

1 Problem sheet 2

Consider the following finite difference method for the diffusion equation

$$-\frac{1}{2}\mu u_{j-1}^{i+1} + (1 + \mu)u_j^{i+1} - \frac{1}{2}\mu u_{j+1}^{i+1} = \frac{1}{2}\mu u_{j-1}^i + (1 - \mu)u_j^i + \frac{1}{2}\mu u_{j+1}^i$$

where $\mu = \tau/h^2$.

1. **[5 marks]** Show that the method has a second-order local truncation error. That is, let $\tilde{u}_j^i = u(x_j, t_i)$, where $x_j = jh$, $t_i = i\tau$, show that the method can be expressed as

$$\frac{u_j^{i+1} - u_j^i}{\tau} = \frac{1}{2} \left(\frac{u_{j+1}^{i+1} - 2u_j^{i+1} + u_{j-1}^{i+1}}{h^2} + \frac{u_{j+1}^i - 2u_j^i + u_{j-1}^i}{h^2} \right)$$

and then show that for $\tau, h \rightarrow 0$ (assuming all the relevant partial derivatives are bounded),

$$\frac{\tilde{u}_j^{i+1} - \tilde{u}_j^i}{\tau} - \frac{1}{2} \left(\frac{\tilde{u}_{j+1}^{i+1} - 2\tilde{u}_j^{i+1} + \tilde{u}_{j-1}^{i+1}}{h^2} + \frac{\tilde{u}_{j+1}^i - 2\tilde{u}_j^i + \tilde{u}_{j-1}^i}{h^2} \right) = \mathcal{O}(\tau^2) + \mathcal{O}(h^2).$$

Hint: use Taylor's theorem to expand the exact solution about $(x, t) = (x_j, t_{i+1/2})$, where $t_{i+1/2} = (i + 1/2)\tau = t_i + \tau/2$.

2. **[5 marks]** Use the Von Neumann method to show that the method is unconditionally stable.
3. **[5 marks]** Suppose the finite difference method is used to approximate the solution to the diffusion equation $u_t = u_{xx}$ for $(x, t) \in (0, 1) \times (0, T)$ subject to $u(0, t) = \varphi_0(t)$, $u(1, t) = \varphi_1(t)$ for $t \in [0, T]$ and $u(x, 0) = f(x)$ for $x \in [0, 1]$. Let $u(x_j, t_i) \approx u_j^i$, where $x_j = jh$, $h = 1/(n_x + 1)$, $j = 0, \dots, n_x + 1$, $t_i = i\tau$, $\tau = \mu h^2$ and set

$$\mathbf{u}^i = \begin{bmatrix} u_1^i \\ \vdots \\ u_{n_x}^i \end{bmatrix} \in \mathbb{R}^{n_x}.$$

The finite difference method can be expressed as

$$L\mathbf{u}^{i+1} = R\mathbf{u}^i + \mathbf{k}^i,$$

where $L, R \in \mathbb{R}^{n_x \times n_x}$ and $\mathbf{k}^i \in \mathbb{R}^{n_x}$. Give the matrices L and R and the vector \mathbf{k}^i .

4. **[5 marks]** What are the eigenvalues of the matrix $A := L^{-1}R$? Find a bound on the spectral radius of A . Hint: consider the eigendecomposition (spectral factorisation) of L and R .
5. **[10 marks]** Let

$$u(x, 0) = f(x) = \sin \frac{1}{2}\pi x + \frac{1}{2} \sin 2\pi x, \quad 0 \leq x \leq 1,$$

and

$$u(0, t) = \varphi_0(t) = 0, \quad u(1, t) = \varphi_1(t) = e^{-\pi^2 t/4}, \quad t \geq 0,$$

then the exact solution to the diffusion equation is

$$u(x, t) = e^{-\pi^2 t/4} \sin \frac{1}{2}\pi x + \frac{1}{2} e^{-4\pi^2 t} \sin 2\pi x, \quad 0 \leq x \leq 1, \quad t \geq 0.$$

Using the method in question 3 and the backward Euler method, approximate the exact solution for $T = 1$, $\mu = n_x$ and plot the maximum error of the two methods for $n_x = 2^k$, $k = 4, 5, \dots, 12$. Comment on your results.