1 Chapter 4: Solutions to Exercises, Part II

Consider the Laguerre polynomials

$$L_n^{(\alpha)}(x) = \frac{(-1)^n}{n!}x^n + O(x^{n-1})$$

where $\alpha > -1$, which are orthogonal with respect to

$$\langle f, g\rangle_\alpha = \int_0^\infty f(x)g(x)x^\alpha \mathrm{e}^{-x}\,\mathrm{d}x.$$

1. Show that the Rodrigues formula holds:

$$L_n^{(\alpha)}(x) = \frac{x^{-\alpha}\mathrm{e}^x}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[x^{\alpha+n}\mathrm{e}^{-x}\right].$$

In other words, prove that $L_n^{(\alpha)}(x)$ (defined by the Rodrigues formula) is

(i) a polynomial of degree exactly n

(ii) orthogonal to all lower degree polynomials

(iii) has leading coefficient $\frac{(-1)^n}{n!}$

Hints: For (i) and (iii), it may be helpful first to prove that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{\alpha+1}\mathrm{e}^{-x}L_n^{(\alpha+1)}(x)\right] = (n+1)x^\alpha \mathrm{e}^{-x}L_{n+1}^{(\alpha)}(x)$$

For (ii), use integration by parts.

Solution Following the hint, we have that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{\alpha+1}\mathrm{e}^{-x}L_n^{(\alpha+1)}(x)\right] &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[x^{\alpha+1+n}\mathrm{e}^{-x}\right]\right] \\
&= (n+1)\frac{1}{(n+1)!}\frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}}\left[x^{\alpha+n+1}\mathrm{e}^{-x}\right] \\
&= (n+1)x^\alpha \mathrm{e}^{-x}L_{n+1}^{(\alpha)}(x),
\end{aligned}$$

hence, differentiating, we find that

$$\left[(\alpha+1)-x\right]x^\alpha \mathrm{e}^{-x}L_n^{(\alpha+1)}(x) + x^{\alpha+1}\mathrm{e}^{-x}\left(L_n^{(\alpha+1)}\right)'(x) = (n+1)x^\alpha \mathrm{e}^{-x}L_{n+1}^{(\alpha)}(x)$$

or

$$(\alpha+1-x)L_n^{(\alpha+1)}(x) + x(L_n^{(\alpha+1)})'(x) = (n+1)L_{n+1}^{(\alpha)}(x).$$

By induction with the fact $L_0^{(\alpha)}(x) = 1$, we therefore get that

$$L_n^{(\alpha)}(x) = \frac{(\alpha+1-x)L_{n-1}^{(\alpha+1)}(x) + x(L_{n-1}^{(\alpha+1)})'(x)}{n}$$

is a degree n polynomial, which proves (i).

To prove (iii), we note that the leading coefficient (coefficient of the highest degree term) is

$$L_n^{(\alpha)}(x) = -\frac{x}{n} L_{n-1}^{(\alpha+1)}(x) + O(x^{n-1}) = \frac{x^2}{n(n-1)} L_{n-2}^{(\alpha+2)}(x) + O(x^{n-1}) = \cdots = \frac{(-1)^n x^n}{n!} L_0^{(\alpha+n)}(x) + O(x^{n-1})$$
$$= \frac{(-1)^n x^n}{n!} + O(x^{n-1})$$

We now prove (ii) using integration by parts: let $p_m(x)$ denote a polynomial of degree $m < n$, then integrating by parts n times,

$$\begin{aligned}
\langle L_n^{(\alpha)}, p_m \rangle_\alpha &= \int_0^\infty L_n^{(\alpha)}(x) p_m(x) x^\alpha \mathrm{e}^{-x} \mathrm{d}x \\
&= \int_0^\infty \frac{1}{n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \left[x^{\alpha+n} \mathrm{e}^{-x} \right] p_m(x) \mathrm{d}x \\
&= -\int_0^\infty \frac{1}{n!} \frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}} \left[x^{\alpha+n} \mathrm{e}^{-x} \right] p_m'(x) \mathrm{d}x \\
&\vdots \\
&= (-1)^n \int_0^\infty \frac{1}{n!} \left[x^{\alpha+n} \mathrm{e}^{-x} \right] p_m^{(n)}(x) \mathrm{d}x = 0
\end{aligned}$$

since $p_m^{(n)}(x) = 0$. Note we used the fact that

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k} \left[x^{\alpha+n} \mathrm{e}^{-x} \right]$$

vanishes at $x = 0$ and as $x \to \infty$ to ignore the boundary terms in integration by parts.

2. Show that

$$(n+1) L_{n+1}^{(\alpha)}(x) = (\alpha + n + 1) L_n^{(\alpha)}(x) - x L_n^{(\alpha+1)}(x)$$

Solution Applying the product rule once, we find that

$$\begin{aligned}
(n+1) L_{n+1}^{(\alpha)}(x) &= \frac{x^{-\alpha} \mathrm{e}^x}{n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \frac{\mathrm{d}}{\mathrm{d}x} \left[x^{\alpha+n+1} \mathrm{e}^{-x} \right] \\
&= \frac{x^{-\alpha} \mathrm{e}^x}{n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \left[(\alpha+n+1) x^{\alpha+n} \mathrm{e}^{-x} - x^{\alpha+n+1} \mathrm{e}^{-x} \right] \\
&= (\alpha + n + 1) L_n^{(\alpha)}(x) - x L_n^{(\alpha+1)}(x)
\end{aligned}$$

3. Show that

$$L_n^{(\alpha+1)}(x) = L_{n-1}^{(\alpha+1)}(x) + L_n^{(\alpha)}(x).$$

Solution Applying the product rule n times, we find that

$$
\begin{aligned}
L_n^{(\alpha+1)}(x) &= \frac{x^{-1-\alpha}\mathrm{e}^x}{n!}\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}\frac{\mathrm{d}}{\mathrm{d}x}\left[xx^{\alpha+n}\mathrm{e}^{-x}\right] \\
&= \frac{x^{-1-\alpha}\mathrm{e}^x}{n!}\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}\left[x^{\alpha+n}\mathrm{e}^{-x}\right] + \frac{x^{-1-\alpha}\mathrm{e}^x}{n!}\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}x\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{\alpha+n}\mathrm{e}^{-x}\right] \\
&= \frac{2}{n}L_{n-1}^{(\alpha+1)}(x) + \frac{x^{-1-\alpha}\mathrm{e}^x}{n!}\frac{\mathrm{d}^{n-2}}{\mathrm{d}x^{n-2}}x\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[x^{\alpha+n}\mathrm{e}^{-x}\right] \\
&= \frac{3}{n}L_{n-1}^{(\alpha+1)}(x) + \frac{x^{-1-\alpha}\mathrm{e}^x}{n!}\frac{\mathrm{d}^{n-3}}{\mathrm{d}x^{n-3}}x\frac{\mathrm{d}^3}{\mathrm{d}x^3}\left[x^{\alpha+n}\mathrm{e}^{-x}\right] \\
&\vdots \\
&= \frac{n}{n}L_{n-1}^{(\alpha+1)}(x) + \frac{x^{-\alpha}\mathrm{e}^x}{n!}\frac{\mathrm{d}^{n}}{\mathrm{d}x^{n}}\left[x^{\alpha+n}\mathrm{e}^{-x}\right] \\
&= L_{n-1}^{(\alpha+1)}(x) + L_n^{(\alpha)}(x)
\end{aligned}
$$

4. Show that the Laguerre polynomials satisfy the following three-term recurrence:

$$xL_n^{(\alpha)}(x) = -(n+\alpha)L_{n-1}^{(\alpha)}(x) + (2n+\alpha+1)L_n^{(\alpha)}(x) - (n+1)L_{n+1}^{(\alpha)}(x)$$

Solution Using the results in questions 2 and 3, we have that

$$
\begin{aligned}
xL_n^{(\alpha)}(x) &= -(n+1)L_{n+1}^{(\alpha-1)}(x) + (n+\alpha)L_n^{(\alpha-1)}(x) \\
&= -(n+1)L_{n+1}^{(\alpha)}(x) + (n+1)L_n^{(\alpha)}(x) + (n+\alpha)L_n^{(\alpha)}(x) - (n+\alpha)L_{n-1}^{(\alpha)}(x) \\
&= -(n+\alpha)L_{n-1}^{(\alpha)}(x) + (2n+\alpha+1)L_n^{(\alpha)}(x) - (n+1)L_{n+1}^{(\alpha)}(x)
\end{aligned}
$$

5. Prove that

$$\frac{\mathrm{d}L_n^{(\alpha)}}{\mathrm{d}x} = -L_{n-1}^{(\alpha+1)}(x)$$

Solution First we show that $\frac{\mathrm{d}L_n^{(\alpha)}}{\mathrm{d}x}$ is orthogonal wrt $\langle\cdot,\cdot\rangle_{\alpha+1}$ to polynomials of degree $\leq n-2$ using integration by parts: let $p_m(x)$ be a polynomial of degree $m \leq n-2$, then

$$
\begin{aligned}
\langle\left(L_n^{(\alpha)}\right)', p_m\rangle_{\alpha+1} &= \int_0^\infty \frac{\mathrm{d}L_n^{(\alpha)}(x)}{\mathrm{d}x}p_m(x)x^{\alpha+1}\mathrm{e}^{-x}\mathrm{d}x \\
&= -\int_0^\infty L_n^{(\alpha)}(x)(xp_m'(x) + (\alpha+1)p_m(x) - xp_m(x))x^\alpha\mathrm{e}^{-x}\mathrm{d}x \\
&= \langle L_n^{(\alpha)}, xp_m + (\alpha+1)p_m - xp_m\rangle_\alpha \\
&= 0
\end{aligned}
$$

since $(xp_m' + (\alpha+1)p_m - xp_m)$ has degree $m+1 < n$. We conclude that $\frac{\mathrm{d}L_n^{(\alpha)}}{\mathrm{d}x} = CL_{n-1}^{(\alpha+1)}$, for some constant C which we can determine by matching leading terms: we have that

$$\frac{\mathrm{d}L_n^{(\alpha)}}{\mathrm{d}x} = \frac{(-1)^n}{(n-1)!}x^{n-1} + \mathcal{O}(x^{n-2}) = CL_{n-1}^{(\alpha+1)} = C\frac{(-1)^{n-1}}{(n-1)!}x^{n-1} + \mathcal{O}(x^{n-2}),$$

which implies that $C = -1$.

6. Let

$$L^{(\alpha)}(x) = \left[L_0^{(\alpha)}(x)|L_1^{(\alpha)}(x)|\cdots\right].$$

Give matrices D_α and S_α such that

$$\frac{\mathrm{d}}{\mathrm{d}x}L^{(\alpha)}(x) = L^{(\alpha+1)}(x)\mathcal{D}_\alpha \quad \text{and} \quad L^{(\alpha)}(x) = L^{(\alpha+1)}(x)S_\alpha$$

Solution It follows from the formulas in questions 5 and 3 that

$$\mathcal{D}_\alpha = \begin{pmatrix} 0 & -1 & & \\ & & -1 & \\ & & & \ddots \end{pmatrix}$$

and

$$\mathcal{S}_\alpha = \begin{pmatrix} 1 & -1 & & \\ & 1 & -1 & \\ & & \ddots & \ddots \end{pmatrix}$$

7. Consider the advection equation on the half line:

$$u_t + u_x = 0, \qquad x \in [0, \infty), \qquad t \geq 0.$$

Suppose the solution has an expansion of the form

$$u(x,t) = \mathrm{e}^{-x/2}\sum_{k=0}^{\infty} u_k(t)L_k^{(0)}(x) = \mathrm{e}^{-x/2}L^{(0)}(x)\mathbf{u}(t)$$

where

$$\mathbf{u}(t) = \begin{bmatrix} u_0(t) \\ u_1(t) \\ \vdots \end{bmatrix}.$$

Show that

$$\mathbf{u}'(t) = A\mathbf{u}(t),$$

where A is a matrix that is expressible in terms of $\mathcal{D}_0$ and $\mathcal{S}_0$ (defined in question 6). Use software of your choice to build a 10×10 version of the matrix A.

Solution Since

$$\begin{aligned} \frac{\partial}{\partial x}u(x,t) &= -\frac{1}{2}\mathrm{e}^{-x/2}L^{(0)}(x)\mathbf{u}(t) + \mathrm{e}^{-x/2}\frac{\mathrm{d}}{\mathrm{d}x}L^{(0)}(x)\mathbf{u}(t) \\ &= -\frac{1}{2}\mathrm{e}^{-x/2}L^{(1)}(x)\mathcal{S}_0\mathbf{u}(t) + \mathrm{e}^{-x/2}L^{(1)}(x)\mathcal{D}_0\mathbf{u}(t) \\ &= \mathrm{e}^{-x/2}L^{(1)}(x)\left[-\frac{1}{2}\mathcal{S}_0 + \mathcal{D}_0\right]\mathbf{u}(t) \end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial t}u(x,t) &= \mathrm{e}^{-x/2}L^{(0)}(x)\mathbf{u}'(t) \\ &= \mathrm{e}^{-x/2}L^{(1)}(x)\mathcal{S}_0\mathbf{u}'(t)\end{aligned}$$

the advection equation implies that

$$u_t + u_x = \mathrm{e}^{-x/2}L^{(1)}(x)\left[\mathcal{S}_0\mathbf{u}'(t) + \left(-\frac{1}{2}\mathcal{S}_0 + \mathcal{D}_0\right)\mathbf{u}(t)\right] = 0$$

hence

$$\mathcal{S}_0\mathbf{u}'(t) = \left(\frac{1}{2}\mathcal{S}_0 - \mathcal{D}_0\right)\mathbf{u}(t)$$

or

$$\mathbf{u}'(t) = A\mathbf{u}(t)$$

where $A = \frac{1}{2}\mathcal{I} - \mathcal{S}_0^{-1}\mathcal{D}_0$ and $\mathcal{I}$ is an infinite identity matrix.

```julia
using ApproxFun, LinearAlgebra

D0 = Derivative() :  Laguerre(0) → Laguerre(1)
real(D0)
```

```
ReOperator : ApproxFunOrthogonalPolynomials.Laguerre{Int64, ApproxFunOrthog
onalPolynomials.Ray{false, Float64}}(0,  0.0,∞ ) → ApproxFunOrthogonalPolyn
omials.Laguerre{Int64, ApproxFunOrthogonalPolynomials.Ray{false, Float64}}(
1,  0.0,∞ )
 0.0  -1.0    ⋅     ⋅     ⋅     ⋅     ⋅     ⋅     ⋅     ⋅    ⋅
  ⋅    0.0  -1.0    ⋅     ⋅     ⋅     ⋅     ⋅     ⋅     ⋅    ⋅
  ⋅     ⋅    0.0  -1.0    ⋅     ⋅     ⋅     ⋅     ⋅     ⋅    ⋅
  ⋅     ⋅     ⋅    0.0  -1.0    ⋅     ⋅     ⋅     ⋅     ⋅    ⋅
  ⋅     ⋅     ⋅     ⋅    0.0  -1.0    ⋅     ⋅     ⋅     ⋅    ⋅
  ⋅     ⋅     ⋅     ⋅     ⋅    0.0  -1.0    ⋅     ⋅     ⋅    ⋅
  ⋅     ⋅     ⋅     ⋅     ⋅     ⋅    0.0  -1.0    ⋅     ⋅    ⋅
  ⋅     ⋅     ⋅     ⋅     ⋅     ⋅     ⋅    0.0  -1.0    ⋅    ⋅
  ⋅     ⋅     ⋅     ⋅     ⋅     ⋅     ⋅     ⋅    0.0  -1.0   ⋅
  ⋅     ⋅     ⋅     ⋅     ⋅     ⋅     ⋅     ⋅     ⋅    0.0   ⋱
  ⋅     ⋅     ⋅     ⋅     ⋅     ⋅     ⋅     ⋅     ⋅     ⋅    ⋱
```

```julia
S0 = I : Laguerre(0) → Laguerre(1)
```

```
ConstantTimesOperator : ApproxFunOrthogonalPolynomials.Laguerre{Int64, Appr
oxFunOrthogonalPolynomials.Ray{false, Float64}}(0,  0.0,∞ ) → ApproxFunOrth
ogonalPolynomials.Laguerre{Int64, ApproxFunOrthogonalPolynomials.Ray{false,
 Float64}}(1,  0.0,∞ )
 1.0  -1.0    ⋅     ⋅     ⋅     ⋅     ⋅     ⋅     ⋅     ⋅    ⋅
  ⋅    1.0  -1.0    ⋅     ⋅     ⋅     ⋅     ⋅     ⋅     ⋅    ⋅
  ⋅     ⋅    1.0  -1.0    ⋅     ⋅     ⋅     ⋅     ⋅     ⋅    ⋅
  ⋅     ⋅     ⋅    1.0  -1.0    ⋅     ⋅     ⋅     ⋅     ⋅    ⋅
  ⋅     ⋅     ⋅     ⋅    1.0  -1.0    ⋅     ⋅     ⋅     ⋅    ⋅
  ⋅     ⋅     ⋅     ⋅     ⋅    1.0  -1.0    ⋅     ⋅     ⋅    ⋅
  ⋅     ⋅     ⋅     ⋅     ⋅     ⋅    1.0  -1.0    ⋅     ⋅    ⋅
```

```
  ⋅    ⋅    ⋅    ⋅    ⋅    ⋅    ⋅   1.0  -1.0    ⋅   ⋅
  ⋅    ⋅    ⋅    ⋅    ⋅    ⋅    ⋅    ⋅    1.0  -1.0  ⋅
  ⋅    ⋅    ⋅    ⋅    ⋅    ⋅    ⋅    ⋅    ⋅    1.0  ⋱
  ⋅    ⋅    ⋅    ⋅    ⋅    ⋅    ⋅    ⋅    ⋅    ⋅    ⋱
```

```
n = 10
A = I/2 - S0[1:n,1:n]\real(D0[1:n,1:n])
```

```
10×10 Matrix{Float64}:
  0.5   1.0   1.0   1.0   1.0   1.0   1.0   1.0   1.0  1.0
 -0.0   0.5   1.0   1.0   1.0   1.0   1.0   1.0   1.0  1.0
 -0.0  -0.0   0.5   1.0   1.0   1.0   1.0   1.0   1.0  1.0
 -0.0  -0.0  -0.0   0.5   1.0   1.0   1.0   1.0   1.0  1.0
 -0.0  -0.0  -0.0  -0.0   0.5   1.0   1.0   1.0   1.0  1.0
 -0.0  -0.0  -0.0  -0.0  -0.0   0.5   1.0   1.0   1.0  1.0
 -0.0  -0.0  -0.0  -0.0  -0.0  -0.0   0.5   1.0   1.0  1.0
 -0.0  -0.0  -0.0  -0.0  -0.0  -0.0  -0.0   0.5   1.0  1.0
 -0.0  -0.0  -0.0  -0.0  -0.0  -0.0  -0.0  -0.0   0.5  1.0
 -0.0  -0.0  -0.0  -0.0  -0.0  -0.0  -0.0  -0.0  -0.0  0.5
```