1 Chapter 2: Exercises

1. Give explicit formulae for the Fourier coefficients c_k and approximate Fourier coefficients $\tilde{c}_k^n$ for the following functions:

$$\cos x, \frac{3}{3 - \mathrm{e}^{\mathrm{i}x}}$$

Hint: You may wish to try the change of variables $z = \mathrm{e}^{\mathrm{i}x}$.

2. Show that the DFT Q_n is symmetric ($Q_n = Q_n^\top$) but not Hermitian ($Q_n \neq$ ăQ_n^*).

3. Show that

$$\sum_{k=-m}^{m} \mathrm{e}^{\mathrm{i}kx} = \begin{cases} \frac{\sin((m+1/2)x)}{\sin(x/2)} & \text{if } x \neq 0 \\ 2m+1 & \text{if } x = 0 \end{cases}$$

4. Prove that the trigonometric interpolant $p_n(x)$ that interpolates f at $x = x_j = jh$, $j = 0, \ldots, n-1$ with $h = 2\pi/n$ and $n = 2m+1$ is unique.

5. Consider the advection equation

$$u_t + u_x = 0, \qquad x \in [0, 2\pi), \qquad t \in [0, T],$$

with $u(x, 0) = f(x) = \mathrm{e}^{-100(x-1)^2}$ and exact solution $u(x, t) = f(x - t)$; also consider (i) the forward-difference-Fourier method

$$\mathbf{u}^{i+1} = \mathbf{u}^i - \tau \mathcal{F}^{-1}\left\{\mathrm{i}(-m\!:\!m) \cdot \mathcal{F}\{\mathbf{u}^i\}\right\}, \qquad i = 0, \ldots, n_t - 1$$

and (ii) the central-difference-Fourier method (aka the leapfrog method)

$$\mathbf{u}^{i+1} = \mathbf{u}^{i-1} - 2\tau \mathcal{F}^{-1}\left\{\mathrm{i}(-m\!:\!m) \cdot \mathcal{F}\{\mathbf{u}^i\}\right\}, \qquad i = 1, \ldots, n_t - 1.$$

For the leapfrog method, set $u_j^1 = f(x_j - \tau)$. For both methods, set $n_x = 401$, $n_t = 500$ and $T = 1.05$ and plot the maximum error for each time step, i.e., plot

$$e_i := \max_{j=0,\ldots,n_x-1} |u(x_j, t_i) - u_j^i|$$

for $i = 1, \ldots, n_t$. Describe the behaviour of e_i for each method.