1 Problem Sheet 1

1. [**5 marks**] What are the Fourier coefficients c_k of $\sin^4 x$?

2. [**5 marks**] Show for $0 \leq k, \ell \leq n-1$

$$\frac{1}{n} \sum_{j=1}^{n} \cos k\theta_j \cos \ell\theta_j = \begin{cases} 1 & k = \ell = 0 \\ 1/2 & k = \ell \\ 0 & \text{otherwise} \end{cases}$$

for $\theta_j = \pi(j - 1/2)/n$. Hint: You may consider replacing cos with complex exponentials:

$$\cos\theta = \frac{\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta}}{2}.$$

3. [**5 marks**] Consider the Discrete Cosine Transform (DCT)

$$C_n := \begin{bmatrix} \sqrt{1/n} & & & \\ & \sqrt{2/n} & & \\ & & \ddots & \\ & & & \sqrt{2/n} \end{bmatrix} \begin{bmatrix} 1 & \cdots & 1 \\ \cos\theta_1 & \cdots & \cos\theta_n \\ \vdots & \ddots & \vdots \\ \cos(n-1)\theta_1 & \cdots & \cos(n-1)\theta_n \end{bmatrix}$$

for $\theta_j = \pi(j - 1/2)/n$. Prove that C_n is orthogonal: $C_n^\top C_n = C_n C_n^\top = I$. Hint: $C_n C_n^\top = I$ might be easier to show than $C_n^\top C_n = I$ using the previous problem.

4. [**10 marks**] Consider the variable-coefficient advection equation

$$u_t + c(x)u_x = 0, \qquad c(x) = \frac{1}{5} + \sin^2(x-1), \qquad x \in [0, 2\pi), \qquad t \in [0, T],$$