1 Fourier methods for periodic functions

We assume that f is 2π-periodic and its Fourier series converges on $x \in [0, 2\pi)$, i.e.,

$$f(x) = \sum_{k=-\infty}^{\infty} c_k \mathrm{e}^{\mathrm{i}kx}$$

where the Fourier coeffients are defined as

$$c_k = \frac{1}{2\pi} \int_0^{2\pi} f(x) \mathrm{e}^{-\mathrm{i}kx} \mathrm{d}x.$$

Suppose we only know the values of $f(x)$ at the n equally spaced points $x_j = jh = \frac{2\pi j}{n}$ for $j = 0, \ldots, n-1$, can we somehow use Fourier series to approximate f and its derivatives?

The idea is to use a truncated Fourier series and approximate Fourier coefficients $\widetilde{c}_k$ to approximate f with

$$f(x) \approx \sum_{k=-(n-1)/2}^{(n-1)/2} \widetilde{c}_k \mathrm{e}^{\mathrm{i}kx},$$

assuming n is odd.

1.1 Trapezoidal rule

We are going to approximate the Fourier coefficients c_k by using the *trapezoidal rule.*

Trapezoidal rule: Let x_j, $j = 0, \ldots, n$ be $n+1$ equally spaced points on the interval $[a, b]$. Hence, $x_j = a + jh$ with $h = (b-a)/n$. The $n+1$-point trapezoidal rule for approximating the integral

$$I[g] = \int_a^b g(x) \mathrm{d}x$$

is denoted by $I_n[g]$ and defined as

$$I_n[g] := \frac{h}{2} \left(g(x_0) + 2g(x_1) + 2g(x_2) + \cdots + 2g(x_{n-1}) + g(x_n) \right)$$

The trapezoidal rule is an example of a quadrature method, which are methods to approximate integrals by weighted sums.

Let's use the trapezoidal rule to approximate the Fourier coefficients. Setting $a = 0$, $b = 2\pi$, $h = 2\pi/n$ and using the fact that $g(x) = f(x)\mathrm{e}^{-\mathrm{i}kx}$ is a 2π-periodic function (since $f(x)$ is assumed to be 2π-periodic), it follows that $g(x_0) = g(x_0 + 2\pi) = g(x_n)$ and we obtain the approximation

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\begin{aligned}
c_k &= \frac{1}{2\pi}\int_0^{2\pi} f(x)\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x && (1)\\
&\approx \frac{1}{2\pi} I_n\left[f\mathrm{e}^{-\mathrm{i}kx}\right] && (2)\\
&= \frac{1}{n}\sum_{j=0}^{n-1} f(x_j)\mathrm{e}^{-\mathrm{i}kx_j} && (3)\\
&:= \widetilde{c}_k && (4)
\end{aligned}
$$