1 Problem sheet 2

The (well-posed) convection-diffusion equation is

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - b\frac{\partial u}{\partial x}, \qquad 0 \le x \le 1,\ 0 \le t \le T, T > 0,$$

where $b > 0$ is given (a constant) and the boundary conditions are $u(0,t) = \varphi_0(t)$ and $u(1,t) = \varphi_1(t)$ for $t \in [0,T]$. Let

$$v_j' = \frac{1}{h^2}\left(v_{j-1} - 2v_j + v_{j+1}\right) - \frac{b}{2h}\left(v_{j+1} - v_{j-1}\right), \qquad j = 1, 2, \ldots, n_x,$$

where $h = \dfrac{1}{n_x + 1}$ and $v_j = v_j(t)$ be a semi-discrete method for the convection-diffusion equation.

1. **[5 marks]** Prove that the semi-discrete method is second-order accurate. That is, let $\widetilde{v}_j(t) = u(x_j, t)$ and show that (assuming all the relevant partial derivatives are bounded on $(x,t) \in [0,1] \times [0,T]$),

$$\widetilde{v}_j' - \frac{1}{h^2}\left(\widetilde{v}_{j-1} - 2\widetilde{v}_j + \widetilde{v}_{j+1}\right) + \frac{b}{2h}\left(\widetilde{v}_{j+1} - \widetilde{v}_{j-1}\right) = \mathcal{O}\left(h^2\right), \qquad h \to 0.$$

2. **[2 marks]** Is the semi-discrete method consistent? Motivate your answer.

3. **[6 marks]** Use the von Neumann method to determine whether the semi-discrete method is stable.

4. **[2 marks]** Is the semi-discrete method convergent? Motivate your answer.

5. **[5 marks]** Show that the semi-discrete method can be expressed as the following system of ordinary differential equations (ODEs):

$$\mathbf{v}' = A\mathbf{v} + \mathbf{h}$$

with $\mathbf{v}, \mathbf{h} \in \mathbb{R}^{n_x}$, $A \in \mathbb{R}^{n_x \times n_x}$ where

$$A = \frac{1}{h^2}\begin{bmatrix} -2 & 1 - \frac{bh}{2} & & & \\ 1 + \frac{bh}{2} & -2 & 1 - \frac{bh}{2} & & \\ & \ddots & \ddots & \ddots & \\ & & 1 + \frac{bh}{2} & -2 & 1 - \frac{bh}{2} \\ & & & 1 + \frac{bh}{2} & -2 \end{bmatrix}, \quad \mathbf{h} = \frac{1}{h^2}\begin{bmatrix} \left(1 + \frac{bh}{2}\right)\varphi_0(t) \\ 0 \\ \vdots \\ 0 \\ \left(1 - \frac{bh}{2}\right)\varphi_1(t) \end{bmatrix}.$$

6. **[10 marks]** Let $n_x = 300$, $T = 0.003$, $b = 100$, $\varphi_0(t) = 0 = \varphi_1(t)$ and $u(x,0) = \mathrm{e}^{-300(x-0.3)^2}$, then solve the ODE system in question 5 with an error tolerance of 10^{-4} using any ODE solver that's available in your programming language of choice. Plot the solution at $t = 0$ and $t = T$ on the same set of axes.