Unit - I
Random Variables

Probability of an event :

$$P(A) = \frac{\text{favourable cases}}{\text{Possible cases}}$$

Random experiment :

All outcomes are known, but we can't Predict the exact outcome.

Trial

Performing an experiment.

Sample Space :

All possible outcomes of an experiment

eg 1) Tossing a coin 2) rolling a die

$S = \{H, T\}$ $S = \{1, 2, 3, 4, 5, 6\}$

Event :

Subset of a Sample Space.

eg In the toss of a coin, Let A be the event of getting head.

Equally Likely.

Cannot be expected to happen in Preferrence to any other.

eg Turning up of the head or tail is equally likely.

Mutually Exclusive:

Occurrence of one of them does not Prevent the occurrence of others.

eg Either head or tail will turn up. Both cannot happen at the same time.

Exhaustive Events:

A set is exhaustive if it includes all possible outcomes of a trial.

Axioms of Probability:

Let S be a Sample Space. To each event A, there is a number $P(A)$ associated, Probability of A Satisfying the following conditions

(i) $P(A) \geq 0$

(ii) $P(S) = 1$

(iii) If $A_1, A_2, \ldots A_n$ are mutually exclusive events, then

Addition theorem:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Conditional Probability:

$$P(A/B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) \neq 0$$

$$P(B/A) = \frac{P(A \cap B)}{P(A)}, \quad P(A) \neq 0$$

Multiplication theorem:

$$P(A \cap B) = P(A) \cdot P(B/A)$$

Independent events

$$P(A \cap B) = P(A) \cdot P(B).$$

Random Variables:

It is a function X which assigns a number to every outcome of a random experiment.

eg Tossing two unbiased coins.

Outcomes : HH, HT, TH, TT

Random Variable X : No. of heads.

(Assigning real no's) : (2, 1, 1, 0)

Mathematical defn : $X: S \longrightarrow R$

① A random Variable X has the following Probability distribution

x	0	1	2	3	4	5	6	7
$P(x)$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

find (i) the value of k

(ii) $P[1.5 < X < 4.5 / X > 2]$

(iii) the Smallest Value of λ for which $P[X \leq \lambda] > 1/2$.

Soln

(i) WKT $\sum P_i = 1$

$$\Rightarrow 0+k+2k+2k+3k+k^2+2k^2+7k^2+k = 1$$

$$9k+10k^2 = 1$$

$$10k^2+9k-1 = 0$$

$$(k+1)(10k-1) = 0$$

$$k=-1 \quad k = 1/10$$

(ii) $$P[1.5 < X < 4.5 / X > 2] = \frac{P[(1.5<X<4.5) \cap (X>2)]}{P[X>2]}$$

$$= \frac{P[2<X<4.5]}{P[X>2)} = \frac{P[X=3]+P[X=4]}{P[X>2]}$$

$$= \frac{\frac{2}{10}+\frac{3}{10}}{\frac{6}{10}+\frac{10}{100}} = \frac{5}{7}$$

(iii) $P[X \leq \lambda] > 1/2$

$\Rightarrow P[X \leq 3] = 1/2$

(2) A random variable X takes the values 1, 2, 3 & 4 such that $2P[X=1] = 3P[X=2] = P[X=3] = 5P[X=4]$ find the Probability distribution and cumulative distribution function of X.

Soln

Let

$2P[X=1] = 3P[X=2] = P[X=3] = 5P[X=4] = k$

$$\left.\begin{array}{l} P[X=1] = k/2 \\ P[X=2] = k/3 \\ P[X=3] = k \\ P[X=4] = k/5 \end{array}\right\} \text{———} (1)$$

WKT $\sum p_i = 1$

$$\frac{k}{2} + \frac{k}{3} + k + \frac{k}{5} = 1$$

$$\frac{15k + 10k + 30k + 6k}{30} = 1$$

$$\frac{61k}{30} = 1$$

$$\boxed{k = \frac{30}{61}}$$

Sub $k = \frac{30}{61}$ in (1)

$$P[X=1] = \frac{30}{61} \times \frac{1}{2} = \frac{15}{61}$$

$$P[X=2] = \frac{30}{61} \times \frac{1}{3} = \frac{10}{61}$$

(i) Probability distribution function is

X	1	2	3	4
$P(x)$	$\frac{15}{61}$	$\frac{10}{61}$	$\frac{30}{61}$	$\frac{6}{61}$

ii) Cummulative distribution function is

X	$F[X] = P[X \le x]$
1	$\frac{15}{61}$
2	$\frac{15}{61} + \frac{10}{61} = \frac{25}{61}$
3	$\frac{15}{61} + \frac{10}{61} + \frac{30}{61} = \frac{55}{61}$
4	$\frac{15}{61} + \frac{10}{61} + \frac{30}{61} + \frac{6}{61} = \frac{61}{61} = 1$

(3) A R.V X has the following probability function

x	0	1	2	3	4	5	6	7	8
$P(x)$	a	$3a$	$5a$	$7a$	$9a$	$11a$	$13a$	$15a$	$17a$

i) Determine a

ii) Evaluate $P(X<3)$, $P(X \ge 4)$, $P(0<X\le 5)$

iii) Find the distribution function of X

Soln

WKT $\sum p(x) = 1$

x	0	1	2	3	4	5	6	7	8
$P(x)$	$1/81$	$3/81$	$5/81$	$7/81$	$9/81$	$11/81$	$13/81$	$15/81$	$17/81$
$F[x]$	$1/81$	$4/81$	$9/81$	$\frac{16}{81}$	$\frac{25}{81}$	$\frac{36}{81}$	$\frac{49}{81}$	$\frac{64}{81}$	1.

$$P(X<3) = P(0) + P(1) + P(2)$$

$$= \frac{1}{81} + \frac{3}{81} + \frac{5}{81} = \frac{9}{81}$$

$$P(X \geq 4) = P(4) + P(5) + P(6) + P(7) + P(8)$$

$$= \frac{9}{81} + \frac{11}{81} + \frac{13}{81} + \frac{15}{81} + \frac{17}{81}$$

$$= \frac{65}{81}$$

$$P(0 < X \leq 5) = P(X = 1, 2, 3, 4, 5)$$

$$= \frac{3}{81} + \frac{5}{81} + \frac{7}{81} + \frac{9}{81} + \frac{11}{81}$$

$$= \frac{25}{81}$$

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4) The probability Mass function of a R.V X is defined as $p(X=0) = 3c^2$, $p(X=1) = 4c - 10c^2$, $p(X=2) = 5c - 1$ where $c > 0$ and $p(X=r) = 0$ if $r = 0, 1, 2$, find

i) the value of c

(iii) The distribution function of X.
(iv) The Largest value of X for which $F[X] < 1/2$

Soln

x	0	1	2
$P(x)$	$3c^2$	$4c-10c^2$	$5c-1$

i) To find c

$$\Sigma p(x) = 1$$

$$3c^2 + 4c - 10c^2 + 5c - 1 = 1$$

$$-7c^2 + 9c - 2 = 0$$

$$7c^2 - 9c + 2 = 0$$

$$(c-1)(7c-2) = 0$$

$$c = 1, 2/7$$

$c \neq 1$

$\therefore \boxed{c = 2/7}$

(iv)

x	0	1	2
$P(x)$	$\frac{12}{49}$	$\frac{16}{49}$	$\frac{3}{7}$
$F(x)$	$\frac{12}{49}$	$\frac{28}{49}$	1

(ii) $P(0 < X < 2 / X > 0) = \dfrac{P[X=1] \cap P[X=1,2]}{P[X=1,2]}$

$$= \frac{P[X=1]}{P[X=1,2]} = \frac{\frac{16}{49}}{\frac{16}{49}+\frac{3}{7}}$$

$$= \frac{16}{49} \times \frac{49}{37}$$

$$= 16/37$$

(iii) $F[x] < 1/2$ is 0.

Continuous Random Variable:

X takes all its possible values in an interval.

Probability density function:

Let X be a continuous R.V then a function $f(x)$ is called pdf if

(i) $f(x) \geq 0$

(ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Cummulative distribution function

$F[x] = P[X \leq x] = \int^{x} f(x)\,dx$

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Note:

i) $f(x) = \frac{d}{dx} F[x]$

(ii) If X is continuous, then $P[a<x<b] = F[b] - F[a]$

1) Find the value of c given that pdf of a r.v X as $f(x) = \frac{c}{x^3}$, $1 < x < \infty$

Soln

W.K.T $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_1^{\infty} \frac{c}{x^3}\,dx = 1$$

$$c\int_1^{\infty} x^{-3}\,dx = 1$$

$$c\left[\frac{x^{-2}}{-2}\right]_1^{\infty} = 1$$

$$\frac{c}{-2}[0-1] = 1$$

$$\boxed{c = 2}$$

2) If X is a discrete r.v taking the values 1, 2, 3, . . . with probability function $P[X=x] = \frac{c^x}{x!}$, $x = 1, 2, \ldots$ then find the value of c.

Soln

$$\Sigma p_i = 1$$

$$\left[\frac{c^1}{1!}+\frac{c^2}{2!}+\frac{c^3}{3!}+\cdots\right]=1$$

$$\left[e^c-1\right]=1 \qquad \left[\because e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots\right]$$

$$e^c=2$$

$$\boxed{c=\log 2}$$

(5) If the pdf of a continuous r.v X is given by
$$f(x)=\begin{cases} ax, & 0\le x\le 1 \\ a, & 1\le x\le 2 \\ 3a-ax, & 2\le x\le 3 \\ 0, & \text{elsewhere}\end{cases}$$

i) find the value of a

ii) find the cdf of X

iii) If x_1, x_2, x_3 are 3 independent observations of X, what is the probability that exactly one of these 3 is greater than 1.5 ?

Soln

i) To find a.

WKT $\int_{-\infty}^{\infty} f(x)\,dx=1$

$$\int_0^1 ax\,dx+\int_1^2 a\,dx+\int_2^3 (3a-ax)\,dx=1$$

$$a\left[\frac{x^2}{2}\right]_0^1+a\left[x\right]_1^2+\left[3ax-\frac{ax^2}{2}\right]_2^3=1$$

$$a\left[\frac{1}{2}-0\right]+a\left[2-1\right]+\left[9a-\frac{9a}{2}\right.$$

$$\frac{a}{2} + a + 5a - \frac{9a}{2} = 1$$

$$\frac{4a}{2} = 1$$

$$\boxed{a = 1/2}$$

$$\therefore f(x) = \begin{cases} \frac{x}{2}, & 0 \le x \le 1 \\ \frac{1}{2}, & 1 \le x \le 2 \\ \frac{1}{2}(3-x), & 2 \le x \le 3 \\ 0, & \text{Otherwise} \end{cases}$$

ii) Cummulative distribution function of X is

$$F[x] = P[X \le x]$$

$$= \int_{-\infty}^{x} f(x)\, dx$$

$$= \int_{-\infty}^{0} f(x)\, dx + \int_{0}^{1} f(x)\, dx + \int_{1}^{2} f(x)\, dx + \int_{3}^{x} f(x)\, dx$$

Case (i) $x \le 0$

$$F[x] = \int_{-\infty}^{0} f(x)\, dx = 0$$

Case (ii) $0 \le x \le 1$

$$F[x] = \int_{-\infty}^{0} f(x)\, dx + \int_{0}^{x} f(x)\, dx$$

$$= 0 + \int_{0}^{x} \left(\frac{x}{2}\right) dx$$

$$= \left[0 + \frac{x^2}{4}\right]^{x} = \frac{x^2}{4}$$

case (iii) $1 \le x \le 2$

$$F[x] = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{x} f(x)\,dx$$

$$= 0 + \int_{0}^{1} \frac{x}{2}\,dx + \int_{1}^{x} \frac{1}{2}\,dx$$

$$= \left[\frac{x^2}{4}\right]_0^1 + \left[\frac{1}{2}x\right]_1^x$$

$$= \left[\frac{1}{4} - 0\right] + \left[\frac{x}{2} - \frac{1}{2}\right]$$

$$= \frac{1}{4} + \frac{x}{2} - \frac{1}{2}$$

$$= \frac{x}{2} - \frac{1}{4}$$

Case (iv) $2 \le x < 3$

$$F[x] = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx + \int_{2}^{x} f(x)\,dx$$

$$= 0 + \int_{0}^{1} \frac{x}{2}\,dx + \int_{1}^{2} \frac{1}{2}\,dx + \int_{2}^{x} \frac{1}{2}(3-x)\,dx$$

$$= \left[\frac{x^2}{4}\right]_0^1 + \left[\frac{x}{2}\right]_1^2 + \frac{1}{2}\left[3x - \frac{x^2}{2}\right]_2^x$$

$$= \left[\frac{1}{4} - 0\right] + \left[\frac{2}{2} - \frac{1}{2}\right] + \frac{1}{2}\left[\left(3x - \frac{x^2}{2}\right) - \left(6 - \frac{4}{2}\right)\right]$$

$$= \frac{1}{4} + \left[1 - \frac{1}{2}\right] + \frac{1}{2}\left[\left(3x - \frac{x^2}{2}\right) - \frac{8}{2}\right]$$

$$= \frac{1}{4} + \frac{1}{2} + \frac{3x}{2} - \frac{x^2}{4} - 2$$

$$= \frac{3x}{2} - \frac{x^2}{4} - \frac{5}{4}$$

Case (v) $x \geq 3$

$$F[x] = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx + \int_{2}^{3} f(x)\,dx + \int_{3}^{x} f(x)\,dx$$

$$F[x] = 1$$

$$F[x] = \begin{cases} 0, & x \leq 0 \\ \frac{x^2}{4}, & 0 \leq x \leq 1 \\ \frac{x}{2} - \frac{1}{4}, & 1 \leq x \leq 2 \\ \frac{3x}{2} - \frac{x^2}{4} - \frac{5}{4}, & 2 \leq x \leq 3 \\ 1, & x > 3 \end{cases}$$

(iii) $P(1 \leq x \leq 2.5) = \int_{1}^{2.5} f(x)\,dx$

$$= \int_{1}^{2} f(x)\,dx + \int_{2}^{2.5} f(x)\,dx$$

$$= \int_{1}^{2} \frac{1}{2}\,dx + \int_{2}^{2.5} \frac{1}{2}(3-x)\,dx$$

$$= \left[\frac{x}{2}\right]_{1}^{2} + \frac{1}{2}\left[3x - \frac{x^2}{2}\right]_{2}^{2.5}$$

$$= \left[1 - \frac{1}{2}\right] + \frac{1}{2}\left[\left(3(2.5) - \frac{5}{2}\right) - \left(6 - \frac{4}{2}\right)\right]$$

$$= \frac{1}{2} + \frac{1}{2}\left[\left(7.5 - \frac{5}{2}\right] - \left(\frac{8}{2}\right)\right]$$

$$= \frac{11}{16}$$

iv $P(X > 1.5) = \int_{1.5}^{\infty} f(x)\,dx$

$= \int_{1.5}^{2} f(x)\,dx + \int_{2}^{3} f(x)\,dx + \int_{3}^{\infty} f(x)\,dx$

$= \int_{1.5}^{2} \frac{1}{2}\,dx + \int_{2}^{3} \frac{1}{2}(3-x)\,dx + 0$

$= \frac{1}{2}[x]_{1.5}^{2} + \frac{1}{2}\left[3x - \frac{x^2}{2}\right]_{2}^{3}$

$= \frac{1}{2}\left[2 - \frac{3}{2}\right] + \frac{1}{2}\left[\left(9 - \frac{9}{2}\right) - \left(6 - \frac{4}{2}\right)\right]$

$= \frac{1}{2}\left(\frac{1}{2}\right) + \left[\left(\frac{9}{2} - \frac{9}{4}\right) - 2\right]$

$= \frac{1}{4} + \left[\frac{9}{4} - 2\right] = \frac{1}{2}$

Assume $p = \frac{1}{2}$ $q = \frac{1}{2}$ $n = 3$

P[exactly one value greater than 1.5]

$= 3C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)$

$= 3\left(\frac{1}{2}\right)\left(\frac{1}{4}\right)$

$= \frac{3}{8}$

4. A Continuous random variable X has the pdf $f(x) = \begin{cases} \frac{k}{1+x^2}, & -\infty < x < \infty \\ 0, & \text{otherwise} \end{cases}$

i) find k (ii) Distribution function of X
iii) $P[X > 0]$

Soln

i)
$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$k \int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = 1$$

$$k\left[\tan^{-1} x\right]_{-\infty}^{\infty} = 1$$

$$k\left[\tan^{-1}\infty - \tan^{-1}(-\infty)\right] = 1$$

$$k\left[\pi/2 + \pi/2\right] = 1$$

$$k \cdot \pi = 1$$

$$k = 1/\pi$$

$$\therefore f(x) = \begin{cases} \frac{1}{\pi}\cdot\frac{1}{1+x^2}, & -\infty < x < \infty \\ 0, & \text{otherwise} \end{cases}$$

ii) In $-\infty < x < \infty$

$$F[x] = \int_{-\infty}^{x} f(x)\,dx = \int_{-\infty}^{x} \frac{1}{\pi}\frac{1}{1+x^2}\,dx$$

$$= \frac{1}{\pi}\left[\tan^{-1} x\right]_{-\infty}^{x}$$

$$= \frac{1}{\pi}\left[\tan^{-1}x + \tan^{-1}\infty\right]$$

$$F[x] = \frac{1}{\pi}\left[\tan^{-1}x + \pi/2\right]$$

iii $P[X > 0] = 1 - P[X \le 0]$

$= 1 - F[0]$

$= 1 - \frac{1}{2} = \frac{1}{2}$

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5) The Probability density function of a R.V X is given by $f(x) = \begin{cases} x, & 0 < x < 1 \\ k(2-x), & 1 \le x < 2 \\ 0, & \text{otherwise} \end{cases}$

i) Find the value of k

ii) Find $P(0.2 < x < 1.2)$

iii) What is $P(0.5 < x < 1.5 / x \ge 1)$

iv) Find the distribution function of X.

Soln

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ k(2-x), & 1 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$$

i) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_0^1 x\,dx + \int_1^2 k(2-x)\,dx = 1$$

$$\left[\frac{x^2}{2}\right]_0^1 + k\left[2x - \frac{x^2}{2}\right]_1^2 = 1$$

$$\left(\frac{1}{2}\right) + k\left[(4-2) - (2-\tfrac{1}{2})\right] = 1$$

$$\frac{1}{2} + k\left[2 - \tfrac{3}{2}\right] = 1$$

$$\tfrac{1}{2} + \tfrac{k}{2} = 1$$

$$\tfrac{k}{2} = 1 - \tfrac{1}{2}$$

$$\boxed{k = 1}$$

$$\therefore f(x) = \begin{cases} x &, 0 < x < 1 \\ 2-x &, 1 \le x \le 2 \\ 0 &, \text{Otherwise} \end{cases}$$

(iv) Distribution function

Case (i) $x \le 0$

$$F[x] = \int_{-\infty}^{x} f(x)\,dx = 0$$

Case (ii) $0 < x < 1$

$$F[x] = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{x} f(x)\,dx$$

$$= 0 + \int_{0}^{x} x\,dx = \left[\frac{x^2}{2}\right]_0^x = \frac{x^2}{2}$$

Case (iii) $1 < x < 2$

$$F[x] = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{x} f(x)\,dx$$

$$= 0 + \int_{0}^{1} x\,dx + \int_{1}^{x} (2-x)\,dx$$

$$= \frac{1}{2} + \left[\left(2x - \frac{x^2}{2}\right) - \left(2 - \tfrac{1}{2}\right)\right]$$

$$= \frac{1}{2} + 2x - \frac{x^2}{2} - \frac{3}{2}$$

$$= 2x - \frac{x^2}{2} - 1$$

Case (iv) $x \geq 2$

$$F[x] = \int_{-\infty}^{x} f(x)\,dx = 1$$

(ii) $P(0.2 < X < 1.2) = F(1.2) - F(0.2)$

$$= \left[2(1.2) - \frac{(1.2)^2}{2} - 1\right] - \left[\cancel{2(0.2)} \; \frac{(0.2)^2}{2}\right]$$

$$= 0.68 - 0.02$$

$$= 0.66$$

(iii) $P(0.5 < X < 1.5 / X \geq 1) = \dfrac{P(0.5 < X < 1.5) \cap P(X \geq 1)}{P(X \geq 1)}$

$$= \frac{P(0.5 < X < 1.5) \cap (1 \leq x \leq 2)}{P(X \geq 1)}$$

$$= \frac{P(1 \leq X \leq 1.5)}{P(X \geq 1)}$$

$$= \frac{F(1.5) - F(1)}{1 - P(X < 1)} =$$

Mathematical Expectation

Let X be a r.v then the Mathematical expectation of X is given by

$$E[X] = \begin{cases} \sum x\, p(x), & X \text{ is discrete} \\ \int x\, f(x)\, dx, & X \text{ is continuous.} \end{cases}$$

Moments about origin

The r^{th} Moment about origin is

$$\mu_r' = E[X^r] = \begin{cases} \sum x^r p(x), & X \text{ is discrete} \\ \int x^r f(x)\, dx, & X \text{ is continuous} \end{cases}$$

Mean $= E[X]$

Variance $= E[X^2] - \{E[X]\}^2$

Moments about Mean [central Moments]

$$\mu_r = E[(X-\bar{X})^r] = \begin{cases} \sum (x-\bar{x})^r p(x), & X \text{ is discrete} \\ \int (x-\bar{x})^r f(x)\, dx, & X \text{ is continuous} \end{cases}$$

	Discrete R.V	Continuous R.V
①	$E[X] = \sum x\, p(x)$	$E[X] = \int_{-\infty}^{\infty} x\, f(x)\, dx$
②	$E[X^r] = \mu_r' = \sum x^r p(x)$	$E[X^r] = \mu_r' = \int_{-\infty}^{\infty} x^r f(x)\, dx$
③	Mean $= \mu_1' = \sum x p(x)$	Mean $\mu_1' = \int_{-\infty}^{\infty} x f(x)\, dx$
④	$\mu_2' = \sum x^2 p(x)$	$\mu_2' = \int_{-\infty}^{\infty} x^2 f(x)\, dx$
⑤	Variance $= \mu_2' - \mu_1'^2$ $= E[X^2] - \{E[X]\}^2$	Variance $= \mu_2' - \mu_1'^2$ $= E[X^2] - \{E[X]\}^2$

Note:

1. $E[aX+b] = a\,E[X] + b$
2. $Var(aX+b) = a^2\, var\, X$
3. $cov(X,Y) = E[XY] - E[X].E[Y]$
4. If X & Y are independent, then $cov(X,Y) = 0$
5. $cov(aX, bY) = ab\, cov(X,Y)$
6. $cov(X+a, Y+b) = cov(X,Y)$
7. $cov(aX+b, cY+d) = ac\, cov(X,Y)$
8. $Var(X_1 + X_2) = Var(X_1) + Var(X_2) + 2\,cov(X_1, X_2)$
9. $Var(X_1 - X_2) = Var(X_1) + Var\, X_2 - 2\,cov(X_1, X_2)$

Note:

1) $E[X+Y] = E[X] + E[Y]$

2) $E[XY] = E[X] \cdot E[Y]$.

① Given the following Probability distribution of X Compute (i) $E[X]$ (ii) $E[X^2]$ (iii) $E[2X \pm 3]$ (iv) $Var(2X \pm 3)$.

X	-3	-2	-1	0	1	2	3
P(x)	0.05	0.10	0.30	0	0.30	0.15	0.10

Soln

(i) $E[X] = \sum_{i=1}^{7} x_i P(x_i)$

$= (-3)(0.05) + (-2)(0.1) + (-1)(0.30) + 0 + 1(0.30) + 2(0.15) + 3(0.10)$

$= 0.25$

(ii) $E[X^2] = \sum_{i=1}^{7} x_i^2 P(x_i)$

$= (-3)^2(0.05) + (-2)^2(0.10) + (-1)^2(0.30) + 0 + 1^2(0.30) + 2^2(0.15) + 3^2(0.10)$

$= 2.95$

(iii) $E[2X \pm 3] = 2E[X] \pm 3$

$= 2(0.25) \pm 3$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$= \int_{-\infty}^{0} x^2 f(x)\, dx + \int_{0}^{1} x^2 \cdot x\, dx + \int_{1}^{2} x^2 (2-x)\, dx + \int_{2}^{\infty} x^2 f(x) dx$$

$$= 0 + \int_{0}^{1} x^3\, dx + \int_{1}^{2} (2x^2 - x^3)\, dx + 0$$

$$= \left[\frac{x^4}{4}\right]_0^1 + \left[2\frac{x^3}{3} - \frac{x^4}{4}\right]_1^2$$

$$= \left[\frac{1}{4} - 0\right] + \left[\left(\frac{16}{3} - \frac{16}{4}\right) - \left(\frac{2}{3} - \frac{1}{4}\right)\right]$$

$$= \frac{1}{4} + \frac{16}{3} - 4 - \frac{2}{3} + \frac{1}{4}$$

$$= \frac{7}{6}$$

$$E[X^2] = 7/6$$

$$Var[X] = E[X^2] - [E[X]]^2$$

$$= \frac{7}{6} - 1 = \frac{1}{6}$$

Moment generating function:

$$M_X(t) = E[e^{tx}] = \begin{cases} \int_{-\infty}^{\infty} e^{tx} f(x)\, dx, & X \text{ is continuous} \\ \sum_{x=-\infty}^{\infty} e^{tx} p(x), & X \text{ is discrete.} \end{cases}$$

① Prove that r^{th} moment of the R.V 'x' about origin is $M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r'$.

Soln

$$M_x(t) = E[e^{tx}]$$

$$= E\left[1 + \frac{tx}{1!} + \frac{(tx)^2}{2!} + \dots + \frac{(tx)^r}{r!} + \dots\right]$$

$$= 1 + t\frac{E[x]}{1!} + t^2\frac{E[x^2]}{2!} + \dots + \frac{t^r}{r!}E[x^r] + \dots$$

$$= 1 + t\mu_1' + t^2\mu_2' + \dots + \frac{t^r}{r!}\mu_r' + \dots$$

$$M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r'$$

Note:

r^{th} Moment = co-efficient of $\frac{t^r}{r!}$

② Find μ_1' and μ_2' from $M_x(t)$.

Soln

WKT $M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r'$

$$M_x(t) = \mu_0' + \frac{t}{1!}\mu_1' + \frac{t^2}{2!}\mu_2' + \dots + \frac{t^r}{r!}\mu_r' + \dots$$

diff w.r.t 't'

$$M_x'(t) = \mu_1' + \frac{2t}{2!}\mu_2' + \dots$$

$M_X'(0) = \mu_1' = \text{Mean}$

$\therefore \text{Mean} = \mu_1' = M_X'(0) = \left[\frac{d}{dt} M_X(t)\right]_{t=0}$

IIIly

$M_X''(t) = \mu_2' + t\mu_3' + \dots\dots$

$\mu_2' = M_X''(0) = \left[\frac{d^2}{dt^2} M_X(t)\right]_{t=0}$.

In General

$\mu_r' = \left[\frac{d^r}{dt^r} M_X(t)\right]_{t=0}$.

(3) Obtain the Mgf of X about the pt $X = a$.

$M_X(t) = E[e^{t(X-a)}]$

$= E\left[1 + \frac{t}{1!}(X-a) + \frac{t^2}{2!}(X-a)^2 + \dots + \frac{t^r}{r!}(X-a)^r + \dots\right]$

$= 1 + tE[X-a] + \frac{t^2}{2!}E(X-a)^2 + \dots + \frac{t^r}{r!}E(X-a)^r + \dots$

$= 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \dots + \frac{t^r}{r!}\mu_r' + \dots$

$[M_X(t)] = 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \dots + \frac{t^r}{}$

(4) Find the MGF of the random variable with the probability law $P(X=x) = q^{x-1} \cdot p \quad x = 1, 2, 3, \ldots$ find the Mean and Variance.

Soln:

$$M_X(t) = E[e^{tx}]$$

$$= \sum_{x=1}^{\infty} e^{tx} \, p(x)$$

$$= \sum_{x=1}^{\infty} e^{tx} \, q^{x-1} \cdot p$$

$$= \sum_{x=1}^{\infty} e^{tx} \cdot q^{x} \cdot \frac{p}{q}$$

$$= \frac{p}{q} \sum_{x=1}^{\infty} (qe^{t})^{x}$$

$$= \frac{p}{q}(qe^{t}) \sum_{x=1}^{\infty} (qe^{t})^{x-1}$$

$$= pe^{t} \left[1 + qe^{t} + (qe^{t})^{2} + \cdots\right]$$

$$= pe^{t} \left[1 - qe^{t}\right]^{-1}$$

M.g.F is $M_X(t) = \frac{pe^{t}}{1-qe^{t}}$

diff w.r.t 't'

$$\frac{d}{dt} M_X(t) = \frac{(1-qe^{t})\,pe^{t} - pe^{t}(-qe^{t})}{\ldots}$$

$$= \frac{pe^t - pqe^{2t} + pqe^{2t}}{(1-qe^t)^2}$$

$$M_x'(t) = \frac{pe^t}{(1-qe^t)^2} \quad \text{——— } (1)$$

To find Mean:

$$\mu_1' \text{ (about origin)} = M_x'(0)$$

$$= \frac{p}{(1-q)^2}$$

$$= \frac{p}{p^2} = \frac{1}{p}$$

$$\mu_1' = \text{Mean} = \frac{1}{p}$$

diff (1) w.r.t 't'

$$M_x''(t) = \frac{(1-qe^t)^2 pe^t - pe^t\, 2(1-qe^t)(-qe^t)}{(1-qe^t)^4}$$

$$= \frac{(1-qe^t)\left[(1-qe^t)pe^t + 2pqe^{2t}\right]}{(1-qe^t)^4}$$

$$= \frac{pe^t - pqe^{2t} + 2pqe^{2t}}{(1-qe^t)^3}$$

$$M_x''(t) = \frac{pe^t(1+qe^t)}{(1-qe^t)^3}$$

μ_2' (about origin) $= M_x''(0)$

$$\mu_2' = \frac{p(1+q)}{(1-q)^3} = \frac{p(1+q)}{p^3} = \frac{1+q}{p^2}$$

$$\boxed{\text{Mean } \mu_1' = \frac{1}{p}}$$

$$\text{Variance} = \mu_2' - \mu_1'^2 = \frac{1+q}{p^2} - \frac{1}{p^2}$$

$$= \frac{1+q-1}{p^2}$$

$$\boxed{\text{Variance} = \frac{q}{p^2}}$$

5) find the MGF of the random Variable X having the probability density function $f(x) = \begin{cases} \frac{x}{4}e^{-x/2}, & x>0 \\ 0, & \text{otherwise} \end{cases}$

Also deduce the first 4 Moments about the origin.

Soln

$$f(x) = \begin{cases} \frac{x}{4} e^{-x/2}, & x > 0 \\ 0 & , \text{elsewhere} \end{cases}$$

MGF

$$M_X(t) = E[e^{tx}]$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\, dx \qquad u = x \quad v = e^{-(\frac{1}{2}-t)x}$$

$$= \int_0^{\infty} e^{tx} \cdot \frac{x}{4} e^{-x/2} dx$$

$$= \frac{1}{4} \int_0^{\infty} x\, e^{-(\frac{1}{2}-t)} dt$$

$$= \frac{1}{4} \left[\frac{x e^{-(\frac{1}{2}-t)x}}{-(\frac{1}{2}-t)} - \frac{e^{-(\frac{1}{2}-t)x}}{-(\frac{1}{2}-t)^2} \right]_0^{\infty}$$

$$= \frac{1}{4} \left[\frac{-2x e^{-(\frac{1-2t}{2})}}{(1-2t)} + \frac{4e^{(\frac{1-2t}{t})x}}{(1-2t)^2} \right]_0^{\infty}$$

$$= \frac{1}{4} \left[(0+0) - \left(0 + \frac{4e^0}{(1-2t)^2}\right) \right]$$

$$M_X(t) = \frac{1}{(1-2t)^2} = (1-2t)^{-2}$$

$$M_X'(t) = (-2)(1-2t)^{-3}(-2)$$
$$= 4(1-2t)^{-3}$$

$$M_X'(0) = 4$$

$$M_X''(t) = 4(-3)(1-2t)^{-4}(-2)$$
$$= 24(1-2t)^{-4}$$

$$M_X''(0) = 24$$

$$M_X'''(t) = 24(-4)(1-2t)^{-5}(-2)$$
$$= 192(1-2t)^{-5}$$

$$M_X'''(0) = 192$$

$$M_X^{IV}(t) = 192(-5)(1-2t)^{-6}(-2)$$
$$= 1920(1-2t)^{-6}$$

$$M_X^{IV}(0) = 1920.$$

6) Let X be a random variable with pdf

$$f(x) = \begin{cases} \frac{1}{3}e^{-x/3}, & x>0 \\ 0, & \text{otherwise} \end{cases}$$ find

a) $P(X>3)$
b) MGF of 'X'
c) $E[X]$ & $\text{var } X$.

Soln

$$f(x) = \begin{cases} \frac{1}{3}e^{-x/3}, & x>0 \end{cases}$$

$M_X'(t) = -(1-3t)^{-2}(-3) = 3(1-3t)^{-2}$

$E[X] = \text{Mean} = M_X'(0) = 3$

$M_X''(t) = -6(1-3t)^{-3} = 18(1-3t)^{-3}$

$M_X''(0) = 18$

$E[X^2] = 18$

$\text{Var } X = E[X^2] - [E[X]]^2$

$= 18 - 3^2$

$\text{Var } X = 9.$

7) A continous random variable X has the pdf $f(x) = kx^2e^{-x}, x \geq 0$ find the r^{th} moment of X about the origin. Hence find the variance of X.

Soln

To find k

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{0}^{\infty} kx^2e^{-x}\,dx = 1$$

$$k\left[-x^2e^{-x}-2xe^{-x}-2e^{-x}\right]_0^{\infty}=1$$

$u=x^2 \quad v=e^{-x}$
$u'=2x \quad v_1=-e^{-x}$
$u''=2 \quad v_2=e^{-x}$
$u'''=0 \quad v_3=-e^{-x}$

$$k\left[(\infty-\infty-2e^{-\infty})-(0-0-2e^{-0})\right]=1$$

$2k=1$

$k=\frac{1}{2}$

$$f(x)=\frac{1}{2}x^2e^{-x}.$$

<u>find r^{th} moment:</u>

$$\mu_r'=E[x^r]=\int_{-\infty}^{\infty}x^r f(x)\,dx$$

$$=\int_0^{\infty}x^r\frac{1}{2}x^2e^{-x}\,dx$$

$$=\frac{1}{2}\int_0^{\infty}x^{r+2}e^{-x}\,dx$$

$u=x^{r+2} \quad v=e^{-x}$
$u'=(r+2)x^{r+1} \quad v_1=-e^{-x}$
$u''=(r+2)(r+1)x^{r} \quad v_2=e^{-x}$
$v_3=-e^{-x}$

$$=\frac{1}{2}\left[-e^{-x}x^{r+2}-(r+2)x^{r+1}e^{-x}-(r+2)(r+1)x^{r}e^{-x}-\dots-(r+2)!e^{-x}\right]_0^{\infty}$$

$$=-\frac{1}{2}\left[e^{-x}\left(x^{r+2}+(r+2)x^{r+1}+\dots(r+2)!\right)\right]_0^{\infty}$$

$$\mu_r'=-\frac{1}{2}\left(-(r+2)!\right)=\frac{1}{2}(r+2)!$$

$$\mu_1'=\frac{3!}{2}=3$$

$$var(x)=E[x^2]-[E[x]]^2$$

$$=12-3^2$$

$$\mu_2'=\frac{4!}{2}=12 \qquad =3$$

8) If the pdf of 'x' is given by

$$f(x) = \begin{cases} 2(1-x), & 0<x<1 \\ 0, & \text{otherwise} \end{cases}$$

a) Show that $E[X^r] = \frac{2}{(r+1)(r+2)}$

b) Using this result, evaluate $E[(2X+1)^2]$

Soln

$$f(x) = 2(1-x)$$

$$E[X^r] = \int_{-\infty}^{\infty} x^r f(x)\, dx$$

$$= \int_0^1 x^r 2(1-x)\, dx$$

$$= 2\int_0^1 (x^r - x^{r+1})\, dx$$

$$= 2\left[\frac{x^{r+1}}{r+1} - \frac{x^{r+2}}{r+2}\right]_0^1$$

$$= 2\left[\left(\frac{1}{r+1} - \frac{1}{r+2}\right) - (0-0)\right]$$

$$= 2\left[\frac{1}{r+1} - \frac{1}{r+2}\right]$$

$$= 2\left[\frac{(r+2)-(r+1)}{(r+1)(r+2)}\right]$$

$$= 2\left[\frac{\cancel{r}+2-\cancel{r}-1}{(r+1)(r+2)}\right]$$

$$E[X^r] = \frac{2}{(r+1)(r+2)}$$

Put $r = 1$

$$E[X] = \frac{2}{(1+1)(1+2)} = \frac{1}{3}$$

$$E[X^2] = \frac{2}{(2+1)(2+2)} = \frac{1}{6}$$

$$E[(2X+1)^2] = E[4X^2+1+4X]$$

$$= E[4X^2] + E[1] + E[4X]$$

$$= 4E[X^2] + 4E[X] + 1$$

$$= \frac{4}{6} + \frac{4}{3} + 1$$

$$= \frac{4+8+6}{6}$$

$$= \frac{18}{6}$$

$E[(2X+1)^2]$

9. Consider a discrete r.v 'X' with probability function $p(X=x) = \begin{cases} \frac{1}{x(x+1)}, & x=1,2,\dots \\ 0, & \text{Otherwise} \end{cases}$

Show that $E[X]$ does not exist even though MGF exist.

Soln

$$p(x) = \frac{1}{x(x+1)}$$

Gn $\quad E[X] = \sum_{x=1}^{\infty} x\,p(x)$

$$= \sum_{x=1}^{\infty} \cancel{x} \frac{1}{\cancel{x}(x+1)}$$

$$= \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

$$= -1 + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

$$= -1 + \sum_{x=1}^{\infty} \frac{1}{x}$$

$\sum_{x=1}^{\infty} \frac{1}{x}$ is a divergent series.

$\therefore E[X]$ does not exist and hence no moment exists.

Now, MGF of X is gn by

$$M_X(t) = \sum_{x=1} e^{tx} p(x) = \sum_{x=1}^{\infty} \frac{e^{tx}}{x(x+1)}$$

Put $z = e^t$

$$= \sum_{x=1}^{\infty} \frac{z^x}{x(x+1)}$$

$$= \frac{z}{1\cdot 2} + \frac{z^2}{2\cdot 3} + \frac{z^3}{3\cdot 4} + \cdots$$

$$= z\left(1 - \frac{1}{2}\right) + z^2\left(\frac{1}{2} - \frac{1}{3}\right) + z^3\left(\frac{1}{3} - \frac{1}{4}\right) + \cdots$$

$$= \left[z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots\right] - \frac{z}{2} - \frac{z^2}{3} - \frac{z^3}{4} - \cdots$$

$$= -\log(1-z) - \frac{1}{z}\left[\frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \cdots\right]$$

$$= -\log(1-z) - \frac{1}{z}\left[-z + z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots\right]$$

$$= -\log(1-z) + 1 - \frac{1}{z}\left[z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots\right]$$

$$= -\log(1-z) + 1 - \frac{1}{z}\left[-\log(1-z)\right]$$

$= 1 + \left(\frac{1}{z} - 1\right) \log(1-z)$

$M_x(t) = 1 + (e^{-t} - 1) \log(1 - e^t)$, $t < 0$

$M_x(t) = 1$, for $t = 0$.

$M_x(t)$ does not exist for $t > 0$

10) A random variable X has pdf

$f(x) = \begin{cases} 2e^{-2x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$ find the MGF

when $t < 2$. find the first 4 Moments about the origin.

Soln

MGF of X is

$M_x(t) = E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$

$= \int_{-\infty}^{\infty} 2e^{-2t} e^{tx}\, dx = 2\int^{\infty} e^{-(2-t)x}\, dx$

$= 2\left[\frac{e^{-(2-t)x}}{-(2-t)}\right]_0^{\infty} = \frac{-2}{2-t}\left[e^{-\infty} - e^0\right]$

$M_x(t) = \frac{2}{2-t}$

On $t < 2$

$\frac{t}{2} < 1$

$\left|\frac{t}{2}\right| < 1$

$M_x(t) = \frac{2}{2(1-\frac{t}{2})} = \left(1 - \frac{t}{2}\right)^{-1}$

$$= 1 + \left(\frac{t}{2}\right) + \left(\frac{t}{2}\right)^2 + \cdots \quad \because \left|\frac{t}{2}\right| < 1$$

$$= 1 + \frac{1}{2}\frac{t}{1!} + \frac{2!}{4}\cdot\frac{t^2}{2!} + \frac{3!}{8}\frac{t^3}{3!} + \frac{4!}{16}\frac{t^4}{4!} + \cdots$$

$$\mu_1' = \frac{1}{2}$$

$$\mu_2' = \frac{2!}{4} = \frac{1}{2}$$

$$\mu_3' = \frac{3!}{8} = \frac{3}{4}$$

$$\mu_4' = \frac{4!}{2^4} = \frac{24}{16} = \frac{3}{2}$$

Note:

If the MGF of X is

$$M_X(t) = 1 + m_1\frac{t}{1!} + m_2\frac{t^2}{2!} + m_3\frac{t^3}{3!} + \cdots$$

μ_1' = Co-efficient of $\frac{t}{1!}$

& μ_2' = Co-efficiet of $\frac{t^2}{2!}$

$\vdots$

$= 0.608$

(iv) $P[\text{B/w 1 and 3 defectives}] = P[1 \le X \le 3]$

$= P[X=1] + P[X=2] + P[X=3]$

$= \left[\cancel{\left(20C_0 (1/10)^0 (9/10)^{20}\right)} + \left(20C_1 (1/10)^1 (9/10)^{19}\right) + \left(20C_2 (1/10)^2 (9/10)^{18}\right) + \left(20C_3 (1/10)^3 (9/10)^{17}\right) \right]$

$= [0.27 + 0.28517 + 0.1901178]$

$= 0.7452$

18.1.13

Poisson distribution:

A random variable 'x' is said to follow poisson distribution, if its probability mass function is

$$P[X=x] = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0,1,2,3\ldots$$

Moment Generating Function:

$$M_x(t) = E[e^{tx}]$$

$$= \sum_{x=0}^{\infty} e^{tx} p(x)$$

$$= \sum^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^x}{}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{e^{tx} \lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!}$$

$$= e^{-\lambda} \left[\frac{(\lambda e^t)^0}{0!} + \frac{(\lambda e^t)^1}{1!} + \frac{(\lambda e^t)^2}{2!} + \cdots \right]$$

$$= e^{-\lambda} \left[1 + \frac{(\lambda e^t)}{1!} + \frac{(\lambda e^t)^2}{2!} + \cdots \right]$$

$$= e^{-\lambda} \left[e^{\lambda e^t} \right]$$

$$= e^{\lambda [e^t - 1]}$$

$$\boxed{\therefore M_x(t) = e^{\lambda(e^t - 1)}}$$

Mean:

$$M_x(t) = e^{\lambda(e^t - 1)}$$

$$M_x'(t) = e^{\lambda(e^t - 1)} \cdot \lambda e^t$$

$$M_x'(0) = e^{\lambda(e^0 - 1)} \cdot \lambda e^0$$

$$= e^0 \cdot \lambda e^0$$

$$= \lambda$$

$$\boxed{\therefore \text{Mean} = \lambda} = E[x]$$

 To find $E[x^2]$:

$$M_x''(t) = \lambda\left[e^{\lambda(e^t-1)} \cdot e^t + e^t \cdot e^{\lambda(e^t-1)} \cdot \lambda e^t\right]$$

$$M_x''(0) = \lambda\left[e^{\lambda(e^0-1)} \cdot e^0 + e^0 \cdot e^{\lambda(e^0-1)} \cdot \lambda e^0\right]$$

$$= \lambda\left[e^0 \cdot e^0 + e^0 \cdot e^0 \cdot \lambda e^0\right]$$

$$= \lambda[1+\lambda]$$

$$= \lambda + \lambda^2$$

$$E[x^2] = \lambda + \lambda^2$$

Variance:

$$Var[x] = E[x^2] - [E[x]]^2$$

$$= \lambda + \cancel{\lambda^2} - \cancel{\lambda^2}$$

$$= \lambda$$

$$\boxed{\therefore Var[x] = \lambda}$$

Note:

In poisson distribution,

Mean = Variance $= \lambda$

27. The atoms of a radioactive elements are disintegrating. If every gram of this elements an average emits 3.9 alphaparticles per second, what is the probability that during the next second the number of alpha particles emitted from 1 gram is

(i) atmost 6

(ii) atleast 2

(iii) atleast 3 and atmost 6

Solution:

Given $\lambda = 3.9$

$$P[X=x] = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

(i) $P[\text{atmost } 6] = P[X \le 6]$

$$= P[X=0]+P[X=1]+P[X=2]+P[X=3]+P[X=4]+P[X=5]+P[X=6]$$

$$= e^{-3.9}\left[\frac{(3.9)^0}{0!}+\frac{(3.9)^1}{1!}+\frac{(3.9)^2}{2!}+\frac{(3.9)^3}{3!}+\frac{(3.9)^4}{4!}+\frac{(3.9)^5}{5!}+\frac{(3.9)^6}{6!}\right]$$

= 0.0202 [

$= 0.0202[44.4365]$

$= 0.8976$

(ii) p(atleast 2) $= P[X \geq 2]$

$= 1 - P[X < 2]$

$= 1 - P[X=0] + P[X=1]$

$= 1 - e^{-3.9}[1 + 3.9]$

$= 1 - 0.0202[4.9]$

$= 1 - 0.09898$

$= 0.90102$

(iii) P[atleast 3 and atmost 6] $= P[3 \leq X \leq 6]$

$= P[X=3] + P[X=4] + P[X=5] + P[X=6]$

$= 0.0202[9.8865 + 9.6393 + 7.5186 + 4.8871]$

$= 0.0202[31.9315]$

$= 0.6450.$

2. Suppose that the number of calls coming into telephone exchange b/w 9.am and 10 am is a poisson random variable with parameter 2, and the number of telephone calls coming b/w 10 AM and 11 A.M is a random variable with parameter 6. If these two random variables are independent. What is the probability that more than 5 calls come in between 9 AM and 11 AM.

Solution:

Let x_1 - calls b/w 9.AM and 10 AM with $\lambda_1 = 2$

x_2 - calls b/w 10 AM and 11 A.M with $\lambda_2 = 6$

WKT,

$X = X_1 + X_2$.

$\lambda = \lambda_1 + \lambda_2$

$\therefore \lambda = 2 + 6 = 8$

$\boxed{\lambda = 8}$

X - calls b/w 9 AM and 11 AM with $\lambda = 8$

$$P[X = x] = \frac{e^{-\lambda} \lambda^x}{x!}$$

$$P[X > 5] = 1 - P[X \leq 5]$$

$$= 1 - \left[P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4) + P(X=5) \right]$$

$$= 1 - e^{-8}\left[\frac{8^0}{0!} + \frac{8^1}{1!} + \frac{8^2}{2!} + \frac{8^3}{3!} + \frac{8^4}{4!} + \frac{8^5}{5!}\right]$$

$$= 1 - 3.354 \times 10^{-4}\left[1 + 8 + 32 + 85.33 + 170.666 + 273.06\right]$$

$$= 1 - 3.354 \times 10^{-4}\left[570.056\right]$$

$$= 1 - 0.1911$$

$$= 0.808$$

29. 3. The MGF of a random variable x be $e^{4(e^t-1)}$. Show that $P[\mu - 2\sigma < x < \mu + 2\sigma] = 0.93$

Solution:

Given,

$$MGF = e^{4(e^t-1)} = M_x(t) \quad \text{--- (1)}$$

In poisson distribution.

$$MGF = M_x(t) = e^{\lambda(e^t-1)} \quad \text{--- (2)}$$

On comparing (1) & (2)

$$\lambda = 4$$

In poisson distribution

$$\text{Mean} = \text{Variance} = \lambda$$

$$\mu = \sigma^2 = 4$$

 prove :

$$P[\mu-2\sigma < X < \mu+2\sigma] = 0.93$$

LHS

$$P[4-2(2) < X < 4+2(2)] = P[0 < x < 8)$$

$$= P[X=1] + P[X=2] + P[X=3] + P[X=4] +$$
$$P[X=5] + P[X=6] + P[X=7)$$

$$= e^{-4}\left[\frac{4^1}{1!} + \frac{4^2}{2!} + \frac{4^3}{3!} + \frac{4^4}{4!} + \frac{4^5}{5!} + \frac{4^6}{6!} + \frac{4^7}{7!}\right)$$

$$= 0.01831\left[4 + 8 + 10.66 + 10.666 + 8.533 + 5.6888 + 3.2507\right]$$

$$= 0.01831\left[50.8045\right)$$

$$= 0.93$$

30\. 4. If X is a poisson random variable such that $P[X=2] = 9P[X=4] + 90P[X=6]$

Find the Variate.

Solution:

$$P[X=x] = \frac{e^{-\lambda}\lambda^x}{x!}$$

$$P[X=2] = 9P[X=4] + 90P[X=6]$$

$$\frac{e^{-\lambda}\lambda^2}{} = \frac{e^{-\lambda}\lambda^{4}}{} + 90\frac{e^{-\lambda}}{}$$

$$\frac{1}{1\times 2} = \frac{3\lambda^2}{1\times 2\times 3\times 4} + \frac{30\lambda^4}{1\times 2\times 3\times 4\times 5\times 6}$$

$$1 = \frac{3\lambda^2}{4} + \frac{\lambda^4}{4}$$

$$\lambda^4 + 3\lambda^2 - 4 = 0$$

$$(\lambda^2)^2 + 3(\lambda^2) - 4 = 0$$

$$(\lambda^2 - 1)(\lambda^2 + 4) = 0$$

$\lambda^2 = 1$ $\qquad$ $\lambda^2 + 4 = 0$

$\lambda = 1$ $\qquad$ $\lambda^2 = -4$

$\lambda^2 = -4$ not possible

$\therefore \lambda = 1$

Variance $= \lambda = 1$

Geometric distribution

A random variable 'X' is said to follow geometric distribution if its probability mass function is

$$P[X=x] = q^{x-1} p, \quad x = 1, 2$$

MGF :

$$M_X(t) = E[e^{tx}]$$

$$= \sum_{x=1}^{\infty} e^{tx} p(x)$$

$$= \sum_{x=1}^{\infty} e^{tx} q^{x-1} \cdot p$$

$$= \sum_{x=1}^{\infty} e^{tx} \cdot q^{x} \cdot q^{-1} \cdot p$$

$$= \frac{p}{q} \sum_{x=1}^{\infty} (qe^t)^x$$

$$= \frac{p}{q} \left[(qe^t) + (qe^t)^2 + (qe^t)^3 + \dots \right]$$

$$= \frac{p}{q} \cdot qe^t \left[1 + (qe^t) + (qe^t)^2 + \dots \right]$$

$$= pe^t \left[1 - qe^t \right]^{-1}$$

$$\boxed{M_X(t) = \frac{pe^t}{1 - qe^t}}$$

Mean :

$$M_x(t) = \frac{Pe^t}{1-qe^t}$$

$$M_x'(t) = \frac{(1-qe^t)Pe^t - Pe^t(-qe^t)}{(1-qe^t)^2}$$

$$= \frac{Pe^t - \cancel{pqe^{2t}} + \cancel{pqe^{2t}}}{(1-qe^t)^2}$$

~~$M_x'(0) = Pe^0 - pqe^0$~~

$$M_x'(t) = \frac{Pe^t}{(1-qe^t)^2}$$

$$M_x'(0) = \frac{Pe^0}{(1-qe^0)^2}$$

$$= \frac{P}{(1-q)^2}$$

$$= \frac{P}{P^2}$$

$$= \frac{1}{P}$$

$\therefore$ Mean $= E[x] = \frac{1}{P}$

To find : $E[x^2]$

$$M_x'(t) = \frac{Pe^t}{(1-qe^t)^2}$$

$$M_x''(t) = \frac{(1-qe^t)^2 Pe^t - Pe^t(2)(1-qe^t)(-qe^t)}{(1-qe^t)^4}$$

$$= \frac{(1-qe^t)\left[(1-qe^t)pe^t + 2pqe^{2t}\right]}{(1-qe^t)^{\cancel{4}\,3}}$$

$$= \frac{pe^t - pqe^{2t} + 2pqe^{2t}}{(1-qe^t)^3}$$

$$= \frac{pe^t + pqe^{2t}}{(1-qe^t)^3}$$

$$M_x''(0) = \frac{pe^0 + pqe^0}{(1-qe^0)^3}$$

$$= \frac{p+pq}{(1-q)^3}$$

$$= \frac{p(1+q)}{p^{\cancel{3}\,2}}$$

$$= \frac{1+q}{p^2}$$

$$M_x''(0) = E[x^2] = \frac{1+q}{p^2}$$

Variance:

$$Var[x] = E[x^2] - (E[x])^2$$

$$= \frac{1+q}{p^2} - \left(\frac{1}{p}\right)^2$$

$$= \frac{\cancel{1}+q-\cancel{1}}{p^2}$$

$$= \frac{q}{p^2}$$

Memoryless property:

The memoryless property is given by $P[x > s+t / x > s] = P[x > t]$ for any $s, t > 0$.

Consider:

$$P[x > s+t] = \sum_{x=s+t+1}^{\infty} pq^{x-1}$$

$$= p\left[q^{s+t} + q^{s+t+1} + q^{s+t+2} + q^{s+t+3} + \dots\right]$$

$$= pq^{s+t}\left[1 + q + q^2 + \dots\right]$$

$$= pq^{s+1}\left[1-q\right]^{-1}$$

$$= \frac{pq^{s+1}}{1-q}$$

$$P[x > s+t] = \frac{pq^{s+t}}{1-q}$$

$$= \frac{pq^{s+t}}{\cancel{p}}$$

$$= q^{s+t}$$

$$\therefore P[x > s+t] = q^{s+t} \quad \text{——} (1)$$

$$\therefore P[x > s] = q^{s} \quad \text{——} (2)$$

$$P[x > t] = q^{t} \quad \text{——} (3)$$

$$P[X>s+t / X>s] = \frac{P[X>s+t \cap X>s]}{P[X>s]}$$

$$= \frac{P[X>s+t]}{P[X>s]}$$

$$= \frac{q^{s+t}}{q^{s}}$$

$$= q^{t}$$

$$= P[X>t]$$

$$\therefore P[X>s+t / X>s] = P[X>t]$$

---

31 1. Let one copy of a magazine out of 10 copies bears a special price following distribution determine its mean and variance.

Solution:

Given

$p = \frac{1}{10}$

$\therefore p+q = 1$

$q = 1-p$

$= 1 - \frac{1}{10}$

$\therefore q = \frac{9}{10}$

Mean:

$= 10$

$\therefore$ Mean $= 10$

Variance:

$$Var[X] = \frac{q}{p^2}$$

$$= \frac{9}{10} \Big/ \left(\frac{1}{10}\right)^2$$

$$= \frac{9 \times 100}{10}$$

$$= 90$$

$\therefore$ Variance $= 90$.

Uniform distribution [rectangular distribution]

A random variable X is said to have a continuous uniform distribution, if its probability density function is given by.

$$f(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0 & \text{Otherwise.} \end{cases}$$

MGF:

$$M_X(t) = E[e^{tx}]$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \int_{a}^{b}$$

$$= \frac{1}{b-a}\int_a^b e^{tx}\,dx$$

$$= \frac{1}{b-a}\left[\frac{e^{tx}}{t}\right]_a^b$$

$$= \frac{1}{(b-a)t}\left[e^{bx} - e^{ax}\right]$$

$$= \frac{e^{bx} - e^{ax}}{(b-a)t}$$

$$M_X(t) = \frac{e^{bx} - e^{ax}}{(b-a)t}$$

Mean:

$$E[X] = \int_{-\infty}^{\infty} x\, f(x)\,dx$$

$$= \int_a^b x \cdot \frac{1}{b-a}\,dx$$

$$= \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b$$

$$= \frac{1}{2(b-a)}\left(b^2 - a^2\right)$$

$$= \frac{(b+a)\cancel{(b-a)}}{2\cancel{(b-a)}}$$

$$= \frac{b+a}{2}$$

$$\therefore E[X] = \text{Mean} = \frac{b+a}{}$$

Variance :

$E[x^2]$

$$E[x^2] = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$= \int_a^b x^2 \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b$$

$$= \frac{1}{3(b-a)}\left[b^3 - a^3\right]$$

$$= \frac{\cancel{(b-a)}(b^2+ab+a^2)}{3\cancel{(b-a)}}$$

$$= \frac{b^2+ab+a^2}{3}.$$

$$E[x^2] = \frac{a^2+ab+b^2}{3}$$

$$Var[x] = E[x^2] - (E[x])^2$$

$$= \frac{a^2+ab+b^2}{3} - \left(\frac{b+a}{2}\right)^2$$

$$= \frac{a^2+ab+b^2}{3} - \frac{(a^2+b^2+2ab)}{4}$$

$$= \frac{4a^2+4ab+4b^2-3a^2-3b^2-6ab}{12}$$

$$= \frac{a^2+b^2-2ab}{12}$$

$$= \frac{(a-b)^2}{}$$

1. If x is uniformly distributed over (0,10) find the probability that

(i) $(x<2)$

(ii) $(x>8)$

(iii) $(3<x<9)$

Solution:

$$f(x) = \begin{cases} \frac{1}{b-a}, & a \le x < b \\ 0, & \text{Otherwise} \end{cases}$$

(0, 10)

$$f(x) = \begin{cases} \frac{1}{10}, & 0 < x < 10 \\ 0, & \text{Otherwise.} \end{cases}$$

(i) $P[x<2]$

$$f(x) = \int_0^2 f(x)\,dx$$

$$= \int_0^2 \frac{1}{10}\,dx$$

$$= \frac{1}{10}\left[x\right]_0^2$$

$$= \frac{1}{10}\left[2-0\right]$$

$$= \frac{2}{10}$$

$$= \frac{1}{5}$$

$\therefore P[x<2] = 1/5$

(ii) $P[x>8]$

$$f(x)=\int_{8}^{10}\frac{1}{10}\,dx$$

$$=\frac{1}{10}\left[x\right]_{8}^{10}$$

$$=\frac{1}{10}\left[10-8\right]$$

$$=\frac{2}{10}$$

$$=\frac{1}{5}$$

$\therefore P[x>8]=\frac{1}{5}$

(iii) $P[3<x<9]$

$$f(x)=\int_{3}^{9}\frac{1}{10}\,dx$$

$$=\frac{1}{10}\left[x\right]_{3}^{9}$$

$$=\frac{(9-3)}{10}$$

$$=\frac{6}{10}$$

$$=3/5$$

$\therefore P[x>8]=3/5$

2. If X is uniformly distributed over $(-\alpha, \alpha)$, $\alpha < 0$ find α so ~~should~~ that

(i) $P[X > 1] = \frac{1}{3}$

(ii) $P[|X| < 1) = P(|X| > 1)$

Solution:

$$f(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & \text{Otherwise.} \end{cases}$$

$a = -\alpha$, $b = \alpha$

$$f(x) = \begin{cases} \frac{1}{2\alpha}, & -\alpha < x < \alpha \\ 0, & \text{otherwise} \end{cases}$$

(i) $P[X > 1] = \frac{1}{3}$

$$\int_1^{\alpha} \frac{1}{2\alpha}\, dx = \frac{1}{3}$$

$$\frac{1}{2\alpha}\left[x\right]_1^{\alpha} = \frac{1}{3}$$

$$\frac{1}{2\alpha}\left[\alpha - 1\right] = \frac{1}{3}$$

$$3\alpha - 3 = 2\alpha$$

$$3\alpha - 2\alpha - 3 = 0$$

$$\alpha - 3 = 0$$

$$\alpha = 3$$

$$\therefore \boxed{\alpha = 3}$$

(ii) $P[|x|<1] = P[|x|>1]$

$$P[|x|<1] = 1-P(|x|<1)$$

$$2P[|x|<1] = 1$$

$$2P[-1<x<1] = 1$$

$$2\int_{-1}^{1}\frac{1}{2\alpha}\,dx = 1$$

$$\frac{2}{2\alpha}\left[x\right]_{-1}^{1} = 1$$

$$\frac{1}{\alpha}\left[1-(-1)\right] = 1$$

$$\frac{1}{\alpha}[2] = 1$$

$$\boxed{\alpha = 2}$$

34.
3. Four buses arrive at a specified stop at 15 min intervals starting at 7 AM. (ie) they arrive at 7, 7.15, 7.30, 7.45. AM and so on.
If a passenger arrives at a time (ie) uniformly distributed between 7 and 7.30 AM. Find the probability that he waits

(a) less than 5 mins for a bus.

(b) more than 10 mins for a bus.

Solution:

Let 'x' denote the number of minutes passed 7, that the passenger arrives bus stop in (0, 30)

$$f(x) = \begin{cases} \frac{1}{30}, & 0<x<30 \\ 0, & \text{Otherwise.} \end{cases}$$

(i) A passenger will have to wait less than 5 minutes, if he arrives between 7.10 and 7.15 and if he arrives between 7.25 and 7.30.

$$P(10<X<15) + P(25<X<30)$$

$$= \int_{10}^{15} \frac{1}{30}\,dx + \int_{25}^{30} \frac{1}{30}\,dx$$

$$= \frac{1}{30}\left[[x]_{10}^{15} + [x]_{25}^{30} \right]$$

$$= \frac{1}{30}\left[(15-10) + (30-25)\right]$$

$$= \frac{1}{30}\left[5+5\right]$$

$$= \frac{10}{30}$$

$$= \frac{1}{3}$$

$$\therefore \left[P[10<X<15) + P(25<X<30)\right] = \frac{1}{3}$$

(ii) A passenger will have to wait more than ten minutes if he arrives b/w 7 and 7.05, (or) b/w 7.15 and 7.20.

$P[0 < X < 5] + P[15 < X < 20]$

$= \int_0^5 \frac{1}{30}\,dx + \int_{15}^{20} \frac{1}{30}\,dx$

$= \frac{1}{30}\left[[x]_0^5 + [x]_{15}^{20}\right]$

$= \frac{1}{30}\left[(5-0) + (20-15)\right]$

$= \frac{1}{30}\left[5+5\right]$

$= \frac{10}{30}$

$= \frac{1}{3}$

$P[0 < X < 5] + P[15 < X < 20] = \frac{1}{3}$.

Exponential distribution

A continuous random variable x is said to follow an exponential distribution with parameter $\lambda > 0$, if its probability density function is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

MGF:

$$M_x(t) = E[e^{tx}]$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \int_{0}^{\infty} e^{tx} \lambda e^{-\lambda x}\, dx$$

$$= \lambda \int_{0}^{\infty} e^{tx} e^{-\lambda x}\, dx$$

$$= \lambda \int_{0}^{\infty} \left[e^{-(\lambda - t)x}\, dx \right]$$

$$= \lambda \left[\frac{e^{-(\lambda - t)x}}{-(\lambda - t)} \right]_0^{\infty}$$

$$= \frac{\lambda}{-(\lambda - t)} \left[e^{-\infty} - e^{0} \right]$$

$$= \frac{\lambda}{\lambda - t}$$

$$M_x(t) = \frac{\lambda}{\lambda - t}, \lambda > t$$

Mean:

$$M_x(t) = \lambda (\lambda - t)^{-1}$$

$$M_x'(t) = \lambda (-1) (\lambda - t)^{-2} \cdot (-1)$$

$$= \lambda (\lambda - t)^{-2}$$

$$= \frac{\lambda}{(\lambda - t)^2}$$

$$M_x'(0) = \frac{\lambda}{(\lambda - 0)^2}$$

$$= \frac{\lambda}{\lambda^2}$$

$$= \frac{1}{\lambda}$$

$$\boxed{\therefore \text{Mean} = E[x] = \frac{1}{\lambda}}$$

To find $E[x^2]$:

$$M_x'(t) = \lambda (\lambda - t)^{-2}$$

$$\cancel{E[x^2]}\ M_x''(t) = \lambda (-2) (\lambda - t)^{-3} \cdot (-1)$$

$$= \frac{2\lambda}{(\lambda - t)^3}$$

$$\text{Variance} = E[x^2] - (E[x])^2$$

$$= \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2$$

$$= \frac{2-1}{\lambda^2}$$

$$= 1/\lambda^2$$

$$\therefore \boxed{\text{Variance} = \frac{1}{\lambda^2}}$$

Memoryless property:

If x is exponentially distributed, then $P[x > s+t / x > s] = P(x > t)$ for any $s, t > 0$.

Solution:

$$P[x > k) = \int_k^\infty f(x)\, dx.$$

$$= \int_k^\infty \lambda e^{-\lambda x} dx$$

$$= \lambda \int_k^\infty e^{-\lambda x} dx$$

$$= \lambda \left[\frac{e^{-\lambda x}}{-\lambda} \right]_k^\infty$$

$$= \frac{\lambda}{-\lambda}\left[-e^{-\lambda k}\right]$$

$$= e^{-\lambda k}.$$

$$P[x > s+t / x > s) = \frac{P[(x>s+t) \cap (x>s)]}{P[x>s]}$$

$$= \frac{P[x > s+t]}{P[x>s]}$$

$$= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}}$$

$$= e^{-\lambda s - \lambda t} \cdot e^{\lambda s}$$

$$= e^{-\lambda s - \lambda t + \lambda s}$$

$$= e^{-\lambda t}$$

$$= P[x > t].$$

$$\therefore P[x > s+t / x > s] = P[x > t]$$

35. 1. The time (in hrs) required to repair a machine is exponentially distributed with parameter $\lambda = 1/2$

(a) what is the probability that the

Given that its duration exceeds 8 Hrs.?

Solution:

Let 'x' be the random Variable which represents the time to repair the machine.

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0, \\ 0, & \text{otherwise} \end{cases}$$

$$f(x) = \begin{cases} \frac{1}{2} e^{-x/2} & x > 0. \end{cases}$$

(i) $P[X > 2] = e^{-\lambda 2}$ $\left[\because P[X > k] = e^{-\lambda k}.\right]$

$= e^{-2/2}$

$= e^{-1}$

$= 0.3678$

(ii) $P[X \geq 11 / X > 8] = P[X > 8+3 / X > 8]$

$= P[X > 3]$ $\quad$ $\left(\because P[X > s+t / X > s] = P[X > t]\right)$

$= e^{-3\lambda}$

$= e^{-3/2}$

$= 0.2231$

36

2) If x is a random variable which follows an exponential distribution with parameter λ with $P[x \le 1) = P[x > 1]$. Find Variance of x?

Solution:

$$f(x) = \lambda e^{-\lambda x}, \quad x > 0$$

$$P(x \le 1) = P(x > 1)$$

$$1 - P(x > 1) = P(x > 1)$$

$$2P(x > 1) = 1$$

$$P(x > 1) = 1/2$$

$$e^{-\lambda} = 1/2 \quad \left[\because P[x > k] = e^{-\lambda k}\right]$$

$$\cancel{\log} \frac{1}{e^{\lambda}} = \frac{1}{2} \Rightarrow e^{\lambda} = 2$$

Taking log on both sides,

$$\lambda = \log_e 2$$

$$Var[x] = \frac{1}{\lambda^2}$$

$$= \frac{1}{(\log_e 2)^2}$$

$$\therefore Var[x] = \frac{1}{(\log_e 2)^2}$$

Gamma distribution:

The continuous random variable 'x' is said to follow a Gamma distribution with parameter λ, if its probability function is given by

$$f(x) = \begin{cases} \dfrac{e^{-x} x^{\lambda-1}}{\Gamma\lambda} & \lambda>0,\ 0<x<\infty \\ 0 & \text{, otherwise} \end{cases}$$

MGF:

$$M_x(t) = E[e^{tx}]$$

$$= \int_0^\infty e^{tx} f(x)\, dx$$

$$= \int_0^\infty e^{tx} \frac{e^{-x} x^{\lambda-1}}{\Gamma\lambda}\, dx$$

$$= \frac{1}{\Gamma\lambda} \int_0^\infty e^{tx} e^{-x} x^{\lambda-1}\, dx$$

$$= \frac{1}{\Gamma\lambda} \int_0^\infty e^{-(1-t)x} x^{\lambda-1}\, dx$$

Put

$u = (1-t)x$	$x \to 0 \Rightarrow u \to 0$
$du = (1-t)\, dx$	$x \to \infty \Rightarrow u \to \infty$

$$= \frac{1}{\Gamma\lambda}\int_0^\infty e^{-u}\left(\frac{u}{1-t}\right)^{\lambda-1}\cdot\frac{du}{1-t}$$

$$= \frac{1}{\Gamma\lambda}\int_0^\infty e^{-u}\frac{u^{\lambda-1}}{(1-t)^{\lambda-1}}\cdot\frac{du}{(1-t)}$$

$$= \frac{1}{\Gamma\lambda}\cdot\frac{1}{(1-t)^\lambda}\int_0^\infty e^{-u}u^{\lambda-1}du.$$

$$= \frac{1}{\cancel{\Gamma\lambda}(1-t)^\lambda}\cdot\cancel{\Gamma\lambda}\quad\left[\because \Gamma n = \int_0^\infty e^{-x}x^{n-1}\cdot dx\right]$$

$$= \frac{1}{(1-t)^\lambda}$$

$$\therefore M_x(t) = \frac{1}{(1-t)^\lambda}$$

<u>Mean</u>:

$$M_x(t) = (1-t)^{-\lambda}$$

$$M_x'(t) = -\lambda(1-t)^{-\lambda-1}\cdot(-1)$$

$$= \lambda(1-t)^{-(\lambda+1)}$$

$$= \frac{\lambda}{(1-t)^{\lambda+1}}$$

$$M_x'(0) = \frac{\lambda}{(1-0)^{\lambda+1}}$$

$$= \frac{\lambda}{(1)^{\lambda+1}} = \frac{\lambda}{1} = \lambda$$

$E[x^2]$:

$$M_x'(t) = \frac{\lambda}{(1-t)^{\lambda+1}}$$

$$= \lambda (1-t)^{-(\lambda+1)}$$

$$M_x''(t) = \lambda \cdot (-(\lambda+1)) (1-t)^{-(\lambda+2)} (-1)$$

$$= \lambda(\lambda+1)(1-t)^{-(\lambda+2)}$$

$$M_x''(0) = \lambda(\lambda+1)(1-0)^{-(\lambda+2)}$$

$$= \lambda(\lambda+1)$$

$$= \lambda^2+\lambda$$

$$E[x^2] = \lambda^2+\lambda$$

Variance:

$$Var[x] = E[x^2] - (E[x])^2$$

$$= \cancel{\lambda^2} + \lambda - \cancel{\lambda^2}$$

$$= \lambda$$

$$\therefore Var[x] = \lambda$$

Normal distribution:

A continuous random variable x is said to follow a normal distribution with mean μ, and variance σ^2, if its density function is given by the probability law,

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \, e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty,$$

$$\sigma > 0, \; -\infty < \mu < \infty.$$

MGF:

$$M_x(t) = E[e^{tx}]$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx.$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} \, e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx.$$

Put $z = \frac{x-\mu}{\sigma}$ | $x \to -\infty \Rightarrow z \to -\infty$

$dz = \frac{dx}{\sigma}$ | $x \to \infty \Rightarrow z \to \infty$

$\sigma\, dz = dx$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\sigma z + \mu)} \cdot e^{-\frac{z^2}{2}} \sigma\, dz$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t\sigma z} e^{-\frac{z^2}{2}} dz$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(z^2 - 2t\sigma z)}{2}} dz$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2 \left(z^2 - 2t\sigma z + (t\sigma)^2\right) + \frac{(t\sigma)^2}{2}} dz$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2 (z - t\sigma)^2} e^{\frac{t^2\sigma^2}{2}} dz$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} e^{\frac{t^2\sigma^2}{2}} \int_{-\infty}^{\infty} e^{-1/2 (z - t\sigma)^2} dz$$

$$= \frac{e^{\mu t + \frac{t^2\sigma^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2/2} du$$

$$= \frac{e^{\mu t + \frac{t^2\sigma^2}{2}}}{\cancel{\sqrt{2\pi}}} \cdot \cancel{\sqrt{2\pi}}$$

$$M_x(t) = e^{\mu t + \frac{t^2\sigma^2}{2}}$$

Mean:

$$M_x(t) = e^{\mu t + \frac{t^2\sigma^2}{2}}$$

$$M_x'(t) = e^{\mu t + \frac{t^2\sigma^2}{2}} (\mu + t\sigma^2)$$

$= \mu$

$$\boxed{Mean = E[x] = \mu.}$$

To find $E[x^2]$:

$$M_x''(t) = e^{\mu t + \frac{t^2\sigma^2}{2}} \cdot \sigma^2 + (\mu + t\sigma^2) e^{\mu t + \frac{t^2\sigma^2}{2}} (\mu + t\sigma^2)$$

$$M_x''(0) = e^{0} \cdot \sigma^2 + (\mu + 0) e^{0} (\mu + 0)$$

$$E[x^2] = \sigma^2 + \mu^2$$

Variance:

$$Var[x] = E[x^2] - (E[x])^2$$

$$= \sigma^2 + \cancel{\mu^2} - \cancel{\mu^2}$$

$$= \sigma^2$$

$$\therefore \boxed{Variance = \sigma^2}$$

Standard normal distribution.

$$Z = \frac{x - \mu}{\sigma} \text{ with parameter}$$

μ and σ

Basic properties:

* Total area under the standard normal curve is equal to 1.

* The standard normal curve is asymptotic to x-axis.

The Standard normal curve is Symmetric about zero, most of the area under the Standard normal curve lies b/w -3 and 3.

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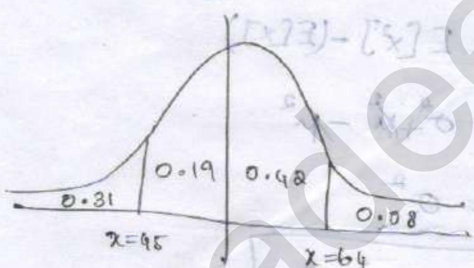
37 1. In a normal distribution 31 % of the items are under 45 and 8 % are over 64. Find the mean and Standard deviation.

Solution:



~~The Value of z corresponding to the area 0.19 is~~

Let the mean and Standard deviation of the given normal distribution be μ and σ.

The Value of z corresponding to the area 0.19 is 0.5 nearly

$$\frac{45-\mu}{\sigma} = -0.5$$

$$-0.5\sigma + \mu = 45 \quad \text{——(1)}$$

The Value of z corresponding to the area 0.42 is 1.4 nearly.

$$\frac{64-\mu}{\sigma} = 1.4$$

$$1.4\sigma + \mu = 64 \quad \text{---} (2)$$

Solve (1) & (2)

$$\begin{aligned} -0.5\sigma + \cancel{\mu} &= 45 \\ 1.4\sigma + \cancel{\mu} &= 64 \\ (-) \qquad (-) \qquad & (-) \\ \hline -1.9\sigma \qquad &= -19 \end{aligned}$$

$$\sigma = \frac{-19}{-1.9}$$

$$\sigma = 10$$

Sub $\sigma = 10$ in (2)

$$(1.4)10 + \mu = 64$$

$$14 + \mu = 64$$

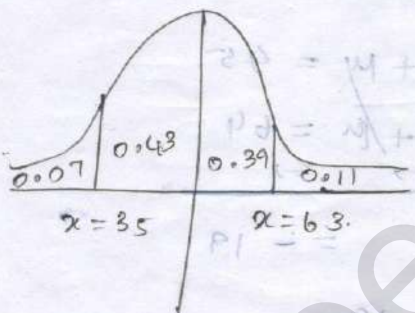
$$\mu = 64 - 14$$

$$\mu = 50$$

$$\therefore \sigma = 10, \mu = 50.$$

2. In a distribution, exactly normal 7% of the items are under 35 and 39% items are under 63. What are the mean and standard deviation of the distribution.

Solution:



Let the mean and standard deviation of the normal distribution be μ and σ.

The value of z corresponding to the area 0.43 is 1.47 nearly.

$$\frac{35-\mu}{\sigma} = -1.47$$

$$-1.47\sigma + \mu = 35 \quad \text{——(1)}$$

The value of z corresponding to the area 0.39 is 1.2 nearly.

$$\frac{63-\mu}{\sigma} = 1.2$$

$$1.2\sigma + \mu = 63 \quad \text{——(2)}$$

Solve (1) & (2)

$$-1.47\sigma + \mu = 35$$
$$1.2\sigma + \mu = 63$$
$$(-) \quad (-) \quad (-)$$

$$-1.67\sigma = 28$$

$$\sigma = \frac{28}{-1.67}$$

$$\sigma = 10.48$$

$$\mu = 50.41.$$

3. The marks obtained by a number of students for a certain subject is assumed to be normally distributed with mean 65 and standard deviation 5. if 3 students are taken at random from this set, what is the probability that exactly ~~two~~ one of them will have marks over ~~70~~ 75?

Let 'x' be the random variable which denotes the marks obtained by students.

Given:

$\mu = 65$

$\sigma = 5$

The standard normal variation is

$$Z = \frac{x-\mu}{\sigma} = \frac{x-65}{5}$$

To find $P(x>70)$

When $x=70$

$$Z = \frac{70-65}{5} = \frac{5}{5} = 1$$

$$P(x>70) = P(z>1)$$

$$= 0.5 - P(0<z<1)$$

$$= 0.5 - 0.3413$$

$$= 0.1587.$$

P(a student score > 70) = 0.1587

$p = 0.1587$

$q = 1 - 0.1587 = 0.8413$

$n = 3.$

$$P(X=x) = nCx\, p^x q^{n-x}$$

 Solution:

Let 'X' be the random variable denoting the life time of a light bulb.

Given

$\mu = 800$

$\sigma = 40$

$Z = \frac{X-\mu}{\sigma}$

$= \frac{X-800}{40}$

(i) P(a bulb burns more than 834 hrs)

$= P(X > 834)$

when $X = 834$ $Z = \frac{834-800}{40} = \frac{34}{40} = 0.85$

$P(X>834) = P(Z>0.85)$

$= 0.5 - P(0 < Z < 0.85)$

$= 0.5 - 0.3023$

$= 0.1977$

(ii) $P(778 < X < 834)$

when $X = 778 \Rightarrow Z = \frac{778-800}{40} = -0.55$

when $X = 834 \Rightarrow Z = 0.85$

$P(778 < X < 834) = P(-0.55 < Z < 0.85)$

31/1/13

TWO DIMENSIONAL RANDOM VARIABLE

Let 'S' be the sample space. Let $X = X(s)$ and $Y = Y(s)$ be the two functions each assigning a real number to each outcome $s \in S$. Then (X, Y) is a two dimensional random variable.

Note:

The two random variables of (X, Y) are said to be independent if,

$$P[X = x_i / Y = y_j] = P[X = x_i) \; P(Y = y_j)$$

$$P_{ij} = P_{i\cdot} \times P_{\cdot j}$$

Problems based on marginal distribution

1. From the following joint distribution of X and Y. Find the marginal distribution.

Y \ X	0	1	2
0	3/28	9/28	3/28

$$P[X=0] = P(0,0) + P(0,1) + P(0,2)$$

$$= \frac{3}{28} + \frac{3}{14} + \frac{1}{28}$$

$$= \frac{10}{28}$$

$$P[X=1] = P(1,0) + P(1,1) + P(1,2)$$

$$= \frac{9}{28} + \frac{3}{14} + 0$$

$$= \frac{15}{28}$$

$$P[X=2] = P(2,0) + P(2,1) + P(2,2)$$

$$= \frac{3}{28}$$

The marginal distribution of Y are

$$P[Y=0] = P[0,0] + P[1,0] + P[2,0]$$

$$= \frac{3}{28} + \frac{9}{28} + \frac{3}{28}$$

$$= \frac{15}{28}$$

$$P[Y=1] = P(0,1) + P(1,1) + P(2,1)$$

$$= \frac{3}{14} + \frac{3}{14}$$

$$= \frac{6}{14}$$

$$P[Y=2] = P(0,2) + P(1,2) + P(2,2)$$

$$= \frac{1}{28}$$

The marginal distribution of X & Y are

Y \ X	0	1	2	P(Y=y)
0	3/28	9/28	3/28	15/28
1	3/14	3/14	0	6/14
2	1/28	0	0	1/28
P(X=x)	5/14	15/28	3/28	1

1/2/13 2. If the joint p.d.f of (X, Y) is given by,

$P(x,y) = K(2x+3y)$, $x = 0, 1, 2$; $y = 1, 2, 3$. Find the marginal distribution. Also find the probability distribution of $(X+Y)$

Solution:

$P(x,y) = K(2x+3y)$

$P(0,1) = K(2(0)+3(1)) = 3K$

$P(0,2) = K(2(0)+3(2)) = 6K$

$P(0,3) = K(2(0)+3(3)) = 9K$

$P(1,1) = K(2(1)+3(1)) = 5K$

$P(1,2) = K(2(1)+3(2)) = 8K$

$P(1,3) = K(2(1)+3(3)) = 11K$

$P(2,2) = K(2(2)+3(2)) = 10K$

$P(2,3) = K(2(2)+3(3)) = 13K$

To find K:

y \ x	0	1	2	P(Y=y)
1	3K	5K	7K	15K
2	6K	8K	10K	24K
3	9K	11K	13K	33K
P(X=x)	18K	24K	30K	72K

$72K = 1$

$$\boxed{K = \frac{1}{72}}$$

The marginal distribution of X & Y is

Y \ X	0	1	2	P(Y=y)
1	$\frac{3}{72}$	$\frac{5}{72}$	$\frac{7}{72}$	$\frac{15}{72}$
2	6/72	8/72	10/72	24/72
3	9/72	11/72	13/72	33/72
P(X=x)	18/72	24/72	$\frac{30}{72}$	$\frac{72}{72} = 1$

X+Y	Probability
1	$P(0,1) = 3/72$
2	$P(0,2)+P(1,1) = 6/72 + 5/72 = \frac{11}{72}$
3	$P(2,1)+P(1,2)+P(0,3) = \frac{7}{72}+\frac{8}{72}+\frac{9}{72} = \frac{24}{72}$
4	$P(2,2)+P(1,3) = \frac{10}{72}+\frac{11}{72} = \frac{21}{72}$
5	$P(2,3) = \frac{13}{72}$

Problems based on Conditional distribution

3. From the following table for by variate distribution (X,Y). Find

(i) $P(X \le 1)$

(ii) $P(Y \le 3)$

(iii) $P(X \le 1, Y \le 3)$

(iv) $P(X \le 1 / Y \le 3)$

(v) $P(Y \le 3 / X \le 1)$

(vi) $P(X+Y \le 4)$

X \ Y	1	2	3	4	5	6
0	0	0	$\frac{1}{32}$	$\frac{2}{32}$	$\frac{2}{32}$	$\frac{3}{32}$
1	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$1/8$
2	$1/32$	$1/32$	$1/64$	$1/64$	0	$2/64$

X \ Y	6	1	2	3	4	5	P(X=x)
0	3/32	0	0	1/32	2/32	2/32	8/32
1	1/8	1/16	1/16	1/8	1/8	1/8	10/16
2	2/64	1/32	1/32	1/64	1/64	0	8/64
P(Y=y)	$\frac{16}{64}$	$\frac{3}{32}$	$\frac{3}{32}$	$\frac{11}{64}$	$\frac{13}{64}$	$\frac{6}{32}$	1

(i) $P(X \le 1) = P(X=0) + P(X=1)$

$$= \frac{8}{32} + \frac{10}{16}$$

$$= \frac{28}{32} = \frac{7}{8}$$

(ii) $P(Y \le 3) = P(Y=1) + P(Y=2) + P(Y=3)$

$$= \frac{3}{32} + \frac{3}{32} + \frac{11}{64}$$

$$= \frac{23}{64}$$

(iii) $P(X \le 1, Y \le 3) = P(0,1) + P(0,2) + P(0,3) + P(1,1) + P(1,2) + P(1,3)$

$$= 0 + 0 + \frac{1}{32} + \frac{1}{16} + \frac{1}{16} + \frac{1}{8}$$

$$= \frac{9}{32}$$

(iv) $P(X \le 1 \mid Y \le 3) = \dfrac{P((X \le 1) \cap P(Y \le 3))}{P(Y \le 3)}$

$= \dfrac{9/32}{\frac{23}{64}}$

$= \dfrac{18}{23}$

(v) $P(Y \le 3 / X \le 1) = \dfrac{P(Y \le 3 \cap X \le 1)}{P(X \le 1)}$

$= \dfrac{9/32}{7/8}$

$= 9/28$

(vi) $P(X+Y \le 4) = P(0,1) + P(0,2) + P(0,3) + P(0,4)$

$+ P(1,1) + P(1,2) + P(1,3) +$

$P(2,1) + P(2,2)$

$= 0 + 0 + \frac{1}{32} + \frac{2}{32} + \frac{1}{16} + \frac{1}{16} + \frac{1}{8}$

$+ \frac{1}{32} + \frac{1}{32}$

$= \frac{13}{32}$

... joint probability mass function of X and Y is

Y \ X	0	1	2
0	0.10	0.04	0.02
1	0.08	0.20	0.06
2	0.06	0.14	0.30

Find the marginal distribution function of X & Y. Also $P(X \le 1, Y \le 1)$ and check if X and Y are independent.

Solution: MDF of X & Y

Y \ X	0	1	2	$P(Y=y)$
0	0.10	0.04	0.02	0.16
1	0.08	0.20	0.06	0.34
2	0.06	0.14	0.30	0.5
$P(X=x)$	0.24	0.38	0.32	1

$$P(X \le 1, Y \le 1) = P(0,0) + P(0,1) + P(1,0) + P(1,1)$$

$$= 0.10 + 0.04 + 0.08 + 0.20$$

$$= 0.42$$

To check X & Y are independent

$$P(X=0) \cdot P(Y=0) = (0.16)(0.24)$$

$$= 0.038$$

$$\neq 0.10$$

Joint Probability distribution function for continuous random variable

$$F[x,y] = \int_{-\infty}^{x}\int_{-\infty}^{y} f(x,y)\,dx\,dy.$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1$$

Marginal distribution functions:

$$F_X(x) = \int_{-\infty}^{x}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dy\,dx$$

$$F_Y(y) = \int_{-\infty}^{y}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy.$$

Joint Probability density function:

$$f_{XY}(x,y) = \frac{\partial^2 F[x,y]}{\partial x\,\partial y}.$$

Marginal Probability density function.

$$f(x) = f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$$

$$f(y) = f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx.$$

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

Conditional probability density function.

$$f(y/x) = \frac{f(x,y)}{f(x)}, \quad f(x) > 0$$

$$f(x/y) = \frac{f(x,y)}{f(y)}, \quad f(y) > 0.$$

5) Show that the function ~~f(x,y)=~~

$$f(x,y) = \begin{cases} \frac{2}{5}(2x+3y), & 0<x<1,\ 0<y<1 \\ 0, & \text{Otherwise.} \end{cases}$$

Find JDF of X & Y.

Solution:

(i) $f(x,y) \geq 0$ in $0 \leq x, y \leq 1$

(ii) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dx\, dy.$

$$= \int_0^1\int_0^1 \frac{2}{5}(2x+3y)\, dx\, dy.$$

$\frac{1}{6} \cdot \frac{1}{6}$

$1 = \frac{K}{4}$

$$= \frac{2}{5}\int_0^1 \cancel{\left[\frac{2x^2}{2}\right]_0^1 dy} \left[\frac{2x^2}{2} + 3xy\right]_0^1 dy$$

$$= \frac{2}{5}\int_0^1 1+3y\ dy$$

$$= \frac{2}{5}\left[y + \frac{3y^2}{2}\right]_0^1$$

$$= \frac{2}{5}\left(\left[1+\frac{3}{2}\right] - 0\right)$$

$$= \frac{2}{5}\left(\frac{5}{2}\right)$$

Joint p.d.f of random variable X and Y is given by

$$f(x,y) = kxy\, e^{-(x^2+y^2)}, \quad x>0,\ y>0$$

Find the value of k and prove also that X and Y are independent.

Solution:

$$F[x,y] = kxy\, e^{-(x^2+y^2)}$$

$$\int_0^\infty \int_0^\infty kxy\, e^{-(x^2+y^2)}\, dx\, dy = 1$$

$$k \int_0^\infty x e^{-(x^2)}\, dx \int_0^\infty y e^{-y^2}\, dy = 1$$

$$K \cdot \frac{1}{2} \cdot \frac{1}{2} = 1$$

$$\frac{K}{4} = 1$$

$$\boxed{K = 4}$$

To prove X and Y independent:

(ie) $f(x) \cdot f(y) = f(x,y)$

$$f(x) = f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$

$$= \int_0^\infty kxy\, e^{-(x^2+y^2)}\, dy$$

$$= 4xe^{-x^2}\int_0^\infty ye^{-y^2}\,dy$$

$$= \cancel{4}^{2} xe^{-x^2}\cdot\frac{1}{2}$$

$$f(x) = 2xe^{-x^2}$$

$$f(y) = f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \int_0^\infty kxy\,e^{-(x^2+y^2)}\,dx$$

$$= 4ye^{-y^2}\int_0^\infty xe^{-x^2}\,dx$$

$$= 4ye^{-y^2}\cdot\frac{1}{2}$$

$$f(y) = 2ye^{-y^2}$$

$$f(x)\cdot f(y) = \underline{2xe^{-x^2}\cdot 2ye^{-y^2}}$$

$$= 4xye^{-(x^2+y^2)}$$

$$= f(x,y)$$

X & Y are independent

 $f(x,y) = 2 \quad 0 < x < y < 1$. Find the MDF. Find the CDF of $(Y/X=x)$

Solution :-

M.D.f of X,

$$f_X(x) = f(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$

$$= \int_x^1 2\, dy$$

$$= 2\,[y]_x^1$$

$$f(x) = 2\,[1-x)$$

M.d.f of Y,

$$f_Y(y) = f(y) = \int_{-\infty}^{\infty} f(x,y)\, dx$$

$$= \int_0^y 2\, dx$$

$$= 2\,[x]_0^y$$

$$= 2y$$

The c.d.f of Y given $X=x$ is $(Y/X=x)$

$$f(y/x) = \frac{f(x,y)}{f(x)}$$

$$= \frac{2}{2(1-x)}$$

$$= \frac{1}{\;}$$

Y is given by

$$f(x,y) = \begin{cases} 8xy, & 0<x<1,\ 0<y<x \\ 0, & \text{Otherwise.} \end{cases}$$

find (i) $f_X(x)$

(ii) $f_Y(y)$

(iii) $f(Y/x)$

(i) $f_X(x) = f(x) = \int_0^x 8xy\, dy$

$= 8x \int_0^x y\, dy$

$= 8x \left[\frac{y^2}{2}\right]_0^x$

$= 8x \cdot \frac{x^2}{2}$

$= 4x^3$

$f(x) = 4x^3$

(ii) $f_Y(y) = f(y) = \int_0^1 8xy\, dx$

$= 8y \left[\frac{x^2}{2}\right]_0^1$

$= 8y \cdot \frac{1}{2}$

$= 4y.$

$f(y) = 4y.$

$$f(y/x) = \frac{f(x,y)}{f(x)}$$

$$\cancel{f(x,y)} = \frac{8xy}{4x^3} = \frac{8xy}{4x^{3}}$$

$$f(y/x) = \frac{2y}{x^2}$$

9. The Joint p.d.f of a two dimensional random Variable (x,y) is given by.

$$f(x,y) = xy^2 + x^2/8, \quad 0 \le x \le 2, \quad 0 \le y \le 1$$

Compute (i) $P(X>1 \mid Y<1/2)$

(ii) $P(Y<1/2 \mid X>1)$

(iii) $P(X<Y)$

(iv) $P(X+Y \le 1)$

Solution :

(i) $$P(X>1/Y<1/2) = \frac{P(X>1, Y<1/2)}{P(Y<1/2)}$$

$$P(X>1, Y<1/2) = \int_1^2 \int_0^{1/2} \left(xy^2 + \frac{x^2}{8}\right) dy\, dx$$

$$= \int_1^2 \left[\frac{xy^3}{3} + \frac{x^2 y}{8}\right]_0^{1/2} dx$$

$$= \int_1^2 \left[\left[\frac{x}{24} + \frac{x^2}{16}\right] - (0+0)\right] dx$$

$$= \int_1^2 \left(\frac{x}{24} + \frac{x^2}{16}\right) dx$$

$$= \left[\frac{x^2}{48} + \frac{x^3}{48}\right]_1^2$$

$$= \frac{1}{48}\left[(4+8) - (1+1)\right]$$

$$= \frac{1}{48}\left[12 - 1\right]$$

$$= \frac{10}{48} = \frac{5}{24}$$

$$P(Y < 1/2) = \int_0^{1/2}\int_0^2 \left(xy^2 + \frac{x^2}{8}\right) dx\, dy$$

$$\cancel{= \left[\frac{xy^3}{3} + \frac{x^2 y}{8}\right]_0^{1/2}}$$

$$\cancel{= \frac{x}{24} + \frac{x^2}{16}}$$

$$= \int_0^{1/2} \left[\frac{x^2y^2}{2} + \frac{x^3}{24}\right]_0^2 dy$$

$$= \int_0^{1/2} \left(\left[\frac{4y^2}{2} + \frac{8}{24}\right] - \left[0+0\right]\right) dy$$

$$= \int_0^{1/2} \left[\frac{4y^2}{2} + \frac{1}{3}\right] dy$$

$$= \left[\frac{2}{3}\left(\frac{1}{2}\right)^3 + \frac{1}{3}\left(\frac{1}{2}\right)\right]$$

$$= \left[\frac{2}{24} + \frac{1}{6}\right]$$

$$= \frac{6}{24} = \frac{1}{4}$$

$$P\left[X>1 / Y<1/2\right] = \frac{P(X>1, Y<1/2)}{P(Y<1/2)}$$

$$= \frac{5/24}{1/4}$$

$$= 5/6$$

$$\therefore P\left[X>1 / Y<1/2\right] = 5/6.$$

(ii) ~~P(X>1 / Y<1/2)~~

$$P(Y<1/2 / X>1) = \frac{P(Y<1/2 ; X>1)}{P(X>1)}$$

$$P(X>1) = \int_0^1 \int_1^2 \left(xy^2 + \frac{x^2}{8}\right) dx\, dy$$

$$= \int_0^1 \left(\frac{x^2}{2} y^2 + \frac{x^3}{24}\right)_1^2 dy$$

$$= \int_0^1 \left[\left[\frac{4y^2}{2} + \frac{8}{24}\right] - \left[\frac{y^2}{2} + \frac{1}{24}\right)\right] dy$$

$$= \left[\frac{2y}{3} + \frac{y}{3} - \frac{y}{6} - \frac{y}{24}\right]_0$$

$$= \left[\frac{2}{3} + \frac{1}{3} - \frac{1}{6} - \frac{1}{24}\right]$$

$$= \left[\frac{3}{3} - \frac{1}{6} - \frac{1}{24}\right]$$

$$= \left[\frac{24 - 4 - 1}{24}\right]$$

$$= \frac{19}{24}$$

$$P(Y < 1/2 / X > 1) = \frac{P(Y < 1/2, X > 1)}{P(X > 1)}$$

$$= \frac{5/24}{19/24}$$

$$= \frac{5}{19}$$

$$\therefore P[Y < 1/2 / X > 1] = \frac{5}{19}$$

(iii) $$P(X < Y) = \int_0^1 \int_0^y \left(xy^2 + \frac{x^2}{8}\right) dx\, dy$$

$$= \int_0^1 \left(\frac{x^2 y^2}{2} + \frac{x^3}{24}\right)_0^y dy$$

$$= \int_0^1 \left(\frac{y^4}{2} + \frac{y^3}{24}\right) dy$$

$$= \left[\frac{y^5}{} + \frac{y^4}{}\right]^1$$

12.2.

Covariance:

If X and Y are random Variables, then covariance between X and Y is defined as

$$\text{Cov}(X,Y) = E[XY] - E[X].E[Y].$$

Note:.

If X and Y are independent then, $E[XY] = E[X].E[Y]$

$\Rightarrow \text{Cov}(X,Y) = 0$

(1) $\text{Cov}(aX, bY) = ab\,\text{Cov}(X,Y)$

(2) $\text{Cov}(X+a, Y+b) = \text{Cov}(X,Y)$

(3) $\text{Cov}(aX+b, cY+d) = ac\,\text{Cov}(X,Y)$

(4) $\text{Var}(X_1+X_2) = \text{Var}\,X_1 + \text{Var}\,X_2 + 2\,\text{Cov}(X_1,X_2)$

(5) $\text{Var}(X_1-X_2) = \text{Var}\,X_1 + \text{Var}\,X_2 - 2\,\text{Cov}(X_1,X_2)$

(6) If X_1 & X_2 are independent, then

$\text{Var}(X_1 \pm X_2) = \text{Var}\,X_1 \pm \text{Var}\,X_2.$

Correlation:

Karl-pearsons coefficient of correlation:.

$$r(X,Y) = \frac{\text{Cov}(X,Y)}{\sigma_X\,\sigma_Y}$$

where,

$$\sigma_X = \sqrt{\frac{1}{n}\left\{\sum x^2 - \bar{x}^2\right\}}, \quad \bar{X} = \frac{\sum x}{n}$$

$$\sigma_Y = \sqrt{\frac{1}{n}\sum y^2 - \bar{y}^2}, \quad \bar{Y} = \frac{\sum y}{n}$$

1) Correlation Coefficient may also be denoted by $\rho(x,y)$ (or) ρ_{xy}

2) If $\rho(x,y) = 0$, we say that x and y are uncorrelated.

3) When $r = 1$, the correlation is Perfect and positive.

Two independent Variables are uncorrelated. Since $Cov(x,y) = 0$ when X and Y are independent

$$r(X,Y) = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} = 0.$$

10.

1. Calculate the Correlation Coefficient for the following ~~types~~ height (in inches) of Fathers (x) and their Sons (y)

X	65	66	67	67	68	69	70	72
Y	67	68	65	68	72	72	69	71

X	Y	XY	X^2	Y^2
65	67	4355	4225	4489
66	68	4488	4356	4624
67	65	4355	4489	4225
67	68	4556	4489	4624
68	72	4896	4624	5184
69	72	4968	4761	5184
70	69	4830	4900	4761
72	71	5112	5184	5041
$\Sigma x = 544$	$\Sigma y = 552$	$\Sigma xy = 37560$	$\Sigma x^2 = 37028$	$\Sigma y^2 = 38132$

$\bar{X} = \frac{\Sigma x}{n} = \frac{544}{8} = 68$

$\bar{Y} = \frac{\Sigma y}{n} = \frac{552}{8} = 69$

$\bar{X}^2 = \frac{\Sigma x^2}{n} = \frac{37028}{8} =$

$\bar{X}\,\bar{Y} = 68 \times 69 = 4692$

$\sigma_x = \sqrt{\frac{1}{n}\Sigma x^2 - \bar{x}^2}$

$= \sqrt{\frac{1}{8}(37028) - (68)^2}$

$= 2.121$

$$\sigma_Y = \sqrt{\frac{1}{n}\sum Y^2 - \bar{Y}^2}$$

$$= \sqrt{\frac{1}{8}(38132) - 69^2}$$

$$= \sqrt{} \quad 2.345$$

$$r(X,Y) = \frac{Cor(X,Y)}{\sigma_X \sigma_Y}$$

$$= \frac{\frac{1}{n}\sum XY - \bar{X}\bar{Y}}{\sigma_X \sigma_Y}$$

$$= \frac{\frac{1}{8}(37560) - 4692}{2.121 \times 2.345}$$

$$= 0.6030.$$

2. Find the correlation coefficient for the following data.

X	10	14	18	22	26	30
Y	18	12	24	6	30	36

Solution:

X	Y	$U = \frac{X-22}{4}$	$V = \frac{Y-24}{6}$	UV	U^2	V^2
10	18	-3	-1	3	9	1
14	12	-2	-2	4	4	4
18	24	-1	0	0	1	0
22	6	0	-3	0	0	9
26	30	1	1	1	1	1
30	36	2	2	4	4	4

$$\bar{u} = \frac{\Sigma u}{n} = \frac{-3}{6} = -0.5$$

$$\vec{v} = \frac{\Sigma v}{n} = \frac{-3}{6} = -0.5$$

$$\bar{U}\bar{V} = -0.5 \times -0.5$$

$$= 0.25$$

$$\sigma_u = \sqrt{\frac{1}{6}\Sigma u^2 - (\bar{u})^2}$$

$$= \sqrt{\frac{1}{6}(19) - (0.25)}$$

$$= 1.70$$

$$\sigma_v = \sqrt{\frac{1}{6}\Sigma v^2 - (\bar{v})^2}$$

$$= \sqrt{\frac{1}{6}(19) - (0.25)}$$

$$= 1.70$$

$$r(x,y) =$$

$$r(u,v) = \frac{Cov(u,v)}{\sigma_u \sigma_v}$$

$$= \frac{\frac{1}{n}\Sigma uv - \bar{u}\bar{v}}{\sigma_u \sigma_v}$$

$$= \frac{\frac{1}{6}(12) - 0.25}{(1.70)(1.70)}$$

$$= \frac{1.75}{2.917266} = 0.6$$

The joint probability mass function of X and Y is

Y / X	-1	1
0	1/8	3/8
1	2/8	2/8

Find the correlation coefficient of X and Y.

Solution:

Y / X	-1	1	P(X=x)
0	1/8 P(0,-1)	3/8 P(0,1)	4/8 P(X=0)
1	2/8 P(1,-1)	2/8 P(1,1)	4/8 P(X=1)
P(Y=y)	3/8 P(Y=-1)	5/8 P(Y=1)	1

$$E[X] = \sum x\, p(x)$$

$$= 0 \times \frac{1}{2} + 1 \times \frac{1}{2}$$

$$= \frac{1}{2}$$

$$E[Y] = \sum y\, p(y)$$

$$= -1 \times \frac{3}{8} + 1 \times \frac{5}{8}$$

$$= \frac{1}{4}$$

$$E[X^2] = \sum x^2\, p(x)$$

$$E[Y^2] = \sum y^2 P(y)$$

$$= (-1)^2 \times \frac{3}{8} + 1^2 \times \frac{5}{8}$$

$$= 1.$$

$$E[XY] = \sum xy \; P(x,y)$$

$$= 0 \times -1 \times \frac{1}{8} + 0 \times 1 \times \frac{3}{8} + 1 \times -1 \times \frac{2}{8} + 1 \times 1 \times \frac{2}{8}$$

$$= 0$$

$$\sigma_x^2 = E[x^2] - [E[x]]^2$$

$$= \frac{1}{2} - \left(\frac{1}{2}\right)^2$$

$$= \frac{1}{4}.$$

$$\sigma_y^2 = E[Y^2] - [E(Y)]^2$$

$$= 1 - \left(\frac{1}{4}\right)^2$$

$$= \frac{15}{16}$$

$$\rho(x,y) = \frac{\operatorname{cov}(x,y)}{\sigma_x \sigma_y} = \frac{E[XY] - E[X]E[Y]}{\sigma_x \sigma_y}$$

$$= \frac{0 - \left(\frac{1}{2}\right)\left(\frac{1}{4}\right)}{\sqrt{\frac{1}{4}}\sqrt{\frac{15}{16}}}$$

If the independent random variable X and Y have the variance 36 and 16 respectively. Find the correlation coefficients b/w X+Y and X−Y.

Solution:

Given:

$Var(X) = 36$

$Var(Y) = 16$

X+Y are independent

$E[XY] = E[X]E[Y]$

Let $u = X+Y$

$v = X-Y$

$Var(u) = Var(X+Y)$

$= 1^2 Var(X) + 1^2 Var(Y)$

$= 1 \times 36 + 1 \times 16 = 36+16$

$= 52$

$\sigma_u^2 = 52$

$Var(v) = Var(X-Y)$

$= 1^2 Var(X) + (-1)^2 Var(Y)$

$= 1 \times 36 + 1 \times 16$

$= 36 + 16$

$= 52$

$\sigma_v^2 = 52$

$\sigma_u = \sqrt{52}, \quad \sigma_v = \sqrt{52}$

$Cov(u,v) = E[uv] - E[u]E[v]$

$E[uv] = E[(x+y)(x-y)]$

$= E[x^2 - y^2]$

$E[uv] = E[x^2] - E[y^2]$

$E[u] = E[x+y]$

$= E[x] + E[y]$

$E[v] = E[x-y]$

$= E[x] - E[y]$

$Cov(u,v) = E[x^2] - E[y^2] - [[E[x]]^2 - [[E(y)]]^2]$

$= E[x^2] - E[y^2] - [(E[x])^2] + [(E(y))^2]$

$= [E[x^2] - [(E[x])^2] - [E[y^2] - \{E(y)\}^2]$

$= Var(x) - Var(y)$

$= 36 - 16$

$= 20.$

$$\rho(u,v) = \frac{Cov(u,v)}{\sigma_u \cdot \sigma_v}$$

$$= \frac{20}{\sqrt{52} \cdot \sqrt{52}}$$

$$= \frac{20}{52}$$

If the joint p.d.f of (X, Y) is given by $f(x,y) = x+y$, $0 \le x, y \le 1$

Find ρ_{xy}.

Solution:

Given

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\ f(x,y)\,dx\,dy$$

$$= \int_0^1\int_0^1 xy(x+y)\,dx\,dy$$

$$= \int_0^1\int_0^1 x^2y + xy^2\,dx\,dy$$

$$= \int_0^1 \left[\frac{x^3y}{3} + \frac{x^2y^2}{2}\right]_0^1 dy$$

$$= \int_0^1 \left(\frac{y}{3} + \frac{y^2}{2}\right) dy$$

$$= \left[\frac{y^2}{6} + \frac{y^3}{6}\right]_0^1$$

$$= \frac{1}{6} + \frac{1}{6}$$

$$= 2/6$$

$$= 1/3$$

$E[X, Y] = 1/3$.

Mdf of X, $f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$

$$= \int^1 x + y\,dy$$

$$= \left[y + \frac{1}{2} \right]_0$$

$$= xy + \frac{1}{2}$$

$$f(x) = x + 1/2$$

Mdf of Y, $f(y) = \int_{-\infty}^{\infty} f(x,y)\, dx$

$$= \int_0^1 (x+y)\, dx$$

$$= \left[\frac{x^2}{2} + xy \right]_0^1$$

$$= \frac{1}{2} + y$$

$$f(y) = y + 1/2$$

$$E[X] = \int_{-\infty}^{\infty} x\, f(x)\, dx.$$

$$= \int_0^1 x(x + 1/2)\, dx$$

$$= \int_0^1 x^2 + \frac{x}{2}\, dx$$

$$= \left[\frac{x^3}{3} + \frac{x^2}{4} \right]_0^1$$

$$= \frac{1}{3} + \frac{1}{4}$$

$$= \frac{7}{12}.$$

$$E[Y] = \int_{-\infty}^{\infty} y \, f(y) \, dy$$

$$= \int_0^1 y \, (y + 1/2) \, dy$$

$$= \int_0^1 \left(y^2 + y/2\right) dy$$

$$= \left[\frac{y^3}{3} + \frac{y^2}{4}\right]_0^1$$

$$= \frac{1}{3} + \frac{1}{4}$$

$$= \frac{7}{12}$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 \, f(x) \, dx$$

$$= \int_0^1 x^2 \, (x + 1/2) \, dx$$

$$= \int_0^1 \left(x^3 + \frac{x^2}{2}\right) dx$$

$$= \left[\frac{x^4}{4} + \frac{x^3}{6}\right]_0^1$$

$$= \frac{1}{4} + \frac{1}{6}$$

$$= \frac{10}{24}$$

$$= 5/12$$

$$E[Y^2] = \int_{-\infty}^{\infty} y^2 \, f(y) \, dy$$

$$= \int_0 \left(y^3 + \frac{y^2}{2}\right) dy$$

$$= \left[\frac{y^4}{4} + \frac{y^3}{6}\right]_0^1$$

$$= \frac{1}{4} + \frac{1}{6}$$

$$= \frac{10}{24}$$

$$= 5/12.$$

$$Var(x) = E[x^2] - [E(x)]^2$$

$$= \frac{5}{12} - \frac{49}{144}$$

$$\sigma_x^2 = \frac{11}{144} \Rightarrow \sigma_x = \sqrt{\frac{11}{144}}$$

$$Var(Y) = E[Y] - (E(Y))^2$$

$$= \frac{7}{12} - \cancel{\frac{25}{144}}\ \frac{49}{144}$$

$$\sigma_y^2 = \frac{49}{144} \Rightarrow \sigma_Y = \cancel{\sqrt{\frac{59}{144}}}\ \sqrt{\frac{11}{144}}$$

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sigma_x \sigma_y}$$

$$= \frac{E[XY] - E[X]\,E[Y]}{\sigma_x \sigma_y}$$

$$= \frac{1}{3} - \frac{7}{12} \cdot \frac{7}{12}$$

$$= \frac{\frac{1}{3} - \frac{49}{144}}{\frac{11}{144}}$$

$$= \frac{48-49}{144} \times \frac{144}{11}$$

$$= \frac{-1}{11}$$

$$\therefore \; P(X,Y) = \frac{-1}{11}$$

If $f(x,y) = \frac{6-x-y}{8}$, $0 \le x \le 2$, $2 \le y \le 4$, find the correlation coefficient between X and Y.

Solution:

Given

$$f(x,y) = \frac{6-x-y}{8} \qquad \begin{array}{l} 0 \le x \le 2 \\ 2 \le y \le 4 \end{array}$$

Mdf of X:

$$f(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$

$$= \int_{2}^{4} \frac{6-x-y}{8}\, dy = \frac{1}{8}\left[6y - xy - y^2/2\right]_2^4$$

$$= \cancel{\left[\frac{6y-xy-y^2}{16}\right]_2^4} = \frac{1}{8}\left[\cancel{24-4x-16}\right]$$

$$= \cancel{\left[\frac{24-4x-16}{16}\right]} - \cancel{\left[\frac{12-2x-4}{16}\right]}$$

$$= \frac{1}{8}\left[\left(24 - 4x - \frac{16}{2}\right) - \left(12 - 2y - 4/2\right)\right]$$

$$= \frac{1}{8}\left[16 - 4x - 10 + 2y\right]$$

$$= \frac{1}{8}\left[6 - 2x\right]$$

$$f(y) = \int_{-\infty}^{\infty} f(x,y)\, dx$$

$$= \int_{0}^{2} \frac{6 - x - y}{8}\, dx$$

$$= \frac{1}{8}\int_{0}^{2} 6 - x - y\, dx$$

$$= \frac{1}{8}\left[6x - \frac{x^2}{2} - xy\right]_0^2$$

$$= \frac{1}{8}\left[\left[\cancel{72 - \frac{144}{2} - 12y}\right] - [0]\right]$$

$$= \frac{1}{8}\left[\cancel{72 - 72 - 12y}\right]$$

$$= \cancel{-\frac{12y}{8}} \qquad = \frac{1}{8}\left[12 - \frac{4}{2} - 2y\right]$$

$$= \cancel{-3y/2} \qquad = \frac{1}{8}\left[10 - 2y\right]$$

$$= \frac{1}{4}(5 - y)$$

$$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\, dx\, dy$$

$$= \int_{2}^{4}\int_{0}^{2} xy\left(\frac{6-x-y}{8}\right) dx\, dy$$

$$= \frac{1}{8}\int_{2}^{4}\int_{0}^{2} 6xy - x^2y - xy^2\, dx\, dy$$

$$= \frac{1}{8}\int_{2}^{4}\left[\frac{6x^2y}{2} - \frac{x^3y}{3} - \frac{x^2y^2}{2}\right]_0^2 dy$$

$$= \frac{1}{8}\int_{2}^{4}\left[\frac{6(4)y}{2} - \frac{8y}{3} - \frac{4y^2}{2}\right] dy$$

$$= \frac{1}{8}\int_{2}^{4} 12y - \frac{8y}{3} - 2y^2\, dy$$

$$= \frac{1}{8}\left[\frac{12y^2}{2} - \frac{8y^2}{6} - \frac{2y^3}{3}\right]_2^4$$

$$= \frac{1}{8}\left[\left[\frac{12\times16}{2} - \frac{8(16)}{6} - \frac{2(64)}{3}\right] - \left[\frac{12\times4}{2} - \frac{8\times4}{6} - \frac{2(8)}{3}\right]\right]$$

$$= \frac{1}{8}\left[\left[96 - \frac{64}{3} - \frac{128}{3}\right] - \left[24 - \frac{16}{3} - \frac{16}{3}\right]\right]$$

$$= \frac{1}{8}\left[32 - \frac{40}{3}\right]$$

$$= 4 - \frac{5}{3} = \frac{12-5}{3} = \frac{7}{3}$$

$$E[X] = \int_{-\infty}^{\infty} x \; f(x) \; dx$$

$$= \int_0^2 x\left(\frac{6-x-y}{8}\right)dx$$

$$= \frac{1}{8}\int_0^2 6x - x^2 - xy \; dx$$

$$= \frac{1}{8}\left[\frac{6x^2}{2} - \frac{x^3}{3} - \frac{x^2 y}{2}\right]_0^2$$

$$= \frac{1}{8}\left[\frac{6\times 4}{2} - \frac{8}{3} - \frac{4y}{2}\right]$$

$$= \frac{1}{8}\left[12 - \frac{8}{3} - y\right]$$

$$= \int_0^2 x\left(\frac{1}{8}(6-2x)\right)dx$$

$$= \frac{1}{8}\int_0^2 6x - 2x^2 \; dx$$

$$= \frac{1}{8}\left[\frac{6x^2}{2} - \frac{2x^3}{3}\right]_0^2$$

$$= \frac{1}{8}\left[\frac{6(4)}{2} - \frac{2(8)}{3}\right]$$

$$= \frac{1}{8}\left[12 - \frac{16}{3}\right]$$

$$= \frac{1}{8}\left[\frac{36-16}{3}\right]$$

$$= \frac{1}{8}\left[\frac{20}{3}\right]$$

$$= 5/6$$

$$E[X] = 5/6$$

$$E[Y] = \int_{-\infty}^{\infty} y \; f(y) \; dy$$

$$= \int_2^4 y \; \frac{1}{4}(5-y) \; dy$$

$$= \frac{1}{4}\int_2^4 5y - y^2 \; dy$$

$$= \frac{1}{4}\left[\left[\frac{5\times 16}{2} - \frac{64}{3}\right] - \left[\frac{5\times 4}{2} - \frac{8}{3}\right]\right]$$

$$= \frac{1}{4}\left[\left[40 - \frac{64}{3}\right] - \left[10 - \frac{8}{3}\right]\right]$$

$$= \frac{1}{4}\left[\left(\frac{120-64}{3}\right) - \left(\frac{30-8}{3}\right)\right]$$

$$= \frac{1}{4}\left[\frac{56}{3} - \frac{22}{3}\right]$$

$$= \frac{1}{\cancel{4}_2} \times \frac{\cancel{34}^{17}}{3}$$

$$= \frac{17}{6}$$

$$E[x^2] = \int_{-\infty}^{\infty} x^2 \, f(x)\, dx$$

$$= \int_0^2 x^2 \left(\frac{6-2x}{8}\right) dx$$

$$= \frac{1}{8}\int_0^2 6x^2 - 2x^3 \, dx$$

$$= \frac{1}{8}\left[\frac{6x^3}{3} - \frac{2x^4}{4}\right]_0^2$$

$$= \frac{1}{8}\left[\frac{\cancel{6}^2\times 8}{\cancel{3}} - \frac{2\times \cancel{16}^4}{\cancel{4}}\right]$$

$$= \frac{1}{8}\left[16 - 8\right]$$

$$= 1$$

$$= \int_{2}^{4} y^2 \left(\frac{5-y}{4}\right) dy$$

$$= \frac{1}{4}\int_{2}^{4} 5y^2 - y^3 \, dy$$

$$= \frac{1}{4}\left[\frac{5y^3}{3} - \frac{y^4}{4}\right]_2^4$$

$$= \frac{1}{4}\left[\left[\frac{5\times 64}{3} - \frac{\cancel{256}^{64}}{\cancel{4}}\right] - \left[\frac{5\times 8}{3} - \frac{\cancel{16}^{4}}{\cancel{4}}\right]\right]$$

$$= \frac{1}{4}\left[\left[\frac{320}{3} - 64\right] - \left[\frac{40}{3} - 4\right]\right]$$

$$= \frac{1}{4}\left[\frac{280}{3} - 60\right]$$

$$= \frac{1}{4}\left[\frac{280-180}{3}\right]$$

$$= \frac{1}{\cancel{4}}\left[\frac{\cancel{100}^{25}}{3}\right]$$

$$= 25/3$$

$$Var(x) = \sigma_x^2 = E[x^2] - (E(x))^2$$

$$= 1 - (5/6)^2$$

$$= 1 - \frac{25}{36}$$

$$= \frac{36-25}{}$$

$$\sigma_X = \sqrt{\frac{11}{36}}$$

$$Var(Y) = \sigma_Y^2 = E[Y^2] - (E(Y))^2$$

$$= \frac{25}{3} - \left(\frac{17}{6}\right)^2$$

$$= \frac{25}{3} - \frac{289}{36}$$

$$= \frac{300-289}{36}$$

$$= \frac{11}{36}$$

$$\sigma_Y = \sqrt{11/36}$$

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sigma_X \cdot \sigma_Y}$$

$$= \frac{E[XY] - E[X]E[Y]}{\sigma_X \cdot \sigma_Y}$$

$$= \frac{\frac{7}{3} - \frac{5}{6} \cdot \frac{17}{6}}{\sqrt{\frac{11}{36}} \cdot \sqrt{\frac{11}{36}}}$$

$$= \frac{\frac{7}{3} - \frac{85}{36}}{\frac{11}{36}}$$

$$= \frac{84-85}{\cancel{36}} \times \frac{\cancel{36}}{11}$$

$$r = \pm\sqrt{b_{xy} \times b_{yx}}$$

$$b_{yx} = \frac{\sum(x-\bar{x})(y-\bar{y})}{\sum(x-\bar{x})^2}$$

$$b_{xy} = \frac{\sum(x-\bar{x})(y-\bar{y})}{\sum(y-\bar{y})^2}$$

1. From the following data, Find

(i) The two regression eqn.

(ii) The Coefficient of correlation between the marks in economics and Statistics.

(iii) The most likely marks in Statistics, when marks in economics 30.

Marks in economics	25	28	35	32	31	36	29	38	34	3
Marks in Statistics	43	46	49	41	36	32	31	30	33	3

$\cancel{b_{yx} = 0.6643}$ $\boxed{b_{yx} = -0.6643}$

$$b_{xy} = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(y-\bar{y})^2}$$

$$= -\frac{93}{398}$$

$$① \cancel{\Rightarrow y} = -0.2337$$

$$\boxed{b_{xy} = -0.2337}$$

$$① \Rightarrow y - 38 = -0.6643(x-32)$$

$$y - 38 = -0.6643 \cancel{x} \times (-0.6643x - 32)$$

$$y = -0.6643x \times 21.2576 + 38$$

$$y = +0.6643x\ 59.2576$$

$$② \Rightarrow x - 32 = -0.2337(y-38)$$

$$x = -0.2337y \times (-0.2337x - 38) + 32$$

$$= -0.2337y \times 8.8806 + 32$$

$$x = 40.8806 - 0.2337y.$$

(ii) Coefficient of Correlation

$$r = \pm\sqrt{b_{yx} \times b_{xy}}$$

$$= \pm\sqrt{(-0.6643)(-0.2337)}$$

$$= \pm\sqrt{0.152}$$

the lines of regression of Y on X and X on Y is 0.6 and $\sigma_x = \frac{1}{2}\sigma_y$. Find the correlation coefficient between X and Y.

$\tan\theta = 0.6 \quad \sigma_x = 0.5\sigma_y$

Solution:

Given:

$\tan\theta = 0.6.$

$\sigma_x = 0.5\,\sigma_y.$

Angle b/w lines of regression is

$$\tan\theta = \frac{1-r^2}{r}\left(\frac{\sigma_x\sigma_y}{\sigma_x^2+\sigma_y^2}\right)$$

$$0.6 = \frac{1-r^2}{r}\left(\frac{(0.5\sigma_y)\,\sigma_y}{(0.5\sigma_y)^2+\sigma_y^2}\right)$$

$$= \frac{1-r^2}{r}\left(\frac{0.5\,\sigma_y^2}{0.25\sigma_y^2+\sigma_y^2}\right)$$

$$0.6 = \frac{1-r^2}{r}\left(\frac{0.5\,\cancel{\sigma_y^2}}{1.25\,\cancel{\sigma_y^2}}\right)$$

$$\frac{1-r^2}{r} = \frac{(1.25)(0.6)}{0.5}$$

$$= \frac{0.75}{0.5}$$

$$r^2 + 1.5r - 1 = 0$$

$$r = \frac{1}{2}, -2 \quad (-2 \text{ is not possible})$$

$$\boxed{r = 1/2}$$

19/2/2013 Transformation of two dimensional random Variable:

$$f_{UV} = f_{XY}(x,y) \left| \frac{\partial(x,y)}{\partial(u,v)} \right|$$

$$f_U(u) = \int_{-\infty}^{\infty} f_{UV}(u,v)\, dv$$

$$f_V(u,v) = \int_{-\infty}^{\infty} f_{UV}(u,v)\, du.$$

If the joint p.d.f of X, Y is given by $f_{XY}(x,y) = x+y$, $0 \le x, y \le 1$. Find the p.d.f of $U = XY$.

Solution:

Step 1:

To find joint p.d.f of X & Y.

Given:

$$f_{XY}(x,y) = x+y.$$

Step 2:

Introduc...

Expressing the above eqn as

$x = g_1(u,v) \qquad y = g_2(u,v)$

$u = xy \qquad v = y$

$u = x \cdot v \qquad y = v$

$x = \frac{u}{v}$

$\frac{\partial x}{\partial u} = \frac{1}{v} \qquad \frac{\partial x}{\partial v} = \frac{-u}{v^2}$

$\frac{\partial y}{\partial u} = 0 \qquad \frac{\partial y}{\partial v} = 1$

Step 4:

Find $|J| = \left| \frac{\partial(x,y)}{\partial(u,v)} \right|$

$$J = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{v} & \frac{-u}{v^2} \\ 0 & 1 \end{vmatrix}$$

$$= \frac{1}{v} - 0$$

$$= \frac{1}{v}$$

$$|J| = \frac{1}{v}$$

Step 5:

To find pdf of (u,v)

$$f_{UV}(u,v) = f_{XY}(x,y)\,|J|$$

$$= \left(\frac{u}{v} + v\right) \cdot \frac{1}{v}$$

Step 6:- Changing the domain, (x, y) into domain (u, v)

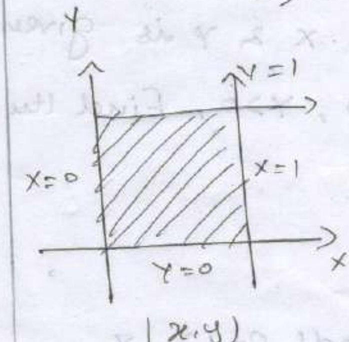
$u = v$

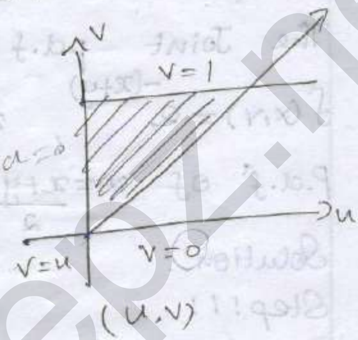
u	0	1	2	3
v	0	1	2	3

$0 \le y \le 1 \Rightarrow 0 \le v \le 1$

$0 \le x \le 1 \Rightarrow 0 \le \frac{u}{v} \le 1$

$\Rightarrow 0 \le u \le v$





P.d.f of (u, v) is given by

$$f_{uv}(u, v) = \frac{1}{v}\left(\frac{u}{v} + v\right) \qquad \begin{array}{l} 0 \le u \le v \\ 0 \le v \le 1 \end{array}$$

Step 7:

To find the p.d.f. of $(u = XY)$

$$f_U(u) = \int_{-\infty}^{\infty} f_{uv}(u, v)\, dv$$

$$= \int_u^1 \frac{1}{v}\left(\frac{u}{v} + v\right) dv$$

$$= \int_u^1 \frac{u}{v^2} + 1 \, dv$$

$$= \left[u \cdot \frac{v^{-1}}{-1} + v \right]_u^1$$

$$= \left[(-u+1) - (-1+u)\right] \quad \begin{matrix} -u+1+1-u \\ 2-2u \\ 2(1-u) \end{matrix}$$

$$= -u+1+1-u$$

$$= 2-2u$$

$$= 2(1-u)$$

$$f_U(u) = 2(1-u) \quad 0 \le u \le 1$$

The Joint p.d.f of X & Y is given by $f(x,y) = e^{-(x+y)}$, $x>0$, $y>0$, Find the p.d.f of $U = \frac{x+y}{2}$

Solution:

Step 1:

To find Joint pdf of X, Y.

Given

$$f(x,y) = e^{-(x+y)}, \quad x>0, \; y>0.$$

Step 2:

Introducing new random Variable.

Given, $U = \frac{x+y}{2}$,

Let $v=y$.

Step 3:

Expressing the above eqn as

$x = g_1(u,v)$ & $y = g_2(u,v)$

$$u = \frac{x+y}{2} \qquad v = y$$

$$u = \frac{x+v}{2} \qquad y = v$$

$$2u = x+v$$

$\frac{\partial x}{\partial u} = 2 \qquad \frac{\partial x}{\partial v} = -1$

$\frac{\partial y}{\partial u} = 0 \qquad \frac{\partial y}{\partial v} = 1$

Step 4 :-

Find $|J| = \left| \frac{\partial(x,y)}{\partial(u,v)} \right|$

$$|J| = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix}$$

$$= 2$$

Step 5:-

To find pdf of (u,v)

$$f_{UV}(u,v) = f_{XY}(x,y)\,|J|$$

$$= \cancel{(x+y)} \frac{e^{-(x+y)}}{2}$$

$$= e^{-(x+y)} \cdot 2$$

$$= 2e^{-2u}$$

$x>0$ $\quad y>0$

$2u - v > 0$ $\quad v > 0$

$2u > v$

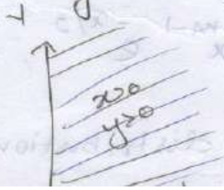
$u > v/2$

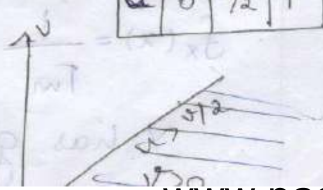
Step 6 : Changing the domain (x,y) into (u,v)

$x > 0 \Rightarrow u > v/2$

$y > 0 \Rightarrow v > 0$

v	0	1	2	3
u	0	1/2	1	1.5





To find the pdf of $u = \frac{x+y}{2}$

$$f_u(u) = \int_{-\infty}^{\infty} f_{uv}(u,v)\, dv$$

$$= \int_0^{2u} \frac{2e^{-2u}}{2u}\, dv$$

$$= 2e^{-2u} \int_0^{2u} dv$$

$$= 2e^{-2u} \left[v\right]_0^{2u}$$

$$= 2e^{-2u} \cdot \left[2u - 0\right]$$

$$= 4ue^{-2u} \quad u>0.$$

The random Variable X and Y are Statistically independent having gamma Variable with parameters $(m, 1/2)$ and $(n, 1/2)$ respectively. Derive the p.d.f of a random variable $U = \frac{X}{X+Y}$

Solution:

Step 1: To find Joint p.d.f of X, Y.

X has a gamma distribution with parameters $(m, 1/2)$

$$f_X(x) = \frac{1}{\Gamma m} \frac{1}{2^m} x^{m-1} e^{-x/2}, \quad x>0.$$

Y has gamma distribution with

$f_{XY}(x,y) = f_X(x) \cdot f_Y(y)$.

$$= \left[\frac{1}{\Gamma m} \frac{1}{2^m} x^{m-1} e^{-x/2}\right]\left[\frac{1}{\Gamma n} \frac{1}{2^n} y^{n-1} e^{-y/2}\right]$$

Step 2: Given Introducing new random variable.

$$u = \frac{x}{x+y}$$

Let $v = x+y$.

Step 3: Expressing the above eqn as

$x = g_1(u,v)$ & $y = g_2(u,v)$

$$u = \frac{x}{x+y} \qquad v = x+y.$$

$$u = \frac{x}{v} \qquad v = uv + y$$

$$x = uv \qquad y = v - uv$$

$$y = v(1-u)$$

$$\frac{\partial x}{\partial u} = v \qquad \frac{\partial x}{\partial v} = u$$

$$\frac{\partial y}{\partial u} = -v \qquad \frac{\partial y}{\partial v} = 1-u.$$

Step 4:

To find $|J| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

$$|J| = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} v & u \\ -v & 1-u \end{vmatrix}$$

$$= v(1-u) - (-uv)$$

$|J| = v$

Step 5: To find pdf of (u,v)

$$f_{UV}(u,v) = f_{XY}(x,y)\,|J|$$

$$= v\left[\frac{1}{\Gamma m}\cdot\frac{1}{2^m}\cdot x^{m-1}\cdot e^{-x/2}\right]\left[\frac{1}{\Gamma n}\frac{1}{2^n}y^{n-1}e^{-y/2}\right]$$

$$= v\left[\frac{1}{\Gamma m\Gamma n}\cdot\frac{1}{2^{m+n}}x^{m-1}\cdot y^{n-1}e^{-\frac{(x+y)}{2}}\right]$$

$$= v\left[\frac{1}{\Gamma m\Gamma n}\cdot\frac{1}{2^{m+n}}(uv)^{m-1}(v(1-u))^{n-1}\cdot e^{-v/2}\right]$$

$$= v\left[\frac{1}{\Gamma m\Gamma n\, 2^{m+n}}u^{m-1}v^{m-1}v^{n-1}(1-u)^{n-1}e^{-v/2}\right]$$

$$= v\left[\frac{1}{\Gamma m\Gamma n\, 2^{m+n}}u^{m-1}\cdot v^{m+n-2}\cdot(1-u)^{n-1}\cdot e^{-v/2}\right]$$

$$f_{UV}(u,v) = \left[\frac{1}{\Gamma m\Gamma n\, 2^{m+n}}u^{m-1}v^{m+n-1}(1-u)^{n-1}e^{-v/2}\right]$$

Step 6:

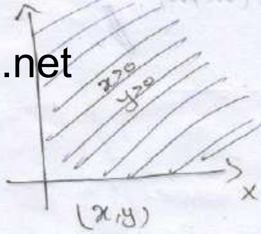
Changing the domain x, y into (u,v)

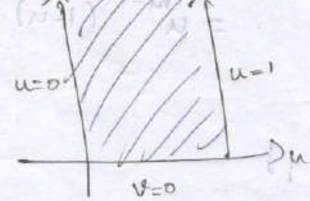
$x > 0 \Rightarrow uv > 0 \Rightarrow v > 0.$

$y > 0 \Rightarrow v - uv > 0 \Rightarrow v > uv$

$\therefore \Rightarrow 1 > u$

$\Rightarrow 0 \le u < 1$





Step I:

To find the p.d.f of $u = \frac{x}{x+y}$

$$f_u(u) = \int_{-\infty}^{\infty} f_{uv}(u,v)\, dv$$

$$= \int_0^{\infty} \frac{1}{\Gamma m \Gamma n\, 2^{m+n}} u^{m-1} v^{m+n-1} e^{-v/2} (1-u)^{n-1} . dv$$

$$= \frac{1}{\Gamma m \Gamma n\, 2^{m+n}} u^{m-1} (1-u)^{n-1} \int_0^{\infty} v^{m+n-1} . e^{-v/2}\, dv$$

Put $v/2 = w \Rightarrow v = 2w$

$dv = 2\,dw$

$v \to 0 \Rightarrow w \to 0$

$v \to \infty \Rightarrow w \to \infty$

$$= \frac{1}{\Gamma m \Gamma n\, 2^{m+n}} u^{m-1} (1-u)^{n-1} \int_0^{\infty} e^{-w} . (2w)^{m+n-1}\, 2\, dw$$

$$= \frac{1}{\Gamma m \Gamma n\, \cancel{2^{m+n}}} u^{m-1} (1-u)^{n-1} \int_0^{\infty} e^{-w} . \cancel{2^{m+n}} . w^{m+n-1}\, dw.$$

$$= \frac{1}{\Gamma m \Gamma n} u^{m-1} (1-u)^{n-1} \int_0^{\infty} e^{-w} w^{(m+n)-1}\, dw.$$

Unit III

Markov Processes and Markov Chains

Random process Introduction:

A random Variable is a Rule that assigns a Real number to every outcome of the Random experiment.

A random process is a Rule that assigns a Real number and time function to every outcome of the random experiment.

Def

A random process is a Collection of random Variables $\{X(s,t)\}$ where $s \in$ Sample Space, S
$t \in$ time parameter, T

Classification of Random Process

① Discrete Random Sequence

If both S and T are discrete then the random process is Called discrete random Sequence

Eg:
Number of books in the library at opening time

② Continuous Random Sequence

If S is Discrete and T is Continuous then the random process is Called Discrete random Process.

Eg: Number of phone calls from 10am to 11am

(4) Continuous Random process.

If S is Continuous and T is Continuous then the random process is Called Continuous random Process

Eg: Continuous Stirring Sugar in Coffee

I Stationary process

Def: A random process is Called a Strongly Stationary process or Strict Sense Stationary [SSS], if all its finite dimensional distributions are invariant under translation of time parameter

Note: $X(t)$ is SSS if (i) $E[X(t)]$ = Constant

(ii) $E[X^2(t)]$ - Constant

[$\Rightarrow$ Var[$X(t)$] - Constant]

A random process is called a Wide Sense Stationary [WSS] or Week Sense Stationary

$X(t)$ & $Y(t)$ are said to be jointly WSS if

(i) each process is individually WSS

(ii) $R_{XY}(\tau)$ is a function of τ.

1. The process $\{X(t)\}$ whose probability distribution under certain conditions is given by

$$P[X(t)=n] = \begin{cases} \dfrac{(at)^{n-1}}{(1+at)^{n+1}} & n=1,2\ldots \\ \dfrac{at}{1+at} & n=0 \end{cases}$$

Show that it is not stationary.

Sol

$$P[X(t)=n] = \begin{cases} \dfrac{(at)^{n-1}}{(1+at)^{n+1}} & n=1,2\ldots\ldots \\ \dfrac{at}{1+at} & n=0. \end{cases}$$

$X(t)$ is stationary if (i) $E[X(t)]=$ Constant

(ii) $E[X^2(t)]=$ Constant

(i) $E[X(t)] = \sum X(t) P[X(t)=n$

$$= 0 \cdot \frac{at}{1+at} + 1 \cdot \frac{1}{(1+at)^2} + 2 \cdot \frac{at}{(1+at)^3} + 3 \cdot \frac{(at)^2}{(1+at)^4} + \cdots$$

$X(t)$ 0 1 2 3

$$= \frac{1}{(1+at)^2} \cdot \left(\frac{1}{1+at}\right)^{-2}$$

$E[X(t)] = 1$ (constant)

(ii) $E[X^2(t)] = \sum X^2(t) P[X(t)] = \sum n^2 p(n)$

$$= \sum [n(n+1) - n]\, p(n)$$

$$= \sum n(n+1) \cdot p(n) - \sum n\, p(n)$$

$$= 0 + 1 \cdot 2 \frac{1}{(1+at)^2} + 2 \cdot 3 \frac{at}{(1+at)^3} + 3 \cdot 4 \frac{(at)^2}{(1+at)^4} + \cdots$$

$$-1$$

$$= \frac{2}{(1+at)^2}\left[1 + 3\left[\frac{at}{1+at}\right] + 6\left[\frac{at}{1+at}\right]^2 + \cdots\right]$$

$$= \frac{2}{(1+at)^2}\left[1 - \left(\frac{at}{1+at}\right)\right]^{-3} - 1$$

$$= \frac{2}{(1+at)^2}\left[\frac{1}{1+at}\right]^{-3} - 1$$

$$= \frac{2(1+at)^3}{(1+at)^2} - 1$$

$$= 2(1+at) - 1$$

$$= 2at + 2 - 1$$

(1) Show that the process $Y(t) = X(t) \cdot \cos(\omega t + \theta)$ where $X(t)$ is WSS, θ is a random variable independent of $X(t)$ and is distributed uniformly in $(-\pi, \pi)$ and ω is a constant is WSS.

Given

$$Y(t) = X(t) \cdot \cos(\omega t + \theta)$$

$$X(t) - WSS \Rightarrow E[X(t)] = \text{Constant}$$

$$\Rightarrow R_{XX}(\tau) - \text{Constant}$$

θ is uniformly distributed in $(-\pi, \pi)$

$$f(\theta) = \frac{1}{\pi - (-\pi)} = \frac{1}{2\pi}$$

To prove : $Y(t)$ is WSS

(i) $E[Y(t)] = E[X(t) \cdot \cos(\omega t + \theta)]$

$$= E[X(t)] \cdot E[\cos(\omega t + \theta)]$$

$$= \text{Constant } (0) = \text{Constant}$$

(ii) $R_{XX}(\tau) = E[Y(t) \cdot Y(t+\tau)]$

$$= E[X(t)\cos(\omega t + \theta) \cdot X(t+\tau)\cos(\omega(t+\tau)+\theta)]$$

$$= E[X(t) X(t+\tau)] \cdot E[\cos(\omega t + \theta)\cos(\omega(t+\tau)+\theta)]$$

$$= R_{XX}(\tau) \cdot \frac{1}{2} E[\cos(2\omega t + \omega\tau + 2\theta) \ldots$$

2. Show that the process $X(t) = A\cos\lambda t + B\sin\lambda t$. ($A, B$ are Random Variables) is WSS, show that if (i) $E(A) = E(B) = 0$ (ii) $E[A^2] = E[B^2]$ (iii) $E[AB] = 0$

Sol Given $X(t) = A\cos\lambda t + B\sin\lambda t$

To prove $X(t)$ is WSS

(i) $E[X(t)] = E[A\cos\lambda t + B\sin\lambda t]$

$= E[A].\cos\lambda t + E(B)\sin\lambda t$

$= 0$ (Constant)

(ii) $R_{XX}(\tau) = E[X(t).X(t+\tau)]$

$= E[(A\cos\lambda t + B\sin\lambda t)(A\cos\lambda(t+\tau) + B\sin\lambda(t+\tau))]$

$= E[A^2\cos\lambda t\cos\lambda(t+\tau) + B^2\sin\lambda t\sin\lambda(t+\tau)$
$+ AB(\sin\lambda t\cos\lambda(t+\tau) + \cos\lambda t.\cos\lambda(t+\tau)]$

$= k[\cos\lambda t.\cos\lambda(t+\tau) + \sin\lambda t\sin\lambda(t+\tau)]$

$= k\cos\lambda\tau$

= Constant

$\Rightarrow X(t)$ is WSS

3. The random Variable $X(t) = A\cos\omega t + B\sin\omega t$,

A, B uncorrelated $\Rightarrow E(AB) = 0$

$E(A) = E(B) = 0$ & $E[A^2] = E[B^2] = k$ (say)

To show $X(t)$ & $Y(t)$ are jointly W.S.S.

$$\left[Var(A) = E[A^2] - E[A]^2 = E[A^2] \right]$$

(i) $X(t)$ is WSS

(ii) $Y(t)$ is WSS

(iii) $R_{XY}(\tau) \to$ a function of τ.

(i) $E[X(t)] = E[A\cos wt + B\sin wt]$

$= E(A)\cos wt + E(B)\sin wt$

$= 0$

$R_{XX}(\tau) = E[X(t) \cdot X(t+\tau)]$

$= E[(A\cos wt + B\sin wt)(A\cos w(t+\tau) + B\sin w(t+\tau))]$

$= E[A^2\cos wt \cdot \cos w(t+\tau) + B^2\sin wt \sin(wt+\tau)$

$+ AB(\sin wt \cos(wt+\tau) + \cos wt \sin w(t+\tau))]$

$= k(\cos wt \cos w\overline{t+\tau} + \sin wt \sin w\overline{t+\tau})$

$= k\cos w\tau$

$\Rightarrow X(t)$ is WSS

(i) $E[Y(t)] = E[B\sin wt - A\sin wt]$

$= k\,E[\cos wt \cos(wt+\tau) + \sin wt \sin \overline{wt+\tau}]$

$= k \cos w\tau$

$\Rightarrow$ $Y(t)$ is WSS

(iii) $R_{xy}(\tau) = E[X(t)\, Y(t+\tau)]$

$= E[(A\cos wt + B\sin wt)(B\cos(\overline{wt+\tau}) - A\sin(\overline{wt+\tau}))]$

$= E[AB\cos wt \cos w\overline{t+\tau} - A^2 \cos wt \sin w\overline{t+\tau} + B^2 \sin wt \cos w\overline{t+\tau} - AB \sin wt \sin w\overline{t+\tau})$

$= k[\sin wt \cos \overline{wt+\tau} - \cos wt \sin w(t+\tau)]$

$= k \sin(-w\tau)$

$\Rightarrow$ $X(t)$ & $Y(t)$ are jointly WSS

Ergodicity

& Def: A random process $\{X(t)\}$ is said to be ergodic if its ensemble average (statistical averages) (i.e.) mean, autocorrelation) are equal to appropriate time averages

Def: If $x(t)$ is a random process, then $\frac{1}{2T}\int_{-T}^{T} x(t)\,dt$ is called the time average of $x(t)$ over $(-T, T)$ and

Problem procedure:

Step 1 Find $\overline{X_T}$

Step 2: Find $E[\overline{X_T}]$

Step 3: Variance of $(\overline{X_T})$ or

$$Var[\overline{X_T}] = \frac{1}{T}\int_{-T}^{T} C_{XX}(\tau)\left[1-\frac{|\tau|}{T}\right]d\tau$$

where $C_{XX}(\tau) = E[X(t)X(t+\tau)] - E[X(t)]\,E[X(t+\tau)]$

Step 4 $\lim_{T\to\infty} Var(\overline{X_T}) = 0$

Def: $X(t)$ is Correlation ergodic if

$$\overline{Z_T} = \frac{1}{2T}\int_{-T}^{T} X(t+\tau)\,X(t)\,dt$$

$= R(\tau)$ as limit $T\to\infty$

1. If the WSS process is given by $X(t) = 10\cos(100t+\theta)$ where θ is uniformly distributed $(-\pi,\pi)$ prove that $X(t)$ is Correlation ergodic

Sol Since θ is uniformly distributed.

$$f(\theta) = \frac{1}{\pi-(-\pi)} = \frac{1}{2\pi}$$

Problem procedure:

Step 1 Find $\overline{X_T}$

Step 2: Find $E[\overline{X_T}]$

Step 3: Variance of $(\overline{X_T})$ or

$$Var[\overline{X_T}] = \frac{1}{T}\int_{-T}^{T} C_{XX}(\tau)\left[1-\frac{|\tau|}{T}\right]d\tau$$

where $C_{XX}(\tau) = E[X(t)X(t+\tau)] - E[X(t)]\,E[X(t+\tau)]$

Step 4 $\lim_{T\to\infty} Var(\overline{X_T}) = 0$

Def: $X(t)$ is Correlation ergodic if

$$\overline{Z_T} = \frac{1}{2T}\int_{-T}^{T} X(t+\tau)X(t)\,dt$$

$= R(\tau)$ as limit $T\to\infty$

1. If the WSS process is given by $X(t) = 10\cos(100t+\theta)$ where θ is uniformly distributed $(-\pi,\pi)$ prove that $X(t)$ is Correlation ergodic

Sol Since θ is uniformly distributed.

$$f(\theta) = \frac{1}{\pi-(-\pi)} = \frac{1}{2\pi}$$

$$\lim_{T\to\infty} \bar{Z}_T = 50\cos 100\tau$$

$$\therefore R_{XX}(\tau) = \lim_{T\to\infty} \bar{Z}_T = 50\cos 100\tau$$

$X(t)$ is a Correlation ergodic

4 Consider the two random processes $X(t) = 3\cos(\omega t+\theta)$ and $Y(t) = 2\cos(\omega t+\theta-\frac{\pi}{2})$ where θ is a Random Variable uniformly distributed in $(0, 2\pi)$. Prove that

$$\sqrt{R_{XX}(0)R_{YY}(0)} \geq |R_{XY}(\tau)|$$

<u>Soln</u> Given $X(t) = 3\cos(\omega t+\theta)$

$$Y(t) = 2\cos(\omega t+\theta-\frac{\pi}{2})$$

Since θ is uniformly distributed in $(0, 2\pi)$

$$f(\theta) = \frac{1}{2\pi}$$

$$R_{XX}(\tau) = E[X(t)X(t+\tau)]$$

$$= E[3\cos(\omega t+\theta)\cdot 3\cos(\omega(t+\tau)+\theta)]$$

$$= 9E[\cos(\omega t+\theta)\cos(\omega t+\omega\tau+\theta)]$$

$$= \frac{9}{2}E[\cos(2\omega t+\omega\tau+2\theta)+\cos\omega\tau]$$

$$= \frac{9}{2}\{E\cos(2\omega t+\omega\tau+\theta)+E(\cos\omega\tau)\}$$

$R_{XX}(0) = \frac{9}{2}$

$R_{YY}(\tau) = E[Y(t) \cdot Y(t+\tau)]$

$= E[2\cos(\omega t + \theta - \frac{\pi}{2}) \, 2\cos(\omega(t+\tau) + \theta - \frac{\pi}{2})]$

$= 4E[\cos(\omega t + \theta - \frac{\pi}{2}) \cos(\omega t + \omega\tau + \theta - \frac{\pi}{2})]$

$= \frac{4}{2} E[\cos(2\omega t + \omega\tau + 2\theta - \pi) + \cos(\omega\tau)]$

$= 2[E\cos(2\omega t + \omega\tau + 2\theta - \pi) + E[\cos\omega\tau]]$

$E[\cos(2\omega t + \omega\tau + 2\theta - \pi)] = \int_0^{2\pi} \cos(2\omega t + \omega\tau + 2\theta - \pi) \cdot \frac{1}{2\pi} \, d\theta$

$= \frac{1}{2\pi} \left[\frac{\sin(2\omega t + \omega\tau + 2\theta - \pi)}{2} \right]_0^{2\pi}$

$= 0$

$R_{YY}(\tau) = 0 + \frac{4}{2}\cos\omega\tau$

$= 2\cos\omega\tau$

$R_{YY}(0) = 2$

$R_{XY}(\tau) = E[X(t) \cdot Y(t+\tau)]$

$= E[3\cos(\omega t + \theta) \cdot 2\cos(\omega(t+\tau) + \theta - \frac{\pi}{2})]$

$= 6E[\cos(\omega t + \theta)\cos(\omega t + \omega\tau + \theta - \frac{\pi}{2})]$

$= \frac{6}{2} E[\cos(2\omega t + \omega\tau + 2\theta - \frac{\pi}{2}) + \cos(\omega\tau - \frac{\pi}{2})]$

$E[\cos(2\omega t+\omega\tau+2\theta-\frac{\pi}{2})]=0$

$$R_{xy}(\tau)=0+3\cos(\omega\tau-\frac{\pi}{2})$$

$$R_{xy}(\tau)=3\sin\omega\tau \longrightarrow (3)$$

$$\sqrt{R_{xx}(0)R_{yy}(0)}=\sqrt{\frac{9}{2}\cdot 2}=\sqrt{9}=3$$

$$|R_{xy}(\tau)|=|3\sin\omega\tau|\leq 3.$$

$$\Rightarrow R_{xy}(\tau)\leq\sqrt{R_{xx}(0)R_{yy}(0)}$$

Markov Process:

Def:

If $P[X_n=a_n / X_{n-1}=a_{n-1}, X_{n-2}=a_{n-2} \ldots X_0=a_0]$

$=P[X_n=a_n / X_{n-1}=a_{n-1}]$, then the process $\{X_n\}$

$n=0,1,2\ldots$ is called Markov chain.

Note: (i) one step transition probability

(ie) $P[X_n=a_j / X_{n-1}=a_i]=P_{ij}^{(1)}$

(ii) n step transition probability

$P[X_n=a_j / X_0=a_i]=P_{ij}^{(n)}$

(iii) The tpm [transition probability matrix] of a Markov chain is a stochastic matrix, since $P_{ij}\geq 0$ &

(v) To find Steady State distribution (or) Stationary State distribution (or) long run probability distribution (or) long run probability distribution we $\pi P = \pi$ where $\Sigma \pi = 1$

(vi) If every State Can be reached from every other State, then Markov chain is Said to be irreducible

(vii) State i is Said to be periodic with period d_i if $d_i > 1$.

State i is said to be aperiodic if $d_i = 1$

(viii) A State i is Said to be recurrent if the return to State i is Certain (ie) $\Sigma P_i = 1$

The State i is Said to be transient if the return to State i is uncertain

(ix) If a Markov chain is irreducible then all States are of the Same type

If the Markov Chain is finite irreducible then all States are non-null persistent.

If a State is non-null persistent and aperiodic then it is ergodic.

Type 1

Finding the nature of the Matrix.

Given $P = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix}$

$$P^2 = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

$$P^3 = P^2 \cdot P = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix}$$

$$P^4 = P^3 \cdot P = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

(i) $P_{00}^{(2)} > 0$, $P_{02}^{(2)} > 0$, $P_{11}^{(2)} > 0$, $P_{20}^{(2)} > 0$, $P_{22}^{(2)} > 0 \Rightarrow$

Markov chain is finite irreducible

(ii) State 0

$P_{00}^{(2)} > 0$, $P_{00}^{(4)} > 0 \ldots$

$\Rightarrow \gcd\{2, 4, \ldots\} = 2$

$\Rightarrow$ period of state 0 is 2.

(iii) State 1

$P_{11}^{(2)} > 0$, $P_{11}^{(4)} > 0 \ldots$

$\Rightarrow \gcd\{2, 4, \dots\} = 2$

$\Rightarrow$ period of state 2 is 2.

Here Markov chain is finite irreducible $\Rightarrow$

All states are non-null persistent

Here, All states are non-null persistent & period 2

$\Rightarrow$ all states are not ergodic

2. A man either drives a car (or) catches a train to go to office each day. he never goes two days in a row by train, but if he drives one day, then the next day he is just as likely to drive again as he is to travel by train. Now suppose that on the first day of the week, the man tossed a fair die and drove to work if and only if a six appear, Find (i) The probability that he takes the train on the third day (ii) The probability that he drives to work in the long run.

Soln Given

$$P = \begin{array}{c|cc} & T & C \\ \hline T & 0 & 1 \\ C & \frac{1}{2} & \frac{1}{2} \end{array}$$

Initial probability distribution is

$$P^{(1)} = \begin{array}{cc} T & C \\ \left(\frac{5}{6}\right. & \left.\frac{1}{6}\right) \end{array}$$

$P^{(2)} \quad P^{(1)} P$

$\Rightarrow P[\text{Train on the third day}] = \frac{11}{24}$

(ii) To find: long run distribution

$\pi P = \pi$ where $\sum \pi = 1$

$$(\pi_1, \pi_2)\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = (\pi_1 \; \pi_2)$$

$$\Rightarrow \left[\frac{\pi_2}{2} \quad \pi_1 + \frac{\pi_2}{2}\right] = [\pi_1 \; \pi_2]$$

$$\Rightarrow \frac{\pi_2}{2} = \pi_1 \qquad \pi_1 + \frac{\pi_2}{2} = \pi_2$$

$$\Rightarrow \pi_1 = \frac{\pi_2}{2}$$

Sub the relation in $\pi_1 + \pi_2 = 1$

$$\Rightarrow \frac{\pi_2}{2} + \pi_2 = 1$$

$$\Rightarrow \frac{3\pi_2}{2} = 1$$

$$\Rightarrow \pi_2 = 2/3$$

$$\Rightarrow \pi_1 = 1 - 2/3 = 1/3$$

$$\therefore \text{Probability (long run distribution)} = \begin{array}{cc} T & C \\ \left(\frac{1}{3}\right. & \left.\frac{2}{3}\right) \end{array}$$

$\Rightarrow P[\text{Drive in the long run}] = 2/3$

3. The transition probability matrix of a

Soln

(i) $P[X_2=3] = \sum_{i=1}^{3} P[X_2=3 / X_0=i] \cdot P[X_0=i]$

$= P[X_2=3 / X_0=1]\ P[X_0=1] +$

$P[X_2=3 / X_0=2]\ P[X_0=2] +$

$P[X_2=3 / X_0=3] \cdot P[X_0=3]$

$= P_{13}^{(2)} P[X_0=1] + P_{23}^{(2)} P[X_0=2] + P_{33}^{(2)} P[X_0=3]$

$$P^2 = \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix} \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$$

$$= \begin{bmatrix} 0.43 & 0.31 & 0.26 \\ 0.24 & 0.42 & 0.34 \\ 0.36 & 0.35 & 0.29 \end{bmatrix}$$

$P[X_2=3] = (0.26)(0.7) + (0.34)(0.2) + (0.29)(0.1)$

$= 0.279$

(ii) $P[X_3=2, X_2=3, X_1=3, X_0=2]$

$= P[X_3=2 / X_2=3, X_1=3, X_0=2] \cdot P[X_2=3, X_1=3, X_0=2]$

$= P[X_3=2 / X_2=3]\ P[X_2=3 / X_1=3, X_0=2]$
$P[X_1=3, X_0=2]$

$= P_{32}^{(1)} P[X_2=3 / X_1=3]\ P[X_1=3 / X_0=2]\ P[X_0=2]$

$= P_{32}^{(1)} \cdot P_{33}^{(1)} P_{23}^{(1)} P[X_0=2]$

Poisson Process

Def: If $x(t)$ represents the number of occurrances of a certain event in $(0,t)$. then discrete random process $\{x(t)\}$ is called Poisson process. if (i) Probability or

$P[\text{one happening in } (t, t+\Delta t)] = \lambda\,\Delta t + o(\Delta t)$

(ii) $P[\text{zero happening in } (t, t+\Delta t)] = (1-\lambda\,\Delta t) + o(\Delta t)$

(iii) $P[\text{two happening in } (t, t+\Delta t)] = o(\Delta t)$

(iv) $x(t)$ is independent

(v) The probability value depends on t but not Δt.

Derivation:

(i) Let λ be the number of occurrances of the event in unit time

$$\text{Let } P_n(t) = P[x(t) = n]$$

$$\text{Now } P_n(t+\Delta t) = P_n(t)[1-\lambda\Delta t] + P_{n-1}(t)(\lambda\Delta t)$$

$$= P_n(t) - \lambda P_n(t)\,\Delta t + P_{n-1}(t)\,\lambda\,\Delta t$$

$$\Rightarrow \frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = \lambda[P_{n-1}(t) - P_n(t)]$$

As $\Delta t \to 0$, $P_n'(t) = \lambda[P_{n-1}(t) - P_n(t)] \longrightarrow (1)$

Let the soln of $P_n(t) = f(t)\cdot\frac{(\lambda t)^n}{n!}$

$$\frac{f'(t)}{f(t)} = -\lambda$$

Integrating, we get

$$\log(f(t)) = -\lambda t + c$$

$$\Rightarrow f(t) = e^{-\lambda t} \cdot k \quad \text{---} (2)$$

W.K.T $P_0(0) = f(0) = 1$

when $t = 0$

$$(2) \Rightarrow k = 1$$

$$\therefore f(t) = e^{-\lambda t}$$

$\therefore$ The solution is $P_n(t) = \dfrac{e^{-\lambda t}(\lambda t)^n}{n!}$, $n = 0, 1, 2 \ldots$

(i) Find the mean & autocorrelation of the Poisson Process

Sol (i) Mean $E[X(t)] = \sum X(t) P[X(t)]$

$$= \sum n P(n)$$

$$= \sum n \cdot e^{-\lambda t} \cdot \frac{(\lambda t)^n}{n!}$$

$$= e^{-\lambda t} \sum_{n=0}^{\infty} \left[\frac{(\lambda t)^n}{n!}\right] \cdot n$$

$$= e^{-\lambda t} \cdot (\lambda t) \sum_{n=0}^{\infty} \frac{(\lambda t)^{n-1}}{(n-1)!}$$

$$= e^{-\lambda t} \cdot \lambda t \cdot e^{\lambda t} = \lambda t$$

$$R_{XX}(t_1,t_2) = E[X(t)]\, E[X(t_2) - X(t_1)] + E[X^2(t)]$$

$$= \lambda t_1 [\lambda t_2 - \lambda t_1] + (\lambda t_1 + \lambda^2 t_1^2)$$

$$= \lambda^2 t_1 t_2 - \cancel{\lambda^2 t_1^2} + \lambda t_1 + \cancel{\lambda^2 t_1^2}$$

$$= \lambda^2 t_1 t_2 + \lambda t_1$$

$$R_{XX}(t_1,t_2) = \lambda^2 t_1 t_2 + \lambda \min(t_1,t_2)$$

Find the Covariance & Correlation of Poisson process

Soln (i) Covariance

$$C_{XX}(t_1,t_2) = R_{XX}(t_1,t_2) - E[X(t_1)]\, E[X(t_2)]$$

$$= [\lambda^2 t_1 t_2 + \lambda \min(t_1,t_2)] - (\lambda t_1, \lambda t_2)$$

$$= \lambda \min(t_1,t_2)$$

(ii) $$R_{XX}(t_1,t_2) = \frac{C_{XX}(t_1,t_2)}{\sqrt{Var[X(t)]}\sqrt{Var(X(t))}}$$

$$= \frac{\lambda \min(t_1,t_2)}{\sqrt{\lambda t_1, \lambda t_2}} = \sqrt{\frac{t_1}{t_2}}$$

1. If $\{X(t)\}$ is a Poisson Process, prove that

$$P\{X(s)=r \mid X(t)=n\} = nC_r \left(\frac{s}{t}\right)^r \left(1-\frac{s}{t}\right)^{n-r} \text{ where } s<t$$

Sol Consider

Since $\{X(t)\}$ is a Poisson Process, we get

$$P\{X(s)=r \mid X(t)=n\} = \frac{\dfrac{e^{-\lambda s}(\lambda s)^r}{r!} \; \dfrac{e^{-\lambda(t-s)}[\lambda(t-s)]^{n-r}}{(n-r)!}}{\dfrac{e^{-\lambda t}(\lambda t)^n}{n!}}$$

$$= \frac{n!}{r!(n-r)!} \frac{(\lambda s)^r [\lambda(t-s)]^{n-r}}{(\lambda t)^n} \frac{e^{-\lambda s} \cdot e^{\lambda s} e^{-\lambda t}}{e^{-\lambda t}}$$

$$= nC_r \frac{s^r (t-s)^{n-r}}{t^n} \cdot \frac{\lambda^r \cdot \lambda^n \cdot \lambda^{-r}}{\lambda^n}$$

$$= nC_r \cdot \frac{s^r \, t^{n-r} \left(1-\frac{s}{t}\right)^{n-r}}{t^n}$$

$$= nC_r \frac{s^r \, t^{n-r} \left(1-s/t\right)^{n-r}}{t^r \cdot t^{n-r}}$$

$$= nC_r \left(\frac{s}{t}\right)^r \left(1-s/t\right)^{n-r}$$

Unit IV

Queueing Theory

Model: I

Single Server Poisson Queue: (M/M/1) : (∞/FIFO)

a) The average arrival rate is constant, $\lambda_n = \lambda \ \forall n$

b) The average service rate is constant $\mu_n = \mu \ \forall n$

c) The average arrival rate is less than the average service rate $\lambda < \mu$ which assumes than an infinite queue will not form.

$$P_n = \frac{\lambda^n}{\mu^n} P_0$$

$$= \left(\frac{\lambda}{\mu}\right)^n P_0$$

where $P_0 = 1 - \frac{\lambda}{\mu}$

$$\therefore P_n = \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right)$$

Note

(1) $P_0 = 1 - \frac{\lambda}{\mu}$ denote the Probability of a system being idle (or) the system is free.

(2) $\rho = \frac{\lambda}{\mu}$ (traffic intensity (or) utilization factor)

1. Average number of Customers in the system

3. Average Number of Customers in Nonempty Queues (L_w)

$$L_w = \frac{\mu}{\mu-\lambda}$$

4. Probability that the number of Customers in the System exceeds k.

$$P(\text{number of Customers in the System} > k) = \left(\frac{\lambda}{\mu}\right)^{k+1}$$

Probability that the System is busy $= 1 - P_0 = \frac{\lambda}{\mu}$

Probability that the System is empty $= P_0 = 1 - \frac{\lambda}{\mu}$

5. Probability Density function of the waiting time in the System.

$$f(w) = (\mu-\lambda)\, e^{-(\mu-\lambda)w}$$

6. Average waiting time of a Customer in the System (W_s)

$$W_s = \frac{1}{\mu-\lambda}$$

7. Probability that the waiting time of a Customer in the System exceeds t.

9. Average waiting time of a Customer in the Queue (W_q)

$$W_q = \frac{\lambda}{\mu(\mu-\lambda)}$$

10. Probability that the waiting time in the Queue Exceeds t.

$$P[W_q > t] = \frac{\lambda}{\mu} e^{-(\mu-\lambda)t}$$

11. Average waiting time of a Customer in the Queue, if che has to wait (W_w)

$$W_w = \frac{1}{\mu-\lambda}$$

Problems on Model-I

$(M|M|1):(\infty|FCFS)$

1. A Self Service Store has one Cashier at its Counter. Nine Customers arrive on an average 5 minutes while the Cashier Can Serve 10 Customers in 5 minutes. Assuming Poisson Distribution for arrival Rate and exponential distribution for Service rate. find (i) Average number of Customers in the System. (ii) Average number of Customers in queue or average queue length (iii) Average time a Customer waits before being Served (iv) Average time a Customer Spend in the System.

Sol one Cashier → Single channel

$$L_s = \frac{\lambda}{\mu-\lambda}$$

$$= \frac{1.8}{2-1.8} = \frac{1.8}{0.2} = 9 \text{ Customers.}$$

(ii) Average number of Customers in the queue

$$L_q = \frac{\lambda^2}{\mu(\mu-\lambda)}$$

$$= \frac{(1.8)^2}{2(2-1.8)}$$

$$= 8.1$$

$$= 8 \text{ Customers.}$$

(iii) Average time a Customer waits in the queue before being Served

$$W_q = \frac{\lambda}{\mu} \cdot \frac{1}{\mu-\lambda}$$

$$= \frac{1.8}{2} \cdot \frac{1}{(2-1.8)}$$

$$= 4.5 \text{ minutes}$$

(iv) Average time a Customer Spends in the System

$$W_s = \frac{1}{\mu-\lambda}$$

$$= \frac{1}{2-1.8}$$

and the Service time is also exponential with an average 36 minutes. Calculate the following

(i) The mean queue size

(ii) The average number of trains in the queue

(iii) The probability that the queue size exceeds 10.

b) If the input of trains increases to an average 33 per day, what will be the change in (i) & (ii)

Soln One yard → Single channel

Arrival of trains → any number (∞)

Model → $(M/M/1) : (\infty/FCFS)$

Mean arrival rate, $\lambda = 30$ trains/day

$$= \frac{30}{60 \times 24} = \frac{1}{48} \text{ trains/minute}$$

Mean Service time $\frac{1}{\mu} = 36$ minutes

Mean Service rate $\mu = \frac{1}{36}$ trains/minutes

a)(i) The mean queue size = the number of trains in the system (L_s)

$$L_s = \frac{\lambda}{\mu - \lambda} = \frac{\frac{1}{48}}{\frac{1}{36} - \frac{1}{48}} = \frac{1}{48} \cdot \frac{1}{\frac{1}{12}\left(\frac{1}{3} - \frac{1}{4}\right)}$$

$$= \frac{1}{4} \cdot \frac{1}{(4-3)}$$

$$= \frac{36}{48} \cdot \frac{1}{48} \cdot \frac{1}{\frac{1}{12}\left(\frac{1}{3}-\frac{1}{4}\right)}$$

$$= \frac{3}{48} \times \frac{1}{4} \cdot \frac{1}{\frac{1}{12}\left(\frac{1}{12}\right)} = \frac{3}{4 \cdot \cancel{4}} \cdot \cancel{12}^{3}$$

$$= \frac{9}{4}$$

$$= 2.25$$

$= 2.25$ trains $= 2$ trains (approximately)

(iii) $P(\text{queues Size} > k) = \left(\frac{\lambda}{\mu}\right)^{k+1}$

$$P(\text{queues Size} > 10) = \left(\frac{\frac{1}{48}}{\frac{1}{36}}\right)^{10+1} = \left(\frac{3}{4}\right)^{10+1} = (0.75)^{11}$$

$$\simeq 0.0422$$

b) If the Input of trains increase to an average 33 per day then

Mean arrival Rate $\lambda = \frac{33}{60 \times 24}$

$$= \frac{11}{480} \text{ trains/minutes}$$

Mean Service Rate, $\mu = \frac{1}{36}$ trains/minute

(iii) % of ladies who has to wait prior to getting into the Chair for hair designs

Soln one beautician → Single channel

Arrival of ladies → any number (∞)

Model → $(M/M/1):(\infty/FCFS)$

Mean Arrival Rate, $\lambda = 5$ person in 1 hour

$= \frac{1}{12}$ per minute

Mean Service time $\frac{1}{\mu} = 10$ minutes

Mean Service rate $\mu = \frac{1}{10}$ per minutes

(i) Average number of ladies in the shop.

$$L_s = \frac{\lambda}{\mu - \lambda}$$

$$= \frac{\frac{1}{12}}{\frac{1}{10} - \frac{1}{12}} = \frac{\frac{1}{12}}{\frac{12-10}{120}} = \frac{1}{12} \times \frac{120}{2}$$

$= 5$ Customers.

(ii) Average number of ladies in the queue

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{\lambda}{\mu} \times \frac{\lambda}{\mu - \lambda}$$

$$= \frac{\frac{1}{12}}{\frac{1}{10}} \times 5 = \frac{10}{12} \times 5$$

$$= \frac{11}{40}\left(\frac{1}{\frac{40-33}{120}}\right)$$

$$= \frac{11}{40} \cdot \frac{120}{7} = \frac{33}{7}$$

$= 4.7$

$\simeq 5$ trains / minutes

$\therefore$ Change in queue Size $= 4.7 - 2.25$

$= 2.45$

$= 2$ trains (approximately)

(ii) $P(\text{queue Size} \geq 10) = \left(\frac{\lambda}{\mu}\right)^{10+1} = \left(\frac{\frac{11}{480}}{\frac{1}{36}}\right)^{10+1}$

$$= \left(\frac{33}{40}\right)^{11}$$

$= (0.825)^{11}$

$= 0.1205$ trains

At a Beauty Parlour Shop, with one beautician, Ladies arrive according to Poisson distribution with mean arrival rate of 5 per hour and hair design was exponentially distributed with an average design taking 10 minutes. As it is a very good parlour, Customers do have

$$P(\text{queue Size} \geq 1) = P(N \geq 1)$$

$$= \left(\frac{1/12}{1/10}\right)^1$$

$$= \frac{10}{12}$$

$$= 0.833$$

(iv) % ladies who have to wait $= 0.833 \times 100 = 83.3$

% ladies who can walk without Waiting $\Big\} = 100 - 83.3$

$= 16.7\%$.

Model II $(M/M/c) : (\infty/FCFS)$

~~probability that an~~

1. Average number of Customers in the Queue (L_q)

$$L_q = \frac{1}{c!\,c}\left(\frac{\lambda}{\mu}\right)^{c+1} \left\{\frac{1}{\left(1 - \frac{\lambda}{\mu c}\right)^2}\right\} P_0$$

2. Average Number of Customers in the System (L_s)

$$L_s = \frac{1}{c!\,c}\left(\frac{\lambda}{\mu}\right)^{c+1} \left(\frac{1}{\left(1 - \frac{\lambda}{\mu c}\right)^2}\right) P_0 + \frac{\lambda}{\mu}$$

3. Average time a Customer Spends in the System (W_s)

$$W_s = \frac{1}{\mu} + \frac{1}{\mu\, c!}\left(\frac{\lambda}{\mu}\right)^{c} \frac{1}{\left(1 - \frac{\lambda}{\mu c}\right)^2} P_0$$

Probability that an arrival has to wait $P[N \geq c]$

$$P[W_s > 0] = P(N \geq c] = \frac{\left(\frac{\lambda}{\mu}\right)^c P_0}{c!\left(1-\frac{\lambda}{\mu c}\right)}$$

Probability that an arrival has to get the Service without waiting.

$$P(\text{system is idle}) = 1 - \frac{\left(\frac{\lambda}{\mu}\right)^c P_0}{c!\left(1-\frac{\lambda}{\mu c}\right)}$$

Probability that Some one will be waiting

$$P(N \geq c+1) = \frac{\left(\frac{\lambda}{\mu}\right)^c \frac{\lambda}{\mu c} P_0}{c!\left(1-\frac{\lambda}{\mu c}\right)}$$

Mean waiting time in the Queue for those who actually wait. (L_w)

$$E[W_q / W_s] = \frac{1}{\mu c - \lambda}$$

Average Number of Customers who have to actually wait

$$L_w = \frac{\lambda}{\mu c - \lambda}.$$

1. A supermarket has two Servers Serving at Counter. The Customer arrive in a poisson fashion at the rate of

Soln

2 Servers → Multiple channel

Arrival of Customers → any number (∞)

Model → $(M/M/c) : (\infty/FCFS)$

No of Servers $c = 2$

Mean arrival rate $\lambda = 10$ per hour

Mean arrival rate $\mu = 4$ minutes per Customer

$= 15$ persons per hour.

$$P_0 = \frac{1}{\sum_{n=0}^{c-1} \frac{\left(\frac{\lambda}{\mu}\right)^n}{n!} + \frac{\left(\frac{\lambda}{\mu}\right)^c}{c!} \cdot \frac{c\mu}{c\mu - \lambda}}$$

$$= \frac{1}{\sum_{n=0}^{2-1} \frac{\left(\frac{\lambda}{\mu}\right)^n}{n!} + \frac{\left(\frac{\lambda}{\mu}\right)^2}{2!} \cdot \frac{2\mu}{2\mu - \lambda}}$$

$$= \frac{1}{\sum_{n=0}^{1} \frac{\left(\frac{\lambda}{\mu}\right)^n}{n!} + \frac{\left(\frac{\lambda}{\mu}\right)^2}{2!} \times \frac{2\mu}{2\mu - \lambda}}$$

$$= \frac{1}{\left[\sum_{n=0}^{1} \frac{\left(\frac{2}{3}\right)^n}{n!} + \frac{\left(\frac{2}{3}\right)^2}{2!} \times \frac{2(15)}{2(15)-10}\right]}$$

$$= \frac{1}{\quad}$$

Probability of Customer has to wait for the Service

$$P(W_s>0) = \frac{\left(\frac{\lambda}{\mu}\right)^c}{c!\left(1-\frac{\lambda}{\mu c}\right)} P_0$$

$$= \frac{\left(\frac{2}{3}\right)^2}{2!\times\left(1-\frac{1}{3}\right)} \times 0.5 = \frac{1}{6}$$

(ii) Average queue length

$$L_q = \frac{1}{c!\,c}\left(\frac{\lambda}{\mu}\right)^{c+1} \frac{1}{\left(1-\frac{\lambda}{\mu c}\right)^2} P_0$$

$$= \frac{1}{2!\times 2}\left(\frac{2}{3}\right)^3 \frac{1}{\left(1-\frac{10}{15\times 2}\right)^2} \times 0.5$$

$$= \frac{1}{4} \times \frac{8}{27} \times \frac{1}{\left(\frac{30-10}{30}\right)^2} \times 0.5$$

$$= \frac{1}{4} \times \frac{8}{27} \times \frac{9}{4} \times \frac{1}{2} = \frac{1}{12}.$$

$= 0.083$ Customer.

(iii) Average waiting time in the queue

$$W_q = \frac{L_q}{\lambda} = \frac{0.083}{10} = 0.0083 \text{ hours.}$$

Soln Three typist → Multiple Channel

Arrived letters → any number (∞)

Model → $(M/M/c):(\infty/FCFS)$

Mean arrival Rate, $\lambda = 15$ per hour.

Mean Service Rate, $\mu = 6$ per hour.

No of Server, $c = 3$

$$\frac{\lambda}{\mu} = \frac{15}{6} = \frac{5}{2}$$

$$P_0 = \frac{1}{\sum_{n=0}^{c-1} \frac{\left(\frac{\lambda}{\mu}\right)^n}{n!} + \frac{\left(\frac{\lambda}{\mu}\right)^c}{c!} \times \frac{c\mu}{c\mu - \lambda}}$$

$$= \frac{1}{\sum_{n=0}^{2} \frac{\left(\frac{5}{2}\right)^n}{n!} + \frac{\left(\frac{5}{2}\right)^3}{3!} \times \frac{18}{18-15}}$$

$$= \frac{1}{\frac{\left(\frac{5}{2}\right)^0}{0!} + \frac{\left(\frac{5}{2}\right)^1}{1!} + \frac{\left(\frac{5}{2}\right)^2}{2!} + \frac{125}{8} \times \frac{6}{3!}}$$

$$= \frac{1}{1 + \frac{5}{2} + \frac{25}{8} + \frac{125}{8}}$$

$$= \frac{\left(\frac{5}{2}\right)^3 \times 0.0449}{6\left(1 - \frac{1}{3}\cdot\frac{15}{6}\right)}$$

$$P(N \geq 3) = 0.7016$$

(2) The average number of letters waiting to be typed

$$L_q = E[N_q]$$

$$= \frac{1}{c!\,c}\left(\frac{\lambda}{\mu}\right)^{c+1} \frac{1}{\left(1-\frac{\lambda}{\mu c}\right)^2} P_0$$

$$= \frac{1}{2!\,3}\left(\frac{5}{2}\right)^4 \frac{1}{\left(1-\frac{5}{2}\left(\frac{1}{3}\right)\right)^2} \times 0.0449$$

$$= 0.5078$$

Model : 3 $(M/M/1):(N/FIFO)$

(i) Probability that the System is idle

$$P_0 = \frac{1-\rho}{1-\rho^{N+1}} \quad \because \rho = \frac{\lambda}{\mu}$$

(ii) Average Number of Customers in the System

$$L_s = \frac{N}{2} \text{ for } \lambda = \mu$$

(iii) Average Number of Customers in the

(v) Average waiting time in the queue

$$W_q = \frac{L_q}{\lambda'}$$

1. Trains arrive at the yard every 15 minutes and the Service time is 33 minutes. If the line Capacity at the yard is limited to trains find

(i) Probability that the yard is empty

(ii) the average number of trains in the system.

Soln One yard - Single change

Arrival of trains → 4 (finite)

Model → (M/M/1) : (N/FCFS)

Mean arrival rate, $\lambda = \frac{1}{15}$ per min

Mean Service time $\frac{1}{\mu} = \frac{1}{33}$ per min

Capacity of the System $N = 4$

$$\rho = \frac{\lambda}{c\mu} = \frac{33}{15} = 2.2$$

(i) probability (yard is empty)

$$P_0 = \frac{1-\rho}{1-\rho^{N+1}}$$

$$= \frac{1-2.2}{1-(2.2)^5}$$

$$L_s = \frac{\rho}{1-\rho} - \frac{(N+1)\rho^{N+1}}{1-\rho^{N+1}}$$

$$= \frac{2\cdot 2}{1-2\cdot 2} - \frac{(4+1)(2\cdot 2)^{4+1}}{1-(2\cdot 2)^{4+1}}$$

$$= 3\cdot 26.$$

In a Railway Marshalling yard, goods trains arrive at the rate of 30 trains per day. Assume that the Interarrival time follows an exponential distribution & the Service time is also to be assumed an exponential with mean of 36 minutes. Calculate (i) the probability that the yard is empty (ii) the average queue length assuming the line capacity of the yard is 9 trains.

Soln

one yard → Single channel

Capacity of the yard → 9 (finite)

Model → (M/M/1) : (N/FCFS)

Mean arrival Rate $\lambda = \frac{30}{60 \times 24} = \frac{1}{48}$ per minutes

Mean Service time $\frac{1}{\mu} = 36$ minutes

Mean Service rate $\mu = \frac{1}{36}$ per minute

Capacity of the System $N = 9$

P(the yard is empty)

$$P_0 = \frac{1-\rho}{1-\rho^{N+1}}$$

$$= \frac{1-0.75}{1-(0.75)^{10}}$$

$$= \frac{0.25}{0.94}$$

$$= 0.27$$

(ii) Average queue length.

$$L_s = \frac{\rho}{1-\rho} - \frac{(N+1)\rho^{N+1}}{1-\rho^{N+1}}$$

$$= \frac{0.75}{1-0.75} - \frac{(9+1)(0.75)^{9+1}}{1-(0.75)^{9+1}}$$

$$= \frac{0.75}{0.25} - \frac{10(0.75)^{10}}{1-(0.75)^{10}}$$

$$= \frac{0.75}{0.25} - \frac{10(0.0563)}{1-0.0563}$$

$$= 2.4$$

3. In a Single Server queueing system with Poisson input and exponential Service times, if the mean arrival rate is 3 callers units per hour, the expected Service time

Soln One Server - Single Channel

Maximum number of Calling units } - 2 (finite

Model → (M/M/1) : (N/FCFS)

Mean arrival Rate, $\lambda = 3$ units/hour.

Mean Service time $\frac{1}{\mu} = 0.25$

Mean Service rate, $\mu = \frac{1}{0.25} = 4$ units/hour.

Capacity of the system $N = 2$

$$\rho = \frac{\lambda}{\mu} = 3/4 = 0.75$$

$$P_0 = \frac{1-\rho}{1-\rho^{N+1}}$$

$$= \frac{1-0.75}{1-(0.75)^3}$$

$$= 0.4324$$

$$P_n = \frac{1-\rho}{1-\rho^{N+1}} \cdot \rho^N = (0.43)(0.75)^n$$

(i) Average number of Calling units in the system

(iii) Average number of Calling units in the Sequence

$$L_q = L_s - \frac{\lambda'}{\mu}$$

$$\lambda' = \mu(1-P_0) = 4(1-0.4324)$$
$$= 2.27$$

$$L_q = 0.81 - \frac{2.27}{4} = 0.24.$$

Model IV $(M/M/C):(k/FCFS)$

Formulas are

(i) $P_0 = \left[\sum_{n=0}^{C-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c \sum_{n=c}^{k}\left(\frac{\lambda}{\mu c}\right)^{n-c}\right]^{-1}$

(ii) $P_n = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0, \quad n \le c$

(iii) $P_n = \frac{1}{c!\, c^{n-c}}\left(\frac{\lambda}{\mu}\right)^n P_0$ for $c \le n \le k$.

(iv) $\rho = \frac{\lambda}{\mu c}$

(v) $L_q = P_0 \left(\frac{\lambda}{\mu}\right)^c \frac{\rho}{c!(1-\rho)^2}\left[1-\rho^{k-c} - (k-c)(1-\rho)\rho^{k-c}\right]$

(vi) $L_s = L_q + c - \left[\sum_{n=0}^{c-1}(c-n)P_n\right]$

(vii) $W_s = \frac{L_s}{\lambda'}$

Example!

A group of engineers has 2 terminals available to aid in their Calculations. The average Computing job requires 20 min of terminal time and each engineer requires some computation distributed according to an exponential distribution. If there are 6 engineers in the group, find

a) the expected number of engineers waiting to use one of the terminals and in the Computing Centre and

b) the total time lost per day

Soln Two terminals → Multiple channel

Engineers → 6 (finite)

model → (M/M/C) : (k/FCFS)

Mean arrival rate, $\lambda = 2$ per hour.

Mean Service rate, $\mu = 3$ per hour.

$C = 2,\ k = 6$

a) $$P_0 = \left[\sum_{n=0}^{C-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{C!}\left(\frac{\lambda}{\mu}\right)^C \sum_{n=C}^{k} \left(\frac{\lambda}{\mu C}\right)^{n-C}\right]^{-1}$$

$$= \left(\sum_{n=0}^{1} \frac{1}{n!}\left(\frac{2}{3}\right)^n + \frac{1}{2!}\left(\frac{2}{3}\right)^2 \sum_{n=2}^{6} \left(\frac{2}{3\times 2}\right)^{n-2}\right)^{-1}$$

$$= \left[\left\{1 + \frac{2}{3} + \frac{1}{2}\times\left(\frac{2}{3}\right)^2 \left\{1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^4\right\}\right.\right]^{-1}$$

$$L_q = P_0 \times \left(\frac{\lambda}{\mu}\right)^c \times \frac{\rho}{c!(1-\rho)^2}\left[1-\rho^{k-c}-(k-c)(1-\rho)\rho^{k-c}\right]$$

Where $\rho = \frac{\lambda}{\mu c} = \frac{2}{2 \cdot 3} = \frac{1}{3}$

$$L_q = 0.5003 \times \left(\frac{2}{3}\right)^2 \times \frac{\frac{1}{3}}{2!\,(1-\frac{1}{3})^2}\left[1-\left(\frac{1}{3}\right)^{6-2}-(6-2)\left(1-\frac{1}{3}\right)\left(\frac{1}{3}\right)^{6-2}\right]$$

$$= 0.5003 \times \left(\frac{2}{3}\right)^2 \times \frac{\frac{1}{3}}{2 \times \left(\frac{2}{3}\right)^2}\left[1-\left(\frac{1}{3}\right)^4 - 4 \times \frac{2}{3} \times \left(\frac{1}{4}\right)^4\right]$$

$$= 0.5003 \times \frac{4}{9} \times \frac{3}{8}\left[1-\frac{1}{81}-\frac{8}{243}\right]$$

$$= 0.0796$$

$$L_s = L_q + c - \left(\sum_{n=0}^{c-1}(c-n)P_n\right)$$

$$= 0.0796 + 2 - \left(\sum_{n=0}^{1}(2-n)P_n\right)$$

$$= 2.0796 - (2P_0 + P_1)$$

$$= 2.0796 - \left(2.(0.5003) + \frac{2}{3}(0.5003)\right)$$

$$= 2.0796 - \frac{8}{3}(0.5003)$$

$$= 2.0796 - 1.3341$$

$$\lambda' = \mu\left[c - \sum_{n=0}^{c-1}(c-n)P_n\right]$$

$$= 3\left[2 - \sum_{n=0}^{1}(2-n)P_n\right]$$

$$= 3\left[2 - (2P_0 + P_1)\right]$$

$$= 3\left[2 - \left(2\times 0.5003 + \frac{2}{3}\times 0.5003\right)\right]$$

$$= 1.9976$$

$$W_q = \frac{L_q}{\lambda'} = \frac{0.0796}{1.9976} = 0.0398\ hr$$

Every time an engineer approaches the Computer Centre he has to lose 0.0398 hr by way of waiting.

If the day consists of 8 working hours, he has to approach the Centre 16 times

∴ Time lost in waiting in a day per engineer

$= 16\times 0.0398 = 0.6368$ hr

∴ Total time lost in waiting in a day by all the 6 Engineers $= 6\times 0.6368$

$= 0.82$ hrs.

2. A 2 person barber shop has 5 chairs to accommodate waiting customers. Potential customers, who arrive when

Sol: Two barbers → Multiple channel

Available chairs → 5 (finite)

Model → $(M/M/c):(k/FCFS)$

Arrival rate $\lambda = 4$ per hour.

Service rate $\mu = 5$ per hour

$$C = 2$$

$$k = 2+5 = 7$$

$$\rho = \frac{\lambda}{\mu} = \frac{4}{5} = 0.8$$

a) $$P_0 = \left[\sum_{n=0}^{c-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c \sum_{n=c}^{k}\left(\frac{\lambda}{\mu c}\right)^{n-c}\right]^{-1}$$

$$= \left[\sum_{n=0}^{1} \frac{1}{n!}\left(\frac{4}{5}\right)^n + \frac{1}{2}\left(\frac{4}{5}\right)^2 * \sum_{n=2}^{7}\left(\frac{2}{5}\right)^{n-2}\right]^{-1}$$

$$= \left[1+\frac{4}{5}+\frac{8}{25}\left\{1+\frac{2}{5}+\left(\frac{2}{5}\right)^2+\left(\frac{2}{5}\right)^3+\left(\frac{2}{4}\right)^4+\left(\frac{2}{5}\right)^5\right\}\right]^{-1}$$

$$= 0.4289$$

b) $P_n = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0$ for $n \le c$

$P_1 = \frac{4}{5} \times 0.4289$

$[\because P_0 = 0.4297)$

d) $L_q = P_0 \times \left(\frac{\lambda}{\mu}\right)^c \times \frac{\rho}{c!\,(1-\rho)^2}\left[1-\rho^{k-c}-(k-c)(1-\rho)\,\rho^{k-c}\right)$

where $\rho = \frac{\lambda}{\mu C} = \frac{4}{10} = 0.4$

$= (0.4299)\,(0.8)^2\,\frac{(0.4)}{2\times(0.6)^2}\left[1-(0.4)^5 - 5\times 0.6\times(0.4)^5\right)$

$= 0.15$ Customer.

e) $L_s = L_q + c - \sum_{n=0}^{c-1}(c-n)\,P_n$

$= 0.1462 + 2 - \sum_{n=0}^{1}(2-n)\,P_n$

$= 2.1426 - (2\cdot P_0 + 1\cdot P_1)$

$= 2.1426 - (2\times 0.4287 + 1\times 0.3430)$

$= 0.95$

$=$ 1 Customer approximately

Now $\lambda' = \mu\left[c - \sum_{n=0}^{c-1}(c-n)\,P_n\right]$

$= 4\left[2-(2\times 0.4287 + 1\times 0.3430)\right]$

$= 3.1984$

Unit V

Non Markovian Queues and Queue Networks

Pollaczek - Khinchine formula:

$$L_s = \lambda E[T] + \frac{\lambda^2 [Var(T) + [E[T]]^2]}{2(1-\lambda E(T))}$$

where $Var(T) = E[T^2] - \{E[T]\}^2$

Characteristics for M|G|1 Model.

$$E[T] = \frac{1}{\mu} \ \& \ \lambda E[T] = \frac{\lambda}{\mu} = \rho$$

$$Var(T) = \sigma^2$$

$$L_s = \frac{\lambda}{\mu} + \frac{\lambda^2 [\sigma^2 + \frac{1}{\mu^2}]}{2(1-\frac{\lambda}{\mu})}$$

$$= \rho + \frac{\lambda^2 \sigma^2 + \rho^2}{2(1-\rho)}$$

Little's Formula:

$$L_q = L_s - \frac{\lambda}{\mu}$$

$$W_q = \frac{L_q}{\lambda}$$

Parking lot if the bay is busy. The parking lot is large enough to accommodate any number of cars. Find the average number of cars waiting in the parking lot, if the time for washing and cleaning a car follows

a) Uniform distribution between 8 and 12 minutes

b) a normal distribution with mean 12 minutes and S.D 3 minutes

c) a discrete distribution with values equal to 4, 8 and 15 minutes and corresponding probabilities 0.2, 0.6, 0.2.

Sol

Mean $\lambda = 4/\text{hour}$

$= \frac{4}{60}$ per minutes

$= \frac{1}{15}$ per minute

a) $E(T) =$ mean of the uniform distribution

$= \frac{1}{2}(a+b)$

$= \frac{1}{2}(8+12)$

$= \frac{1}{2}(20)$

$= 10$

$\mu = \frac{1}{E(T)}$

$= \frac{1}{10}$ per minutes

$$\sigma^2 = \frac{1}{12}(4)^2$$

$$= \frac{1}{12}(16)$$

$$= \frac{4}{3}$$

By the pollaczek - khinchine formula

$$L_s = \lambda E(T) + \frac{\lambda^2 [Var(T) + E(T)^2]}{2(1-\lambda E(T))}$$

$$= \frac{1}{15} \cdot 10 + \frac{\left(\frac{1}{15}\right)^2 \left[\frac{4}{3} + (10)^2\right]}{2\left(1 - \frac{1}{15} \cdot 10\right)}$$

$$= \frac{2}{3} + \frac{\frac{1}{225}\left(\frac{4}{3} + 100\right)}{2\left(1 - \frac{2}{3}\right)}$$

$$= \frac{2}{3} + \frac{\frac{1}{225} \cdot \frac{304}{3}}{2\left(\frac{1}{3}\right)}$$

$$= \frac{2}{3} + \frac{1}{225} \cdot \frac{\overset{152}{\cancel{304}}}{\cancel{3}} \cdot \frac{\cancel{3}}{\cancel{2}_1}$$

$$= \frac{2}{3} + \frac{152}{225}$$

$$= \frac{150 + 152}{225}$$

$$= \frac{302}{}$$

$$L_q = L_s - \frac{\lambda}{\mu}$$

$$= 1.342 - \frac{1}{15} \Big/ \frac{1}{10}$$

$$= 1.342 - 2/3$$

$$= 0.675 \text{ } \cancel{\text{chairs}} \text{ Cars}$$

$$\simeq 1 \text{ Cars}$$

(b) $\lambda = \frac{1}{15}$ per minute, $E(T) = 12$ min &

$Var(T) = 9, \Rightarrow \sigma^2 = 9$ (S.D = 3)

$$\mu = \frac{1}{E(T)} = \frac{1}{12}$$

By P.K. formula.

$$L_s = \lambda \cdot E(T) + \frac{\lambda^2 \{Var(T) + (E(T))^2\}}{2(1 - \lambda E(T)]}$$

$$= \frac{1}{15} \cdot 12 + \frac{\left(\frac{1}{15}\right)^2 (9 + 12^2)}{2\left(1 - \frac{1}{15} \cdot 12\right)}$$

$$= \frac{4}{5} + \frac{\frac{1}{225} \cdot 153}{2\left(\frac{3}{15}\right)}$$

$$= \frac{4}{5} + \frac{153}{225} \cdot \frac{15}{6}$$

$$= \frac{4}{5} + \frac{153}{90}$$

$$L_q = L_s - \frac{\lambda}{\mu}$$

$$= 1.342 - \frac{1}{15}\Big/\frac{1}{10}$$

$$= 1.342 - 2/3$$

$= 0.675$ ~~chairs~~ Cars

≈ 1 Cars

(b) $\lambda = \frac{1}{15}$ per minute, $E(T) = 12$ min &

$Var(T) = 9, \Rightarrow \sigma^2 = 9 \quad (S.D = 3)$

$$\mu = \frac{1}{E(T)} = \frac{1}{12}$$

By P.K. formula.

$$L_s = \lambda \cdot E(T) + \frac{\lambda^2 \{Var(T) + (E(T))^2\}}{2[1 - \lambda E(T)]}$$

$$= \frac{1}{15} \cdot 12 + \frac{\left(\frac{1}{15}\right)^2 (9 + 12^2)}{2\left(1 - \frac{1}{15} \cdot 12\right)}$$

$$= \frac{4}{5} + \frac{\frac{1}{225} \cdot 153}{2\left(\frac{3}{15}\right)}$$

$$= \frac{4}{5} + \frac{153}{225} \cdot \frac{15}{6}$$

$$= \frac{4}{5} + \frac{153}{90}$$

$= 0.573 + 0.451$

$= 1.024$

$\simeq 1$ Car.

By Little's formula

$L_q = L_s - \frac{\lambda}{\mu}$

$= 1.024 - \frac{1/15}{1/8.1}$

$= 1.025 - \frac{8.1}{15}$

$= 0.45$ Car.

2. Automatic Car wash facility operates with only one bay. Cars arrive according to a Poisson process with the mean of 4 cars per hour and may wait in the facility's parking lot if the bay is busy. If the Service times for all cars is Constant and equal to 10 minutes. Determine L_s, L_q, W_s, W_q

Sol

$\lambda = 4$ cars per hour

$= \frac{4}{60}$ per minute

$= \frac{1}{15}$ per minutes

By P.K. formula

$= \frac{10}{15} \times \frac{100}{\cancel{225}_{15}} \times \frac{\cancel{15}^{1}}{\cancel{10}}$

$= \frac{10}{15} + \frac{10}{15}$

$= \frac{20}{15} = \frac{4}{3}$

$= 1.333$ Cars

$L_q = L_s - \frac{\lambda}{\mu}$

$= 1.333 - \frac{1/15}{1/10}$

$= 1.333 - 10/15$

$= 0.667$ Car

By Little formula

$W_s = \frac{L_s}{\lambda} = \frac{4/3}{1/15} = 15 \times \frac{4}{3} = 20$ minutes

$W_q = \frac{L_q}{\lambda}$

$= \frac{0.667}{1/15} = 0.667 \times 15$

$= 10.005$ minutes

Networks:

enter the System from outside and after Service at one or more queues, eventually leave the system.

b) Closed Networks

A closed queueing network does not have any external arrivals (or) departures. It represents a situation where a fixed number of jobs circulate in the system, moving from one queue to the next, getting served at individual queues. No jobs enter the system nor does any job leave the system.

Two Stage Series Queue

$P_{mn} = P$(m customers in the I station and n customers in the II station)

$$= \left(\frac{\lambda}{\mu_1}\right)^m \left(1-\frac{\lambda}{\mu_1}\right)\left(\frac{\lambda}{\mu_2}\right)^n \left(1-\frac{\lambda}{\mu_2}\right).$$

1. In an oncology clinic, there are two sections, the first section for knowing the basic complaints of the patients for writing it in the prescription and the other one for suggesting further treatment. Patient arrive at the oncology clinic in a Poisson fashion at the rate of 3 per hour. The doctor in the first section takes nearly

find the average number of patients in the entire clinic and average waiting time of a patient in the clinic.

Soln Mean arrival rate $\lambda = 3$/hour.

Service time in the Doctor Section $= \frac{1}{\mu_1} = 15$ minutes

$= \frac{1}{4}$ hr

$\therefore$ Service rate in the Doctor Section, $\mu_1 = 4$/hr

Service time in the chief Doctor Section $= \frac{1}{\mu_2} = 6$ minutes

$= \frac{1}{10}$ hours.

$\therefore$ Service rate in the Doctor Section $\mu_2 = 10$/hr

(i) we know that

$P_{mn} = P$(m customers in the I station & n Customers in the II Station)

$$= \left(\frac{\lambda}{\mu_1}\right)^m \left(1-\frac{\lambda}{\mu_1}\right) \left(\frac{\lambda}{\mu_2}\right)^n \left(1-\frac{\lambda}{\mu_2}\right)$$

$$P_{32} = \left(\frac{3}{4}\right)^3 \left(1-\frac{3}{4}\right) \left(\frac{3}{10}\right)^2 \left(1-\frac{3}{10}\right)$$

$$= \frac{27}{64} \cdot \frac{1}{4} \cdot \frac{9}{100} \cdot \frac{7}{10}$$

$$= \frac{1701}{\ }$$

$$= \frac{\lambda}{\mu_1 - \lambda} + \frac{\lambda}{\mu_2 - \lambda}$$

$$= \frac{3}{4-3} + \frac{3}{10-3}$$

$$= 3 + \frac{3}{7}$$

$$= \frac{24}{7} = 3.42 \simeq 3$$

2) Average waiting time of a patient in the Clinic W_s

$$= W_{s_1} + W_{s_2}$$

(Station 1) + (Station 2)

$$= \frac{1}{\mu_1 - \lambda} + \frac{1}{\mu_2 - \lambda}$$

$$= \frac{1}{4-3} + \frac{1}{10-3}$$

$$= 1 + \frac{1}{7}$$

$$= 8/7 \text{ hours}$$

$$= \frac{480}{7} \text{ minutes}$$

$$= 68.6 \text{ minutes (approximately)}$$

2. A TVS Company in Chennai Containing a repair Section shared by a large number of machines has 2 sequential stations with respective Service Rates of 3 per hour and 4 per hour. The Cumulative failure rate of all machines is 1 per hour. Assuming that the system

Soln Cumulative failure rate $\lambda = 1/hr$

Service rate of Station I $\mu_1 = 3/hr$

Service rate of Station II $\mu_2 = 4/hr$

(i) P(m customers in the I station & n customers in the II station)

$$P_{mn} = \left(\frac{\lambda}{\mu_1}\right)^m \left(1-\frac{\lambda}{\mu_1}\right)\left(\frac{\lambda}{\mu_2}\right)^n\left(1-\frac{\lambda}{\mu_2}\right)$$

$$P_{00} = \left(\frac{1}{3}\right)^0\left(1-\frac{1}{3}\right)\left(\frac{1}{4}\right)^0\left(1-\frac{1}{4}\right)$$

$$= 2/3 \times \frac{3}{4}$$

$$= \frac{1}{2}$$

(ii) Average repair time

$$= \frac{1}{\mu_1-\lambda} + \frac{1}{\mu_2-\lambda}$$

$$= \frac{1}{3-1} + \frac{1}{4-1}$$

$$= \frac{1}{2} + \frac{1}{3}$$

$$= \frac{5}{6} \text{ hours}$$

$$= 50 \text{ minutes}$$

(iii) Since $\frac{\lambda}{\mu_1} = \frac{1}{3}$, $\frac{\lambda}{\mu_2} = \frac{1}{4}$.

issues the bill. On finishing the job with any one of the bill Collectors, the Customer goes to the next Section where the Supervisor issues the items needed for the Customers. The Customers enter the hotel at a Poisson fashion at the rate of 6 per hour. Each bill Collector takes 15 minutes for noting the items and issuing the bill in an exponential fashion. The Supervisor issues the items who takes on the average 3 minutes per Customer to pack and give it to the Customers.

(1) Find the average number of Customers in the hotel

(2) Find the average waiting time of a customer in the hotel

Soln

Mean arrival rate $\lambda = 6/\text{hour}$

Service time of first section $= \frac{1}{\mu_1} = 15$ minutes

Service rate of first section $\mu_1 = \frac{1}{15}/\text{min} = 4/\text{hr}$

Service time of second section $= \frac{1}{\mu_2} = 3$ minutes

$\therefore$ Service rate of second section, $\mu_2 = \frac{1}{3}/\text{min} = 20/\text{hr}$

$C = 3$

(i) we know that

$$P_0 = \left[\sum_{n=0}^{C-1} \frac{1}{n!}\left[\frac{\lambda}{\mu_1}\right]^n + \frac{1}{C!}\left(\frac{\lambda}{\mu_1}\right)^C\left(\frac{C}{C-\frac{\lambda}{\mu_1}}\right)\right]^{-1}$$

$$= \left[\sum_{n=0}^{2} \frac{1}{n!}\left(\frac{6}{4}\right)^n + \frac{1}{3!}\left(\frac{6}{4}\right)^3\left(\frac{3}{3-\frac{6}{4}}\right)\right]^{-1}$$

$$P_0 = \left[1+\frac{6}{4}+\frac{9}{8}+\frac{9}{8}\right]^{-1}$$

$$= \left[\frac{38}{8}\right]^{-1}$$

$$= 0.21$$

Average number of Customers in the billing section

$$L_{s_1} = \frac{\lambda\mu_1\left(\frac{\lambda}{\mu_1}\right)^c}{(c-1)!\,(c\mu_1-\lambda)^2}P_0+\frac{\lambda}{\mu_1}$$

$$= \frac{(6)(4)\left(\frac{3}{2}\right)^3}{2!\,(12-6)^2}\cdot 0.21+\frac{3}{2}$$

$$= 0.24+1.5$$

$$= 1.74$$

Average number of Customers in the dispatch Section

$$L_{s_2} = \frac{\lambda}{\mu_2-\lambda} = \frac{6}{20-6} = \frac{6}{14}$$

Average number of Customers in the System (L_s)

$$L_s = L_{s_1}+L_{s_2}$$

$$= 1.74+0.43$$

$$= 2.17$$

$$W_s = \frac{L_{s1}}{\lambda} + \frac{L_{s2}}{\lambda}$$

$$= \frac{1.74}{6} + \frac{0.43}{6}$$

$$= \frac{2.17}{6}$$

$$= 0.362 \, hr$$

$$= 21.7 \, min$$

Series Queues with blocking:

If a customer is in Station 2 and Service is Completed at Station 1, the Customers at Station 1 Should wait unit the Service is Completed for Station 2 Customers (The system is blocked).

Steady State Probabilities

The possible States for this model are given below

n_1, n_2	Detail
0, 0	System is empty
1, 0	Customer in process at station 1 & Station 2 is empty
0, 1	Station 1 is empty and Customer in process at station 2.
1, 1	Customers in ~~empty~~ ~~and custom~~

$$\lambda P_{0,0} = \mu_2 P_{0,1}$$

$$\mu_1 P_{1,0} = \mu_2 P_{1,1} + \lambda P_{0,0}$$

$$(\lambda+\mu_2) P_{0,1} = \mu_1 P_{1,0} + \mu_2 P_{b,1}$$

$$(\mu_1+\mu_2) P_{1,1} = \lambda P_{0,1}$$

$$\mu_2 P_{b,1} = \mu_1 P_{1,1}$$

Since $P_{0,0} + P_{1,1} + P_{0,1} + P_{b,1} = 1$

$$P_{0,1} = \frac{\lambda P_{0,0}}{\mu_2}$$

$$P_{1,1} = \frac{\lambda^2 P_{0,0}}{\mu_2(\mu_1+\mu_2)}$$

$$P_{1,0} = \frac{\mu_2[\lambda^2 + \lambda(\mu_1+\mu_2)] P_{0,0}}{\mu_1\mu_2(\mu_1+\mu_2)}$$

$$= \frac{[\lambda^2 + \lambda(\mu_1+\mu_2)] P_{0,0}}{\mu_1[\mu_1+\mu_2]}$$

$$P_{b,1} = \frac{\mu_1 \lambda^2 P_{0,0}}{\mu_2^2(\mu_1+\mu_2)}$$

Solving problems on Series Queues with blocking

1. There are 2 salesman in a Supermarket out of the 2 salesman one is incharge of billing and receiving payment

Process at rate of 1 per hr and the Service times of 2 clerks are independent & have exponential rates of 3/hr and 2/hr. find

(i) The proportion of Customers who enter the Supermarket

(ii) The average number of Customers in the Supermarket

(iii) The average amount of time a Customer spends in the Shop

Soln

Arrival rate $\lambda = 1/hr$

Service rate of I Salesman $\mu_1 = 3/hr$

" " " II Salesman $\mu_2 = 2/hr$

$$\lambda P_{0,0} = \mu_2 P_{0,1}$$

$$\mu_1 P_{1,0} = \mu_2 P_{1,1} + \lambda P_{0,0}$$

$$(\lambda + \mu_2) P_{0,1} = \mu_1 P_{1,0} + \mu_2 P_{b,1}$$

$$(\mu_1 + \mu_2) P_{1,1} = \lambda P_{0,1}$$

$$\mu_2 P_{b,1} = \mu_1 P_{1,1}$$

$\lambda = 1$, $\mu_1 = 3$ $\mu_2 = 2$

$$P_{0,0} = 2P_{0,1}$$

$$3P_{1,0} = P_{0,0} + 2P_{1,1}$$

$$3P_{0,1} = 3P_{1,0} + 2P_{b,1}$$

$$P_{1,1} = \frac{1}{5}P_{0,1} = \frac{1}{10}P_{0,0}$$

$$P_{b,1} = \frac{3}{2}P_{1,1} = \frac{3}{20}P_{0,0}$$

$$P_{1,0} = \frac{1}{3}P_{0,0} + \frac{2}{3}P_{1,1}$$

$$= \frac{1}{3}P_{0,0} + \frac{2}{30}P_{0,0}$$

$$P_{1,0} = \frac{2}{5}P_{0,0}$$

$$P_{0,0} + \frac{2}{5}P_{0,0} + \frac{1}{2}P_{0,0} + \frac{1}{10}P_{0,0} + \frac{3}{20}P_{0,0} = 1$$

$$P_{0,0}\left[1 + 2/5 + 1/2 + 1/10 + \frac{3}{20}\right] = 1$$

$$P_{0,0}\left[\frac{20+8+10+2+3}{20}\right] = 1$$

$$P_{0,0}\left[\frac{43}{20}\right] = 1$$

$$P_{0,0} = \frac{20}{43}$$

$$P_{0,1} = \frac{10}{43}$$

$$P_{1,0} = \frac{8}{43}$$

$$P_{1,1} = \frac{2}{43}$$

$$P_{b,1} = 3/43$$

(i) proportion of Customers entering the shop

(ii) The average number of Customers in the Super market is given by

$$L_s = 0 \cdot P_{0,0} + 1\,(P_{1,0} + P_{0,1}) + 2(P_{1,1} + P_{b,1})$$

$$= \frac{10}{43} + \frac{8}{43} + 2\left(\frac{2}{43} + \frac{3}{43}\right)$$

$$= \frac{28}{43}$$

(iii) Average waiting time that a Customer Spends in the System

$$W_s = \frac{L_s}{\lambda_A},$$

λ_A = average rate of Customers entering the Supermarket

$$= \lambda\,(P_{0,0} + P_{0,1})$$

$$= 1 \times \frac{30}{43}$$

$$W_s = \frac{28}{43} \Big/ \frac{30}{43} = \frac{14}{15} \text{ hours (or) } \frac{14}{15} \times 60 \text{ minutes}$$

$$= 56 \text{ minutes}$$

2. There are 2 Salesman in a Supermarket. out of the 2 Salesman one is incharge of ~~wei~~ billing & receiving payment while the other Salesman is incharge of

(i) The proportion of Customers who enter the Supermarket

(ii) The average number of Customers in the Supermarket

(iii) The average waiting time a Customer Spends in the Shop.

Soln Mean arrival Rate $\lambda = 5$/hour

Service time of I Salesman = 6 minutes = $1/10$ hour.

Service Rate of I Salesman $\mu_1 = 10$/hr.

Service time of II Salesman = 6 minutes = $1/10$ hr

Service Rate of II Salesman $\mu_2 = 10$/hr

Steady State equations are

$$\lambda P_{0,0} = \mu_2 P_{0,1}$$

$$\mu_1 P_{1,0} = \mu_2 P_{1,1} + \lambda P_{0,0}$$

$$(\lambda + \mu_2) P_{0,1} = \mu_1 P_{1,0} + \mu_2 P_{b,1}$$

$$(\mu_1 + \mu_2) P_{1,1} = \lambda P_{0,1}$$

$$\mu_2 P_{b,1} = \mu_1 P_{1,1}$$

Sub $\lambda = 5$, $\mu_1 = 10$, $\mu_2 = 10$

$$5 P_{0,0} = 10 P_{0,1}$$

$$10 P_{1,0} = 5 P_{0,0} + 10 P_{1,1}$$

$$15 P_{0,1} = 10 P_{1,0} + 10 P_{b,1}$$

$$20 P_{1,1} = 5 P_{0,1}$$

$$P_{b,1} = P_{1,1} = \frac{1}{8} P_{0,0}$$

$$P_{1,0} = \frac{1}{2} P_{0,0} + P_{1,1}$$

$$= \frac{1}{2} P_{0,0} + \frac{1}{8} P_{0,0}$$

$$= \frac{5}{8} P_{0,0}$$

$$P_{0,0} + \frac{5}{8} P_{0,0} + \frac{1}{2} P_{0,0} + \frac{1}{8} P_{0,0} + \frac{1}{8} P_{0,0} = 1$$

$$P_{0,0} \left[1 + \frac{5}{8} + \frac{1}{2} + \frac{1}{8} + \frac{1}{8} \right] = 1$$

$$P_{0,0} \left[\frac{19}{8} \right] = 1$$

$$P_{0,0} = \frac{8}{19}.$$

$$P_{0,1} = \frac{1}{2} \cdot \frac{8}{19} = \frac{4}{19}$$

$$P_{1,0} = \frac{5}{8} \cdot \frac{8}{19} = \frac{5}{19}.$$

$$P_{1,1} = \frac{1}{8} \times \frac{8}{19} = \frac{1}{19}$$

$$P_{b,1} = \frac{1}{8} \times \frac{8}{19} = \frac{1}{19}$$

(i) proportion of Customers entering the Shop

$$= P_{0,0} + P_{0,1}$$

$$= \frac{8}{19} + \frac{4}{19}$$

$$L_s = \left(\frac{5}{19}+\frac{4}{19}\right) + 2\left(\frac{1}{19}+\frac{1}{19}\right)$$

$$= \frac{9}{19}+\frac{4}{19}$$

$$= \frac{13}{19}.$$

(iii) Average waiting time that a customer spends in the system

$$W_s = \frac{L_s}{\lambda_A},$$

λ_A = Average rate of entering of customers in the Super market

$$= \lambda\left(P_{0,0}+P_{0,1}\right)$$

$$= 5\left[\frac{8}{19}+\frac{4}{19}\right] = \frac{60}{19}$$

$$\therefore W_s = \frac{\frac{13}{19}}{\frac{60}{19}} = \frac{13}{60} \text{ hr}$$

$$= \frac{13}{60}\times 60 = 13 \text{ minutes}$$

Jackson Networks:

1. Arrivals from outside through node i follow a Poisson Process with mean arrival rate r_i

2. Service times at each channel at node i are independent

Open Jackson Networks:

All other general cases where $r_i \neq 0$ for any i or $r_{i0} \neq 0$ for any i are referred to as open Jackson Networks.

Open Networks where

$$r_i = \begin{cases} \lambda & i=1 \\ 0 & \text{elsewhere} \end{cases}$$

& $$r_{ij} = \begin{cases} 1 & j=i+1,\ 1 \le i \le k-1 \\ 1 & i=k,\ j=0 \\ 0 & \text{elsewhere.} \end{cases}$$

1. In a book shop there are 2 sections, one for Engineering books and the other section for Mathematics books. There is only one salesman in each ~~Station~~ section. Customers from outside arrive at the Engineering book section at a Poisson rate of 4 per hour and at the Mathematics book section at a Poisson rate of 3 per hour. The service rates of the Engineering book section and Mathematics book section are 8 and 10 per hour respectively. A customer after service at Engineering book section is equally likely to go to the Mathematics book section or to leave the book shop. However a customer upon completion of service at Mathematics book section will go the Engineering book section with probability $\frac{1}{3}$ and will leave the book-shop otherwise. Find the

(i) ... steady state probability that there are 3 customers

Soln

S_1 = Engineering book section

S_2 = Mathematics book section

λ_1 = Total arrival rate of S_1

λ_2 = Total arrival rate of S_2.

To find λ_1 and λ_2

From the Jackson's flow balance equation for a open Queueing network, we get

$$\lambda_i = r_j + \sum_{i=1}^{k} \lambda_i p_{ij} \quad j = 1, 2 \ldots k$$

$$\lambda_j = r_j + \sum_{i=1}^{2} \lambda_i p_{ij} \quad (k=2) \longrightarrow (1)$$

when $j=1$

$$(1) \Rightarrow \lambda_1 = r_1 + \sum_{i=1}^{2} \lambda_i p_{i1}$$

$$\Rightarrow \lambda_1 = r_1 + \lambda_1 p_{11} + \lambda_2 p_{21}$$

r_1 = arrival rate of the I section

$= 4/hr$

$p_{11} = 0$ (customer after visiting the Engineering book section will not enter again)

$p_{21} = 1/3$ (probability of a customer entering the Engineering book section after entering the Mathematics book section)

$\lambda_1 = 4 + \frac{1}{3}\lambda_2$

r_2 = arrival rate of the II Section

$= 3/hr$

But $P_{22} = 0$ & $P_{12} = \frac{1}{2}$

$$\lambda_2 = 3 + \frac{1}{2}\lambda_1 \Rightarrow -\lambda_1 + 2\lambda_2 = 6$$

$$-\lambda_1 + 2\lambda_2 = 6$$

(1)×(2) $\Rightarrow 6\lambda_1 - 2\lambda_2 = 24$

(2)+(3) $\Rightarrow 5\lambda_1 = 30$

$$\lambda_1 = 6$$

Sub $\lambda_1 = 6$ in (1) we get

$$18 - \lambda_2 = 12 \Rightarrow \lambda_2 = 6$$

Total arrival rate at Station 1 (S_1) $\lambda_1 = 6/hr$

Total arrival rate at Station 2 (S_2) $\lambda_2 = 6/hr$

Total Service rate at Station 1 $(S_1) \Rightarrow \mu_1 = 8/hr$

Total Service rate at Station 2 $(S_2) \Rightarrow \mu_2 = 10/hr$

P (m Customers in the Station S_1 and n Customers in the Station S_2)

$$P_{mn} = \left(\frac{\lambda_1}{\mu_1}\right)^m \left(1 - \frac{\lambda_1}{\mu_1}\right) \left(\frac{\lambda_2}{\mu_2}\right)^n \left(1 - \frac{\lambda_2}{\mu_2}\right)$$

$$P_{32} = \left(\frac{6}{8}\right)^3 \left(1 - \frac{6}{8}\right) \left(\frac{6}{10}\right)^2 \left(1 - \frac{6}{10}\right)$$

(2) Average number of Customers in the System

L_s = (Average number of Customers in the Engineering Section) + (Average number of Customers in the Mathematical Section)

$= L_{s_1} + L_{s_2}$

$= \frac{\lambda}{\mu_1 - \lambda_1} + \frac{\lambda_2}{\mu_2 - \lambda_2}$

$= \frac{6}{8-6} + \frac{6}{10-6} = 3 + 1.5$

$= 4.5$

3) Average waiting time of a Customer in the system

$W_s = \frac{L_s}{\lambda}$, $\lambda = r_1 + r_2 = 4 + 3 = 7$

$W_s = \frac{4.5}{7} = 0.64$ hr

$= 38.5$ minutes

2. In a Textile shop, there are 2 Sections, one for silk Sarees and the other Section for Dhoties. There is only one Salesman in each Section. Customers from outside arrive at the silk Sarees Section at a Poisson rate of 10/hour and at the Dhoties Section at a poisson rate of 8/hour. The Service rates

(i) Jointly Steady State probability that there are 1 Customers in the Silk Sarees Section and 2 in the Dhoties Section.

(ii) Average number of Customers in the Shop

(iii) Average waiting time of a Customer in the Shop

Soln S_1 = Saree Section

S_2 = Dhoties Section

λ_1 = Total arrival Rate at S_1

λ_2 = Total arrival rate at S_2

To find λ_1 and λ_2.

$$\lambda_j = r_j + \sum_{i=1}^{k} \lambda_i p_{ij} \quad j=1,2 \ldots k$$

Here $k=2$.

$$\lambda_j = r_j + \sum_{i=1}^{2} \lambda_i p_{ij} \quad \text{—— (A)}$$

put $j=1$ in (A),

$$\lambda_1 = r_1 + \sum_{i=1}^{2} \lambda_i p_{i1}$$

put $j=1$ in (A),

$$\lambda_1 = r_1 + \sum_{i=1}^{2} \lambda_i p_{i1}$$

$$\lambda_1 = r_1 + \lambda_1 p_{11} + \lambda_2 p_{21}$$

But r_1 = arrival Rate at the Saree Section

$= 10/hr$

$p_{11} = 0$

$\lambda_2 = r_2 + \lambda_1 p_{12} + \lambda_2 p_{22}$

r_2 = arrival rate at the dhobi's section

$= 8/hr$

$p_{22} = 0$ & $p_{12} = \frac{1}{2}$

$\lambda_2 = 8 + \frac{1}{2}\lambda_1$ (or) $-\lambda_1 + 2\lambda_2 = 16 \longrightarrow (2)$

$$6\lambda_1 - 2\lambda_2 = 60$$
$$-\lambda_1 + 2\lambda_2 = 16$$
$$5\lambda_1 = 76$$
$$\lambda_1 = \frac{76}{5}$$

Sub $\lambda_1 = \frac{76}{5}$ in (1) we get

$$\frac{228}{5} - \lambda_2 = 30$$
$$\lambda_2 = \frac{78}{5}$$

Total arrival rate at Station 1, $\lambda_1 = \frac{76}{5}$ /hr

Total arrival rate at " 2, $\lambda_2 = \frac{78}{5}$ /hr

Total Service rate at Station 1, $\mu_1 = 18$/hr

" " " " " 2, $\mu_2 = 26$/hr

$$P_{mn} = \left(\frac{\lambda_1}{\mu_1}\right)^m \left(1 - \frac{\lambda_1}{\mu_1}\right) \left(\frac{\lambda_2}{\mu_2}\right)^n \left(1 - \frac{\lambda_2}{\mu_2}\right)$$

(1) $P_{12} = \frac{38}{45}\left(1 - \frac{38}{45}\right)\left(\frac{39}{52}\right)^2\left(1 - \frac{39}{52}\right)$

2) Average number of Customers in the textile shop

$$L_s = \frac{\lambda_1}{\mu_1-\lambda_1} + \frac{\lambda_2}{\mu_2-\lambda_2}$$

$$= \frac{\frac{76}{5}}{18-\frac{76}{5}} + \frac{\frac{78}{5}}{20-\frac{78}{5}}$$

$$= \frac{76}{14} + \frac{78}{22}$$

$$= 5.4285 + 3.5454$$

$$= 8.9739$$

$L_s = 9$ customers

3) Average waiting time of a customer in the textile shop

$$W_s = \frac{L_s}{\lambda}, \quad \lambda = r_1 + r_2 = 10+8 = 18$$

$$\therefore W_s = \frac{9}{18} = \frac{1}{2} = 0.5 \text{ hr} = 30 \text{ minutes}$$

Closed Jackson Networks:

Cases for which $r_i = 0$ for all i (ie) no customers enter the system from outside) and $r_{i0} = 0$

There are 2 clerks in a bank, one processing Educational loan and the other processing ATM Cards applications. While processing, they get doubts according to an exponential distribution each with a mean of $\frac{1}{2}$. To get clarification, a clerk goes to the deputy manager with probability $\frac{3}{4}$ and to the Senior manager with probability $\frac{1}{4}$. After completing the job with Deputy Manager, a clerk goes to senior manager with probability $\frac{1}{3}$ and returns to his seat otherwise. On completing the job with Senior Manager a clerk always returns to his seat. If the deputy manager clarifies the doubts and advises a clerk according to an exponential distribution with parameter 1 and the Senior manager with parameter 3. Find

(i) the steady state probabilities P_{n_1, n_2, n_3} of all possible Values of n_1, n_2, n_3

(ii) The probability that both the managers are idle

(iii) The probability that atleast one manager is idle.

Sol $\lambda = 2$

Service rate $\mu_1 = 2$ also $\mu_1 = \lambda$.

Deputy Manager $\mu_2 = 1$

Senior manager $\mu_3 = 3$

$\lambda_j = \sum^{3} \lambda_i p_{ij} \quad (j = 1, 2, 3)$

$P_{21} = 1 - P_{23} = 1 - \frac{1}{3} = \frac{2}{3}$

$P_{31} = 1$

W.K.T $\quad P_i = \frac{\lambda_i}{\mu_i}$

$P_1 = \frac{\lambda_1}{\mu_1} = \frac{\lambda_1}{2}$

$\lambda_1 = P_1 \lambda = 2P_1$

Illy $\quad \lambda_2 = P_2 \mu_2 = P_2$

$\lambda_3 = P_3 \mu_3 = 3P_3$

$$(2P_1, P_2, 3P_3) = (2P_1, P_2, 3P_3)\begin{pmatrix} 0 & 3/4 & 1/4 \\ 2/3 & 0 & 1/3 \\ 1 & 0 & 0 \end{pmatrix}$$

$$(2P_1 \quad P_2 \quad 3P_3) = \left[\frac{2}{3}P_2 + 3P_3 \quad \frac{3}{2}P_1 \quad \frac{1}{2}P_1 + \frac{1}{3}P_2\right]$$

$\frac{2}{3}P_2 + 3P_3 = 2P_1$

$\frac{3}{2}P_1 = P_2$

$\frac{1}{2}P_1 + \frac{1}{3}P_2 = 3P_3$

Assume $P_2 = 1$

$P_1 = \frac{2}{3}P_2$

$P_1 = \frac{2}{3}$

Sub $P_1 = \frac{2}{3}$ & $P_2 = 1$

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Now $P_{n_1,n_2,n_3} = D_N \dfrac{\rho_1^{n_1}}{a_1(n_1)} \rho_2^{n_2} \rho_3^{n_3}$

where $N = n_1 + n_2 + n_3 = 2$

$$D_N = \left[\sum_{n_1+n_2+n_3=2} \frac{\rho_1^{n_1}}{n_1!} \rho_2^{n_2} \rho_3^{n_3}\right]^{-1}, \text{ Since}$$

$$a_1(n_1) = \begin{cases} n_1! & \text{if } n_1 < C_1 \\ C_1!\, C_1^{n_1-C_1} & \text{if } n_1 \geq C_1 \end{cases}$$

$n_1 + n_2 + n_3 = 2$, $\rho_i = \dfrac{\lambda_i}{\mu_i}$

Hence $D_N = \left[\dfrac{\rho_1^2}{2!} \rho_2^0 \rho_3^0 + \dfrac{\rho_1^0}{0!} \rho_2^2 \rho_3^0 + \dfrac{\rho_1^0}{0!} \rho_2^0 \rho_3^2\right.$

$\left. + \dfrac{\rho_1^1}{1!} \rho_2^1 \rho_3^0 + \dfrac{\rho_1^1}{1!} \rho_2^0 \rho_3^1 + \dfrac{\rho_1^0}{0!} \rho_2^1 \rho_3^1\right)^{-1}$

$\because [\rho_2^0 = \rho_3^0 = \rho_1^0 = 1]$

$$= \left[\frac{\rho_1^2}{2!} + \rho_2^2 + \rho_3^2 + \rho_1 \rho_2 + \rho_1 \rho_3 + \rho_2 \rho_3\right]^{-1}$$

$$= \left[\left(\tfrac{2}{3}\right)^2\left(\tfrac{1}{2}\right) + (1)^2 + \left(\tfrac{2}{9}\right)^2 + \left(\tfrac{2}{3}\right)(1) + \left(\tfrac{2}{3}\right)\left(\tfrac{2}{9}\right) + (1)\left(\tfrac{2}{9}\right)\right]^{-1}$$

$$= \left[\frac{4}{18} + 1 + \frac{4}{81} + \frac{2}{3} + \frac{2}{3} \times \frac{2}{9} + 1 \times \frac{2}{9}\right]^{-1}$$

$$= \frac{81}{187}$$

$$P_{2,0,0} = 0.4332 \times \frac{(2/3)^2}{2!} 1^0 \left(\frac{2}{9}\right)^0$$

$$= 0.4332 \times \frac{4}{9} \times \frac{1}{2}$$

$$= 0.0962$$

$$P_{0,2,0} = 0.4332 \times \frac{(2/3)^0}{0!} (1)^2 \times \left(\frac{2}{9}\right)^0$$

$$= 0.4332 \times 1$$

$$= 0.4332$$

$$P_{0,0,2} = 0.4332 \times \frac{(2/3)^0}{0!} (1)^0 \times \left(\frac{2}{9}\right)^2$$

$$= 0.4332 \times 4/81$$

$$= 0.0214$$

$$P_{1,1,0} = 0.4332 \times \frac{(2/3)^1}{1!} (1)^1 \times \left(\frac{2}{9}\right)^0$$

$$= 0.4332 \times 2/3$$

$$= 0.2888$$

$$P_{1,0,1} = 0.4332 \times \frac{(2/3)^1}{1!} (1)^0 \times \left(\frac{2}{9}\right)^1$$

$$= 0.4332 \times 2/3 \times 2/9$$

$$= 0.0642$$

P(both the managers are idle) $= P_{2,0,0}$

$= 0.0962$

P(atleast one manager is idle) $= P_{1,1,0} + P_{1,0,1} + P_{2,0,0}$

$= 0.2888 + 0.0642 + 0.0962$

$= 0.4492$

Mean Value analysis

$L_S = \frac{\lambda}{\mu - \lambda}$, $W_S = \frac{1}{\mu - \lambda} \Rightarrow W_S = \frac{1 + L_S}{\mu}$

$W_i(N) = \frac{1 + L_i(N-1)}{\mu_i}$

$W_i(N) =$ mean waiting time at node i for the closed network containing 'N' Customers

$\mu_i =$ mean Service rate for the Single Server at node i

$L_i(N-1) =$ mean number of Customers at node i in a closed network with $(N-1)$ Customers.

Mean Value Algorithm for A. k-node, Single server-per-Node Network

Step 1 $\quad V_j = \sum_{i=1}^{k} V_i P_{ij} \quad (j = 1, 2 \ldots k)$

$V_j = \frac{\lambda_i}{\lambda_l}$ So that $V_l = 1$ for some l

(i) $W_i(r) = \dfrac{1 + L_i(r-1)}{\mu_i}$ $i = 1, 2 \ldots k$

(ii) $\lambda_l(r) = \dfrac{r}{\sum_{i=1}^{k} V_i W_i(r)}$ where $V_l = 1$

(iii) $\lambda_i(r) = V_i \lambda_l(r)$ $i = 1, 2 \ldots k,\ i \neq l$

(iv) $L_i(r) = \lambda_i(r) W_i(r)$ $i = 1, 2 \ldots k$

Mean Value analysis Algorithm for find $L_i(N)$ & $W_i(N)$ in a k-Node, Multiple-Server

$$p_i(m, N) = \frac{\lambda_i(N)}{\alpha_i(m)\,\mu_i}\, p_i(m-1, N-1), \text{ for } i \le m \le N-1$$

where $\alpha_i(m) = \dfrac{\alpha_i(m)}{\alpha_i(m-1)} = \begin{cases} m & \text{for } m \le c_i \\ c_i & \text{for } m \ge c_i \end{cases}$

Step 1 $V_j = \sum_{i=1}^{k} V_i p_{ij}$ $(j = 1, 2 \ldots k)$

Step 2 $L_i(0) = 0,\ p_i(0, 0) = p_i(m, 0) = 0$

Step 3 for $r = 1$ to N

(i) $W_i(r) = \dfrac{1}{c_i \mu_i}\left[1 + L_i(r-1) + \sum_{m=0}^{c_i-2}(c_i - 1 - m)\, p_i(m, r-1)\right]$

for $i = 1, 2 \ldots k$.

(ii) $\lambda_l(r) = \dfrac{r}{\sum_{i=1}^{k} V_i W_i(r)}$ where $V_l = r$

In a bank one clerk is processing applications of educational loans to students while processing the application he gets doubts according to an exponential distribution with mean '1/2'. To clarify his doubts he goes to the assistant manager with probability $\frac{2}{4}$ and to the manager with probability $\frac{1}{4}$. After discussing $3/4$ and to the manager ~~the clerk~~ with probability $1/4$. After discussing with the assistant manager with probability $1/3$ or else return to his seat. If the clarification of doubts by the assistant manager and the manager follows exponential distributions with parameters 1 and 3 respectively. without finding the probabilities find the average length's at the nodes by using mean value analysis algorithm

Soln Arrival rate $\lambda = 2$.

Service rate $\mu_1 = 2$

$\mu_2 = 1$

$\mu_3 = 3$

$$(\lambda_1, \lambda_2\, \lambda_3) = (\lambda_1\, \lambda_2\, \lambda_3)\begin{bmatrix} 0 & p_{12} & p_{13} \\ p_{21} & 0 & p_{23} \\ p_{31} & p_{32} & 0 \end{bmatrix} \quad (1)$$

$p_{12} = 3/4$, $p_{13} = \frac{1}{4}$, $p_{23} = \frac{1}{3}$, $p_{21} = 1 - p_{23} = 1 - \frac{1}{3} = \frac{2}{3}$

$p_{31} = 1$

$V_2 = 1$ & $\lambda_2 = 1$

$V_1 = \lambda_1$, $V_2 = \lambda_2 = 1$, $V_3 = \lambda_3$

Eqn ② becomes

$$(V_1 \; 1 \; V_3) = (V_1 \; 1 \; V_3) \begin{bmatrix} 0 & 3/4 & 1/4 \\ 2/3 & 0 & 1/3 \\ 1 & 0 & 0 \end{bmatrix}$$

$$V_1 = 2/3 + V_3, \quad 1 = 3/4 V_1 \Rightarrow V_1 = 4/3$$

Sub $V_1 = 4/3$

$$4/3 = 2/3 + V_3 \Rightarrow V_3 = 2/3$$

& $V_2 = 1$

Assuming $L_1(0) = L_2(0) = L_3(0) = 0$

$$W_i(r) = \frac{1 + L_i(r-1)}{\mu_i} \quad i = 1, 2 \ldots k$$

For $r = 1$, $W_1(1) = \frac{1}{\mu_1} = \frac{1}{2} = \frac{1}{2}$

$$W_2(1) = \frac{1}{\mu_2} = 1$$

$$W_3(1) = \frac{1}{\mu_3} = \frac{1}{3}$$

$$\lambda_l(r) = \frac{r}{\sum_{i=1}^{k} V_i W_i(r)} \quad \text{when } V_l = 1$$

For $r = 1$, $l = 2$ we get

$$= \frac{1}{\frac{4}{3} \times \frac{1}{2} + 1 \times 1 + \frac{2}{3} \times \frac{1}{3}}$$

$$= \frac{1}{\frac{2}{3} + 1 + \frac{2}{9}} = \frac{9}{17}$$

$\lambda_i(r) = V_i \lambda_l(r) \quad i = 1, 2 \ldots k \quad i \neq l.$

for $r=1$ we get

$\lambda_1(1) = V_1 \lambda_2(1)$

$= \frac{4}{3} \times \frac{9}{17}$

$= \frac{12}{17}$

$\lambda_3(1) = V_3 \lambda_2(1)$

$= \frac{2}{3} \times \frac{9}{17} = \frac{6}{17}$

$L_i(r) = \lambda_i(r) W_i(r) \quad i = 1, 2 \ldots k.$

for $r=1$, we get

$L_1(1) = \lambda_1(1) W_1(1)$

$= \frac{12}{17} \times \frac{1}{2} = \frac{6}{17}$

$L_2(1) = \lambda_2(1) W_2(1) = \frac{9}{17} \times 1 = \frac{9}{17}$

$L_3(1) = \lambda_3(1) W_3(1) = \frac{6}{17} \times \frac{1}{3} = \frac{2}{17}$

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