1 Chapter 4: Exercises Part I

1. Construct $p_0(x), p_1(x), p_2(x), p_3(x)$, monic OPs for the weight $\sqrt{1-x^2}$ on $[-1,1]$.

   Hint: first compute $\int_{-1}^1 x^k \sqrt{1-x^2}\mathrm{d}x$ for $0 \le k \le 2$ using a change-of-variables.

2. Let $w(x)$ be a weight function on $x \in [-b,b]$, $b > 0$ satisfying $w(-x) = w(x)$. Show that $a_n = 0$, $n \ge 0$ and conclude that $a_n = 0$ for the Chebyshev polynomials $\{T_n\}$ and the ultraspherical polynomials $\{C_n^{(\lambda)}\}$.

   Hint: first show that the OPs $p_{2n}(x)$ are even and $p_{2n+1}(x)$ are odd.

3. Consider orthogonal polynomials with respect to $\sqrt{1-x^2}$ on $[-1,1]$ with the normalisation
$$U_n(x) = 2^n x^n + O(x^{n-1})$$
   Prove that
$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta},$$
   where $x = \cos\theta$.

4. Show that $\mathcal{S}_{n-1}\cdots\mathcal{S}_1\mathcal{S}_0$ has bandwidths $(0, 2n)$.

5. We showed that when we differentiate the Chebyshev polynomials and change basis, we obtain a sparse differentiation matrix. That is,
$$\frac{\mathrm{d}}{\mathrm{d}x}\left[T_0(x)|T_1(x)|\cdots\right] = \left[C_0^{(1)}(x)|C_1^{(1)}(x)|\cdots\right]\mathcal{D}_0$$
   where $\mathcal{D}_0$ has bandwidths $(-1,1)$. Suppose we don't change basis when differentiating, i.e., let
$$\frac{\mathrm{d}}{\mathrm{d}x}\left[T_0(x)|T_1(x)|\cdots\right] = \left[T_0(x)|T_1(x)|\cdots\right]\mathcal{D}.$$
   What are the bandwidths of $\mathcal{D}$?