}^r,\mathbf{u}^r) \approx \frac{\partial}{\partial \mathbf{x}}\boldsymbol{f}(\mathbf{x}^r,\mathbf{u}^r)(\mathbf{x}-\mathbf{x}^r) + \frac{\partial}{\partial \mathbf{u}}\boldsymbol{f}(\mathbf{x}^r,\mathbf{u}^r)(\mathbf{u}-\mathbf{u}^r).$$

Continuous nonlinear system linearization



$$\dot{x} = \frac{dx}{dt} = v cos(\psi), \quad \dot{y} = \frac{dy}{dt} = v sin(\psi), \quad \dot{\psi} = \frac{d\psi}{dt} = \frac{v}{l} tan(\delta) \qquad (10)$$

Lima, P. F., Mårtensson, J., Wahlberg, B. (2017, December). Stability conditions for linear time-varying model predictive control in autonomous driving. In 2017 IEEE 56th Annual Conference on Decision and Control (CDC) (pp. 2775-2782). IEEE.

Consider a car-like robot $\mathbf{x}^r(t) = (x^r(t), y^r(t), \theta^r(t), \delta^r(t))$ and $\mathbf{u}^r(t) = (v^r(t), \omega^r(t))$, hence

$$
\begin{aligned}
\frac{\partial}{\partial \mathbf{x}} \boldsymbol{f}(\mathbf{x}^r, \mathbf{u}^r) &= \begin{bmatrix} 0 & 0 & -\sin(\theta^r)v^r & 0 \\ 0 & 0 & \cos(\theta^r)v^r & 0 \\ 0 & 0 & 0 & \frac{v^r}{l\cos(\delta^r)^2} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\
\frac{\partial}{\partial \mathbf{u}} \boldsymbol{f}(\mathbf{x}^r, \mathbf{u}^r) &= \begin{bmatrix} \cos(\theta^r) & 0 \\ \sin(\theta^r) & 0 \\ \frac{\tan(\delta^r)}{l} & 0 \\ 0 & 1 \end{bmatrix}
\end{aligned}
\tag{11}
$$

Continuous nonlinear system linearization

Forward Euler discretization leads to

$$\frac{\mathbf{x}_{k+1}-\mathbf{x}_k}{\Delta T}-\frac{\mathbf{x}^r_{k+1}-\mathbf{x}^r_k}{\Delta T}\approx\frac{\partial}{\partial\mathbf{x}}\boldsymbol{f}(\mathbf{x}^r_k,\mathbf{u}^r_k)(\mathbf{x}_k-\mathbf{x}^r_k)+\frac{\partial}{\partial\mathbf{u}}\boldsymbol{f}(\mathbf{x}^r_k,\mathbf{u}^r_k)(\mathbf{u}_k-\mathbf{u}^r_k).$$

$$\mathbf{x}_{k+1}-\mathbf{x}^r_{k+1}+\mathbf{x}^r_k-\mathbf{x}_k\approx\Delta T\frac{\partial}{\partial\mathbf{x}}\boldsymbol{f}(\mathbf{x}^r_k,\mathbf{u}^r_k)(\mathbf{x}_k-\mathbf{x}^r_k)+\Delta T\frac{\partial}{\partial\mathbf{u}}\boldsymbol{f}(\mathbf{x}^r_k,\mathbf{u}^r_k)(\mathbf{u}_k-\mathbf{u}^r_k).$$

$$\underbrace{\mathbf{x}_{k+1}-\mathbf{x}^r_{k+1}}_{\boldsymbol{x}_{k+1}}\approx\underbrace{\left(\Delta T\frac{\partial}{\partial\mathbf{x}}\boldsymbol{f}(\mathbf{x}^r_k,\mathbf{u}^r_k)+\boldsymbol{I}\right)}_{\boldsymbol{A}_k}\underbrace{(\mathbf{x}_k-\mathbf{x}^r_k)}_{\boldsymbol{x}_k}+\underbrace{\Delta T\frac{\partial}{\partial\mathbf{u}}\boldsymbol{f}(\mathbf{x}^r_k,\mathbf{u}^r_k)}_{\boldsymbol{B}_k}\underbrace{(\mathbf{u}_k-\mathbf{u}^r_k)}_{\boldsymbol{u}_k}.$$

This produces a linear time-varying dynamical system

$$\boldsymbol{x}_{k+1}=\boldsymbol{A}_k\boldsymbol{x}_k+\boldsymbol{B}_k\boldsymbol{u}_k,\quad k=0,\dots,N_e-1.$$

However, these type of problem formulation could lead into a **infeasible solution** time to time, i.e., no control inputs that satisfied given constraints. Additionally, **feasibility** does **not ensure** the **asymptotically stability**.

$$
\begin{aligned}
\min_{\mathbf{u}} \quad & \sum_{k=0}^{N_e-1} \mathbf{x}_k^\top \mathbf{Q_k} \mathbf{x}_k + \mathbf{u}_k^\top \mathbf{R_k} \mathbf{u}_k + \mathbf{x}_{N_e}^\top \mathbf{Q_{N_e}} \mathbf{x}_{N_e}, \\
\text{s.t.} \quad & \boldsymbol{x}_{k+1} = \boldsymbol{A}_k \boldsymbol{x}_k + \boldsymbol{B}_k \boldsymbol{u}_k, \quad k = 0, \dots, N_e - 1 \\
& \boldsymbol{x}_0 \text{ is given}, \\
& \boldsymbol{G}_k \boldsymbol{x}_k \leq \boldsymbol{g}_k - \boldsymbol{G}_k \mathbf{x}_{\boldsymbol{k}}^{\boldsymbol{r}}, \quad k = 1, \dots, N_e \\
& \boldsymbol{H}_k \boldsymbol{u}_k \leq \boldsymbol{h}_k - \boldsymbol{H}_k \mathbf{u}_{\boldsymbol{k}}^{\boldsymbol{r}}, \quad k = 0, \dots, N_e - 1
\end{aligned}
\tag{12}
$$

In **receding-horizon control, stability** and **feasibility** are not ensured all the time.

$$\begin{aligned} \min_{\mathbf{u}} \quad & \sum_{k=0}^{N_e-1} \mathbf{x}_k^\top \mathbf{Q_k} \mathbf{x}_k + \mathbf{u}_k^\top \mathbf{R_k} \mathbf{u}_k + \mathbf{x}_{N_e}^\top \mathbf{Q_{N_e}} \mathbf{x}_{N_e}, \\ \text{s.t.} \quad & \boldsymbol{x}_{k+1} = \boldsymbol{A}_k \boldsymbol{x}_k + \boldsymbol{B}_k \boldsymbol{u}_k, \quad k = 0, \ldots, N_e - 1 \\ & \mathbf{x}_0 \text{ is given} \end{aligned}$$

where $\mathbf{Q_k} = \mathbf{Q_k}^\top \succeq 0, \mathbf{R_k} = \mathbf{R_k}^\top \succ 0.$ For a **linearaized** system, the state prediction sequence can be written in a compact sequence as follows:

$$\mathbf{X} = M x_0 + C\mathbf{U},$$

$$M = \begin{bmatrix} A_0 \\ A_1 A_0 \\ \vdots \\ A_{N_e-1} \ldots A_0 \end{bmatrix}, C = \begin{bmatrix} B_0 & & & \\ A_1 B_0 & B_1 & & \\ \vdots & \vdots & \ddots & \\ A_{N_e-1} \ldots A_1 B_0 & A_{N_e-1} \ldots A_2 B_1 & \ldots & B_{N_e-1} \end{bmatrix} \tag{13}$$

The defined quadratic cost (13) can be written in terms of $\mathbf{x}$ and $\mathbf{u}$ as

$$J = \mathbf{X}^\top \tilde{Q}\mathbf{X} + \mathbf{U}^\top \tilde{R}\mathbf{U} = \mathbf{U}^\top H\mathbf{U} + 2x_0^\top F^\top \mathbf{U} + x_0^\top G x_0 \tag{14}$$

where $H = C^\top \tilde{Q}C + \tilde{R}$, $F = C^\top \tilde{Q}M$, and $G = M^\top \tilde{Q}M$

$$\begin{aligned} \min_{\mathbf{U}} \quad & \mathbf{U}^\top H\mathbf{U} + 2x_0^\top F^\top \mathbf{U}, \\ \text{s.t.} \quad & \boldsymbol{x}_{k+1} = \boldsymbol{A}_k \boldsymbol{x}_k + \boldsymbol{B}_k \boldsymbol{u}_k, \quad k = 0, \ldots, N_e - 1 \\ & \boldsymbol{x}_0 \text{ is given}, \\ & \boldsymbol{G}_k \boldsymbol{x}_k \le \boldsymbol{g}_k - \boldsymbol{G}_k \mathbf{x}^{\boldsymbol{r}}_{\boldsymbol{k}}, \quad k = 1, \ldots, N_e \\ & \boldsymbol{H}_k \boldsymbol{u}_k \le \boldsymbol{h}_k - \boldsymbol{H}_k \mathbf{u}^{\boldsymbol{r}}_{\boldsymbol{k}}, \quad k = 0, \ldots, N_e - 1 \end{aligned}$$

Since $x_0^\top G x_0$ is a constant value, can be discarded from the objective function

The constraints $\mathbf{GX} \leq \mathbf{g}$ becomes $\mathbf{GCU} \leq \mathbf{g} - \mathbf{GM}x_0$

$$\begin{aligned} \min_{\mathbf{U}} \quad & \mathbf{U}^\top H\mathbf{U} + 2x_0^\top F^\top \mathbf{U}, \\ \text{s.t.} \quad & \begin{bmatrix} \boldsymbol{GC} \\ \boldsymbol{H} \end{bmatrix} \boldsymbol{U} \leq \begin{bmatrix} \boldsymbol{g} - \boldsymbol{GM}x_0 \\ \boldsymbol{h} \end{bmatrix}. \end{aligned}$$

- Boundary conditions on the velocities
  Let $v_{min} \leq v \leq v_{max}$ be allowed velocity range. Since the model was linearized

$$v_{min} - v_k^r \leq v_k - v_k^r \leq v_{max} - v_k^r, \quad k = 0, \ldots, N_e - 1 \tag{15}$$

- Boundary conditions on the steering angle
  Let $\delta_{min} \leq \delta \leq \delta_{max}$ be allowed steering limits.

$$\mathbf{1}\delta_{min} - \begin{bmatrix} \delta_1^r \\ \vdots \\ \delta_D^r \end{bmatrix} \leq \boldsymbol{GX} \leq \mathbf{1}\delta_{max} - \begin{bmatrix} \delta_1^r \\ \vdots \\ \delta_{N_e}^r \end{bmatrix}, \tag{16}$$

where $\boldsymbol{G} = \begin{bmatrix} 0 & 0 & 0 & 1 & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & & & & & & & & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & 0 & 0 & 0 & 1 \end{bmatrix}$.

- Term $\boldsymbol{GC}$ is constant, depends only on the sampling interval ΔT due the structure of the $\mathbf{C}$, i.e., check eq.(13) and eq.(11)

$$\boldsymbol{GC} = \begin{bmatrix} 0 & \Delta T & 0 & 0 & \dots & 0 & 0 \\ 0 & \Delta T & 0 & \Delta T & & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & 0 & 0 \\ 0 & \Delta T & 0 & \Delta T & & 0 & \Delta T \end{bmatrix}. \tag{17}$$

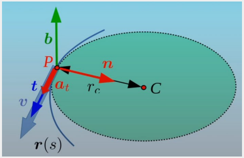
Velocity and acceleration in Frenent-Serret frame

- Derivative of $\mathbf{r}(s)$ with respect to time:

$$\dot{\mathbf{r}} = \dot{s}\mathbf{r}', \quad v = \dot{s}, \quad \mathbf{t} = \mathbf{r}' \rightarrow \mathbf{v} = v\mathbf{t}$$

$$\ddot{\mathbf{r}} = \dot{s}^2\mathbf{r}'' + \ddot{s}\mathbf{r}', \quad a = \ddot{s} = \dot{v}, \quad \mathbf{t}' = \mathbf{r}''$$

$$\rightarrow \mathbf{a} = a\mathbf{t} + v^2\mathbf{t}'$$



- Applying the first Frenent-Serret formula: $\mathbf{t}' = k\mathbf{n}$

$$\mathbf{a} = a\mathbf{t} + v^2 k\mathbf{n} = a\mathbf{t} + (v^2/r_c)\mathbf{n}$$

https://www.youtube.com/watch?v=aFCMIt63pgc

- Constraints on the tangential acceleration

$$a_k = (v_k - v_{k-1})/\Delta T$$

- Constraints on the tangential acceleration

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- Considering the linearized system,

$$a_{min} - a_k^r \le a_k - a_k^r \le a_{max} - a_k^r, \quad k = 0, \ldots, N_e - 1 \qquad (18)$$

Continuous nonlinear system linearization

- Constraints on the tangential acceleration

$$a_k = (v_k - v_{k-1})/\Delta T$$

- Considering the linearized system,

$$a_{min} - a_k^r \leq a_k - a_k^r \leq a_{max} - a_k^r, \quad k = 0, \dots, N_e - 1 \tag{18}$$

-

$$\Delta T a_{min} - v_k^r + v_{k-1}^r \leq (v_k - v_k^r) - (v_{k-1} - v_{k-1}^r) \leq \Delta T a_{max} - v_k^r + v_{k-1}^r. \tag{19}$$

- Constraints on the tangential acceleration

$$a_k = (v_k - v_{k-1})/\Delta T$$

- Considering the linearized system,

$$a_{min} - a^r_k \leq a_k - a^r_k \leq a_{max} - a^r_k, \quad k = 0, \dots, N_e - 1 \tag{18}$$

-

$$\Delta T a_{min} - v^r_k + v^r_{k-1} \leq (v_k - v^r_k) - (v_{k-1} - v^r_{k-1}) \leq \Delta T a_{max} - v^r_k + v^r_{k-1}. \tag{19}$$

- The first constraint, where $k = 0$ is

$$\Delta T a_{min} - v^r_0 + v_{-1} \leq \begin{bmatrix} 1 & 0 \end{bmatrix} \boldsymbol{u}_0 \leq \Delta T a_{max} - v^r_0 + v_{-1}. \tag{20}$$

- The remaining constraints $(v_k - v_k^r) - (v_{k-1} - v_{k-1}^r) := \mathbf{HU}$

$$\mathbf{1}\Delta T a_{min} - \begin{bmatrix} v_1^r \\ \vdots \\ v_{N_e-1}^r \end{bmatrix} + \begin{bmatrix} v_0^r \\ \vdots \\ v_{N_e-2}^r \end{bmatrix} \leq \boldsymbol{HU} \leq \mathbf{1}\Delta T a_{max} - \begin{bmatrix} v_1^r \\ \vdots \\ v_{N_e-1}^r \end{bmatrix} + \begin{bmatrix} v_0^r \\ \vdots \\ v_{N_e-2}^r \end{bmatrix}, \tag{21}$$

where

$$\boldsymbol{H} = \begin{bmatrix} -1 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -1 & 0 & 1 & 0 \end{bmatrix}. \tag{22}$$

- The magnitude of centripetal acceleration

$$a^c = \frac{v^2}{R} = |v||\omega|, \tag{23}$$

where $R = |v^r|/|\omega^r|$ is the radius of the curve, which can be assumed to be independent from changes of the control inputs and ω is the angular velocity

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- The bounds on tangential velocity

$$v_{max} = \sqrt{\frac{a^c_{max}|v^r|}{|\omega^r|}}, \quad v_{min} = -\sqrt{\frac{a^c_{max}|v^r|}{|\omega^r|}}, \tag{24}$$

where $a^c_{max} = \frac{v^2|\omega^r|}{|v^r|}$

- The constraints on tangential velocity

$$-\sqrt{\frac{a^c_{max}|v^r_k|}{|\omega^r_k|}} \leq v_k \leq \sqrt{\frac{a^c_{max}|v^r_k|}{|\omega^r_k|}}, \quad \text{where} \quad \omega^r_k = \frac{\theta^r_{k+1} - \theta^r_k}{\Delta T}, \quad (25)$$

- The constraints on tangential velocity

$$-\sqrt{\frac{a_{max}^{c}|v_k^r|}{|\omega_k^r|}} \leq v_k \leq \sqrt{\frac{a_{max}^{c}|v_k^r|}{|\omega_k^r|}}, \quad \text{where} \quad \omega_k^r = \frac{\theta_{k+1}^r - \theta_k^r}{\Delta T}, \quad (25)$$

- Considering the linearized system,

$$-\sqrt{\frac{a_{max}^{c}|v_k^r|}{|\omega_k^r|}} - v_k^r \leq v_k - v_k^r \leq \sqrt{\frac{a_{max}^{c}|v_k^r|}{|\omega_k^r|}} - v_k^r, \quad \omega_k^r \neq 0 \qquad (26)$$

ω^r is zero when **a car follows a straight line** and **hence does not have centripetal acceleration**.

Discrete-time nonlinear system linearization

Consider discrete-time nonlinear system

$$\mathbf{x}(k+1) = \boldsymbol{f}_c(\mathbf{x}(k), \mathbf{u}(k)),$$

where $\boldsymbol{f}_c : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$, and reference trajectory

$$\mathbf{x}^r(k+1) = \boldsymbol{f}_c(\mathbf{x}^r(k), \mathbf{u}^r(k)).$$

If $\mathbf{x}^r(k)$ is "close to" $\mathbf{x}(k)$,

$$\boldsymbol{f}(\mathbf{x},\mathbf{u}) \approx \boldsymbol{f}(\mathbf{x}^r,\mathbf{u}^r) + \frac{\partial}{\partial \mathbf{x}}\boldsymbol{f}(\mathbf{x}^r,\mathbf{u}^r)(\mathbf{x}-\mathbf{x}^r) + \frac{\partial}{\partial \mathbf{u}}\boldsymbol{f}(\mathbf{x}^r,\mathbf{u}^r)(\mathbf{u}-\mathbf{u}^r),$$

Hence,

$$\begin{aligned}\mathbf{x}(k+1) - \mathbf{x}^r(k+1) &= \boldsymbol{f}(\mathbf{x},\mathbf{u}) - \boldsymbol{f}(\mathbf{x}^r,\mathbf{u}^r) \\ &\approx \frac{\partial}{\partial \mathbf{x}}\boldsymbol{f}(\mathbf{x}^r,\mathbf{u}^r)(\mathbf{x}-\mathbf{x}^r) + \frac{\partial}{\partial \mathbf{u}}\boldsymbol{f}(\mathbf{x}^r,\mathbf{u}^r)(\mathbf{u}-\mathbf{u}^r).\end{aligned} \tag{27}$$

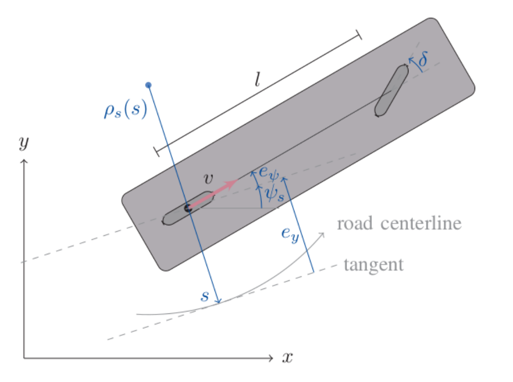
$$\underbrace{\mathbf{x}_{k+1}-\mathbf{x}^{r}_{k+1}}_{\boldsymbol{x}_{k+1}} \approx \underbrace{\left(\frac{\partial}{\partial \mathbf{x}}\boldsymbol{f}(\mathbf{x}^{r}_{k},\mathbf{u}^{r}_{k})\right)}_{\boldsymbol{A}_k}\underbrace{(\mathbf{x}_{k}-\mathbf{x}^{r}_{k})}_{\boldsymbol{x}_k}+\underbrace{\frac{\partial}{\partial \mathbf{u}}\boldsymbol{f}(\mathbf{x}^{r}_{k},\mathbf{u}^{r}_{k})}_{\boldsymbol{B}_k}\underbrace{(\mathbf{u}_{k}-\mathbf{u}^{r}_{k})}_{\boldsymbol{u}_k}.$$

This produces **a first order approximation** of the considered **nonlinear system** as a **linear time-varying** dynamical system.

$$\boldsymbol{x}_{k+1}=\boldsymbol{A}_k\boldsymbol{x}_k+\boldsymbol{B}_k\boldsymbol{u}_k, \quad k=0,\dots,N_e-1.$$

When system is subject to **feedback control law** $\mathbf{u}=K(k)\mathbf{x}(k)$, where $K(k)$ is the **time varying feedback control gain**. Therefor, **discrete-time autonomous system** can be desgined

$$\mathbf{x}(k+1)=f(\mathbf{x}(k),K(k))$$



$$\dot{x} = \frac{dx}{dt} = v_x cos(\psi), \quad \dot{y} = \frac{dy}{dt} = v_x sin(\psi), \quad \dot{\psi} = \frac{d\psi}{dt} = \frac{v_x}{l} tan(\delta) \quad (28)$$

Lima, P. F., Mårtensson, J., Wahlberg, B. (2017, December). Stability conditions for linear time-varying model predictive control in autonomous driving. In 2017 IEEE 56th Annual Conference on Decision and Control (CDC) (pp. 2775-2782). IEEE.

By considering the Linear Time-Varying Model Predictive Control, the following set of expressions can be derived.

$$\dot{e}_y = v_x sin(e_\psi),$$
$$\dot{e}_\psi = \dot{\psi} - \dot{\psi}_s,$$
$$\dot{s} = \omega \cdot \frac{1}{k_r} = \frac{v_x cos(e_\psi)}{k_r(1/k_r - d(t))} = \frac{v_x cos(e_\psi)}{1 - k_r d(t)} = \frac{\rho_s v cos(e_\psi)}{\rho_s - e_y}, \quad \omega = \frac{v_x cos(e\psi)}{(\frac{1}{k_r} - d(t))} \tag{29}$$

where ρ_s is the radius of curvature of the road ψ_s is the road heading angle.

The next step is to derive with respect to s, i.e., $\frac{d(\cdot)}{ds} = \frac{d(\cdot)}{dt}\frac{dt}{ds} = \frac{d(\cdot)}{dt}\frac{1}{\dot{s}}$.

$$e'_y = \frac{\dot{e}_y}{\dot{s}} = \frac{v_x sin(e_\psi)}{v_x cos(e_\psi)}(1-d(t)k_r) = \Big(1-d(t)k_r\Big)tan(e_\psi)$$

$$e'_\psi = \frac{\frac{v_x}{k} - \frac{v_x cos(e_\psi)}{1/k_r - d(t)}}{\frac{v_x cos(e_\psi)}{1-k_r d(t)}} = \frac{k(\rho_s - d(t))}{\rho_s cos(e_\psi)} - \frac{1}{\rho_s} = \frac{k(\rho_s - d(t))}{\rho_s cos(e_\psi)} - \psi'_s, \tag{30}$$

$$\psi'_s = \frac{\dot{\psi}_s}{\dot{s}} = \frac{\frac{v_x cos(e_\psi)k_r}{1-d(t)k_r}}{\frac{v_x cos(e_\psi)k_r}{1-d(t)}} = \frac{1}{\rho_s}$$

After linearizing the nonlinear model (eq.30), around the reference trajectory $[e_{y,r}(k)\, e_{\psi,r}(k)] = [0\, 0]$ for all $k \geq 0$, the reason for having zeros is that those were already considered when defining the state sequence $[e_y(k)\, e_\psi(k)]$

$$\begin{bmatrix} e_y(k+1) \\ e_\psi(k+1) \end{bmatrix} = \begin{bmatrix} 1 & \Delta_s \\ -k_r^2(k)\Delta_s & 1 \end{bmatrix} \begin{bmatrix} e_y(k) \\ e_\psi(k) \end{bmatrix} + \begin{bmatrix} 0) \\ \Delta_s \end{bmatrix} (k(k) - k_r(k)) \tag{31}$$

where $\Delta_s = vT_s$, T_s is the sampling time and v is a constant, i.e., vehicle traversal at constant speed, road curvature, denoted $k_r = \frac{1}{\rho_s}$. The state vector of the derived system $\bar{z}(k) = [e_y(k)\, e_\psi(k)]$ and the control inputs the system is given by $\bar{u}(k) = \bar{k} = k(k) - k_r(k)$

Lima, P. F., Mårtensson, J., Wahlberg, B. (2017, December). Stability conditions for linear time-varying model predictive control in autonomous driving. In 2017 IEEE 56th Annual Conference on Decision and Control (CDC) (pp. 2775-2782). IEEE.

$$\min_{\bar{U}_t} \quad \sum_{k=t}^{t+N_e-1} \bar{z}_{k|t}^\top \mathbf{Q_k} \bar{z}_{k|t} + \bar{u}_{k|t}^\top \mathbf{R_k} \bar{z}_{k|t} + \bar{z}_{t+N_e|t}^\top \mathbf{Q_{N_e}} \bar{z}_{t+N_e|t}, \tag{32}$$

$$\begin{aligned} \text{s.t.} \quad & \bar{z}_{k+1|t} = \boldsymbol{A}_{k|t} \bar{z}_{k|t} + \boldsymbol{B}_k \bar{u}_{k|t}, \quad k = t, \ldots, t+N_e-1 \\ & \bar{z}_{t|t} \text{ is given,} \\ & \bar{z}_{k|t} \in \bar{Z}, \quad k = t+1, \ldots, t+N_e \\ & ,\bar{u}_{k|t} \in \bar{U} \quad k = t, \ldots, t+N_e-1 \end{aligned}$$

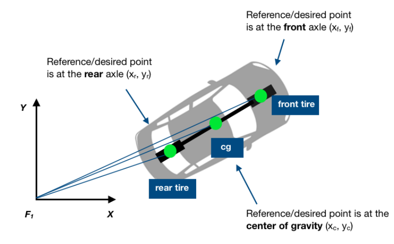
where $\bar{U}_t = \{\bar{u}(t|t), \ldots, \bar{u}(t+N_e-1|t)\}$, boundary constraints for states and controls, denoted $\bar{Z}$, $\bar{U}$

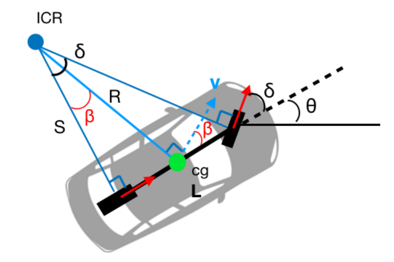
Path tracking control

- Model-free control: PID feedback controller, feedback and feedforward controller
- Model-based control: motion model (kinematics model)
- Differential flatness
- Model-predictive control: MPC, MPCC
- Linear quadratic regulator (LQR): Lateral control: controlling lateral error, rate of change of lateral error, heading error, and rate of change of heading with the control input: steering, acceleration and braking

PATH TRACKING CONTROL WITH MPC: KINEMATIC MODEL

The desired point is at the cg (center of gravity)





Assume that there is no side slippage, i.e., $\beta = 0$, therefore,

$$\dot{x} = vcos(\theta), \quad \dot{y} = vsin(\theta), \quad \dot{\theta} = \frac{v}{R} = \frac{vtan(\delta)}{L}$$

https://www.shuffleai.blog/blog/Simple_Understanding_of_Kinematic_Bicycle_Model.html

Path tracking control with MPC: trajectory generation

Obtain the next 6 waypoints of the reference path from the current pose of the vehicle, and use those 6 waypoints to fit a 3-rd order polynomial.

$$y = f(x) = a_3x^3 + a_2x^2 + a_1x + a_0$$

$$tan(\theta)_d = \frac{dy}{dx} = 3a_3x^2 + 2a_2x + a_1$$

Path tracking control with MPC: dynamic model

- $$x_{t+1} = x_t + v_t cos(\theta) dt$$
  $$y_{t+1} = y_t + v_t sin(\theta) dt$$
  $$\theta_{t+1} = \theta_t + \frac{v_t}{L} \delta_t dt$$
  $$v_{t+1} = v_t + a_t dt$$

- $$x_{t+1} = x_t + v_t cos(\theta) dt$$
  $$y_{t+1} = y_t + v_t sin(\theta) dt$$
  $$\theta_{t+1} = \theta_t + \frac{v_t}{L} \delta_t dt$$
  $$v_{t+1} = v_t + a_t dt$$
- Also, **the residual between the current pose and desired pose** is added to the model as part of the **states**:

$$e_{y,t+1} = f(x_t) - y_t + v_t sin(e_{\theta,t}) dt$$
$$e_{\theta,t+1} = \theta_t - \theta_{d,t} + \frac{v_t}{L} \delta_t dt$$

Path tracking control with MPC: cost

$$J = \Sigma_{t=1}^{N} \lambda_{res} \left\| e_{y,t} \right\|^2 + \lambda_{e_\theta} \left\| e_{\theta,t} \right\|^2 + \lambda_v \left\| v_t - v_{d,t} \right\|^2 + \lambda_\delta \left\| \delta_t \right\|^2 + \lambda_a \left\| a_t \right\|^2 + \lambda_{rate_a} \left\| a_t - a_{t-1} \right\|^2 + \lambda_{rate_\delta} \left\| \delta_t - \delta_{t-1} \right\|^2$$

- $\lambda_{res} \left\| e_{y,t} \right\|^2$ residual error
- $\lambda_{e_\theta} \left\| e_{\theta,t} \right\|^2$ heading error
- $\lambda_v \left\| v_t - v_{d,t} \right\|^2$ speed cost
- $\lambda_\delta \left\| \delta_t \right\|^2$ steering cost
- $\lambda_a \left\| a_t \right\|^2$ acceleration cost
- $\lambda_{rate_a} \left\| a_t - a_{t-1} \right\|^2$ acceleration rate cost
- $\lambda_{rate_\delta} \left\| \delta_t - \delta_{t-1} \right\|^2$ steering rate cost

where λ_* are weighting parameters

Path tracking control with MPC: MPC formulation

$$
\begin{aligned}
\min_{\mathbf{u}=a,\delta} \quad & J, \\
\text{s.t.} \quad & \boldsymbol{x}_{t+1} = f(x_t, y_t, \theta_t, v_t, e_{\theta,t}, \theta_{d,t}, e_{y,t}, a_t, dt), \quad t = 0, \ldots, N_e \\
& \boldsymbol{x}_0 = (x_0, y_0, \theta_0, v_0, e_{\theta,0}, e_{y,0}) \text{ is given}, \\
& -\pi/6 \le \delta \le \pi/6 \\
& a_{min} \le a \le a_{max}
\end{aligned} \tag{33}
$$