

Calculating Biological Quantities

CSCI 2897

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Lecture 3 Plan

- ✓ 1. A little notation & vocabulary
- ✓ 2. What does it mean to “solve” a differential equation?
- ✓ 3. Checking an analytical solution
- ✓ 4. Creating a numerical solution

Notation

- “Leibniz” Notation: $\frac{dy}{dt} + y = 2021$

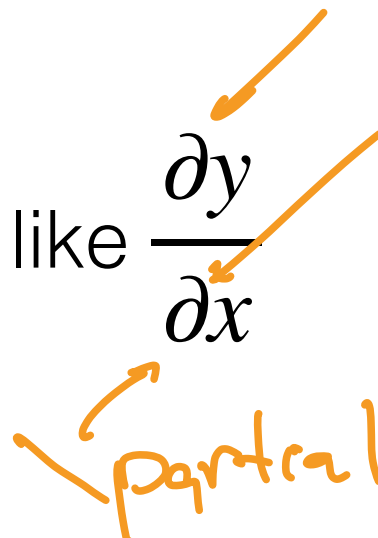
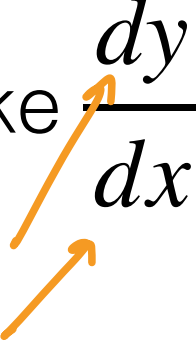
- Prime Notation: $y' + y = 2021$

- Dot Notation: $\dot{y} + y = 2021$  *physicists
dot means derivative
w.r.t. time.*

- Note: $\frac{d^2y}{dt^2} = y'' = \ddot{y}$

Vocab: ODE

- An **ODE** is an ordinary differential equation.
- A **PDE** is a partial differential equation.
- ODEs have ordinary derivatives in them. PDEs have partial derivatives in them.
- Note: partial derivatives come up in Calc 3, but tbh they're not that complicated. Ask me in office hours!

- Ordinary derivatives look like $\frac{dy}{dx}$ while partial derivatives look like $\frac{\partial y}{\partial x}$


Vocab: Order

- The **order** of a differential equation is the highest derivative.
- Examples:

- $y' + y = \pi$

$y' \rightarrow 1 \text{ deriv} \rightarrow \text{first order.}$

- $\ddot{z} - \dot{z} = z$

$\ddot{z} = \frac{d^3 z}{dt^3} \rightarrow \text{third order.}$

- $\frac{d^2 y}{dx^2} + 5 \left(\frac{dy}{dx} \right)^3 = 0$

$\rightarrow \text{second order.}$

Linearity

- A n th order ODE is **linear** if we can write the ODE in this form:

$$\underbrace{a_n(t)}_{\text{wavy}} \underbrace{\frac{d^n y}{dt^n}}_{\text{wavy}} + \underbrace{a_{n-1}(t)}_{\text{wavy}} \underbrace{\frac{d^{n-1} y}{dt^{n-1}}}_{\text{wavy}} + \dots + \underbrace{a_1(t)}_{\text{wavy}} \underbrace{\frac{dy}{dt}}_{\text{wavy}} + \underbrace{a_0(t)}_{\text{wavy}} \underbrace{y}_{\text{wavy}} = \underbrace{g(t)}_{\text{wavy}}$$

just some
function of t

- Two special cases that come up often are linear first order:

$$a_1(t)y' + a_0(t)y = g(t)$$

- and linear second order:

$$a_2(t)y'' + a_1(t)y' + a_0(t)y = g(t)$$

$$y'' + y^2 = 2$$

- A **nonlinear** ODE is simply one which is not linear.

$$y' + y \cdot y' = 19$$

Deterministic vs Stochastic dynamical models

- **Deterministic** models assume that the future is entirely predicted (i.e. determined) by the model.
- **Stochastic** models assume that random (stochastic) events affect the system.

What does it mean to “solve” an ODE?

- What does it mean to solve $x + 3 = 9$?

Find value of x such that,
when I plug it in, I get equality.

$$\begin{aligned}x &= 6 \\ 6 + 3 &= 9 \\ 9 &= 9 \\ &\checkmark\end{aligned}$$

$$\begin{aligned}x &= 2 \\ 2 + 3 &= 9 \\ 5 &= 9 \\ &\times\end{aligned}$$

- Suppose that I give you $\sqrt{z} + z^2 - e^{z-4} = 17$. Is $z = 1$ a solution?

Check:

$$\begin{aligned}\sqrt{1} + 1^2 - e^{1-4} &= 17 \\ 1 + 1 - e^{-3} &= 17 \\ 2 - e^{-3} &= 17\end{aligned}$$

\times No.

- What is the solution above? How do we know?

$z = 4$.
plug it in

$$\begin{aligned}\sqrt{4} + 4^2 - e^{4-4} &= 17 \\ 2 + 16 - 1 &= 17 \\ 17 &= 17\end{aligned}$$

\checkmark

ODEs are the same: solving means satisfying

- Example: $\dot{y} = y$. Show that $y = e^t$ is a solution, but that $y = e^{2t}$ is not.

$$\frac{dy}{dt} = y$$

$$y = e^t$$

$$e^t = e^t \quad \checkmark$$

$$\frac{dy}{dt} = ? = e^t$$

$$y = e^t$$

$$\frac{dy}{dt} = \frac{d}{dt}(e^t)$$

$$= e^t$$

$$y = e^{2t}$$

$$\dot{y} = 2e^{2t}$$

plug in.

$$2e^{2t} = e^{2t}$$

nope!

ODEs are the same: solving means satisfying

- Example: $\frac{dy}{dx} = x\sqrt{y}$. Show that $y = \frac{1}{16}x^4$ is a solution.

till next time.

① compute derivatives as needed

② plug in!

③ simplify to see if =

Install Python

tips? → Slack.

ODEs are the same: solving means satisfying

- Ex: $y'' - 2y' + y = 0$. For what values of the constant k is $y = kte^t$ a solution?

Exercise: DIY ODEs

1. Write down a solution to an ODE that has not yet been written down. In other words, write down a function.
2. Take a couple derivatives and write those down.
3. Combine them in an equation to create your own ODE.
4. Then swap with someone else, and **verify** (meaning confirm) the solution.

Challenge: DIY recurrence equations?

- On the last slide, we made up our own ODEs and solutions. Can you puzzle out how to do the same kind of thing, but with a **recurrence** equation?
- Recall that a recurrence equation looks like: $n(t + 1) = \text{some function of } n(t)$

Numerical Solutions to *initial value problems*

- Remember this? Can we write down a recipe for *approximately* solving this?
- Ex 4: (A) Sketch the derivative vs time. (B) Sketch the variable vs time.

$$\frac{dn(t)}{dt} = \sqrt{n}, \quad n(0) = 1$$

Numerical Solutions to *initial value problems*

- Goal of numerical solution: generate a set of points $(t_i, n(t_i))$ that approximate the analytical solution.
- Why might we want to do this?
- There are many ways to *numerically solve differential equations*, but here is one, referred to as **Euler's Method**.

To solve $y' = f(x, y)$, with $y(x_0) = y_0$, use the formula $y_{n+1} = y_n + hf(x_n, y_n)$

Notebook time!

To solve $y' = f(x, y)$, with $y(x_0) = y_0$, use the formula $y_{n+1} = y_n + hf(x_n, y_n)$

Example: $y' = 2xy, \quad y(1) = 1$

Analytical solution: $y(x) = e^{x^2-1}$

