1 Problem sheet 2

Consider the following finite difference method for the diffusion equation

$$-\frac{1}{2}\mu u_{j-1}^{i+1}+(1+\mu)u_j^{i+1}-\frac{1}{2}\mu u_{j+1}^{i+1}=\frac{1}{2}\mu u_{j-1}^{i}+(1-\mu)u_j^{i}+\frac{1}{2}\mu u_{j+1}^{i} \tag{1}$$

where $\mu=\tau/h^2$.

1. **[5 marks]** Show that (1) has a second-order local truncation error. That is, let $\tilde{u}_j^i=u(x_j,t_i)$, where $x_j=jh$, $t_i=i\tau$, show that (1) can be expressed as

$$\frac{u_j^{i+1}-u_j^i}{\tau}=\frac{1}{2}\left(\frac{u_{j+1}^{i+1}-2u_j^{i+1}+u_{j-1}^{i+1}}{h^2}+\frac{u_{j+1}^{i}-2u_j^{i}+u_{j-1}^{i}}{h^2}\right)$$

and then show that for $\tau,h\to 0$ (assuming all the relevant partial derivatives are bounded),

$$\frac{\tilde{u}_j^{i+1}-\tilde{u}_j^i}{\tau}-\frac{1}{2}\left(\frac{\tilde{u}_{j+1}^{i+1}-2\tilde{u}_j^{i+1}+\tilde{u}_{j-1}^{i+1}}{h^2}+\frac{\tilde{u}_{j+1}^{i}-2\tilde{u}_j^{i}+\tilde{u}_{j-1}^{i}}{h^2}\right)=\mathcal{O}(\tau^2)+\mathcal{O}(h^2).$$

Hint: use Taylor's theorem to expand the exact solution about $(x,t)=(x_j,t_{i+1/2})$, where $t_{i+1/2}=(i+1/2)\tau=t_i+\tau/2$.

2. **[5 marks]** Use the Von Neumann method to show that (1) is unconditionally stable.

3. **[5 marks]** Suppose the finite difference method is used to approximate the solution to the diffusion equation $u_t=u_{xx}$ for $(x,t)\in(0,1)\times(0,T)$ subject to $u(0,t)=\varphi_0(t)$, $u(1,t)=\varphi_1(t)$ for $t\in[0,T]$ and $u(x,0)=f(x)$ for $x\in[0,1]$. Let $u(x_j,t_i)\approx u_j^i$, where $x_j=jh$, $h=1/(n_x+1)$, $j=0,\dots,n_x+1$, $t_i=i\tau$, $\tau=\mu h^2$ and set

$$\mathbf{u}^i=\begin{bmatrix}u_1^i\\ \vdots\\ u_{n_x}^i\end{bmatrix}\in\mathbb{R}^{n_x}.$$

The finite difference method (1) can be expressed as

$$L\mathbf{u}^{i+1}=R\mathbf{u}^i+\mathbf{k}^i,$$

where $L,R\in\mathbb{R}^{n_x\times n_x}$ and $\mathbf{k}^i\in\mathbb{R}^{n_x}$. Give the matrices L and R and the vector $\mathbf{k}^i$.

4. **[5 marks]** What are the eigenvalues of the matrix $A:=L^{-1}R$? Find a bound on the spectral radius of A. Hint: consider the eigendecomposition (spectral factorisation) of L and R.

5. **[10 marks]** Let

$$u(x,0)=f(x)=\sin\frac{1}{2}\pi x+\frac{1}{2}\sin 2\pi x,\qquad 0\le x\le 1,$$

and

$$u(0,t)=\varphi_0(t)=0,\qquad u(1,t)=\varphi_1(t)=\mathrm{e}^{-\pi^2 t/4},\qquad t\ge 0,$$

then the exact solution to the diffusion equation is

$$u(x,t) = \mathrm{e}^{-\pi^2 t/4} \sin \frac{1}{2}\pi x + \frac{1}{2}\mathrm{e}^{-4\pi^2 t} \sin 2\pi x, \qquad 0 \le x \le 1, \qquad t \ge 0.$$

Using the method in question 3 and the backward Euler method, approximate the exact solution for $T = 1$, $\mu = n_x$ and plot the maximum error of the two methods for $n_x = 2^k$, $k = 4, 5, \ldots, 12$. Comment on your results.