1 Problem sheet 2: Solutions

The (well-posed) convection-diffusion equation is

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - b\frac{\partial u}{\partial x}, \qquad 0 \le x \le 1,\ 0 \le t \le T, T > 0,$$

where $b > 0$ is given (a constant) and the boundary conditions are $u(0,t) = \varphi_0(t)$ and $u(1,t) = \varphi_1(t)$ for $t \in [0,T]$. Let

$$v_j' = \frac{1}{h^2}\left(v_{j-1} - 2v_j + v_{j+1}\right) - \frac{b}{2h}\left(v_{j+1} - v_{j-1}\right), \qquad j = 1, 2, \ldots, n_x,$$

where $h = \dfrac{1}{n_x + 1}$ and $v_j = v_j(t)$ be a semi-discrete method for the convection-diffusion equation.

1. **[5 marks]** Prove that the semi-discrete method is second-order accurate. That is, let $\widetilde{v}_j(t) = u(x_j, t)$ and show that (assuming all the relevant partial derivatives are bounded on $(x,t) \in [0,1] \times [0,T]$),

$$\widetilde{v}_j' - \frac{1}{h^2}\left(\widetilde{v}_{j-1} - 2\widetilde{v}_j + \widetilde{v}_{j+1}\right) + \frac{b}{2h}\left(\widetilde{v}_{j+1} - \widetilde{v}_{j-1}\right) = \mathcal{O}\left(h^2\right), \qquad h \to 0.$$

Solution Using Taylor's theorem (see Chapter 1), there exists a $\xi_+ \in (x_j, x_j + h)$ such that

$$\begin{aligned}
\widetilde{v}_{j+1} &= u(x_{j+1}, t) \\
&= u(x_j + h, t) \\
&= u(x_j, t) + hu_x(x_j, t) + \frac{h^2}{2}u_{xx}(x_j, t) + \frac{h^3}{6}u_{xxx}(x_j, t) + \frac{h^4}{24}u_{xxxx}(\xi_+, t) \\
&= \widetilde{v}_j + hu_x(x_j, t) + \frac{h^2}{2}u_{xx}(x_j, t) + \frac{h^3}{6}u_{xxx}(x_j, t) + \frac{h^4}{24}u_{xxxx}(\xi_+, t).
\end{aligned}$$

Similarly, by replacing h with $-h$ in the equations above, there exists a $\xi_- \in (x_j - h, x_j)$ such that

$$\begin{aligned}
\widetilde{v}_{j-1} &= u(x_{j-1}, t) \\
&= u(x_j - h, t) \\
&= u(x_j, t) - hu_x(x_j, t) + \frac{h^2}{2}u_{xx}(x_j, t) - \frac{h^3}{6}u_{xxx}(x_j, t) + \frac{h^4}{24}u_{xxxx}(\xi_-, t) \\
&= \widetilde{v}_j - hu_x(x_j, t) + \frac{h^2}{2}u_{xx}(x_j, t) - \frac{h^3}{6}u_{xxx}(x_j, t) + \frac{h^4}{24}u_{xxxx}(\xi_-, t).
\end{aligned}$$

Hence, it follows that

$$\frac{1}{h^2}\left(\widetilde{v}_{j-1} - 2\widetilde{v}_j + \widetilde{v}_{j+1}\right) = u_{xx}(x_j, t) + \frac{h^2}{24}\left(u_{xxxx}(\xi_+, t) + u_{xxxx}(\xi_-, t)\right)$$

and we conclude that (since we assume that u_{xxxx} is bounded)

$$\frac{1}{h^2}\left(\widetilde{v}_{j-1}-2\widetilde{v}_j+\widetilde{v}_{j+1}\right)=u_{xx}(x_j,t)+\mathcal{O}\left(h^2\right), \qquad h\to 0. \qquad (\star) \tag{1}$$

Also, it follows that

$$\frac{1}{2h}\left(\widetilde{v}_{j+1}-\widetilde{v}_{j-1}\right)=u_x(x_j,t)+\frac{h^2}{6}u_{xxx}(x_j,t)+\frac{h^3}{48}\left(u_{xxxx}(\xi_+,t)-u_{xxxx}(\xi_-,t)\right)$$

and we conclude that (since we assume that u_{xxx} and u_{xxxx} are bounded)

$$\frac{1}{2h}\left(\widetilde{v}_{j+1}-\widetilde{v}_{j-1}\right)=u_x(x_j,t)+\mathcal{O}\left(h^2\right), \qquad h\to 0. \qquad (\diamond) \tag{2}$$

Using $(\star)$ and $(\diamond)$ and the advection-diffusion equation, it follows that as $h\to 0$,

$$\begin{aligned}
&\widetilde{v}_j'-\frac{1}{h^2}\left(\widetilde{v}_{j-1}-2\widetilde{v}_j+\widetilde{v}_{j+1}\right)+\frac{b}{2h}\left(\widetilde{v}_{j+1}-\widetilde{v}_{j-1}\right)\\
&=\underbrace{u_t(x_j,t)-u_{xx}(x_j,t)+bu_x(x_j,t)}_{=0}+\mathcal{O}\left(h^2\right)\\
&=\mathcal{O}\left(h^2\right).
\end{aligned}$$

2. **[2 marks]** Is the semi-discrete method consistent? Motivate your answer.

   Solution Yes, since the method is second-order, the local error tends to zero as $h\to 0$ and therefore the method is consistent.

3. **[6 marks]** Use the von Neumann method to determine whether the semi-discrete method is stable.

   Solution

Substituting

$$v_j(t)=v_j=\lambda(t)\mathrm{e}^{\mathrm{i}kx_j}=\lambda\mathrm{e}^{\mathrm{i}kx_j},$$

where $x_j=jh$ and $k\in\mathbb{Z}$, into the semi-discrete method and using the fact that $v_{j+1}=\lambda\mathrm{e}^{\mathrm{i}kx_{j+1}}=\lambda\mathrm{e}^{\mathrm{i}k(x_j+h)}=\mathrm{e}^{\mathrm{i}kh}v_j$, we find that

$$\begin{aligned}
v_j' &= \frac{1}{h^2}\left(\mathrm{e}^{-\mathrm{i}kh}-2+\mathrm{e}^{\mathrm{i}kh}\right)v_j-\frac{b}{2h}\left(\mathrm{e}^{\mathrm{i}kh}-\mathrm{e}^{-\mathrm{i}kh}\right)v_j\\
&= \frac{1}{h^2}\left(2\cos(kh)-2\right)v_j-\frac{b}{2h}\left(2\mathrm{i}\sin(kh)\right)v_j\\
&= -\frac{2}{h^2}\left(1-\cos(kh)\right)v_j-\frac{b\mathrm{i}}{h}\sin(kh)v_j\\
&= -\left[\frac{4}{h^2}\sin^2(kh/2)+\frac{b\mathrm{i}}{h}\sin(kh)\right]v_j\\
\implies \lambda' &= -\left[\frac{4}{h^2}\sin^2(kh/2)+\frac{b\mathrm{i}}{h}\sin(kh)\right]\lambda\\
\implies \lambda(t) &= \exp\left(-\left[\frac{4}{h^2}\sin^2(kh/2)+\frac{b\mathrm{i}}{h}\sin(kh)\right]t\right)\lambda(0).
\end{aligned}$$

Since, as $h \to 0$,

$$\frac{4}{h^2}\sin^2(kh/2) = k^2 + \mathcal{O}\left(h^2\right), \qquad \frac{1}{h}\sin(kh) = k + \mathcal{O}\left(h^2\right),$$

we have that for $t \in [0, T]$,

$$\begin{aligned}
\lim_{h\to 0} |\lambda(t)| &= \lim_{h\to 0} \left| \exp\left(-\left[\frac{4}{h^2}\sin^2(kh/2) + \frac{b\mathrm{i}}{h}\sin(kh) \right] t \right) \lambda(0) \right| \\
&= \lim_{h\to 0} \exp\left(-\frac{4}{h^2}\sin^2(kh/2) t \right) |\lambda(0)| \\
&= \mathrm{e}^{-k^2 t} |\lambda(0)| \\
&\leq |\lambda(0)| \\
&< \infty
\end{aligned}$$

hence the method is stable.

4. **[2 marks]** Is the semi-discrete method convergent?

   Solution Yes, since the advection-diffusion equation is well-posed and the semi-discrete method is consistent and stable, it is convergent by the Lax equivalence theorem.

5. **[5 marks]** Show that the semi-discrete method can be expressed as the following system of ordinary differential equations (ODEs):

$$\mathbf{v}' = A\mathbf{v} + \mathbf{h}$$

with $\mathbf{v}, \mathbf{h} \in \mathbb{R}^{n_x}$, $A \in \mathbb{R}^{n_x \times n_x}$ where

$$A = \frac{1}{h^2}\begin{bmatrix} -2 & 1-\frac{bh}{2} & & & \\ 1+\frac{bh}{2} & -2 & 1-\frac{bh}{2} & & \\ & \ddots & \ddots & \ddots & \\ & & 1+\frac{bh}{2} & -2 & 1-\frac{bh}{2} \\ & & & 1+\frac{bh}{2} & -2 \end{bmatrix}, \qquad \mathbf{h} = \frac{1}{h^2}\begin{bmatrix} \left(1+\frac{bh}{2}\right)\varphi_0(t) \\ 0 \\ \vdots \\ 0 \\ \left(1-\frac{bh}{2}\right)\varphi_1(t) \end{bmatrix}$$

Solution Rewriting the semi-discrete method

$$\begin{aligned}
v_j' &= \frac{1}{h^2}\left(v_{j-1} - 2v_j + v_{j+1}\right) - \frac{b}{2h}\left(v_{j+1} - v_{j-1}\right) \\
&= \frac{1}{h^2}\left[\left(1 + \frac{bh}{2}\right) v_{j-1} - 2v_j + \left(1 - \frac{bh}{2}\right) v_{j+1} \right], \qquad j = 1, \dots, n_x,
\end{aligned}$$

it follows that

$$\mathbf{v}' = A\mathbf{v} + \mathbf{h}$$

where

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n_x} \end{pmatrix}, \quad A = \frac{1}{h^2}\begin{bmatrix} -2 & 1-\frac{bh}{2} & & & \\ 1+\frac{bh}{2} & -2 & 1-\frac{bh}{2} & & \\ & \ddots & \ddots & \ddots & \\ & & 1+\frac{bh}{2} & -2 & 1-\frac{bh}{2} \\ & & & 1+\frac{bh}{2} & -2 \end{bmatrix}, \quad \mathbf{h} = \frac{1}{h^2}\begin{pmatrix} \left(1+\frac{bh}{2}\right)\varphi_0(t) \\ 0 \\ \vdots \\ 0 \\ \left(1-\frac{bh}{2}\right)\varphi_1(t) \end{pmatrix}$$

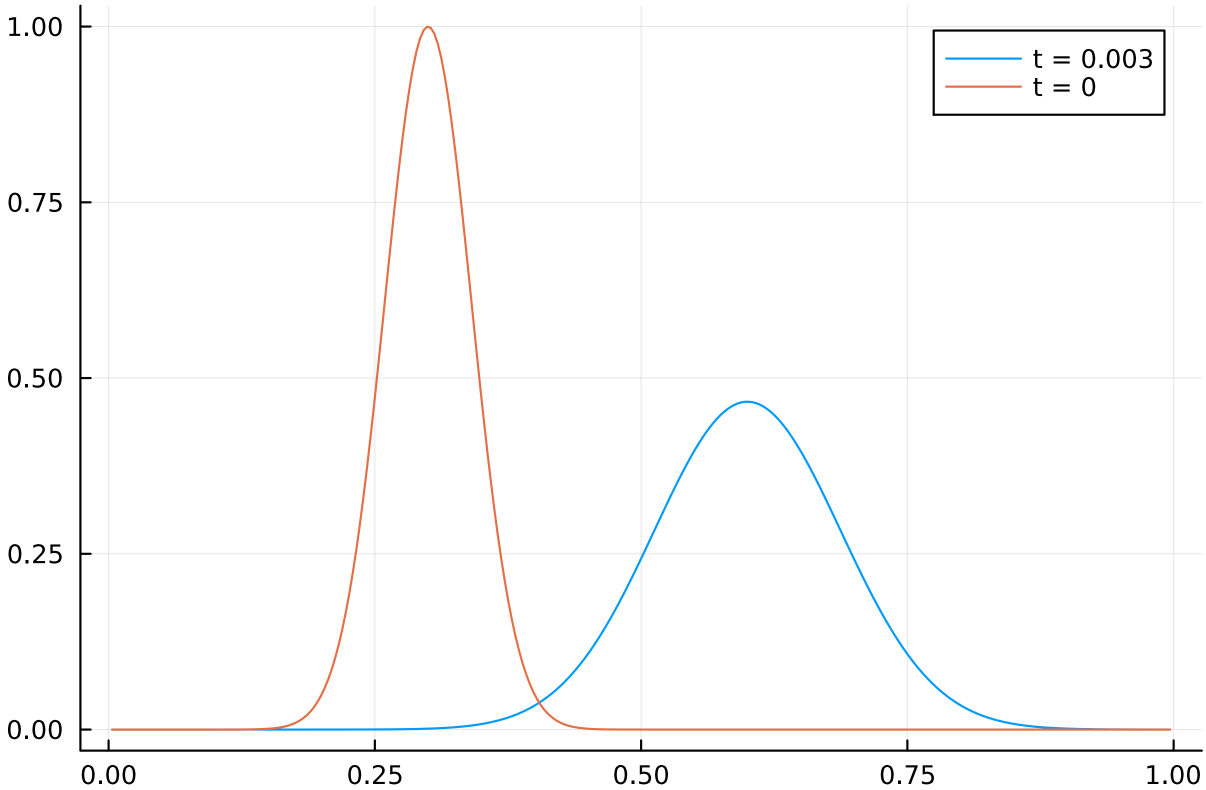
6. **[10 marks]** Let $n_x = 300$, $T = 0.003$, $b = 100$, $\varphi_0(t) = 0 = \varphi_1(t)$ and $u(x,0) = \mathrm{e}^{-300(x-0.3)^2}$, then solve the ODE system in question 5 with an error tolerance of 10^{-4} using any ODE solver that's available in your programming language of choice. Plot the solution at $t = 0$ and $t = T$ on the same set of axes.

Solution

```julia
using LinearAlgebra, Plots, OrdinaryDiffEq

nx = 300
x = range(0,1,nx+2)
f = x -> exp(-300*(x-0.3).^2)
h = 1/(nx+1)
T = 0.003
b = 100
A = Tridiagonal(fill(1+b*h/2,nx-1),fill(-2.0,nx),fill(1-b*h/2,nx-1))/h^2
F = (v,p,t) -> A*v
x = range(0,1,nx + 2)
prob = ODEProblem(F, f.(x[2:end-1]), (0.0, T))
#soln =  solve(prob, RK4(),abstol=1e-4)
#soln =  solve(prob, Rodas4(),abstol=1e-4);
soln =  solve(prob, Rodas4P(),abstol=1e-4)
@show nt = length(soln.t)
plot(x[2:nx+1],soln.u[nt],label = "t = 0.003")
plot!(x[2:nx+1],soln.u[1],label = "t = 0")

nt = length(soln.t) = 14
```


t = 0.003
t = 0
1.00
0.75
0.50
0.25
0.00
0.00
0.25
0.50
0.75
1.00