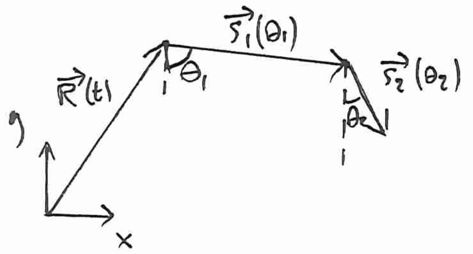

Definitions:

$$\vec{r}_1(\theta_1,t) = \vec{R}(t) + \vec{s}_1(\theta_1) \qquad (A)$$

$$\vec{r}_2(\theta_1,\theta_2,t) = \vec{r}_1(\theta_1,t) + \vec{s}_2(\theta_2) \qquad (B)$$

$$\vec{s}_i(\theta_i) = l_i \begin{bmatrix} \cancel{\cos}\ \sin\theta_i \\ -\cos\theta_i \end{bmatrix} \qquad (C)$$

Properties:

$$|\vec{s}_i(\theta_i)|^2 = l_i^2 \qquad (1)$$

$$\vec{s}_i{}'(\theta_i) = l_i \begin{bmatrix} \cos\theta_i \\ +\sin\theta_i \end{bmatrix} \Rightarrow |\vec{s}_i{}'(\theta_i)|^2 = l_i^2 \quad (2)\ ;\quad \vec{s}_i{}''(\theta_i) = -l_i \begin{bmatrix} \sin\theta_i \\ -\cos\theta_i \end{bmatrix} \Rightarrow |\vec{s}_i{}''(\theta_i)|^2 = l_i^2 \qquad (3)$$

$$\frac{d}{d\theta_0} |\vec{a}(\theta)|^2 = 2\, \vec{a}^{+}(\theta)\, \vec{a}\,'(\theta). \qquad (4)$$

$$\vec{s}_1^{+\prime}(\theta_1) \cdot \vec{s}_2{}'(\theta_2) = l_1 l_2 \cos(\theta_1 - \theta_2) \quad ; \quad \vec{s}_1^{+\prime\prime}(\theta_1) \cdot \vec{s}_2{}'(\theta_2) = l_1 l_2 \sin(\theta_2 - \theta_1) \qquad (5)$$

$$\dot{\vec{r}}_1(\theta_1,t) = \vec{V}(t) + \dot{\theta}_1 \vec{s}_1{}'(\theta_1)$$

$$\dot{\vec{r}}_1^{\,2}(\theta_1,t) = V^2(t) + l_1^2 \dot{\theta}_1^2 + 2\dot{\theta}_1 \vec{V}^{+}(t)\, \vec{s}_1{}'(\theta_1)$$

$$\dot{\vec{r}}_2^{\,2}(\theta_1,\theta_2,t) = \left(\dot{\vec{r}}_1(\theta_1,t) + \dot{\vec{s}}_2(\theta_2)\right)^2 =$$

$$= \dot{\vec{r}}_1^{\,2}(\theta_1,t) + \dot{\vec{s}}_2^{\,2}(\theta_2) + 2\dot{\vec{r}}_1^{\,+}(\theta_1,t)\, \dot{\vec{s}}_2(\theta_2)$$

$$= \dot{\vec{r}}_1^{\,2}(\theta_1,t) + l_2^2 \dot{\theta}_2^2 + 2\dot{\vec{r}}_1^{\,+}(\theta_1,t)\, \vec{s}_2{}'(\theta_2)\, \dot{\theta}_2$$

$$= V^2(t) + l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2\, \vec{s}_1^{+\prime}(\theta_1)\, \vec{s}_2{}'(\theta_2)\, \dot{\theta}_1\dot{\theta}_2 + 2\vec{V}^{+}(t)\, \vec{s}_2{}'(\theta_2)\dot{\theta}_2 + 2\vec{V}^{+}(t)\, \vec{s}_1{}'(\theta_1)\dot{\theta}_1$$

$$T = \frac{m_1+m_2}{2} V^2(t) + \frac{m_1+m_2}{2} l_1^2\dot{\theta}_1^2 + \frac{m_2}{2} l_2^2 \dot{\theta}_2^2 + m_2\, \vec{s}_1^{+\prime}(\theta_1)\, \vec{s}_2{}'(\theta_2)\dot{\theta}_1\dot{\theta}_2 +$$

$$+ (m_1+m_2)\vec{V}^{+}(t)\, \vec{s}_1{}'(\theta_1)\dot{\theta}_1 + m_2 \vec{V}^{+}(t)\, \vec{s}_2{}'(\theta_2)\, \dot{\theta}_2$$

$$V = m_1 g (R_x - l_1\cos\theta_1) + m_2 g (R_x - l_1\cos\theta_1 - l_2\cos\theta_2)$$

$$= (m_1+m_2) g (R_x - l_1\cos\theta_1) - m_2 g \cos\theta_2$$

$$\frac{\partial \mathcal{L}}{\partial \dot\theta_1} = \frac{\partial T}{\partial \dot\theta_1} = (m_1+m_2) l_1^2 \dot\theta_1 + m_2 \vec{s}_1^{\,+\prime}(\theta_1)\, \vec{s}_2^{\,\prime}(\theta_2)\, \dot\theta_2 + (m_1+m_2) \vec{V}^{+}(t)\, \vec{s}_1^{\,\prime}(\theta_1)$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot\theta_1}\right) = M l_1^2 \ddot\theta_1 + m_2 \vec{s}_1^{\,+\prime}(\theta_1)\, \vec{s}_2^{\,\prime}(\theta_2)\, \ddot\theta_2 + m_2 \vec{s}_1^{\,+\prime\prime}(\theta_1)\, \vec{s}_2^{\,\prime}(\theta_2)\, \dot\theta_1 \dot\theta_2 + m_2 \vec{s}_1^{\,+\prime}(\theta_1)\, \vec{s}_2^{\,\prime\prime}(\theta_2)\, \dot\theta_2^2 + M \vec{A}^{+}(t)\, \vec{s}_1^{\,\prime}(\theta_1) + M \vec{V}^{+}(t)\, \vec{s}_1^{\,\prime\prime}(\theta_1)\, \dot\theta_1$$

$$\frac{\partial \mathcal{L}}{\partial \theta_1} = m_2 \vec{s}_1^{\,+\prime\prime}(\theta_1)\, \vec{s}_2^{\,\prime}(\theta_2)\, \dot\theta_1 \dot\theta_2 + M \vec{V}^{+}(t)\, \vec{s}_1^{\,\prime\prime}(\theta_1)\, \dot\theta_1 \mp M g l_1 \sin\theta_1$$

$$\boxed{M l_1^2 \ddot\theta_1 + m_2 \vec{s}_1^{\,+\prime}(\theta_1)\, \vec{s}_2^{\,\prime}(\theta_2)\, \ddot\theta_2 + m_2 \vec{s}_1^{\,+\prime}(\theta_1)\, \vec{s}_2^{\,\prime\prime}(\theta_2)\, \dot\theta_2^2 + M g l_1 \sin\theta_1 = -M \vec{A}^{+}(t)\, \vec{s}_1^{\,\prime}(\theta_1)}$$

$$\frac{\partial \mathcal{L}}{\partial \dot\theta_2} = \frac{\partial T}{\partial \dot\theta_2} = m_2 l_2^2 \dot\theta_2 + m_2 \vec{s}_1^{\,+\prime}(\theta_1)\, \vec{s}_2^{\,\prime}(\theta_2)\, \dot\theta_1 + m_2 \vec{V}^{+}(t)\, \vec{s}_2^{\,\prime}(\theta_2)$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot\theta_2}\right) = m_2 l_2^2 \ddot\theta_2 + m_2 \vec{s}_1^{\,+\prime}(\theta_1)\, \vec{s}_2^{\,\prime}(\theta_2)\, \ddot\theta_1 + m_2 \vec{s}_1^{\,+\prime\prime}(\theta_1)\, \vec{s}_2^{\,\prime}(\theta_2)\, \dot\theta_1^2 + m_2 \vec{s}_1^{\,+\prime}(\theta_1)\, \vec{s}_2^{\,\prime\prime}(\theta_2)\, \dot\theta_1 \dot\theta_2 + m_2 \vec{A}^{+}(t)\, \vec{s}_2^{\,\prime}(\theta_2) + m_2 \vec{V}^{+}(t)\, \vec{s}_2^{\,\prime\prime}(\theta_2)\, \dot\theta_2$$

$$\frac{\partial \mathcal{L}}{\partial \theta_2} = m_2 \vec{s}_1^{\,+\prime}(\theta_1)\, \vec{s}_2^{\,\prime\prime}(\theta_2)\, \dot\theta_1 \dot\theta_2 + m_2 \vec{V}^{+}(t)\, \vec{s}_2^{\,\prime\prime}(\theta_2)\, \dot\theta_2 - m_2 g \sin\theta_2$$

$$\boxed{m_2 l_2^2 \ddot\theta_2 + m_2 \vec{s}_1^{\,+\prime\prime}(\theta_1)\, \vec{s}_2^{\,\prime}(\theta_2)\, \dot\theta_1^2 + m_2 \vec{s}_1^{\,+\prime}(\theta_1)\, \vec{s}_2^{\,\prime}(\theta_2)\, \ddot\theta_1 + m_2 g \sin\theta_2 = -m_2 \vec{A}^{+}(t)\, \vec{s}_2^{\,\prime}(\theta_2)}$$

$$\begin{bmatrix} M_o l_1^2 & m_2 \vec{s}_1^{\,+\prime}(\theta_1)\, \vec{s}_2^{\,\prime}(\theta_2) \\ m_2 \vec{s}_1^{\,+\prime}(\theta_1)\, \vec{s}_2^{\,\prime}(\theta_2) & m_2 l_2^2 \end{bmatrix} \begin{bmatrix} \ddot\theta_1 \\ \ddot\theta_2 \end{bmatrix} = \begin{bmatrix} -m_2 \vec{s}_1^{\,+\prime}(\theta_1)\, \vec{s}_2^{\,\prime\prime}(\theta_2)\, \dot\theta_2^2 - M g l_1 \sin\theta_1 - M \vec{A}^{+}(t)\, \vec{s}_1^{\,\prime}(\theta_1) \\ -m_2 \vec{s}_1^{\,+\prime\prime}(\theta_1)\, \vec{s}_2^{\,\prime}(\theta_2)\, \dot\theta_1^2 - m_2 g \sin\theta_2 - m_2 \vec{A}^{+}(t)\, \vec{s}_2^{\,\prime}(\theta_2) \end{bmatrix}$$

Using properties (5) and (6):

$$\begin{bmatrix} M l_1^2 & m_2 l_1 l_2 \cos(\theta_1-\theta_2) \\ m_2 l_1 l_2 \cos(\theta_1-\theta_2) & m_2 l_2^2 \end{bmatrix}\begin{bmatrix}\ddot\theta_1\\ \ddot\theta_2\end{bmatrix} = \begin{bmatrix} -m_2 l_1 l_2 \sin(\theta_1-\theta_2)\dot\theta_2^2 - M g l_1 \sin\theta_1 \\ -m_2 l_1 l_2 \sin(\theta_2-\theta_1)\dot\theta_1^2 - m_2 g \sin\theta_2 \end{bmatrix} +$$

$$+ \begin{bmatrix} -M l_1 \left(A_x(t)\cos\theta_1 + A_y(t)\sin\theta_1\right) \\ -m_2 l_2 \left(A_x(t)\cos\theta_2 + A_y(t)\sin\theta_1\right) \end{bmatrix}$$

$$\boxed{\begin{aligned}\Delta &= M m_2 l_1^2 l_2^2 - m_2^2 l_1^2 l_2^2 \cos^2(\theta_1-\theta_2) = \\ &= m_2 l_1^2 l_2^2 \left[M - m_2 \cos^2(\theta_1-\theta_2)\right] = \\ &= m_2 l_1^2 l_2^2 \left[m_1 + m_2 \sin^2(\theta_1-\theta_2)\right]\end{aligned}}$$

$$l_1 l_2 \left(\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2\right) = l_1 l_2 \cos(\theta_1-\theta_2)$$

$$l_1 l_2 \begin{bmatrix} -s_1 & c_1 \end{bmatrix}\begin{bmatrix} c_2 \\ s_2 \end{bmatrix} = + l_1 l_2 \left(c_1 s_2 - c_2 s_1\right) = - l_1 l_2 \left(s_1 c_2 - s_2 c_1\right) =$$

$$= - l_1 l_2 \sin(\theta_1-\theta_2)$$

$$\begin{bmatrix}\ddot\theta_1\\ \ddot\theta_2\end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} \underbrace{m_2 l_2^2}_{a} & \underbrace{-m_2 l_1 l_2 \cos(\theta_1-\theta_2)}_{b} \\ \underbrace{-m_2 l_1 l_2 \cos(\theta_1-\theta_2)}_{b} & \underbrace{M l_1^2}_{d} \end{bmatrix}\begin{bmatrix} \overbrace{-m_2 l_1 l_2 \sin(\theta_1-\theta_2)\dot\theta_2^2 - M g l_1 \sin\theta_1 - M l_1 (A_x c_1 + A_y s_1)}^{F_1} \\ \underbrace{-m_2 l_1 l_2 \sin(\theta_1-\theta_2)\dot\theta_1^2 - m_2 g \sin\theta_2 - m_2 l_2 (A_x c_2 + A_y s_2)}_{F_2} \end{bmatrix}$$

$$\begin{bmatrix}\dot\theta_1\\ \dot\omega_1 \\ \dot\theta_2 \\ \dot\omega_2\end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{a}{\Delta} & 0 & \frac{b}{\Delta} \\ 0 & 0 & 1 & 0 \\ 0 & \frac{b}{\Delta} & 0 & \frac{d}{\Delta} \end{bmatrix}\begin{bmatrix}\omega_1 \\ F_1 \\ \omega_2 \\ F_2\end{bmatrix}$$