1 Chapter 2: Solutions to exercises

1. Give explicit formulae for the Fourier coefficients c_k and approximate Fourier coefficients $\tilde{c}_k^n$ for the following functions:

$$\cos x, \frac{3}{3 - \mathrm{e}^{\mathrm{i}x}}$$

Hint: You may wish to try the change of variables $z = \mathrm{e}^{\mathrm{i}x}$.

For $f(x) = \cos x$, since

$$f(x) = \frac{1}{2}\mathrm{e}^{\mathrm{i}x} + \frac{1}{2}\mathrm{e}^{-\mathrm{i}x}$$

we have that

$$c_{-1} = c_1 = \frac{1}{2}, \qquad c_k = 0, \qquad k \neq -1, 1.$$

To find $\tilde{c}_k^n$, we use the aliasing formula:

$$\tilde{c}_k^n = \cdots + c_{k-2n} + c_{k-n} + c_k + c_{k+n} + c_{k+2n} + \cdots$$

we also note that

$$\tilde{c}_{k+np}^n = \tilde{c}_k^n, \qquad p \in \mathbb{Z}.$$

Therefore for $p \in \mathbb{Z}$ we have

$$\tilde{c}_k^1 = c_1 + c_{-1} = 1$$
$$\tilde{c}_{2p}^2 = 0, \quad \tilde{c}_{2p+1}^2 = c_1 + c_{-1} = 1$$

and for $n \geq 3$,

$$\tilde{c}_{1+np}^n = \tilde{c}_{-1+np}^n = 1/2, \qquad \tilde{c}_k^n = 0 \text{ otherwise}$$

For $f(x) = \frac{3}{3-\mathrm{e}^{\mathrm{i}x}}$, under the change of variables $z = \mathrm{e}^{\mathrm{i}x}$ we can use geometric series to determine

$$f = \frac{3}{3-z} = \frac{1}{1 - z/3} = \sum_{k=0}^{\infty} \frac{z^k}{3^k}$$

That is $c_k = 1/3^k$ for $k \geq 0$, and $c_k = 0$ for $k < 0$. We then have for $0 \leq$ ă$k \leq$ ă$n - 1$

$$\tilde{c}_{k+pn}^n = \sum_{\ell=0}^{\infty} \frac{1}{3^{k+\ell n}} = \frac{1}{3^k} \frac{1}{1 - 1/3^n} = \frac{3^n}{3^{n+k} - 3^k}$$

2. Show that the DFT Q_n is symmetric ($Q_n = Q_n^\top$) but not Hermitian ($Q_n \neq$ ăQ_n^*).

$$Q_n := \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{-ix_1} & e^{-ix_2} & \cdots & e^{-ix_{n-1}} \\ 1 & e^{-i2x_1} & e^{-i2x_2} & \cdots & e^{-i2x_{n-1}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & e^{-i(n-1)x_1} & e^{-i(n-1)x_2} & \cdots & e^{-i(n-1)x_{n-1}} \end{bmatrix}$$

where $x_j = 2\pi j/n$ for $j = 0, 1, \ldots, n$ and $\omega := e^{ix_1} = e^{\frac{2\pi i}{n}}$ are n th roots of unity in the sense that $\omega^n = 1$. So $e^{ix_j} = e^{\frac{2\pi ij}{n}} = \omega^j$. Note that $x_j = 2\pi(j-1)/n + 2\pi/n = x_{j-1} + x_1$. By completing this recurrence we find that $x_j = jx_1$, from which the following symmetric version follows immediately

$$Q_n = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \cdots & \omega^{-(n-1)} \\ 1 & \omega^{-2} & \omega^{-4} & \cdots & \omega^{-2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{-(n-1)} & \omega^{-2(n-1)} & \cdots & \omega^{-(n-1)^2} \end{bmatrix}.$$

Now $Q_n^\star$ is found to be

$$Q_n^\star = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{ix_1} & e^{i2x_1} & \cdots & e^{i(n-1)x_1} \\ 1 & e^{ix_2} & e^{i2x_2} & \cdots & e^{i(n-1)x_2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & e^{ix_{n-1}} & e^{i2x_{n-1}} & \cdots & e^{i(n-1)x_{n-1}} \end{bmatrix} = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^{1} & \omega^{2} & \cdots & \omega^{(n-1)} \\ 1 & \omega^{2} & \omega^{4} & \cdots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{(n-1)} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)^2} \end{bmatrix}$$

using the above arguments. Evidently, $Q_n^\star \neq Q_n$ since $\omega \neq \omega^{-1}$.

3. Show that

$$\sum_{k=-m}^{m} e^{ikx} = \begin{cases} \frac{\sin((m+1/2)x)}{\sin(x/2)} & \text{if } x \neq 0 \\ 2m+1 & \text{if } x = 0 \end{cases}$$

If $x = 0 \pmod{2\pi}$), then $e^{ikx} = 1$ and thus

$$\sum_{k=-m}^{m} e^{ikx} = \sum_{k=-m}^{m} 1 = 2m+1$$

otherwise (for $x \neq 0 \pmod{2\pi}$ and thus $e^{ikx} \neq 1$)

$$\begin{aligned} \sum_{k=-m}^{m} e^{ikx} &= e^{-imx} \sum_{k=0}^{2m} e^{ikx} \\ &= e^{-imx} \frac{1 - e^{i(2m+1)x}}{1 - e^{ix}} \\ &= \frac{e^{-i(m+1/2)x} - e^{i(m+1/2)x}}{e^{-ix/2} - e^{ix/2}} \\ &= \frac{\sin((m+1/2)x)}{\sin(x/2)} \end{aligned}$$

4. Prove that the trigonometric interpolant $p_n(x)$ that interpolates f at $x = x_j = jh$, $j = 0, \dots, n-1$ with $h = 2\pi/n$ is unique.

Let

$$p_n(x) = \sum_{m=-k}^{m} \tilde{c}_k^n \mathrm{e}^{\mathrm{i}kx}$$

and suppose there is some other trigonometric interpolant $q_n(x)$ with

$$q_n(x) = \sum_{k=-m}^{m} \tilde{b}_k^n \mathrm{e}^{\mathrm{i}kx}$$

that also satisfies $q_n(x_j) = f(x_j)$, $j = 0, \dots, n-1$, then

$$\mathbf{p} = \begin{pmatrix} p_n(x_0) \\ p_n(x_1) \\ \vdots \\ p_n(x_{n-2}) \\ p_n(x_{n-1}) \end{pmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ \mathrm{e}^{-\mathrm{i}mx_1} & \mathrm{e}^{-\mathrm{i}(m-1)x_1} & \mathrm{e}^{-\mathrm{i}(m-2)x_1} & \cdots & \mathrm{e}^{\mathrm{i}mx_1} \\ \mathrm{e}^{-\mathrm{i}mx_2} & \mathrm{e}^{-\mathrm{i}(m-1)x_2} & \mathrm{e}^{-\mathrm{i}(m-2)x_2} & \cdots & \mathrm{e}^{\mathrm{i}mx_2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathrm{e}^{-\mathrm{i}mx_{n-1}} & \mathrm{e}^{-\mathrm{i}(m-1)x_{n-1}} & \mathrm{e}^{-\mathrm{i}(m-2)x_{n-1}} & \cdots & \mathrm{e}^{\mathrm{i}mx_{n-1}} \end{bmatrix}}_{V} \begin{pmatrix} \tilde{c}_{-m}^n \\ \tilde{c}_{-m+1}^n \\ \vdots \\ \tilde{c}_{m-1}^n \\ \tilde{c}_m^n \end{pmatrix}$$

and

$$\mathbf{q} = \begin{pmatrix} q_n(x_0) \\ q_n(x_1) \\ \vdots \\ q_n(x_{n-2}) \\ q_n(x_{n-1}) \end{pmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ \mathrm{e}^{-\mathrm{i}mx_1} & \mathrm{e}^{-\mathrm{i}(m-1)x_1} & \mathrm{e}^{-\mathrm{i}(m-2)x_1} & \cdots & \mathrm{e}^{\mathrm{i}mx_1} \\ \mathrm{e}^{-\mathrm{i}mx_2} & \mathrm{e}^{-\mathrm{i}(m-1)x_2} & \mathrm{e}^{-\mathrm{i}(m-2)x_2} & \cdots & \mathrm{e}^{\mathrm{i}mx_2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathrm{e}^{-\mathrm{i}mx_{n-1}} & \mathrm{e}^{-\mathrm{i}(m-1)x_{n-1}} & \mathrm{e}^{-\mathrm{i}(m-2)x_{n-1}} & \cdots & \mathrm{e}^{\mathrm{i}mx_{n-1}} \end{bmatrix}}_{V} \begin{pmatrix} \tilde{b}_{-m}^n \\ \tilde{b}_{-m+1}^n \\ \vdots \\ \tilde{b}_{m-1}^n \\ \tilde{b}_m^n \end{pmatrix}.$$

We have that $\mathbf{p} = \mathbf{q}$ because $p_n(x_j) = f(x_j) = q_n(x_j)$, $j = 0, \dots, n-1$. As shown in the notes of Chapter 2,

$$V = \sqrt{n}\, Q_n^* P^\top$$

and $(Q_n^*)^{-1} = Q_n$ and $\left(P^\top\right)^{-1} = P$, therefore V is invertible and $V^{-1} = PQ_n/\sqrt{n}$. Multiplying the equations for $\mathbf{p}$ and $\mathbf{q}$ above by V^{-1}, i.e., $V^{-1}\mathbf{p} = V^{-1}\mathbf{q}$, it follows that $\tilde{c}_k^n = \tilde{b}_k^n$, $k = -m, \dots, m$ and therefore $p_n(x) = q_n(x)$.

5. Consider the advection equation

$$u_t + u_x = 0, \qquad x \in [0, 2\pi), \qquad t \in [0, T],$$

with $u(x, 0) = f(x) = \mathrm{e}^{-100(x-1)^2}$ and exact solution $u(x,t) = f(x-t)$; also consider (i) the forward-difference-Fourier method

$$\mathbf{u}^{i+1} = \mathbf{u}^i - \tau\mathcal{F}^{-1}\left\{\mathrm{i}(-m\!:\!m)\cdot\mathcal{F}\{\mathbf{u}^i\}\right\}, \qquad i = 0,\ldots,n_t - 1$$

and (ii) the central-difference-Fourier method (aka the leapfrog method)

$$\mathbf{u}^{i+1} = \mathbf{u}^{i-1} - 2\tau\mathcal{F}^{-1}\left\{\mathrm{i}(-m\!:\!m)\cdot\mathcal{F}\{\mathbf{u}^i\}\right\}, \qquad i = 1,\ldots,n_t - 1.$$

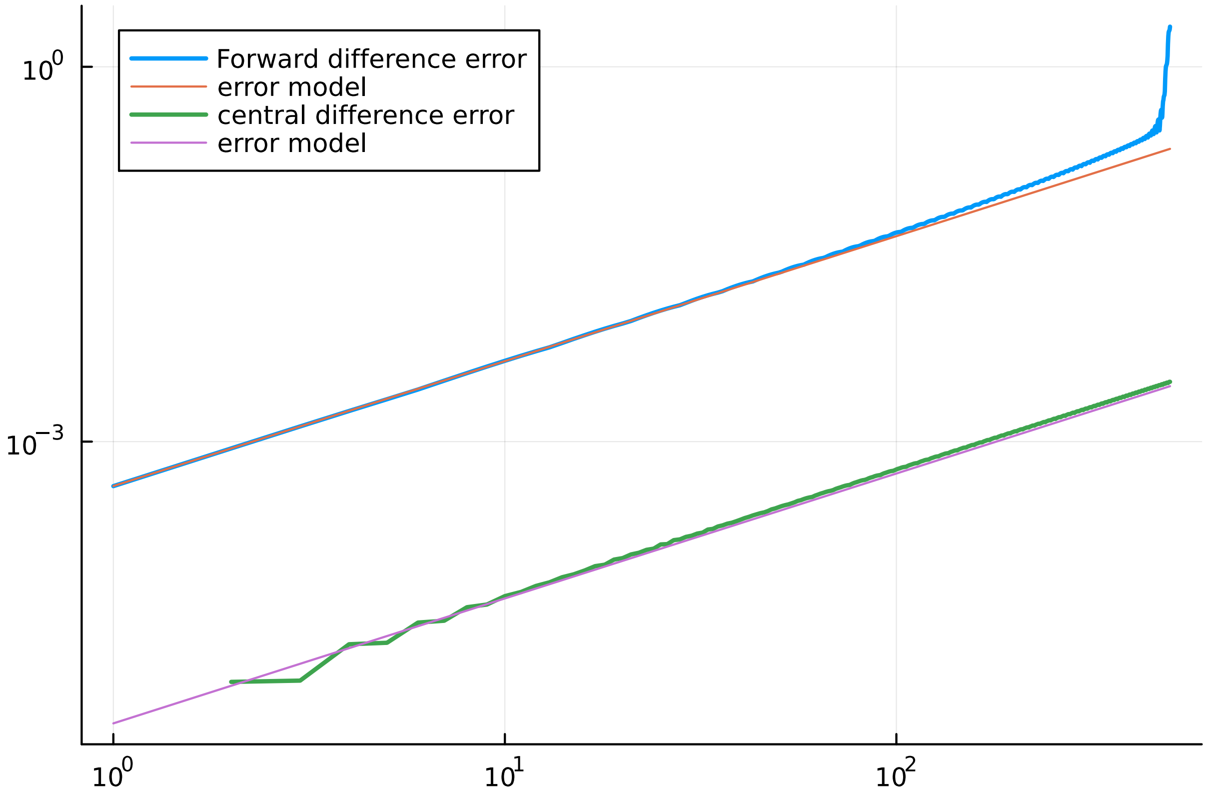
For the leapfrog method, set $u_j^1 = f(x_j - \tau)$. For both methods, set $n_x = 401$, $n_t = 500$ and $T = 1.05$ and plot the maximum error for each time step, i.e., plot

$$e_i := \max_{j=0,\ldots,n_x-1} |u(x_j, t_i) - u_j^i|$$

for $i = 1,\ldots,n_t$. Describe the behaviour of e_i for each method.

```
using LinearAlgebra, FFTW, Plots
f = x -> exp(-100*(x-1)^2)
n_𝑥 = 401
m = (n_𝑥 - 1)÷2
x = range(0,2π;length=n_𝑥+1)[1:end-1] # the equispaced grid in the x-direction
n_𝑡 = 500
T = 1.05
τ = T/n_𝑡
u = zeros(n_𝑡 + 1,n_𝑥)
ulf = zeros(n_𝑡+1,n_𝑥)
maxe = zeros(n_𝑡)
maxelf = zeros(n_𝑡-1)
u[1,:] = ulf[1,:] = f.(x)  # initial data
ulf[2,:] = f.(x .- τ)
for n = 1:n_𝑡
    exact = f.(x .- n*τ)
    u[n+1,:] = real.(u[n,:] - τ*ifft(ifftshift(im*(-m:m)).*fft(u[n,:])))
    maxe[n] = norm(u[n+1,:] - exact,Inf)
    if n > 1
        ulf[n+1,:] = real.(ulf[n-1,:] - 2τ*ifft(ifftshift(im*(-m:m)).*fft(ulf[n,:])))
        maxelf[n-1] = norm(ulf[n+1,:] - exact,Inf)
    end
end

v = 1:n_𝑡
plot(v,maxe;yscale=:log10,xscale=:log10,lw=2,
label="Forward difference error",legend=:topleft)
plot!(v,100*v*τ^2;label="error model")
plot!(v[2:end],maxelf,label="central difference error",lw=2)
plot!(v,600*v*τ^3;label="error model")
```



For the forward-difference-Fourier method, we have $e_i \approx 100 i \tau^2$, however e_i grows explosively after roughly $t = 1$. For the central-difference-Fourier method, $e_i \approx 600 i \tau^3$. Later in this module, we'll see where these error models come from, however you might be able to derive these using Taylor expansions.