1 Problem Sheet 1 Solutions

1. **[5 marks]** What are the Fourier coefficients c_k of $\sin^4 x$?

$$\begin{aligned}
(\sin x)^4 &= \left(\frac{\exp(\mathrm{i}x) - \exp(-\mathrm{i}x)}{2\mathrm{i}}\right)^4 \\
&= \left(\frac{\exp(2\mathrm{i}x) - 2 + \exp(-2\mathrm{i}x)}{-4}\right)^2 \\
&= \frac{\exp(4\mathrm{i}x) - 4\exp(2\mathrm{i}x) + 6 - 4\exp(-2\mathrm{i}x) + \exp(-4\mathrm{i}x)}{16}
\end{aligned}$$

hence, $c_{-4} = c_4 = 1/16$, $c_{-2} = c_2 = -1/4$, $c_0 = 3/8$ and $c_k = 0$ otherwise.

2. **[5 marks]** Show for $0 \leq k, \ell \leq n-1$

$$\frac{1}{n}\sum_{j=1}^{n} \cos k\theta_j \cos \ell\theta_j = \begin{cases} 1 & k = \ell = 0 \\ 1/2 & k = \ell \\ 0 & \text{otherwise} \end{cases}$$

for $\theta_j = \pi(j - 1/2)/n$. Hint: You may consider replacing cos with complex exponentials:

$$\cos\theta = \frac{\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta}}{2}.$$

We have,

$$\begin{aligned}
\frac{1}{n}\sum_{j=1}^{n}\cos(k\theta_j)\cos(l\theta_j) &= \frac{1}{4n}\sum_{j=1}^{n}\left(e^{i(k+l)\theta_j} + e^{-i(k+l)\theta_j} + e^{i(k-l)\theta_j} + e^{-i(k-l)\theta_j}\right) \\
&= \frac{1}{4n}\sum_{j=1}^{n}\left(e^{ia_{kl}\theta_j} + e^{-ia_{kl}\theta_j} + e^{ib_{kl}\theta_j} + e^{-ib_{kl}\theta_j}\right),
\end{aligned}$$

where we have defined $a_{kl} = k + l$ and $b_{kl} = k - l$. Now consider, for $a \in \mathbb{Z}$, $a \neq 2kn$ for some $k \in \mathbb{Z}$,

$$\begin{aligned}
\sum_{j=1}^{n} e^{ia\theta_j} &= \sum_{j=1}^{n} e^{ia\pi(j-\frac{1}{2})/n} \\
&= e^{-ia\pi/2n}\sum_{j=1}^{n} e^{iaj\pi/n} \\
&= e^{-ia\pi/2n}\sum_{j=1}^{n}(e^{ia\pi/n})^j \\
&= e^{-ia\pi/2n}e^{ia\pi/n}\frac{(e^{ia\pi/n})^n - 1}{e^{ia\pi/n} - 1} \\
&= e^{ia\pi/2n}\frac{e^{ia\pi} - 1}{e^{ia\pi/n} - 1},
\end{aligned}$$

where our assumptions on a ensure that we are not dividing by 0. Then we have, for a as above,

$$\begin{aligned}
\sum_{j=1}^{n}\left(e^{ia\theta_j}+e^{-ia\theta_j}\right) &= e^{ia\pi/2n}\frac{e^{ia\pi}-1}{e^{ia\pi/n}-1}+e^{-ia\pi/2n}\frac{e^{-ia\pi}-1}{e^{-ia\pi/n}-1} \\
&= e^{ia\pi/2n}\frac{e^{ia\pi}-1}{e^{ia\pi/n}-1}+e^{-ia\pi/2n}\cdot\frac{e^{ia\pi/n}}{e^{ia\pi/n}}\cdot\frac{e^{-ia\pi}-1}{e^{-ia\pi/n}-1} \\
&= e^{ia\pi/2n}\frac{e^{ia\pi}-1}{e^{ia\pi/n}-1}+e^{ia\pi/2n}\frac{e^{-ia\pi}-1}{1-e^{ia\pi/n}} \\
&= e^{ia\pi/2n}\frac{e^{ia\pi}-1}{e^{ia\pi/n}-1}-e^{ia\pi/2n}\frac{e^{-ia\pi}-1}{e^{ia\pi/n}-1} \\
&= \frac{e^{ia\pi/2n}}{e^{ia\pi/n-1}}\left(e^{ia\pi}-1-e^{-ia\pi}+1\right) \\
&= \frac{e^{ia\pi/2n}}{e^{ia\pi/n-1}}\frac{1}{2i}\sin(a\pi),
\end{aligned}$$

which is 0 for a an integer.

Now, when $k=l=0$, we have $a_{kl}=b_{kl}=0$, and,

$$\frac{1}{n}\sum_{j=1}^{n}\cos(k\theta_j)\cos(l\theta_j)=\frac{1}{4n}\sum_{j=1}^{n}(1+1+1+1)=1.$$

When $k=l\neq 0$, we have $0<a_{kl}=2k<2n$, and $b_{kl}=0$. Hence,

$$\begin{aligned}
\frac{1}{n}\sum_{j=1}^{n}\cos(k\theta_j)\cos(l\theta_j) &= \frac{1}{4n}\sum_{j=1}^{n}(e^{ia_{kl}\theta_j}+e^{-ia_{kl}\theta_j}+1+1) \\
&= \frac{1}{4n}\left[\sum_{j=1}^{n}\left(e^{ia_{kl}\theta_j}+e^{-ia_{kl}\theta_j}\right)+2n\right] \\
&= \frac{1}{2},
\end{aligned}$$

since a_{kl} meets the conditions for the sum considered above.

When $k\neq l$, we have, $-2n<a_{kl},b_{kl}<2n$ and $a_{kl},b_{kl}\neq 0$, hence,

$$\begin{aligned}
\frac{1}{n}\sum_{j=1}^{n}\cos(k\theta_j)\cos(l\theta_j) &= \frac{1}{4n}\sum_{j=1}^{n}(e^{ia_{kl}\theta_j}+e^{-ia_{kl}\theta_j}+e^{ib_{kl}\theta_j}+e^{-ib_{kl}\theta_j}) \\
&= \frac{1}{4n}\left[\sum_{j=1}^{n}(e^{ia_{kl}\theta_j}+e^{-ia_{kl}\theta_j})+\sum_{j=1}^{n}(e^{ib_{kl}\theta_j}+e^{-ib_{kl}\theta_j})\right] \\
&= 0.
\end{aligned}$$

3. **[5 marks]** Consider the Discrete Cosine Transform (DCT)

$$C_n := \begin{bmatrix} \sqrt{1/n} & & & \\ & \sqrt{2/n} & & \\ & & \ddots & \\ & & & \sqrt{2/n} \end{bmatrix} \begin{bmatrix} 1 & \cdots & 1 \\ \cos\theta_1 & \cdots & \cos\theta_n \\ \vdots & \ddots & \vdots \\ \cos(n-1)\theta_1 & \cdots & \cos(n-1)\theta_n \end{bmatrix}$$

for $\theta_j = \pi(j-1/2)/n$. Prove that C_n is orthogonal: $C_n^\top C_n = C_n C_n^\top = I$. Hint: $C_n C_n^\top = I$ might be easier to show than $C_n^\top C_n = I$ using the previous problem.

The components of C without the diagonal matrix, which we may call $\hat{C}$ are

$$\hat{C}_{ij} = \cos((j-1)\theta_{i-1}),$$

where $\theta_j = \pi(j-1/2)/n$. Recalling that the elements of matrix multiplication are given by

$$(ab)_{ij} := \sum_{k=1}^{n} a_{ik} b_{kj}$$

we find that

$$(\hat{C}_n \hat{C}_n^\top)_{ij} = \sum_{k=1}^{n} \cos((i-1)\theta_{k-1}) \cos((k-1)\theta_{j-1}).$$

By using the previous problem and the terms on the diagonal matrices which ensure that the 1/2 terms become 1 we know how to compute all of these entries and find that it is the identity.

Here is a computer-based demonstration:

```julia
using LinearAlgebra, Plots, FFTW
n = 5
θ = range(π/(2n); step=π/n, length=n) # n evenly spaced points starting at π/(2n) with
step size π/n
C = Diagonal([1/sqrt(n); fill(sqrt(2/n), n-1)]) * [cos((k-1)*θ[j]) for k=1:n, j=1:n]
C'C
```

```
5×5 Matrix{Float64}:
  1.0           -4.85266e-18  -2.82901e-17   1.68455e-17   3.95658e-17
 -4.85266e-18    1.0           4.68569e-17  -6.18283e-17   4.0019e-17
 -2.82901e-17    4.68569e-17   1.0           6.05122e-18  -1.76076e-16
  1.68455e-17   -6.18283e-17   6.05122e-18   1.0           1.03351e-16
  3.95658e-17    4.0019e-17   -1.76076e-16   1.03351e-16   1.0
```

4. **[10 marks]** Consider the variable-coefficient advection equation

$$u_t + c(x)u_x = 0, \qquad c(x) = \frac{1}{5} + \sin^2(x-1), \qquad x \in [0, 2\pi), \qquad t \in [0, T],$$

with $u(x,0) = f(x) = \mathrm{e}^{-100(x-1)^2}$, which we approximate with the leapfrog method

$$\mathbf{u}^{i+1} = \mathbf{u}^{i-1} - 2\tau\mathcal{F}^{-1}\left\{\mathrm{i}(-m:m) \cdot \mathcal{F}\{\mathbf{u}^i\}\right\}, \qquad i = 0, \ldots, n_t - 1.$$

Note that one needs $\mathbf{u}^{-1}$ to initialise the leapfrog method. Let the entries of $\mathbf{u}^{-1}$ be $u_j^{-1} = f(x_j + \tau/5)$, $j = 0, \ldots, n_x - 1$. The exact solution is periodic in time, i.e.,

$$u(x, t+T) = u(x,t)$$

where

$$T = \int_0^{2\pi} \frac{1}{c(x)} \mathrm{d}x = 12.8254983016186\ldots$$

Compute T using the Trapezoidal rule and confirm that you get the value stated above. Then compute

$$e(n_x) = \max_{j=0,\ldots,n_x-1} |u_j^0 - u_j^{n_t}|$$

where $n_t = 8n_x$ and $\tau n_t = T$ and plot $e(n_x)$ for $n_x = 2^k + 1$, with $k = 5, 6, \ldots, 10$. Comment on the behaviour of $e(n_x)$.

As discussed in the notes, the trapezoidal rule approximation to an integral is

$$\int_a^b g(x)\,\mathrm{d}x \approx \frac{h}{2}\left(g(x_0) + 2g(x_1) + 2g(x_2) + \cdots + 2g(x_{n-1}) + g(x_n)\right).$$

To compute T, we set $a = 0$, $b = 2\pi$ and $g(x) = 1/c(x)$ and note that since $c(x)$ is 2π-periodic, $g(x_0) = g(x_n)$, hence

$$\int_0^{2\pi} \frac{1}{c(x)}\,\mathrm{d}x \approx \frac{2\pi}{n} \sum_{j=0}^{n-1} \frac{1}{c(x_j)}$$

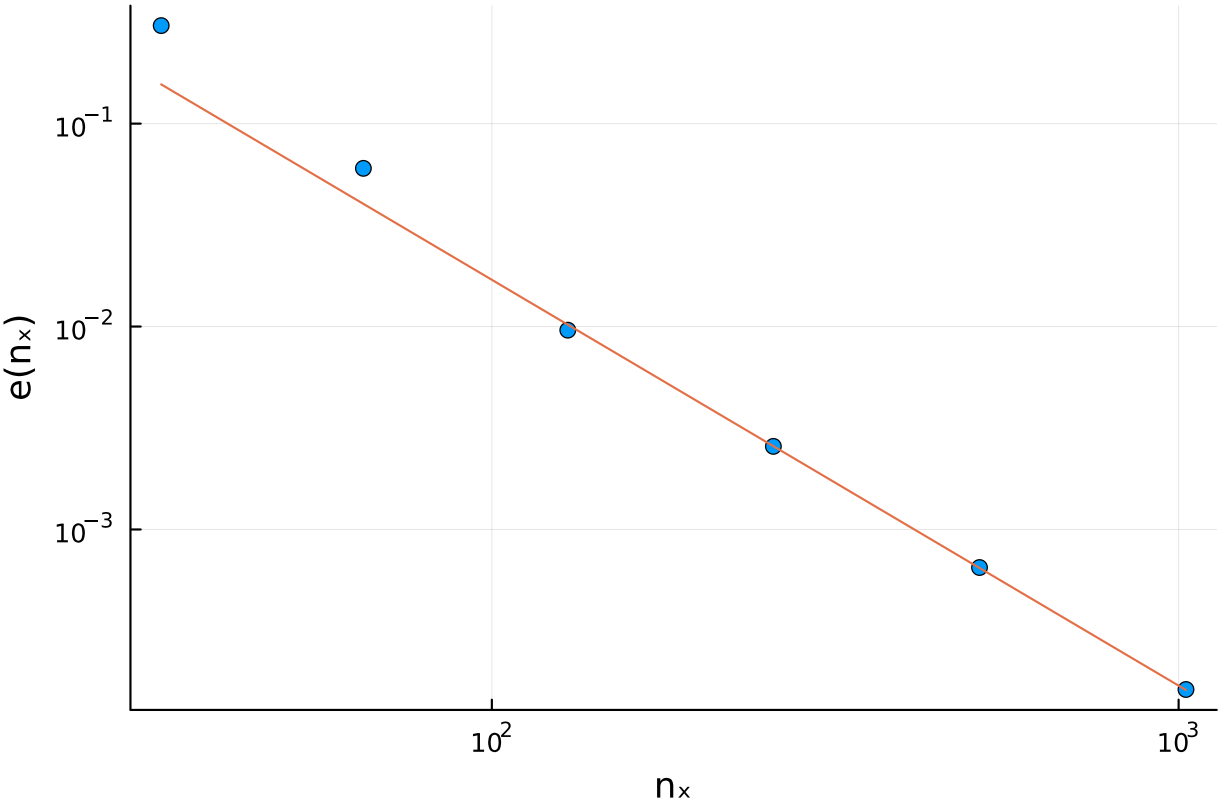
```
c = x -> 0.2 + sin(x - 1)^2
n = 80
xx = range(0,2π;length=n+1)[1:end-1]
T = 2π/n*sum(1 ./c.(xx))
```

```
12.825498301618637
```

```
c = x -> 0.2 + sin(x - 1)^2
f = x -> exp(-100*(x-1)^2)
nxv = 2 .^(5:10) .+ 1
maxerr = []
for n_x = nxv
    m = (n_x-1)÷2
    h = 2π/n_x
    x = h*(0:n_x-1)
    tv = range(0,T;length=8n_x+1)
    τ = tv[2]-tv[1]
    u = f.(x)
    uold = f.(x .+ 0.2τ)
    @time begin
    for n = 1:8n_x
        unew = real.(uold - 2τ*c.(x).*ifft(ifftshift(im*(-m:m)).*fft(u)))
        uold = u
        u = unew
    end
    end
    global maxerr
    maxerr = [maxerr;norm(u-f.(x),Inf)]
end
```

```
0.096903 seconds (159.31 k allocations: 9.735 MiB, 61.45% compilation tim
e)
  0.126623 seconds (38.50 k allocations: 6.919 MiB)
  0.151179 seconds (77.41 k allocations: 22.644 MiB)
  0.765419 seconds (155.24 k allocations: 80.015 MiB, 6.99% gc time)
  1.672286 seconds (310.88 k allocations: 295.999 MiB, 3.96% gc time)
  5.037372 seconds (679.58 k allocations: 1.105 GiB, 3.22% gc time)
```

```
scatter(nxv,maxerr;yscale=:log10,xscale=:log10,legend=false,
xlabel="n_𝑥",ylabel="e(n_𝑥)")
plot!(nxv,170 ./nxv.^2)
```



The plot indicates that $e(n_x) = \mathcal{O}(n_x^{-2})$, $n_x \to \infty$.