Calculating Biological Quantities

CSCI 2897

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Lecture 3 Plan

1. **A little notation & vocabulary**
2. **What does it mean to “solve” a differential equation?**
3. **Checking an analytical solution**
4. **Creating a numerical solution**

Notation

- "Leibniz" Notation: $\dfrac{dy}{dt} + y = 2021$

- Prime Notation: $y' + y = 2021$

- Dot Notation: $\dot{y} + y = 2021$ ← physicists

  dot means derivative
  w.r.t. time.

- Note: $\dfrac{d^2y}{dt^2} = y'' = \ddot{y}$

Vocab: ODE

- An **ODE** is an ordinary differential equation.
- A **PDE** is a partial differential equation.

- ODEs have ordinary derivatives in them. PDEs have partial derivatives in them.
- Note: partial derivatives come up in Calc 3, but tbh they're not that complicated. Ask me in office hours!

- Ordinary derivatives look like $\dfrac{dy}{dx}$ while partial derivates look like $\dfrac{\partial y}{\partial x}$

partial

Vocab: Order

- The **order** of a differential equation is the highest derivative.
- Examples:
  - $y' + y = \pi$ $y' \longrightarrow$ 1 deriv $\longrightarrow$ first order.
  - $\dddot{z} - \ddot{z} = z$ $\dddot{z} = \dfrac{d^3 z}{dt^3} \longrightarrow$ third order.
  - $\dfrac{d^2y}{dx^2} + 5\left(\dfrac{dy}{dx}\right)^3 = 0$ $\longrightarrow$ second order.

Linearity

- A nth order ODE is **linear** if we can write the ODE in this form:

just some function of t

$$\underline{a_n(t)}\frac{d^n y}{dt^n} + \underline{a_{n-1}(t)}\frac{d^{n-1}y}{dt^{n-1}} + \ldots + \underline{a_x(t)}\frac{dy}{dt} + \underline{a_0(t)}y = \underline{g(t)}$$

- Two special cases that come up often are linear first order:

$$a_1(t)y' + a_0(t)y = g(t)$$

- and linear second order:

$$a_2(t)y'' + a_1(t)y' + a_0(t)y = g(t)$$

$y'' + y^2 = 2$

- A **nonlinear** ODE is simply one which is not linear.

$y' + y \cdot y' = 19$

Deterministic vs Stochastic dynamical models

- **Deterministic** models assume that the future is entirely predicted (i.e. determined) by the model.

- **Stochastic** models assume that random (stochastic) events affect the system.

What does it mean to "solve" an ODE?

- What does it mean to solve $x + 3 = 9$?

Find value of x such that, when I plug it in, I get equality.

$x = 6$
$6+3=9$
$9=9$ ✓

$x = 2$
$2+3=9$
$5=9$ ✗

- Suppose that I give you $\sqrt{z} + z^2 - e^{z-4} = 17$. Is $z = 1$ a solution?

Check:

$$\sqrt{1} + 1^2 - e^{1-4} = 17$$
$$1 + 1 - e^{-3} = 17$$
$$2 - e^{-3} = 17$$ ✗

No.

- What *is* the solution above? How do we know?

$z=4$.

plug it in

$$\sqrt{4} + 4^2 - e^{4-4} = 17$$
$$2 + 16 - 1 = 17$$

$17 = 17$ ✓

ODEs are the same: solving means satisfying

- Example: $\dot{y} = y$. Show that $y = e^t$ is a solution, but that $y = e^{2t}$ is not.

$$\frac{dy}{dt} = y$$

$$y = e^t$$

$$\frac{dy}{dt} = ? = e^t$$

$$y = e^t$$

$$\frac{dy}{dt} = \frac{d}{dt}(e^t)$$

$$= e^t$$

$$e^t = e^t \checkmark$$

$$y = e^{2t}$$

$$\dot{y} = 2e^{2t}$$

plug in.

$$2e^{2t} = e^{2t}$$

nope!

ODEs are the same: solving means satisfying

- Example: $\dfrac{dy}{dx} = x\sqrt{y}$. Show that $y = \dfrac{1}{16}x^4$ is a solution.

till next time.

1. compute derivatives as needed
2. plug in!
3. simplify to see if =

Install Python

tips? → Slack.

ODEs are the same: solving means satisfying

- Ex: $y'' - 2y' + y = 0$. For what values of the constant k is $y = kte^t$ a solution?

Exercise: DIY ODEs

1. Write down a solution to an ODE that has not yet been written down. In other words, write down a function.
2. Take a couple derivatives and write those down.
3. Combine them in an equation to create your own ODE.
4. Then swap with someone else, and **verify** (meaning confirm) the solution.

Challenge: DIY recurrence equations?

- On the last slide, we made up our own ODEs and solutions. Can you puzzle out how to do the same kind of thing, but with a **recurrence** equation?
- Recall that a recurrence equation looks like: $n(t+1) =$ some function of $n(t)$

Numerical Solutions to *initial value problems*

- Remember this? Can we write down a recipe for *approximately* solving this?
- Ex 4: (A) Sketch the derivative vs time. (B) Sketch the variable vs time.

$$\frac{dn(t)}{dt} = \sqrt{n}, \quad n(0) = 1$$

Numerical Solutions to *initial value problems*

- Goal of numerical solution: generate a set of points $\left(t_i, n(t_i)\right)$ that approximate the analytical solution.
- Why might we want to do this?

- There are many ways to *numerically solve differential equations*, but here is one, referred to as **Euler's Method.**

To solve $y' = f(x, y)$**, with** $y(x_0) = y_0$**, use the formula** $y_{n+1} = y_n + hf(x_n, y_n)$

Notebook time!

To solve $y' = f(x, y)$**, with** $y(x_0) = y_0$**, use the formula** $y_{n+1} = y_n + hf(x_n, y_n)$

Example: $y' = 2xy, \quad y(1) = 1$

Analytical solution: $y(x) = e^{x^2-1}$