1 Chapter 4: Exercises, Part II

Consider the Laguerre polynomials

$$L_n^{(\alpha)}(x) = \frac{(-1)^n}{n!} x^n + O(x^{n-1})$$

where $\alpha > -1$, which are orthogonal with respect to

$$\langle f, g \rangle_\alpha = \int_0^\infty f(x) g(x) x^\alpha \mathrm{e}^{-x} \, \mathrm{d}x.$$

1. Show that the Rodrigues formula holds:

$$L_n^{(\alpha)}(x) = \frac{x^{-\alpha} \mathrm{e}^x}{n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \left[x^{\alpha+n} \mathrm{e}^{-x} \right].$$

In other words, prove that $L_n^{(\alpha)}(x)$ (defined by the Rodrigues formula)

(i) is a polynomial of degree exactly n

(ii) is orthogonal to all lower degree polynomials

(iii) has leading coefficient $\frac{(-1)^n}{n!}$

Hints: For (i) and (iii), it may be helpful first to prove that

$$\frac{\mathrm{d}}{\mathrm{d}x} \left[x^{\alpha+1} \mathrm{e}^{-x} L_n^{(\alpha+1)}(x) \right] = (n+1) x^\alpha \mathrm{e}^{-x} L_{n+1}^{(\alpha)}(x)$$

For (ii), use integration by parts.

2. Show that

$$(n+1) L_{n+1}^{(\alpha)}(x) = (\alpha + n + 1) L_n^{(\alpha)}(x) - x L_n^{(\alpha+1)}(x)$$

3. Show that

$$L_n^{(\alpha+1)}(x) = L_{n-1}^{(\alpha+1)}(x) + L_n^{(\alpha)}(x).$$

4. Show that the Laguerre polynomials satisfy the following three-term recurrence:

$$x L_n^{(\alpha)}(x) = -(n+\alpha) L_{n-1}^{(\alpha)}(x) + (2n + \alpha + 1) L_n^{(\alpha)}(x) - (n+1) L_{n+1}^{(\alpha)}(x)$$

5. Prove that

$$\frac{\mathrm{d}L_n^{(\alpha)}}{\mathrm{d}x} = -L_{n-1}^{(\alpha+1)}(x)$$

6. Let

$$L^{(\alpha)}(x) = \left[L_0^{(\alpha)}(x)|L_1^{(\alpha)}(x)|\cdots\right].$$

Give matrices D_α and S_α such that

$$\frac{\mathrm{d}}{\mathrm{d}x}L^{(\alpha)}(x) = L^{(\alpha+1)}(x)\mathcal{D}_\alpha \quad \text{and} \quad L^{(\alpha)}(x) = L^{(\alpha+1)}(x)S_\alpha$$

7. Consider the advection equation on the half line:

$$u_t + u_x = 0, \qquad x \in [0,\infty), \qquad t \geq 0.$$

Suppose the solution has an expansion of the form

$$u(x,t) = \mathrm{e}^{-x/2}\sum_{k=0}^{\infty} u_k(t)L_k^{(0)}(x) = \mathrm{e}^{-x/2}L^{(0)}(x)\mathbf{u}(t)$$

where

$$\mathbf{u}(t) = \begin{bmatrix} u_0(t) \\ u_1(t) \\ \vdots \end{bmatrix}.$$

Show that

$$\mathbf{u}'(t) = A\mathbf{u}(t),$$

where A is a matrix that is expressible in terms of $\mathcal{D}_0$ and $\mathcal{S}_0$ (defined in question 6). Use software of your choice to build a 10×10 version of the matrix A.