1 Chapter 5: Exercises

Consider the following finite difference method for the diffusion equation

$$-\frac{1}{2}\mu u_{j-1}^{i+1} + (1+\mu)u_j^{i+1} - \frac{1}{2}\mu u_{j+1}^{i+1} = \frac{1}{2}\mu u_{j-1}^{i} + (1-\mu)u_j^{i} + \frac{1}{2}\mu u_{j+1}^{i} \tag{1}$$

where $\mu = \tau/h^2$.

1. Show that (1) has a second-order local truncation error. That is, let $\tilde{u}_j^i = u(x_j, t_i)$, where $x_j = jh$, $t_i = i\tau$, show that the method can be expressed as

$$\frac{u_j^{i+1} - u_j^i}{\tau} = \frac{1}{2}\left(\frac{u_{j+1}^{i+1} - 2u_j^{i+1} + u_{j-1}^{i+1}}{h^2} + \frac{u_{j+1}^{i} - 2u_j^{i} + u_{j-1}^{i}}{h^2}\right)$$

and then show that for $\tau, h \to 0$ (assuming all the relevant partial derivatives are bounded),

$$\frac{\tilde{u}_j^{i+1} - \tilde{u}_j^i}{\tau} - \frac{1}{2}\left(\frac{\tilde{u}_{j+1}^{i+1} - 2\tilde{u}_j^{i+1} + \tilde{u}_{j-1}^{i+1}}{h^2} + \frac{\tilde{u}_{j+1}^{i} - 2\tilde{u}_j^{i} + \tilde{u}_{j-1}^{i}}{h^2}\right) = \mathcal{O}(\tau^2) + \mathcal{O}(h^2).$$

Hint: use Taylor's theorem to expand the exact solution about $(x,t) = (x_j, t_{i+1/2})$, where $t_{i+1/2} = (i+1/2)\tau = t_i + \tau/2$.

2. Use the Von Neumann method to show that the method is unconditionally stable.

3. Suppose the finite difference method is used to approximate the solution to the diffusion equation $u_t = u_{xx}$ for $(x,t) \in (0,1) \times (0,T)$ subject to $u(0,t) = \varphi_0(t)$, $u(1,t) = \varphi_1(t)$ for $t \in [0,T]$ and $u(x,0) = f(x)$ for $x \in [0,1]$. Let $u(x_j, t_i) \approx u_j^i$, where $x_j = jh$, $h = 1/(n_x+1)$, $j = 0, \ldots, n_x+1$, $t_i = i\tau$, $\tau = \mu h^2$ and set

$$\mathbf{u}^i = \begin{bmatrix} u_1^i \\ \vdots \\ u_{n_x}^i \end{bmatrix} \in \mathbb{R}^{n_x}.$$

The finite difference method can be expressed as

$$L\mathbf{u}^{i+1} = R\mathbf{u}^i + \mathbf{k}^i,$$

where $L, R \in \mathbb{R}^{n_x \times n_x}$ and $\mathbf{k}^i \in \mathbb{R}^{n_x}$. Give the matrices L and R and the vector $\mathbf{k}^i$.

4. What are the eigenvalues of the matrix $A := L^{-1}R$? Find a bound on the spectral radius of A. Hint: consider the eigendecomposition (spectral factorisation) of L and R.

5. Let

$$u(x,0) = f(x) = \sin \frac{1}{2}\pi x + \frac{1}{2}\sin 2\pi x, \qquad 0 \le x \le 1,$$

and

$$u(0,t) = \varphi_0(t) = 0, \qquad u(1,t) = \varphi_1(t) = \mathrm{e}^{-\pi^2 t/4}, \qquad t \ge 0,$$

then the exact solution to the diffusion equation is

$$u(x,t) = \mathrm{e}^{-\pi^2 t/4} \sin \frac{1}{2}\pi x + \frac{1}{2}\mathrm{e}^{-4\pi^2 t} \sin 2\pi x, \qquad 0 \le x \le 1, \qquad t \ge 0.$$

Using the method in question 3 and the backward Euler method, approximate the exact solution for $T = 1$, $\mu = n_x$ and plot the maximum error of the two methods for $n_x = 2^k$, $k = 4, 5, \ldots, 12$. Comment on your results.