

# 1 Problem Sheet 1

1. **[5 marks]** What are the Fourier coefficients  $c_k$  of  $\sin^4 x$ ?
2. **[5 marks]** Show for  $0 \leq k, \ell \leq n-1$

$$\frac{1}{n} \sum_{j=1}^n \cos k\theta_j \cos \ell\theta_j = \begin{cases} 1 & k = \ell = 0 \\ 1/2 & k = \ell \\ 0 & \text{otherwise} \end{cases}$$

for  $\theta_j = \pi(j-1/2)/n$ . Hint: You may consider replacing  $\cos$  with complex exponentials:

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}.$$

3. **[5 marks]** Consider the Discrete Cosine Transform (DCT)

$$C_n := \begin{bmatrix} \sqrt{1/n} & & & \\ & \sqrt{2/n} & & \\ & & \ddots & \\ & & & \sqrt{2/n} \end{bmatrix} \begin{bmatrix} 1 & \cdots & 1 \\ \cos \theta_1 & \cdots & \cos \theta_n \\ \vdots & \ddots & \vdots \\ \cos(n-1)\theta_1 & \cdots & \cos(n-1)\theta_n \end{bmatrix}$$

for  $\theta_j = \pi(j-1/2)/n$ . Prove that  $C_n$  is orthogonal:  $C_n^\top C_n = C_n C_n^\top = I$ . Hint:  $C_n C_n^\top = I$  might be easier to show than  $C_n^\top C_n = I$  using the previous problem.

4. **[10 marks]** Consider the variable-coefficient advection equation

$$u_t + c(x)u_x = 0, \quad c(x) = \frac{1}{5} + \sin^2(x-1), \quad x \in [0, 2\pi], \quad t \in [0, T],$$

with  $u(x, 0) = f(x) = e^{-100(x-1)^2}$ , which we approximate with the leapfrog method

$$\mathbf{u}^{i+1} = \mathbf{u}^{i-1} - 2\tau c(\mathbf{x}) \cdot \mathcal{F}^{-1} \left\{ \mathbf{i}(-m:m) \cdot \mathcal{F}\{\mathbf{u}^i\} \right\}, \quad i = 0, \dots, n_t - 1.$$

Note that one needs  $\mathbf{u}^{-1}$  to initialise the leapfrog method. Let the entries of  $\mathbf{u}^{-1}$  be  $u_j^{-1} = f(x_j + \tau/5)$ ,  $j = 0, \dots, n_x - 1$ . The exact solution is periodic in time, i.e.,

$$u(x, t+T) = u(x, t)$$

where

$$T = \int_0^{2\pi} \frac{1}{c(x)} dx = 12.8254983016186 \dots$$

Compute  $T$  using the Trapezoidal rule and confirm that you get the value stated above. Then compute

$$e(n_x) = \max_{j=0, \dots, n_x-1} |u_j^0 - u_j^{n_t}|$$

where  $n_t = 8n_x$  and  $\tau n_t = T$  and plot  $e(n_x)$  for  $n_x = 2^k + 1$ , with  $k = 5, 6, \dots, 10$ . Comment on the behaviour of  $e(n_x)$ .