1 Problem Sheet 1

1. **[5 marks]** What are the Fourier coefficients c_k of $\sin^4 x$?

2. **[5 marks]** Show for $0 \leq k, \ell \leq n-1$

$$\frac{1}{n}\sum_{j=1}^{n} \cos k\theta_j \cos \ell\theta_j = \begin{cases} 1 & k = \ell = 0 \\ 1/2 & k = \ell \\ 0 & \text{otherwise} \end{cases}$$

for $\theta_j = \pi(j-1/2)/n$. Hint: You may consider replacing cos with complex exponentials:

$$\cos\theta = \frac{\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta}}{2}.$$

3. **[5 marks]** Consider the Discrete Cosine Transform (DCT)

$$C_n := \begin{bmatrix} \sqrt{1/n} & & & \\ & \sqrt{2/n} & & \\ & & \ddots & \\ & & & \sqrt{2/n} \end{bmatrix} \begin{bmatrix} 1 & \cdots & 1 \\ \cos\theta_1 & \cdots & \cos\theta_n \\ \vdots & \ddots & \vdots \\ \cos(n-1)\theta_1 & \cdots & \cos(n-1)\theta_n \end{bmatrix}$$

for $\theta_j = \pi(j-1/2)/n$. Prove that C_n is orthogonal: $C_n^\top C_n = C_n C_n^\top = I$. Hint: $C_n C_n^\top = I$ might be easier to show than $C_n^\top C_n = I$ using the previous problem.

4. **[10 marks]** Consider the variable-coefficient advection equation

$$u_t + c(x)u_x = 0, \qquad c(x) = \frac{1}{5} + \sin^2(x-1), \qquad x \in [0, 2\pi), \qquad t \in [0, T],$$

with $u(x,0) = f(x) = \mathrm{e}^{-100(x-1)^2}$, which we approximate with the leapfrog method

$$\mathbf{u}^{i+1} = \mathbf{u}^{i-1} - 2\tau c(\mathbf{x}) \cdot \mathcal{F}^{-1}\left\{\mathrm{i}(-m\!:\!m) \cdot \mathcal{F}\{\mathbf{u}^i\}\right\}, \qquad i = 0, \ldots, n_t - 1.$$

Note that one needs $\mathbf{u}^{-1}$ to initialise the leapfrog method. Let the entries of $\mathbf{u}^{-1}$ be $u_j^{-1} = f(x_j + \tau/5)$, $j = 0, \ldots, n_x - 1$. The exact solution is periodic in time, i.e.,

$$u(x, t+T) = u(x,t)$$

where

$$T = \int_0^{2\pi} \frac{1}{c(x)} \mathrm{d}x = 12.8254983016186\ldots$$

Compute T using the Trapezoidal rule and confirm that you get the value stated above. Then compute

$$e(n_x) = \max_{j=0,\ldots,n_x-1} |u_j^0 - u_j^{n_t}|$$

where $n_t = 8n_x$ and $\tau n_t = T$ and plot $e(n_x)$ for $n_x = 2^k + 1$, with $k = 5, 6, \ldots, 10$. Comment on the behaviour of $e(n_x)$.